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Categorical Abstract Algebraic Logic: Leibniz Equality and Homomorphism Theorems

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Abstract The study of structure systems, an abstraction of the concept of first-order structures, is continued. Structure systems have algebraic systems rather than universal algebras as their algebraic reducts. Moreover, their relational component consists of a collection of relation systems on the underlying functors rather than simply a system of relations on a single set. Congruence systems of structure systems are introduced and the Leibniz congruence system of a structure system is defined. Analogs of the Homomorphism, the Second Isomorphism and the Correspondence Theorems of Universal Algebra are provided in this more abstract context. These results generalize corresponding results of Elgueta for equality-free first-order logic. Finally, a version of Gödel's Completeness Theorem is provided with reference to structure systems.

Key words structure systems · congruence systems · Leibniz congruence systems · strong system morphisms · homomorphism theorem · second isomorphism theorem · correspondence theorem · Gödel's completeness theorem

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1 Introduction

The theory of logical matrices plays a major role in the model theory of sentential logics. It is particularly important in the context of abstract algebraic logic, since different classes of sentential logics in the abstract algebraic hierarchy of logics, also known as the Leibniz hierarchy, may be characterized by closure properties of their

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matrix model classes. For an overview of results in this area the reader may consult [20] or the more comprehensive treatise [8]. On the other hand, due to Bloom's work [3] (see also an exposition in [8]), the theory of logical matrices may be perceived as part of the theory of equality-free first-order logic and, more specifically, of universal Horn logic without equality. It is this connection that led Raimon Elgueta [14–17] (and, in part, in joint work with Czelakowski [9] and with Jansana [18]) and Pillar Dellunde [10, 11] (and, in part, jointly with Casanovas and Jansana [6] and with Jansana [12]) to consider, in the context of abstract algebraic logic, first-order logic without equality and its model theory.

Since a short summary of the work of Elgueta in [14] was given in the Introduction to [31], which is a precursor to the present paper, it will not be repeated here. Suffices it to say that the present work has been inspired by the aforementioned work of Elgueta and of Dellunde and that the focus here is the part of the work dealing with defining congruences on first-order structures, especially Leibniz congruences and in proving analogs of the homomorphism theorems of Universal Algebra in this more abstract setting. In what follows we describe, on the one hand, the direction of the present work as compared to that of the work of Elgueta and, on the other, give a brief summary of the work presented in [31], which is the basis for the present work and almost all of whose notation will be adopted and used throughout the present work.

In Voutsadakis [26–28], parts of the theory of algebraizability of sentential logics based on the Tarski operator (see, e.g., [19]) were generalized to cover the case of logics formulated as π-institutions. In subsequent work [29] the well-known theory of the Leibniz operator (see [2]) was also abstracted to the level of π-institutions. It has been made clear, especially from the treatment in [28], that the role that algebras play in the theory of algebraizability of sentential logics is replaced in this more abstract framework by algebraic systems. An algebraic system is a set-valued functor endowed with a category of natural transformations on the functor, which gives the functor a distinctive algebraic flavor. Following this work, another major part of the theory of abstract algebraic logic, this time the one pertaining to the algebraizability of the connective of logical implication rather than that of logical equivalence as presented in [25], was also abstracted [30] so that partially ordered algebraic systems instead of partially ordered algebras were at the focus. Partially ordered algebraic systems are of course algebraic systems in the preceding sense endowed with partial ordering systems, i.e., collections of partial orderings that are preserved by signature changing morphisms. These systems are no longer special cases of first-order structures, as are the partially ordered algebras of [25]. But first-ordered structures were generalized in [31] to what were called structure systems, abstractions of first-order structures, and, now, partially-ordered algebraic systems are indeed special cases of structure systems. These successive abstractions led to the detailed study of structure systems along the lines of the study of equality-free first-order structures carried out by Elgueta and Dellunde in their work mentioned in the previous paragraph.

Recall from [31] that a (structure system) language $\mathcal{L} = \langle \mathbf{F}, R, \rho \rangle$ consists of a category $\mathbf{F}$ of natural transformations on a given sentence functor SEN, a nonempty collection R of relation symbols and an arity function $\rho : R \to \omega$ giving the arity of a relation symbol in R. Then, an $\mathcal{L}$-term over a denumerable set of variables V is constructed out of the natural transformations in $\mathbf{F}$, viewed as fundamental operation symbols, and the variables in V in the ordinary way. The collection of all these terms is denoted by $\mathrm{Te}_{\mathcal{L}}(V)$. Note the difference between this definition and the one for $\mathcal{L}$-terms introduced in [31]. Because the category $\mathbf{F}$ is an entire clone of natural transformations, this difference is only superficial as will be detailed upon at the beginning of Section 3. Now, an $\mathcal{L}$-formula over a given denumerable set of variables V is constructed in the usual way, using the relation symbols in R and the terms in $\mathrm{Te}_{\mathcal{L}}(V)$. The models used to study this syntactic framework are $\mathcal{L}$-(structure) systems. An $\mathcal{L}$-structure system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ consists of a functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Set}$, a category $\mathbf{N}^{\mathfrak{A}}$ of natural transformations on $\mathrm{SEN}^{\mathfrak{A}}$ and a surjective functor $F^{\mathfrak{A}} : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}}$, that preserves all projections and, as a result, preserves also the arity of all natural transformations.

Using these basic concepts, in [31] the notions of a subsystem, of a filter extension, of an $\mathcal{L}$-morphism and of a reduced product of $\mathcal{L}$-systems are introduced. The reader should consult Section 3 of [31] for precise definitions and also [14] to find out more about the paradigms used from equality-free first-order logic. The final and main section of [31] deals with the formulation and proof of an analog of Łos' Ultraproduct Theorem in the context of $\mathcal{L}$-systems.

Having briefly described some basic concepts and the contents of [31], we turn now to a short overview of the contents of the present paper. In Section 2, congruence systems of structure systems are introduced, generalizing both the concept of a congruence system on a sentence functor endowed with a category of natural transformations from [26] and the concept of a congruence of an equality-free first-order structure from [14]. In Section 3, the Leibniz congruence system of a structure system is defined, abstracting both the Leibniz congruence system associated with a theory system of a given π-institution from [29] and the Leibniz congruence of a given equality-free first-order structure from [14]. In Section 4 the notion of a quotient system of a given structure system is introduced. Both quotient sentence functors from [26] and quotient equality-free first-order structures from [14] are special cases of this more general construction. Also in Section 4 the main theorems of the paper, analogs of the Homomorphism, the Second Isomorphism and the Correspondence Theorem of Universal algebra, abstracting generalized versions in [14], are presented and proven. Section 5 deals with reduced structure systems, which are those structure systems that are reduced modulo their Leibniz congruence systems and, as a consequence, have themselves identity Leibniz congruence systems. Finally, Section 6 presents a version of Gödel's Completeness Theorem dealing with the concept of logical implication based on structure systems rather than on first-order structures.

Further work is to follow the current developments. Many of the issues and results on the model theory of equality-free first-order logic, as investigated by Elgueta, are currently under investigation in the framework of structure systems.

For general concepts and notation from category theory the reader is referred to any of [1, 4, 22]. For an overview of the current state of affairs in abstract algebraic logic the review article [20], the monograph [19] and the book [8] are all excellent references. To follow recent developments on the categorical side of the subject the reader may refer to the series of papers [26–29] (see also additional references therein). Standard references on model theory are the books by Chang and Keisler [7], Hodges [21], Marker [23] and Doets [13], whereas references for Universal Algebra, both of which include the Homomorphism Theorem, the Second

Isomorphism Theorem and the Correspondence Theorem, are the books by Burris and Sankappanavar [5] and by McKenzie et al. [24].

2 Congruence Systems of Structure Systems

Recall from [31] that a *clone category* is a category $\mathbf{F}$, with objects all finite natural numbers, that is isomorphic to the category of natural transformations $\mathbf{N}$ on a given functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ (see [29] for the definition of a category of natural transformations on a set-valued functor) via an isomorphism that preserves projections, and, as a consequence, also preserves objects. A (*structure system*) *language* is a triple $\mathcal{L} = \langle \mathbf{F}, R, \rho \rangle$, where $\mathbf{F}$ is a clone category, R is a nonempty set of relation symbols and $\rho : R \to \omega$ is an arity function. An $\mathcal{L}$- (*structure*) *system* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ is a triple consisting of

- A functor $\mathrm{SEN}^{\mathfrak{A}} : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Set}$,
- a category of natural transformations $\mathbf{N}^{\mathfrak{A}}$ on $\mathrm{SEN}^{\mathfrak{A}}$, such that $F : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}}$ is a surjective functor that preserves all projections $p^{kl} : k \to 1, k \in \omega, l < k$, and
- $R^{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$ a family of relation systems on $\mathrm{SEN}^{\mathfrak{A}}$ indexed by R, such that $r^{\mathfrak{A}}$ is n-ary if $\rho(r) = n$.

Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-system. A (binary) relation system $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ is said to be an $\mathbf{N}^{\mathfrak{A}}$- *congruence system of* $\mathfrak{A}$ if θ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$ that satisfies, for all $r \in R^{\mathfrak{A}}$, with $\rho(r) = n$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi} \in r^{\mathfrak{A}}_\Sigma \quad \text{and} \quad \vec{\phi}\, \theta^n_\Sigma\, \vec{\psi} \quad \text{imply} \quad \vec{\psi} \in r^{\mathfrak{A}}_\Sigma.$$

The collection of all $\mathbf{N}^{\mathfrak{A}}$-congruence systems on $\mathfrak{A}$ will be denoted by $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.

By the definition of $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, we obviously have $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \subseteq \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$. The following proposition forms an analog of Proposition 2.1 of [14] and describes $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ as it sits inside the larger class $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$.

It should also be mentioned that Proposition 1 generalizes Proposition 2.3 of [29] in the same sense that its precursor, Proposition 2.1 of [14], generalizes the classical result of Blok and Pigozzi [2].

Proposition 1 For every $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, the partially ordered set $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) = \langle \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}), \leq \rangle$ is a principal ideal of the complete lattice $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$.

Proof It is clear that, if $\theta \leq \eta$ and $\eta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, then also $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Thus, it suffices to show that, for any collection $\{\theta^i\}_{i \in I}$ of $\mathbf{N}^{\mathfrak{A}}$-congruences of $\mathfrak{A}$, their join $\bigvee_{i \in I} \theta^i$ in $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$ is in $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.

Indeed, the join $\bigvee_{i \in I} \theta^i$ is computed signature-wise, where, for each signature $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\bigvee_{i \in I} \theta^i_\Sigma$ is given in the usual universal algebraic way. Therefore, the compatibility of $\bigvee_{i \in I} \theta^i_\Sigma$ with $r^{\mathfrak{A}}_\Sigma$ follows by the compatibility of $\theta^i_\Sigma, i \in I$, with $r^{\mathfrak{A}}_\Sigma$, for every $r \in R$. This holding for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ shows that $\bigvee_{i \in I} \theta^i$ is compatible with all $r^{\mathfrak{A}}$, for $r \in R$, i.e., that $\bigvee_{i \in I} \theta^i$ indeed belongs to $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. □

The largest $\mathbf{N}^{\mathfrak{A}}$-congruence system of the system $\mathfrak{A}$ will be denoted by $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. With that notation, Proposition 1 says that

$$\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) = \left\{\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}}) : \theta \leq \Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})\right\}.$$

Several observations may be made about $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.

– If, for some relation symbol $r \in R$, $r^{\mathfrak{A}} \neq \nabla^{\mathrm{SEN}^{\mathfrak{A}}}$ and $r^{\mathfrak{A}} \neq \emptyset$, then $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \neq \nabla^{\mathrm{SEN}^{\mathfrak{A}}}$.
– If $\mathfrak{A}$ contains a binary relation system that satisfies the axioms of equality, then $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ coincides with that relation system.

A system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ is said to be $\mathbf{N}^{\mathfrak{A}}$- *reduced* if

$$\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) = \left\{\Delta^{\mathrm{SEN}^{\mathfrak{A}}}\right\}.$$

This is equivalent to $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) = \Delta^{\mathrm{SEN}^{\mathfrak{A}}}$.

Notice, also, that, if $\mathcal{L}$ has equality, then any $\mathcal{L}$-system is reduced.

Recall from [31] the notion of an $\mathcal{L}$-system morphism $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ from an $\mathcal{L}$-system $\mathfrak{A}$ to an $\mathcal{L}$-system $\mathfrak{B}$ and that of a strict (called strong in [31]; the two terms will be used interchangeably) $\mathcal{L}$-system morphism $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$. If $\mathcal{L}$ is clear from context, the terms system morphism and strict system morphism are sometimes used in place of $\mathcal{L}$-system morphism and strict $\mathcal{L}$-system morphism, respectively.

Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ are two $\mathcal{L}$-systems and $\langle F, \alpha\rangle : \mathfrak{A} \to \mathfrak{B}$ is an $\mathcal{L}$-system morphism. The *kernel* of $\langle F, \alpha\rangle$ is defined by

$$\mathrm{Ker}(\langle F, \alpha\rangle) = \alpha^{-1}\left(\Delta^{\mathrm{SEN}^{\mathfrak{B}}}\right) = \left\{\alpha_\Sigma^{-1}\left(\Delta^{\mathrm{SEN}^{\mathfrak{B}}}_{F(\Sigma)}\right)\right\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}.$$

A characterization of the notion of a congruence system in terms of the notion of kernel of system morphisms is given now. The first lemma provides the first part of this characterization. It forms an analog of Lemma 2.2 of [14] in the context of structure systems.

Lemma 2 Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-structure systems and $\langle F, \alpha\rangle : \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle\rangle \to \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle\rangle$ an algebraic $\mathcal{L}$-morphism. If $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$, then $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Conversely, if $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, and $\alpha_\Sigma(r^{\mathfrak{A}}_\Sigma) = r^{\mathfrak{B}}_{F(\Sigma)}$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $r \in R$, then $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$.

Proof Essentially as in the proof of Proposition 26 of [26] we may show that $\mathrm{Ker}(\langle F, \alpha\rangle)$ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$. So it suffices to show that if $\langle F, \alpha\rangle$ is strict, then $\mathrm{Ker}(\langle F, \alpha\rangle)$ is also an $\mathbf{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$. To this end, suppose that $r \in R$, such that $\rho(r) = n$, and $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\phi} \in r^{\mathfrak{A}}_\Sigma$ and $\vec{\phi}\ \mathrm{Ker}(\langle F, \alpha\rangle)^n_\Sigma\ \vec{\psi}$. Then $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$ and $\alpha_\Sigma(\vec{\phi}) = \alpha_\Sigma(\vec{\psi})$. Therefore $\alpha_\Sigma(\vec{\psi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$, whence, by the strictness of $\langle F, \alpha\rangle$, we obtain that $\vec{\psi} \in r^{\mathfrak{A}}_\Sigma$. This proves that $\mathrm{Ker}(\langle F, \alpha\rangle)$ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$.

Suppose, conversely, that $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, and $\alpha_\Sigma(r^{\mathfrak{A}}_\Sigma) = r^{\mathfrak{B}}_{F(\Sigma)}$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $r \in R$.

If $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma}$, then $\alpha_\Sigma(\vec{\phi}) \in \alpha_\Sigma(r^{\mathfrak{A}}_{\Sigma}) = r^{\mathfrak{B}}_{F(\Sigma)}$.

Conversely, if $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$, then $\alpha_\Sigma(\vec{\phi}) \in \alpha_\Sigma(r^{\mathfrak{A}}_{\Sigma})$, whence, there exists $\vec{\psi} \in r^{\mathfrak{A}}_{\Sigma}$, such that $\alpha_\Sigma(\vec{\phi}) = \alpha_\Sigma(\vec{\psi})$. Hence, by the compatibility of $\mathrm{Ker}(\langle F, \alpha\rangle)$ with $r^{\mathfrak{A}}$, we get that $\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma}$.

Thus $\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma}$ if and only if $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$ showing that $\langle F, \alpha\rangle$ is a strict structure morphism. □

The following lemma reveals a relationship between the congruence systems of a given system and those of a subsystem. It forms an analog for systems of Lemma 2.3 of [14].

Lemma 3 Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ be an $\mathcal{L}$-system and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle \subseteq \mathfrak{A}$. For every binary relation system $\theta^{\mathfrak{A}}$ on $\mathrm{SEN}^{\mathfrak{A}}$, define $\theta^{\mathfrak{B}} = \{\theta^{\mathfrak{B}}_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{B}}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{B}}|$,

$$\theta^{\mathfrak{B}}_{\Sigma} := \theta^{\mathfrak{A}}_{\Sigma} \cap \nabla^{\mathrm{SEN}^{\mathfrak{B}}}_{\Sigma}.$$

Then $\theta^{\mathfrak{A}} \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ implies $\theta^{\mathfrak{B}} \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$.

Proof By the definition of $\theta^{\mathfrak{B}}$ and that of a subfunctor, it is fairly easy to see that $\theta^{\mathfrak{B}}$ is an $\mathbf{N}^{\mathfrak{B}}$-congruence system on $\mathrm{SEN}^{\mathfrak{B}}$. Now the compatibility of $\theta^{\mathfrak{A}}$ with $r^{\mathfrak{A}}$ together with the definition of a subsystem yield that $\theta^{\mathfrak{B}}$ is compatible with $r^{\mathfrak{B}}$, for every $r \in R$. Therefore $\theta^{\mathfrak{B}}$ is indeed an $\mathbf{N}^{\mathfrak{B}}$-congruence system of $\mathfrak{B}$. □

In the next lemma, it is shown, roughly speaking, that strict $\mathcal{L}$-system morphisms carry congruence systems on their codomain to congruence systems on their domain and that reductive $\mathcal{L}$-system morphisms, with isomorphic functor components, carry congruence systems on their domain that include their kernel to congruence systems on their codomain. This result forms an analog of Lemma 2.4 of [14] in the context of $\mathcal{L}$-systems.

Lemma 4 Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-structure systems and suppose that $\langle F, \alpha\rangle : \mathfrak{A} \to_s \mathfrak{B}$ is a strict system morphism.

1. If $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$, then $\alpha^{-1}(\theta) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.
2. If, moreover, $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive morphism, with F an isomorphism, then $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ and $\mathrm{Ker}(\langle F, \alpha\rangle) \leq \theta$ imply that $\alpha(\theta) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$.

Proof

1. It is not difficult to verify that, if $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{B}}}(\mathrm{SEN}^{\mathfrak{B}})$, then $\alpha^{-1}(\theta) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$. Suppose, in addition, that $\theta^{\mathfrak{B}}$ is compatible with $r^{\mathfrak{B}}$, for all $r \in R$. Now, let $r \in R$, $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\phi} \in r^{\mathfrak{A}}_{\Sigma}$ and $\vec{\phi}\alpha^{-1}_{\Sigma}(\theta^{\mathfrak{B}}_{F(\Sigma)})\vec{\psi}$. Then, we have that $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$ and $\alpha_\Sigma(\vec{\phi})\theta^{\mathfrak{B}}_{F(\Sigma)}\alpha_\Sigma(\vec{\psi})$. Hence, by the compatibility of $\theta^{\mathfrak{B}}$, we get that $\alpha_\Sigma(\vec{\psi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$. Therefore, by the strictness of $\langle F, \alpha\rangle$, we obtain that $\vec{\psi} \in r^{\mathfrak{A}}_{\Sigma}$. Hence $\theta^{\mathfrak{A}}$ is also compatible with $r^{\mathfrak{A}}$, for all $r \in R$, implying that $\theta^{\mathfrak{A}} \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.

2. We leave up to the reader in this case also the fact that, if $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$, then $\alpha(\theta) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{B}})$. As in the proof of Lemma 2.4 of [14], the hypothesis $\mathrm{Ker}(\langle F, \alpha\rangle) \leq \theta$ is used to show the transitivity of the binary relation system $\alpha(\theta)$. To show that $\alpha(\theta)$ is an $\mathbf{N}^{\mathfrak{B}}$-congruence system of $\mathfrak{B}$, suppose that $r \in R$, with $\rho(r) = n$, and by surjectivity, that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$ and $\vec{\phi}\theta^{\mathfrak{A}}_\Sigma\vec{\psi}$. Thus, by strictness, $\vec{\phi} \in r^{\mathfrak{A}}_\Sigma$ and $\vec{\phi}\theta^{\mathfrak{A}}_\Sigma\vec{\psi}$, whence, by the compatibility of $\theta^{\mathfrak{A}}$ with $r^{\mathfrak{A}}$, we get that $\vec{\psi} \in r^{\mathfrak{A}}_\Sigma$, showing that $\alpha_\Sigma(\vec{\psi}) \in r^{\mathfrak{B}}_{F(\Sigma)}$. Thus $\alpha_\Sigma(\theta^{\mathfrak{A}}_\Sigma)$ is in fact compatible with $r^{\mathfrak{B}}_{F(\Sigma)}$, for all $r \in R$ and all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. □

Finally, in an analog of Theorem 2.5 of [14], it is shown that, a reductive $\mathcal{L}$-system morphism, with an isomorphic functor component, carries largest congruence systems to largest congruence systems in both directions, i.e., more specifically, it carries the largest congruence system on its domain to the largest congruence system on its codomain and vice-versa.

Theorem 5 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-structure systems. If $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive system morphism, with F an isomorphism, then the following hold:*

1. $\alpha^{-1}(\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})) = \Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.
2. $\alpha(\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})) = \Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$.

Proof

1. We have, by Part 1 of Lemma 4, that $\alpha^{-1}(\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. It suffices to show that it is in fact the largest $\mathbf{N}^{\mathfrak{A}}$-congruence system in $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Suppose to this end, that $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. By Lemma 2, we also have that $\mathrm{Ker}(\langle F, \alpha\rangle) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Thus, we may consider the join $\theta' = \theta \vee \mathrm{Ker}(\langle F, \alpha\rangle)$ in $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Now, by Part 2 of Lemma 4, we have that $\alpha(\theta') \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$. Hence $\alpha(\theta) \leq \alpha(\theta') \leq \Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$. This implies that $\theta \leq \alpha^{-1}(\alpha(\theta)) \leq \alpha^{-1}(\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}))$, showing that $\alpha^{-1}(\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}))$ is indeed the largest $\mathbf{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$ and, therefore, that $\alpha^{-1}(\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})) = \Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.
2. For this part, notice, first, that, by Part 2 of Lemma 4, $\alpha(\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}))$ is an $\mathbf{N}^{\mathfrak{B}}$-congruence system on $\mathfrak{B}$. Thus, it suffices to show that it is the largest $\mathbf{N}^{\mathfrak{B}}$-congruence system on $\mathfrak{B}$. Let, to this end, $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})$. Then, by Part 1 of Lemma 4, $\alpha^{-1}(\theta) \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, and, therefore, $\alpha^{-1}(\theta) \leq \Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Thus, using the surjectivity of $\langle F, \alpha\rangle$, we get that $\theta = \alpha(\alpha^{-1}(\theta)) \leq \alpha(\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}))$. Hence $\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}) = \alpha(\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}))$. □

3 Leibniz Equality

In this section, following the work of Elgueta [14], it is shown that the concept of the largest congruence system of a given structure system is the algebraic counterpart of the logical concept of equality in the sense of Leibniz.

We use the notation and terminology of [31], which followed that of [14], except that instead of considering terms of the form $t \in \mathbf{F}(n, 1)$, we endow the language with a countable collection of variables $V = \{v_0, v_1, v_2, \ldots\}$, that will be denoted metatheoretically, as customary, by $x, y, z, \ldots$, and we consider terms of the form $t(\vec{x})$, for $t \in \mathbf{F}(n, 1)$ and $\vec{x}$ a vector consisting of n variables. The quantifiers will refer, accordingly, to the variables instead of to the place in the term as was the case in [31]. Let us mention that the seemingly more rigid framework of [31] does not induce loss of generality because one may change the number of arguments of a given natural transformation by introducing 'dummy' variables and, as a result, may refer to the same argument by two different places in two different, but essentially equivalent, natural transformations.

With this notation, a *Leibniz formula* over $\mathcal{L}$ or a *Leibniz $\mathcal{L}$-formula* is a formula of the form $\beta(x, y)$ with two free variables, such that, for some atomic formula $\gamma(x, \vec{z})$ with at least one free variable x,

$$\beta(x, y) := (\forall \vec{z})(\gamma(x, \vec{z}) \leftrightarrow \gamma(y, \vec{z})),$$

where by $(\forall \vec{z})$ is denoted the string of universal quantifications $(\forall z_0) \ldots (\forall z_{k-1})$, where k is the length of the vector $\vec{z}$.

We are now ready to prove an analog of Theorem 2.6 of [14] to the effect that, for every $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, every $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, ϕ and ψ are in the same Σ-class modulo $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ iff all their images under $\mathbf{Sign}^{\mathfrak{A}}$-morphisms satisfy all Leibniz $\mathcal{L}$-formulas in $\mathfrak{A}$.

Theorem 6 *Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-system, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. Then $\langle \phi, \psi \rangle \in \Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma}(\mathfrak{A})$ if and only if, for all Leibniz $\mathcal{L}$-formulas $\beta(x, y)$*

$$\begin{array}{l}\mathfrak{A} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)],\\ \textit{for all } \Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|,\ f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma').\end{array}$$

Proof Define the relation system $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$\begin{array}{rl}\theta_\Sigma = \{\langle \phi, \psi \rangle : & \mathfrak{A} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)],\\ & \text{for all } \Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|,\ f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma'), \beta(x, y) \text{ Leibniz}\}.\end{array}$$

It is not difficult to see that θ_Σ is an equivalence relation on $\mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$. To see that it is also an $\mathbf{N}^{\mathfrak{A}}$-congruence, we use the method used in the proof of Theorem 2.6 of [14]. Suppose that $\sigma : (\mathrm{SEN}^{\mathfrak{A}})^n \to \mathrm{SEN}^{\mathfrak{A}}$ is a natural transformation in $\mathbf{N}^{\mathfrak{A}}$ and that $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\phi}\ \theta^n_\Sigma\ \vec{\psi}$. By the transitivity of θ_Σ, it suffices to prove that, for all $i < n$,

$$\sigma_\Sigma(\psi_0, \ldots, \psi_{i-1}, \phi_i, \ldots, \phi_{n-1})\ \theta_\Sigma\ \sigma_\Sigma(\psi_0, \ldots, \psi_i, \phi_{i+1}, \ldots, \phi_{n-1}). \qquad (1)$$

Consider, to this end, a Leibniz $\mathcal{L}$-formula $\beta(x, y)$. Let $w_0, \ldots, w_{n-1}$ be a list of distinct variables not in β. Let δ be the formula that results from β by simultaneously substituting $\sigma(w_0, \ldots, w_{i-1}, x, w_{i+1}, \ldots, w_{n-1})$ for x and $\sigma(w_0, \ldots, w_{i-1}, y, w_{i+1}, \ldots, w_{n-1})$ for y in β. Then, we have that

$$\begin{array}{l}\mathfrak{A} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}^{\mathfrak{A}}(f)(\sigma_\Sigma(\psi_0, \ldots, \psi_{i-1}, \phi_i, \ldots, \phi_{n-1})),\\ \qquad\qquad \mathrm{SEN}^{\mathfrak{A}}(f)(\sigma_\Sigma(\psi_0, \ldots, \psi_i, \phi_{i+1}, \ldots, \phi_{n-1}))]\\ \text{if and only if}\\ \mathfrak{A} \models_{\Sigma'} \delta(x, y, w_0, \ldots, w_{n-1})[\mathrm{SEN}^{\mathfrak{A}}(f)(\psi_i), \mathrm{SEN}^{\mathfrak{A}}(f)(\phi_i), \mathrm{SEN}^{\mathfrak{A}}(f)(\psi_0),\\ \qquad \ldots, \mathrm{SEN}^{\mathfrak{A}}(f)(\psi_{i-1}), \mathrm{SEN}^{\mathfrak{A}}(f)(\phi_{i+1}), \ldots, \mathrm{SEN}^{\mathfrak{A}}(f)(\phi_{n-1})],\end{array}$$

which is true, since $(\forall\vec{w})\delta(x, y, \vec{w})$ is a Leibniz $\mathcal{L}$-formula and $\langle\phi_i, \psi_i\rangle \in \theta_\Sigma$. Thus Condition (1) has been proved, which shows that, in fact, θ_Σ is an $\mathbf{N}^{\mathfrak{A}}$-congruence on $\mathrm{SEN}^{\mathfrak{A}}(\Sigma)$.

Next, to show that θ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1, \Sigma_2)$ and $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)$, such that $\langle\phi, \psi\rangle \in \theta_{\Sigma_1}$. Thus, for all $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, $g \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1, \Sigma')$, and all Leibniz $\mathcal{L}$-formulas $\beta(x, y)$, we have

$$\mathfrak{A} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}^{\mathfrak{A}}(g)(\phi), \mathrm{SEN}^{\mathfrak{A}}(g)(\psi)].$$

$$\begin{tikzcd}
\Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\
& \Sigma' &
\end{tikzcd}$$

Thus, for all $h \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_2, \Sigma')$, we get that

$$\mathfrak{A} \models_{\Sigma'} \beta(x, y)\left[\mathrm{SEN}^{\mathfrak{A}}(hf)(\phi), \mathrm{SEN}^{\mathfrak{A}}(hf)(\psi)\right],$$

whence, for all $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, $h \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_2, \Sigma')$,

$$\mathfrak{A} \models_{\Sigma'} \beta(x, y)\left[\mathrm{SEN}^{\mathfrak{A}}(h)(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi)), \mathrm{SEN}^{\mathfrak{A}}(h)(\mathrm{SEN}^{\mathfrak{A}}(f)(\psi))\right],$$

and, therefore, $\langle\mathrm{SEN}^{\mathfrak{A}}(f)(\phi), \mathrm{SEN}^{\mathfrak{A}}(f)(\psi)\rangle \in \theta_{\Sigma_2}$, i.e., θ is indeed an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathrm{SEN}^{\mathfrak{A}}$.

Finally, to see that θ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\vec{\phi} \in r^{\mathfrak{A}}_\Sigma, \vec{\phi}\ \theta^n_\Sigma\ \vec{\psi}$. Consider the atomic formula

$$\gamma(x, \vec{z}) = r(z_0, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_{n-1})$$

and let, as before, $\beta(x, y) = (\forall\vec{z})(\gamma(x, \vec{z}) \leftrightarrow \gamma(y, \vec{z}))$. Then, since $\langle\phi_i, \psi_i\rangle \in \theta_\Sigma$, we have that, for every $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$,

$$\mathfrak{A} \models_{\Sigma'} \beta(x, y)\left[\mathrm{SEN}^{\mathfrak{A}}(f)(\phi_i), \mathrm{SEN}^{\mathfrak{A}}(f)(\psi_i)\right].$$

Therefore

$$\langle\psi_0, \ldots, \psi_{i-1}, \phi_i, \ldots, \phi_{n-1}\rangle \in r^{\mathfrak{A}}_\Sigma \text{ iff } \langle\psi_0, \ldots, \psi_i, \phi_{i+1}, \ldots, \phi_{n-1}\rangle \in r^{\mathfrak{A}}_\Sigma.$$

This holding for all $i < n$ yields that $\vec{\phi} \in r^{\mathfrak{A}}_\Sigma$ if and only if $\vec{\psi} \in r^{\mathfrak{A}}_\Sigma$, whence θ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$. And $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ being the largest $\mathbf{N}^{\mathfrak{A}}$-congruence system of $\mathfrak{A}$ yields that $\theta \leq \Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$.

To show the system of reverse inclusions, suppose that $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\langle\phi, \psi\rangle \in \Omega^{\mathbf{N}^{\mathfrak{A}}}_\Sigma(\mathfrak{A})$. Let, also $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma')^{n-1}$. If $t_0(x, \vec{z}), \ldots, t_{k-1}(x, \vec{z})$ are terms, whose variables are among $x, z_0, \ldots, z_{n-1}$, then we have

$$\left\langle t^{\mathfrak{A}}_{i_{\Sigma'}}\left(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi), \vec{\chi}\right), t^{\mathfrak{A}}_{i_{\Sigma'}}\left(\mathrm{SEN}^{\mathfrak{A}}(f)(\psi), \vec{\chi}\right)\right\rangle \in \Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma'}(\mathfrak{A}), \quad \text{for all } i < k,$$

whence, the compatibility of $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ with relation systems implies that, for any k-ary relation symbol $r \in R$,

$$\begin{aligned}\mathfrak{A} \models_{\Sigma'} r(t_0(x, \vec{z}), \ldots, t_{k-1}(x, \vec{z})) \leftrightarrow \\ r(t_0(y, \vec{z}), \ldots, t_{k-1}(y, \vec{z})) \left[\mathrm{SEN}^{\mathfrak{A}}(f)(\phi), \mathrm{SEN}^{\mathfrak{A}}(f)(\psi), \vec{\chi}\right],\end{aligned}$$

and, therefore, we obtain $\mathfrak{A} \models_{\Sigma'} \beta(x, y)[\mathrm{SEN}^{\mathfrak{A}}(f)(\phi), \mathrm{SEN}^{\mathfrak{A}}(f)(\psi)]$, for every Leibniz $\mathcal{L}$-formula $\beta(x, y)$. Hence, we conclude that $\langle \phi, \psi \rangle \in \theta_\Sigma$, which proves that $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \leq \theta$. Thus, we have that $\theta = \Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, as was to be shown. □

Theorem 6 has as a corollary an analog of Corollary 2.7 of [14], stating that the atomic formula $\gamma(x, \vec{z})$ in the Leibniz formulas may be replaced by arbitrary elementary predicates.

Corollary 7 Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-system, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, and ϕ, $\psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$. Then $\langle \phi, \psi \rangle \in \Omega_\Sigma^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ if and only if, for all $\mathcal{L}$-formulas $\alpha(x, \vec{z})$

$$\mathfrak{A} \models_{\Sigma'} \alpha(x, \vec{z})[\mathrm{SEN}(f)(\phi), \vec{\chi}] \quad \text{iff} \quad \mathfrak{A} \models_{\Sigma'} \alpha(x, \vec{z})[\mathrm{SEN}(f)(\psi), \vec{\chi}]$$

for all $\Sigma' \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma, \Sigma')$ and $\vec{\chi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma')^n$, where n is the length of $\vec{z}$.

$\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ will be called the *Leibniz* $\mathbf{N}^{\mathfrak{A}}$*- congruence system* of $\mathfrak{A}$, following the terminology of [14] and the close analogies between universal abstract algebraic logic, that inspired the developments in [14], and categorical abstract algebraic logic, the basis of the current developments.

4 Quotient Systems and Homomorphism Theorems

Given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ and an $\mathbf{N}^{\mathfrak{A}}$-congruence system θ of $\mathfrak{A}$, we define the triple $\mathfrak{A}^\theta = \langle \mathrm{SEN}^{\mathfrak{A}^\theta}, \langle \mathbf{N}^{\mathfrak{A}^\theta}, F^{\mathfrak{A}^\theta} \rangle, R^{\mathfrak{A}^\theta} \rangle$ by setting:

- $\mathrm{SEN}^{\mathfrak{A}^\theta}$ and $\mathbf{N}^{\mathfrak{A}^\theta}$ are as already defined in [26].
- $F^{\mathfrak{A}^\theta} : \mathbf{F} \to \mathbf{N}^{\mathfrak{A}^\theta}$ is given by $F^{\mathfrak{A}^\theta} = (F^{\mathfrak{A}})^\theta$, where ${}^\theta : \mathbf{N}^{\mathfrak{A}} \to \mathbf{N}^{\mathfrak{A}^\theta}$ denotes the quotient map, mapping a natural transformation to its quotient, as defined in [26].
- Finally, for all $r \in R$, with $\rho(r) = n$, we define $r^{\mathfrak{A}^\theta}$ by setting, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$r_\Sigma^{\mathfrak{A}^\theta} = \left\{\vec{\phi}/\theta_\Sigma : \vec{\phi} \in r_\Sigma^{\mathfrak{A}}\right\}.$$

Because θ is assumed to be in $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, all the definitions above are independent of the choice of representatives and the triple $\mathfrak{A}^\theta$ is also an $\mathcal{L}$-system. It is called the *quotient system of* $\mathfrak{A}$ *modulo* θ and is sometimes also denoted by $\mathfrak{A}/\theta$.

The next proposition forms a generalization of Proposition 3.1 of [14] and is closely related to Proposition 23 of [26]. It states that, given an $\mathcal{L}$-system and one of its congruence systems, there is a natural projection morphism from the system to its quotient system which is reductive and whose kernel is the congruence system θ.

Proposition 8 Let $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ be an $\mathcal{L}$-system and $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$. Then, the natural projection translation $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\theta} \rangle : \mathrm{SEN}^{\mathfrak{A}} \to \mathrm{SEN}^{\mathfrak{A}^{\theta}}$ is a reductive morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\theta} \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}^{\theta}$, such that $\mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\theta} \rangle) = \theta$.

Proof It is easy to see that the claim is true, since the part dealing with the $(\mathbf{N}^{\mathfrak{A}}, \mathbf{N}^{\mathfrak{A}^{\theta}})$-epimorphic property has already been established in [26], the part dealing with the reduction is clear by the definition of the relation systems in $\mathfrak{A}^{\theta}$ and the part dealing with the kernel follows easily from the definitions of the kernel and of π^{θ}. □

Finally, a lemma is presented that will be used to prove the analog of the Homomorphism Theorem for $\mathcal{L}$-systems. This lemma forms an analog of Lemma 3.2 of [14].

Lemma 9 Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ and $\mathfrak{C} = \langle \mathrm{SEN}^{\mathfrak{C}}, \langle \mathbf{N}^{\mathfrak{C}}, F^{\mathfrak{C}} \rangle, R^{\mathfrak{C}} \rangle$ are $\mathcal{L}$-systems, such that $\mathbf{Sign}^{\mathfrak{A}} = \mathbf{Sign}^{\mathfrak{B}} = \mathbf{Sign}$. Let also $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ be a reductive system morphism and $\langle G, \beta \rangle : \mathfrak{A} \to \mathfrak{C}$ a system morphism, such that $\mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle) \leq \mathrm{Ker}(\langle G, \beta \rangle)$. Then, there exists a system morphism $\langle G, \gamma \rangle : \mathfrak{B} \to \mathfrak{C}$, such that $\langle G, \beta \rangle = \langle G, \gamma \rangle \langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle$. Moreover, $\langle G, \gamma \rangle$ is strong if and only if $\langle G, \beta \rangle$ is strong.

$$\begin{tikzcd}
\mathfrak{A} \arrow[rr, "\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle"] \arrow[dr, "\langle G, \beta \rangle"'] & & \mathfrak{B} \arrow[dl, "\langle G, \gamma \rangle"] \\
& \mathfrak{C} &
\end{tikzcd}$$

Proof Define $\gamma : \mathrm{SEN}^{\mathfrak{B}} \to \mathrm{SEN}^{\mathfrak{C}} \circ G$, by letting, for al $\Sigma \in |\mathbf{Sign}|$, $\gamma_{\Sigma} : \mathrm{SEN}^{\mathfrak{B}}(\Sigma) \to \mathrm{SEN}^{\mathfrak{C}}(G(\Sigma))$ be given, for all $\phi \in \mathrm{SEN}^{\mathfrak{B}}(\Sigma)$, by

$$\gamma_{\Sigma}(\phi) = \beta_{\Sigma}(\psi), \text{ where } \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma) \text{ is such that } \alpha_{\Sigma}(\psi) = \phi.$$

By surjectivity of $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle$, such a ψ always exists and, by the condition $\mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle) \leq \mathrm{Ker}(\langle G, \beta \rangle)$, the value $\gamma_{\Sigma}(\phi)$ is independent of the choice of $\psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\alpha_{\Sigma}(\psi) = \phi$.

It is now shown that γ is a natural transformation. To this end, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $\phi \in \mathrm{SEN}^{\mathfrak{B}}(\Sigma_1)$. Then, for $\psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)$, such that $\alpha_{\Sigma_1}(\psi) = \phi$,

$$\begin{tikzcd}
\mathrm{SEN}^{\mathfrak{B}}(\Sigma_1) \arrow[r, "\gamma_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}^{\mathfrak{B}}(f)"'] & \mathrm{SEN}^{\mathfrak{C}}(G(\Sigma_1)) \arrow[d, "\mathrm{SEN}^{\mathfrak{C}}(G(f))"] \\
\mathrm{SEN}^{\mathfrak{B}}(\Sigma_2) \arrow[r, "\gamma_{\Sigma_2}"'] & \mathrm{SEN}^{\mathfrak{C}}(G(\Sigma_2))
\end{tikzcd}$$

$$\mathrm{SEN}^{\mathfrak{C}}(G(f))(\gamma_{\Sigma_1}(\phi)) = \mathrm{SEN}^{\mathfrak{C}}(G(f))(\beta_{\Sigma_1}(\psi))$$

$$\begin{array}{ccc} \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1) & \xrightarrow{\beta_{\Sigma_1}} & \mathrm{SEN}^{\mathfrak{C}}(G(\Sigma_1)) \\ {\scriptstyle \mathrm{SEN}^{\mathfrak{A}}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}^{\mathfrak{C}}(G(f))} \\ \mathrm{SEN}^{\mathfrak{A}}(\Sigma_2) & \xrightarrow[\beta_{\Sigma_2}]{} & \mathrm{SEN}^{\mathfrak{C}}(G(\Sigma_2)) \end{array}$$

$$\begin{aligned} &= \beta_{\Sigma_2}\left(\mathrm{SEN}^{\mathfrak{A}}(f)(\psi)\right) \\ &= \gamma_{\Sigma_2}\left(\mathrm{SEN}^{\mathfrak{B}}(f)(\phi)\right), \end{aligned}$$

where the last equation follows from

$$\begin{array}{ccc} \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1) & \xrightarrow{\alpha_{\Sigma_1}} & \mathrm{SEN}^{\mathfrak{B}}(\Sigma_1) \\ {\scriptstyle \mathrm{SEN}^{\mathfrak{A}}(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{SEN}^{\mathfrak{B}}(f)} \\ \mathrm{SEN}^{\mathfrak{A}}(\Sigma_2) & \xrightarrow[\alpha_{\Sigma_2}]{} & \mathrm{SEN}^{\mathfrak{B}}(\Sigma_2) \end{array}$$

$$\begin{aligned} \alpha_{\Sigma_2}\left(\mathrm{SEN}^{\mathfrak{A}}(f)(\psi)\right) &= \mathrm{SEN}^{\mathfrak{B}}(f)(\alpha_{\Sigma_1}(\psi)) \\ &= \mathrm{SEN}^{\mathfrak{B}}(f)(\phi). \end{aligned}$$

To see that $\langle G, \gamma\rangle : \mathrm{SEN}^{\mathfrak{B}} \to \mathrm{SEN}^{\mathfrak{C}}$ is an $(\mathbf{N}^{\mathfrak{B}}, \mathbf{N}^{\mathfrak{C}})$-epimorphic translation, consider $\sigma \in \mathbf{F}(n, 1)$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{B}}(\Sigma)^n$. Let $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\alpha_\Sigma(\vec{\psi}) = \vec{\phi}$. Then, we have

$$\begin{aligned} \gamma_\Sigma\left(\sigma^{\mathfrak{B}}_\Sigma\left(\vec{\phi}\right)\right) &= \gamma_\Sigma\left(\sigma^{\mathfrak{B}}_\Sigma\left(\alpha_\Sigma\left(\vec{\psi}\right)\right)\right) \\ &= \gamma_\Sigma\left(\alpha_\Sigma\left(\sigma^{\mathfrak{A}}_\Sigma\left(\vec{\psi}\right)\right)\right) \\ &= \beta_\Sigma\left(\sigma^{\mathfrak{A}}_\Sigma\left(\vec{\psi}\right)\right) \\ &= \sigma^{\mathfrak{C}}_{G(\Sigma)}\left(\beta_\Sigma\left(\vec{\psi}\right)\right) \\ &= \sigma^{\mathfrak{C}}_{G(\Sigma)}\left(\gamma_\Sigma\left(\vec{\phi}\right)\right). \end{aligned}$$

To conclude this part of the proof, $\langle G, \gamma\rangle : \mathfrak{B} \to \mathfrak{C}$ must be shown to be a system morphism. To this end, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{B}}(\Sigma)^n$. Suppose, also, that $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\alpha_\Sigma(\vec{\psi}) = \vec{\phi}$. Now, if $\vec{\phi} \in r^{\mathfrak{B}}_\Sigma$, then $\alpha_\Sigma(\vec{\psi}) \in r^{\mathfrak{B}}_\Sigma$, whence, by the strength of $\langle \mathbf{I}_{\mathbf{Sign}}, \alpha\rangle$, $\vec{\psi} \in r^{\mathfrak{A}}_\Sigma$ and, therefore, since $\langle G, \beta\rangle$ is a system morphism, $\beta_\Sigma(\vec{\psi}) \in r^{\mathfrak{C}}_{G(\Sigma)}$. But this yields, by the definition of γ, that $\gamma_\Sigma(\vec{\phi}) \in r^{\mathfrak{C}}_{G(\Sigma)}$ and, hence, $\langle G, \gamma\rangle$ is also a system morphism.

Finally, for the last statement, it must be shown that $\langle G, \gamma\rangle : \mathfrak{B} \to \mathfrak{C}$ is strong if and only if $\langle G, \beta\rangle : \mathfrak{A} \to \mathfrak{C}$ is strong.

Suppose, first, that $\langle G, \beta\rangle$ is strong and let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{B}}(\Sigma)^n$ and $\vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\alpha_\Sigma(\vec{\psi}) = \vec{\phi}$. If $\gamma_\Sigma(\vec{\phi}) \in r^{\mathfrak{C}}_{G(\Sigma)}$, then, by the definition of γ, $\beta_\Sigma(\vec{\psi}) \in r^{\mathfrak{C}}_{G(\Sigma)}$. Thus, by the strength of $\langle G, \beta\rangle$, $\vec{\psi} \in r^{\mathfrak{A}}_\Sigma$. Therefore, since $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$ is a system morphism, $\alpha_\Sigma(\vec{\psi}) \in r^{\mathfrak{B}}_\Sigma$, whence $\vec{\phi} \in r^{\mathfrak{B}}_\Sigma$. Thus $\langle G, \gamma\rangle$ is also a strong system morphism.

Suppose, conversely, that $\langle G, \gamma\rangle$ is strong and let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, such that $\beta_\Sigma(\vec{\phi}) \in r^{\mathfrak{C}}_{G(\Sigma)}$. Then, by the definition of γ, $\gamma_\Sigma(\alpha_\Sigma(\vec{\phi})) \in r^{\mathfrak{C}}_{G(\Sigma)}$. Hence, by the strength of $\langle G, \gamma\rangle$, $\alpha_\Sigma(\vec{\phi}) \in r^{\mathfrak{B}}_\Sigma$. Therefore, by the strength of $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$, $\vec{\phi} \in r^{\mathfrak{A}}_\Sigma$, which shows that $\langle G, \beta\rangle$ must also be strong. □

Next, the Homomorphism Theorem, an analog of Theorem 3.3 of [14], says that, given two $\mathcal{L}$-systems $\mathfrak{A}$ and $\mathfrak{B}$, and a reductive system morphism $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, there exists a reductive system morphism $\langle F, \beta\rangle : \mathfrak{A}^\theta \twoheadrightarrow_s \mathfrak{B}$, where by θ is denoted the $\mathrm{Ker}(\langle F, \alpha\rangle)$ to simplify notation.

Theorem 10 (Homomorphism Theorem) *Suppose that* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$, $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ *are two* $\mathcal{L}$*-systems and* $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *a reductive system morphism. Then, there exists a reductive system morphism* $\langle F, \beta\rangle : \mathfrak{A}^{\mathrm{Ker}(\langle F, \alpha\rangle)} \twoheadrightarrow_s \mathfrak{B}$*, such that the following triangle commutes:*

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "\langle F, \alpha\rangle"] \arrow[dr, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\langle F,\alpha\rangle}\rangle"'] & & \mathfrak{B} \\ & \mathfrak{A}/\mathrm{Ker}(\langle F, \alpha\rangle) \arrow[ur, "\langle F, \beta\rangle"'] & \end{tikzcd}$$

Proof It is not difficult to verify that, if $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ is a reductive system morphism, then $\mathrm{Ker}(\langle F, \alpha\rangle)$ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathfrak{A}$. Therefore, by Proposition 8, $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\langle F,\alpha\rangle}\rangle : \mathfrak{A} \to \mathfrak{A}/\mathrm{Ker}(\langle F, \alpha\rangle)$ is a reductive system morphism, such that $\mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\langle F,\alpha\rangle}\rangle) = \mathrm{Ker}(\langle F, \alpha\rangle)$. Hence, the existence of $\langle F, \beta\rangle$ follows by applying Lemma 9. □

Now, an analog of the second Isomorphism Theorem will be given for the framework of $\mathcal{L}$-systems. It says that, given two $\mathbf{N}^{\mathfrak{A}}$-congruence systems θ, η of an $\mathcal{L}$-system $\mathfrak{A}$, such that $\theta \le \eta$, the pair $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \alpha\rangle : (\mathfrak{A}/\theta)/(\eta/\theta) \to \mathfrak{A}/\eta$, given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, by

$$\alpha_\Sigma((\phi/\theta_\Sigma)/(\eta_\Sigma/\theta_\Sigma)) = \phi/\eta_\Sigma,$$

is an isomorphism of $\mathcal{L}$-systems.

A lemma is needed first to the effect that η/θ is an $\mathbf{N}^{\mathfrak{A}}$-congruence system on $\mathfrak{A}/\theta$.

Lemma 11 Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ is an $\mathcal{L}$-system and $\theta, \eta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ two $\mathbf{N}^{\mathfrak{A}}$-congruence systems of $\mathfrak{A}$, such that $\theta \le \eta$. Then, the family $\eta/\theta = \{\eta_\Sigma/\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|}$ is an $\mathbf{N}^{\mathfrak{A}^\theta}$-congruence system of $\mathfrak{A}/\theta$.

Proof By Part 1 of Theorem 28 of [26], it suffices to show that, given $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$, if $\langle \vec{\phi}/\theta_\Sigma, \vec{\psi}/\theta_\Sigma\rangle \in \eta_\Sigma/\theta_\Sigma$ and $\vec{\phi}/\theta_\Sigma \in r^{\mathfrak{A}^\theta}_\Sigma$, then also $\vec{\psi}/\theta_\Sigma \in r^{\mathfrak{A}^\theta}_\Sigma$. So, suppose that $\langle \vec{\phi}/\theta_\Sigma, \vec{\psi}/\theta_\Sigma\rangle \in \eta_\Sigma/\theta_\Sigma$ and $\vec{\phi}/\theta_\Sigma \in r^{\mathfrak{A}^\theta}_\Sigma$. Then $\langle \vec{\phi}, \vec{\psi}\rangle \in \eta_\Sigma$ and $\vec{\phi} \in r^{\mathfrak{A}}_\Sigma$. Hence, since $\eta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, we get that $\vec{\psi} \in r^{\mathfrak{A}}_\Sigma$, whence $\vec{\psi}/\theta_\Sigma \in r^{\mathfrak{A}^\theta}_\Sigma$. Thus η/θ is in fact an $\mathbf{N}^{\mathfrak{A}^\theta}$-congruence system of $\mathfrak{A}^\theta$. □

With Lemma 11 at hand, we are now ready to prove the Second Isomorphism Theorem, paralleling the Second Isomorphism Theorem of Universal Algebra (see, e.g., Theorem 6.15 of [5]).

Theorem 12 (Second Isomorphism Theorem) *Let* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ *be an* $\mathcal{L}$*-system and* $\theta, \eta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ *two* $\mathbf{N}^{\mathfrak{A}}$*-congruence systems of* $\mathfrak{A}$*, such that* $\theta \leq \eta$*. Then, the pair* $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \alpha \rangle : (\mathfrak{A}/\theta)/(\eta/\theta) \to \mathfrak{A}/\eta$*, defined, by letting, for all* $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$,

$$\alpha_\Sigma : (\mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\theta_\Sigma)/(\eta_\Sigma/\theta_\Sigma) \to \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\eta_\Sigma$$

be given by

$$\alpha_\Sigma((\phi/\theta_\Sigma)/(\eta_\Sigma/\theta_\Sigma)) = \phi/\eta_\Sigma, \quad \textit{for all } \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma),$$

is an $\mathcal{L}$*-system isomorphism.*

Proof It has been shown in Part 2 of Theorem 28 of [26] that $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \alpha \rangle : \mathrm{SEN}^{\mathfrak{A}^\theta}/(\eta/\theta) \to \mathrm{SEN}^{\mathfrak{A}^\eta}$ is an $(\mathbf{N}^{\theta^{\eta/\theta}}, \mathbf{N}^\eta)$-epimorphic translation. Thus, it suffices to show that, given $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$,

$$\vec{\phi}/\theta_\Sigma \in r_\Sigma^{\mathfrak{A}^\theta} \quad \text{iff} \quad (\vec{\phi}/\theta_\Sigma)(\eta_\Sigma/\theta_\Sigma) \in r_\Sigma^{\mathfrak{A}^\theta/(\eta/\theta)}. \tag{2}$$

But, given Lemma 11, Equivalence (2) follows directly from the definition of a quotient $\mathcal{L}$-system. □

Finally, the Correspondence Theorem, an analog of the classical Correspondence Theorem of Universal Algebra for $\mathcal{L}$-systems (see, e.g., Theorem 6.20 of [5]), is now presented.

Theorem 13 (Correspondence Theorem) *Suppose that* $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ *is an* $\mathcal{L}$*-system and* $\theta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ *an* $\mathbf{N}^{\mathfrak{A}}$*-congruence system of* $\mathfrak{A}$*. Then, the mapping* $\kappa : [\theta, \nabla] \to \mathbf{Con}^{\mathbf{N}^{\mathfrak{A}^\theta}}(\mathfrak{A}/\theta)$*, defined by*

$$\kappa(\eta) = \eta/\theta, \quad \textit{for all } \eta \in [\theta, \nabla],$$

is an isomorphism between the sublattice $[\theta,\nabla]$ *of* $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ *and the lattice* $\mathbf{Con}^{\mathbf{N}^{\mathfrak{A}^\theta}}(\mathfrak{A}/\theta)$.

Proof Lemma 11 shows that this mapping is well-defined. To see that it is one-to-one, suppose that $\eta, \zeta \in [\theta, \nabla]$, such that $\eta \neq \zeta$. Then, there exists, without loss of generality, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\langle \phi, \psi \rangle \in \eta_\Sigma$ but $\langle \phi, \psi \rangle \notin \zeta_\Sigma$. Hence, we have $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma \rangle \in \eta_\Sigma/\theta_\Sigma$, but $\langle \phi/\theta_\Sigma, \psi/\theta_\Sigma \rangle \notin \zeta_\Sigma/\theta_\Sigma$. This shows that $\eta/\theta \neq \zeta/\theta$. Thus κ is indeed one-to-one.

To see that κ is also onto, suppose that $\zeta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}^\theta}}(\mathfrak{A}/\theta)$. Then, let

$$\mathfrak{A} \xrightarrow{\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\theta \rangle} \mathfrak{A}/\theta \xrightarrow{\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\zeta \rangle} \mathfrak{A}^\theta/\zeta$$

$\eta = \mathrm{Ker}(\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\zeta \rangle \circ \langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^\theta \rangle)$. Then $\eta \in \mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ and we have, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$,

$$\begin{aligned} \langle \phi/\theta_\Sigma, \psi/\theta_\Sigma \rangle \in \eta_\Sigma/\theta_\Sigma &\text{ iff } \langle \phi, \psi \rangle \in \eta_\Sigma \\ &\text{ iff } \langle \phi/\theta_\Sigma, \psi/\theta_\Sigma \rangle \in \zeta_\Sigma, \end{aligned}$$

whence $\zeta = \eta/\theta = \kappa(\eta)$ and κ is onto.

Finally, if $\eta, \zeta \in [\theta, \nabla]$, we have $\eta \leq \zeta$ if and only if $\eta/\theta \leq \zeta/\theta$ if and only if $\kappa(\eta) \leq \kappa(\zeta)$, whence $\kappa : [\theta, \nabla] \to \mathbf{Con}^{\mathbf{N}^{\mathfrak{A}^\theta}}(\mathfrak{A}/\theta)$ is a lattice isomorphism. □

5 Leibniz Quotient of a Structure System

Given an $\mathcal{L}$-system $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, the quotient system of $\mathfrak{A}$ modulo $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ is called the *Leibniz quotient of* $\mathfrak{A}$. Adopting the notation used in [26, 29], we denote $\mathfrak{A}/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ by $\mathfrak{A}^{\mathbf{N}^{\mathfrak{A}}}$. Sometimes, to simplify notation and following [2, 14, 19], we may use $\mathfrak{A}^*$ instead of $\mathfrak{A}^{\mathbf{N}^{\mathfrak{A}}}$. This deviation from the established notation Ω^N for the Leibniz N-congruence system operator of [29] has a firm justification. In the case of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with a category N of natural transformations on SEN, the category N is not part of the data of the π-institution, and, therefore, it has to be part of the notation in the Leibniz operator. However, $\mathbf{N}^{\mathfrak{A}}$ is part of the definition of an $\mathcal{L}$-system and, as a consequence, it need not be made explicit when considering the Leibniz quotient of $\mathfrak{A}$. So the notation $\mathfrak{A}^*$ suffices.

At this point another remark concerning notation is in order. The same convention, as above, could have been adopted for the Leibniz $\mathbf{N}^{\mathfrak{A}}$-congruence system of an $\mathcal{L}$-system $\mathfrak{A}$. So $\Omega(\mathfrak{A})$ may be used in place of $\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$, since $\mathbf{N}^{\mathfrak{A}}$ is an integral part of the tuple in the definition of $\mathfrak{A}$. Similarly $\mathrm{Con}(\mathfrak{A})$ may be used instead of $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})$ for the same reasons. However, one may not replace $\mathrm{Con}^{\mathbf{N}^{\mathfrak{A}}}(\mathrm{SEN}^{\mathfrak{A}})$ by $\mathrm{Con}(\mathrm{SEN}^{\mathfrak{A}})$. It is for this reason that the more traditional and, in this case, somewhat redundant, notation was adopted and used in the previous sections.

We will also use the notation ϕ^* to denote the element $\phi/\Omega_\Sigma(\mathfrak{A}) \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\Omega_\Sigma(\mathfrak{A})$, for $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$.

Proposition 8 has the following corollary:

Corollary 14 Given an $\mathcal{L}$-system $\mathfrak{A}$, the Leibniz quotient $\mathfrak{A}^*$ is a reduction of $\mathfrak{A}$ via the reductive system morphism $\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\Omega(\mathfrak{A})} \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}^*$.

Also, the Correspondence Theorem 13 has the following corollary:

Corollary 15 Given an $\mathcal{L}$-system $\mathfrak{A}$, the Leibniz quotient $\mathfrak{A}^*$ is a reduced system. Therefore $\mathfrak{A}^{**} \cong \mathfrak{A}^*$.

Following Elgueta [14], we provide, next, an analog of Proposition 3.4 of [14] for $\mathcal{L}$-systems, showing that, in the present context as well, the Leibniz quotient $\mathfrak{A}^*$ of an $\mathcal{L}$-system $\mathfrak{A}$ is minimal in that it forms a reduction of any other reduction of $\mathfrak{A}$.

Proposition 16 Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}}\rangle, R^{\mathfrak{B}}\rangle$ are $\mathcal{L}$-systems and $\langle F, \alpha\rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ a reductive system morphism, with $F : \mathbf{Sign}^{\mathfrak{A}} \to \mathbf{Sign}^{\mathfrak{B}}$ an isomorphism. Define $\langle F^*, \alpha^*\rangle$ by letting $F^* = F$ and

$$\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}) \circ F^*$$

be given, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, by

$$\alpha^*_\Sigma(\phi^*) = \alpha_\Sigma(\phi)^*, \quad \text{for all } \phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma).$$

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\langle F, \alpha\rangle"] \arrow[d, "\langle I_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\mathbf{N}^{\mathfrak{A}}}\rangle"'] & \mathfrak{B} \arrow[d, "\langle I_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\mathbf{N}^{\mathfrak{B}}}\rangle"] \\
\mathfrak{A}^* \arrow[r, "\langle F^*, \alpha^*\rangle"'] & \mathfrak{B}^*
\end{tikzcd}$$

Then $\langle F^*, \alpha^*\rangle : \mathfrak{A}^* \to \mathfrak{B}^*$ is an isomorphism of $\mathcal{L}$-systems.

Proof We first show that $\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}) \circ F^*$ is well defined. Indeed, if $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|, \phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\phi^* = \psi^*$, then $\langle \phi, \psi\rangle \in \Omega^{\mathbf{N}^{\mathfrak{A}}}_\Sigma(\mathfrak{A})$, whence, by Part 2 of Theorem 5, $\langle \alpha_\Sigma(\phi), \alpha_\Sigma(\psi)\rangle \in \Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma)}(\mathfrak{B})$ and, hence $\alpha_\Sigma(\phi)^* = \alpha_\Sigma(\psi)^*$. Therefore, $\alpha^*_\Sigma(\phi^*)$ is independent of the choice of the representative from the class of ϕ^* and α^*_Σ is well-defined.

Next, to see that $\alpha^* : \mathrm{SEN}^{\mathfrak{A}}/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}) \circ F^*$ is a natural transformation, consider $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}^{\mathfrak{A}}|$, $f \in \mathbf{Sign}^{\mathfrak{A}}(\Sigma_1, \Sigma_2)$ and $\phi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)$. We then have

$$\begin{tikzcd}
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_1)/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \arrow[r, "\alpha^*_{\Sigma_1}"] \arrow[d, "\mathrm{SEN}^{\mathfrak{A}}(f)/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})"'] & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_1))/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B}) \arrow[d, "\mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})"] \\
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_2)/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A}) \arrow[r, "\alpha^*_{\Sigma_2}"'] & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_2))/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})
\end{tikzcd}$$

$$\begin{aligned}
&\mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})(\alpha^*_{\Sigma_1}(\phi^*)) \\
&\qquad = \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})\left(\alpha^*_{\Sigma_1}\left(\phi/\Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma_1}(\mathfrak{A})\right)\right) \\
&\qquad = \mathrm{SEN}^{\mathfrak{B}}(F(f))/\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})\left(\alpha_{\Sigma_1}(\phi)/\Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma_1)}(\mathfrak{B})\right) \\
&\qquad = \mathrm{SEN}^{\mathfrak{B}}(F(f))(\alpha_{\Sigma_1}(\phi))/\Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma_2)}(\mathfrak{B})
\end{aligned}$$

$$\begin{array}{ccc}
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_1) & \xrightarrow{\alpha_{\Sigma_1}} & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_1)) \\
{\scriptstyle \mathrm{SEN}^{\mathfrak{A}}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{SEN}^{\mathfrak{B}}(F(f))} \\
\mathrm{SEN}^{\mathfrak{A}}(\Sigma_2) & \xrightarrow[\alpha_{\Sigma_2}]{} & \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma_2))
\end{array}$$

$$\begin{aligned}
&= \alpha_{\Sigma_2}(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi))/\Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma_2)}(\mathfrak{B}) \\
&= \alpha^*_{\Sigma_2}\left(\mathrm{SEN}^{\mathfrak{A}}(f)(\phi)/\Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma_2}(\mathfrak{A})\right) \\
&= \alpha^*_{\Sigma_2}\left(\mathrm{SEN}^{\mathfrak{A}}(f)/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})\left(\phi/\Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma_1}(\mathfrak{A})\right)\right) \\
&= \alpha^*_{\Sigma_2}\left(\mathrm{SEN}^{\mathfrak{A}}(f)/\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})(\phi^*)\right).
\end{aligned}$$

To see that $\langle F^*, \alpha^*\rangle : \mathrm{SEN}^{\mathfrak{A}}/\Omega(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}/\Omega(\mathfrak{B})$ is an $(\mathbf{N}^{\mathfrak{A}^{\Omega(\mathfrak{A})}}, \mathbf{N}^{\mathfrak{B}^{\Omega(\mathfrak{B})}})$-epimorphic translation, let $\sigma \in \mathbf{F}(n, 1)$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then, we have

$$\begin{aligned}
\alpha^*_{\Sigma}\left(\sigma^{\Omega^{\mathbf{N}^{\mathfrak{A}}}(\mathfrak{A})}_{\Sigma}(\vec{\phi}^*)\right) &= \alpha^*_{\Sigma}\left(\sigma^{\mathfrak{A}}_{\Sigma}(\vec{\phi})/\Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma}(\mathfrak{A})\right) \\
&= \alpha_{\Sigma}\left(\sigma^{\mathfrak{A}}_{\Sigma}(\vec{\phi})\right)/\Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma)}(\mathfrak{B}) \\
&= \sigma^{\mathfrak{B}}_{F(\Sigma)}\left(\alpha_{\Sigma}(\vec{\phi})\right)/\Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma)}(\mathfrak{B}) \\
&= \sigma^{\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})}_{F(\Sigma)}\left(\alpha_{\Sigma}(\vec{\phi})/\Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma)}(\mathfrak{B})\right) \\
&= \sigma^{\Omega^{\mathbf{N}^{\mathfrak{B}}}(\mathfrak{B})}_{F(\Sigma)}\left(\alpha^*_{\Sigma}(\vec{\phi}^*)\right).
\end{aligned}$$

To see that $\langle F, \alpha^*\rangle : \mathfrak{A}^* \to \mathfrak{B}^*$ is strong, let $r \in R$, with $\rho(r) = n$, $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^n$. Then, we have

$$\begin{aligned}
\vec{\phi}^* \in r^{\mathfrak{A}^*}_{\Sigma} \ &\text{iff}\ \vec{\phi} \in r^{\mathfrak{A}}_{\Sigma} \\
&\text{iff}\ \alpha_{\Sigma}(\vec{\phi}) \in r^{\mathfrak{B}}_{F(\Sigma)} \\
&\text{iff}\ \alpha_{\Sigma}(\vec{\phi})^* \in r^{\mathfrak{B}^*}_{F(\Sigma)} \\
&\text{iff}\ \alpha^*_{\Sigma}(\vec{\phi}^*) \in r^{\mathfrak{B}^*}_{F(\Sigma)}.
\end{aligned}$$

Finally, $\alpha_{\Sigma} : \mathrm{SEN}^{\mathfrak{A}}(\Sigma)/\Omega_{\Sigma}(\mathfrak{A}) \to \mathrm{SEN}^{\mathfrak{B}}(F(\Sigma))/\Omega_{F(\Sigma)}(\mathfrak{B})$ is a bijection, for all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, since, if $\phi, \psi \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)$, such that $\alpha^*_{\Sigma}(\phi^*) = \alpha^*_{\Sigma}(\psi^*)$, we have $\alpha_{\Sigma}(\phi)^* = \alpha_{\Sigma}(\psi)^*$, whence we obtain $\langle\alpha_{\Sigma}(\phi), \alpha_{\Sigma}(\psi)\rangle \in \Omega^{\mathbf{N}^{\mathfrak{B}}}_{F(\Sigma)}(\mathfrak{B})$, and, therefore, by Part 1 of Theorem 2.5, $\langle\phi, \psi\rangle \in \Omega^{\mathbf{N}^{\mathfrak{A}}}_{\Sigma}(\mathfrak{A})$, i.e., $\phi^* = \psi^*$. ◻

Corollary 17 Suppose that $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \mathrm{SEN}^{\mathfrak{B}}, \langle \mathbf{N}^{\mathfrak{B}}, F^{\mathfrak{B}} \rangle, R^{\mathfrak{B}} \rangle$ are $\mathcal{L}$-systems. If $\mathfrak{B}$ is a reduction of $\mathfrak{A}$ via an $\langle F, \alpha \rangle : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, with F an isomorphism, then $\mathfrak{A}^*$ is also a reduction of $\mathfrak{B}$. Also, if $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A}^* \cong \mathfrak{B}^*$.

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\langle F, \alpha \rangle"] \arrow[d, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{A}}}, \pi^{\mathbf{N}^{\mathfrak{A}}} \rangle"'] & \mathfrak{B} \arrow[d, "\langle \mathrm{I}_{\mathbf{Sign}^{\mathfrak{B}}}, \pi^{\mathbf{N}^{\mathfrak{B}}} \rangle"] \\
\mathfrak{A}^* \arrow[r, leftrightarrow, "\langle F^*, \alpha^* \rangle"'] & \mathfrak{B}^*
\end{tikzcd}$$

6 The Completeness Theorem for $\mathcal{L}$-Systems

Recall from [31] that, given a set Γ of $\mathcal{L}$-formulas, we define

$$\mathrm{Mod}(\Gamma) = \{\mathfrak{A} : \mathfrak{A} \models \gamma \text{ for all } \gamma \in \Gamma\}.$$

Now taking after [14], define, also,

$$\mathrm{Mod}^*(\Gamma) = \{\mathfrak{A} \in \mathrm{Mod}(\Gamma) : \mathfrak{A} \text{ is reduced}\}.$$

We call $\mathrm{Mod}(\Gamma)$ and $\mathrm{Mod}^*(\Gamma)$ the *abstract model class* and the *reduced model class* of Γ, respectively. As in the case of ordinary first-order structures, if one introduces the class operator

$$\mathrm{L}(\mathtt{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{B}^* \text{ for some } \mathfrak{B} \in \mathtt{K}\},$$

then, by Proposition 6 of [31], we get that $\mathrm{Mod}^*(\Gamma) = \mathrm{L}(\mathrm{Mod}(\Gamma))$.

The operator L is called the *reduction operator*. A class $\mathtt{K}$ of $\mathcal{L}$-systems is said to be an *abstract class* if it is closed under expansions and reductions. It is called a *reduced class* if it is obtained by some class by applying the reduction operator. The entire class of reduced $\mathcal{L}$-systems is called a *reduced semantics* to differentiate it from the entire class of all $\mathcal{L}$-systems, which is referred to as *abstract semantics*.

In the same way that the notation $\Gamma \models \alpha$ is used to indicate that, for all $\mathcal{L}$-systems $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$,

$$\mathfrak{A} \models_{\Sigma} \Gamma[\vec{\phi}] \quad \text{implies} \quad \mathfrak{A} \models_{\Sigma} \alpha[\vec{\phi}],$$

we use the notation $\Gamma \models^* \alpha$ to indicate that for all reduced $\mathcal{L}$-systems $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}} \rangle, R^{\mathfrak{A}} \rangle$, all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$ and all $\vec{\phi} \in \mathrm{SEN}^{\mathfrak{A}}(\Sigma)^{\omega}$,

$$\mathfrak{A} \models_{\Sigma} \Gamma[\vec{\phi}] \quad \text{implies} \quad \mathfrak{A} \models_{\Sigma} \alpha[\vec{\phi}],$$

Then Gödel's Completeness Theorem has the following alternative formulation:

Theorem 18 (Completeness Theorem) *Let* $\Gamma \cup \{\alpha\} \subseteq \mathrm{Fm}_{\mathcal{L}}(V)$ *be a collection of* $\mathcal{L}$-*sentences. Then*

$$\Gamma \vdash \alpha \quad \textit{iff} \quad \Gamma \models \alpha \quad \textit{iff} \quad \Gamma \models^{*} \alpha.$$

Proof The second equivalence follows from Proposition 6 of [31]. By the soundness part of Gödel's Completeness Theorem, we have that $\Gamma \vdash \alpha$ implies, for all $\mathfrak{A} = \langle \mathrm{SEN}^{\mathfrak{A}}, \langle \mathbf{N}^{\mathfrak{A}}, F^{\mathfrak{A}}\rangle, R^{\mathfrak{A}}\rangle$ and all $\Sigma \in |\mathbf{Sign}^{\mathfrak{A}}|$, $\mathfrak{A} \models_{\Sigma} \Gamma$ implies that $\mathfrak{A} \models_{\Sigma} \alpha$. Therefore, we obtain, by the definition of $\models$ that $\Gamma \models \alpha$. Conversely, if $\Gamma \models \alpha$, then, since ordinary first-order $\mathcal{L}$-structures form a subclass of the class of all $\mathcal{L}$-systems, we get that $\Gamma \models \alpha$ in the ordinary first-order sense and, therefore, by the completeness part of Gödel's Completeness Theorem, we obtain that $\Gamma \vdash \alpha$. □

We intend to continue the work presented in this paper with the goal of abstracting several of Elgueta's results to the present framework. Elgueta's results generalize well-known results of the theory of models of first-order logic to the equality-free context. The present framework leads to further generalization of these results to a multi-signature equality-free context.

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