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n-closure systems and n-closure operators

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This paper is dedicated to Walter Taylor.

Abstract. It is very well known and permeating the whole of mathematics that a closure operator on a given set gives rise to a closure system, whose constituent sets form a complete lattice under inclusion, and vice-versa. Recent work of Wille on triadic concept analysis and subsequent work by the author on polyadic concept analysis led to the introduction of complete trilattices and complete n-lattices, respectively, that generalize complete lattices and capture the order-theoretic structure of the collection of concepts associated with polyadic formal contexts. In the present paper, polyadic closure operators and polyadic closure systems are introduced and they are shown to be in a relationship similar to the one that exists between ordinary (dyadic) closure operators and ordinary (dyadic) closure systems. Finally, the algebraic case is given some special consideration.

1. Background: Closure operators and n-ordered Sets

This section contains a brief account of the well-known correspondence between closure operators and closure systems and of the notion of an n-ordered set. Our main source for the former is [3] (see also [2]) and for the latter [10] (see also [9] for the triadic case).

Given a set X, a family $\mathcal{L}$ of subsets of X, such that

- $X \in \mathcal{L}$ and
- $\{A_i : i \in I\} \subseteq \mathcal{L}$ implies $\bigcap_{i \in I} A_i \in \mathcal{L}$,

is said to be a *closure system* or a *topped intersection structure* and it forms a complete lattice under inclusion. The meet and join, respectively, are given by

$$\bigwedge_{i \in I} A_i = \bigcap_{i \in I} A_i, \qquad \bigvee_{i \in I} A_i = \bigcap \big\{B \in \mathcal{L} : \bigcup_{i \in I} A_i \subseteq B\big\}.$$

On the other hand, given a set X, a *closure operator* on X is a function $C \colon \mathcal{P}(X) \to \mathcal{P}(X)$, such that, for all $A, B \subseteq X$,

- $A \subseteq C(A)$, (C is *inflationary*)

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- $A \subseteq B$ implies $C(A) \subseteq C(B)$ (C is *monotonic*) and
- $C(C(A)) = C(A)$ (C is *idempotent*).

Theorem 2.21 of [3] establishes a correspondence between closure systems and closure operators given by sending a closure operator C to the closure system

$$\mathcal{L}_C = \{A \subseteq X : C(A) = A\}$$

and the closure system $\mathcal{L}$ to the closure operator $C_{\mathcal{L}}$, such that for all $A \subseteq X$,

$$C_{\mathcal{L}}(A) = \bigcap\{B \in \mathcal{L} : A \subseteq B\}.$$

This correspondence restricts to a correspondence between algebraic closure systems and algebraic closure operators. A closure system is said to be *algebraic* if the union of any directed subfamily of sets in the closure system is also closed. A closure operator is *algebraic* if, for all $A \subseteq X$,

$$C(A) = \bigcup\{C(B) : B \subseteq_\omega A\},$$

where $\subseteq_\omega$ denotes the "finite subset" relation. Theorem 3.8 of [3] shows that C is an algebraic closure operator on X if and only if $\mathcal{L}_C$, as defined above, is an algebraic closure system.

Besides this correspondence between closure systems and closure operators, also of interest to us will be a specific way of generating a closure operator using what are called Galois connections. Given two sets X and Y, two mappings $f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ and $g\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ are said to *form a Galois connection* if and only if, for all $A \subseteq X$ and $B \subseteq Y$, we have

$$A \subseteq g(B) \quad \text{iff} \quad B \subseteq f(A).$$

Another equivalent formulation of a Galois connection is that f and g satisfy the following four conditions:

- $A_1 \subseteq A_2$ implies $f(A_2) \subseteq f(A_1)$ for all $A_1, A_2 \subseteq X$,
- $B_1 \subseteq B_2$ implies $g(B_2) \subseteq g(B_1)$ for all $B_1, B_2 \subseteq Y$,
- $A \subseteq g(f(A))$ for all $A \subseteq X$,
- $B \subseteq f(g(B))$ for all $B \subseteq Y$.

Every Galois connection gives rise to a closure operator on X, defined by $C(A) = g(f(A))$ for all $A \subseteq X$. The reader familiar with formal concept analysis will recognize in Galois connections the process that, starting from a formal context, gives rise to the concept lattice associated with the formal context (see [7, 8] and also [4]). An extension of this process [9] gives rise, starting from a triadic formal context, to the complete trilattice of formal concepts associated to the triadic context and, more generally in [10], starting from an n-adic formal context, to the complete n-lattice of its n-adic formal concepts.

A brief account of some aspects of n-ordered sets that will be needed in what follows is now given. For more details on the material reviewed here the reader is referred to [9] and [1] for the triadic case and to [10, 11] for the general n-adic case. Related to the topic is the more recent work [12].

An *ordinal structure* $\mathbf{P} = \langle P, \lesssim_1, \lesssim_2, \ldots, \lesssim_n \rangle$ is a relational structure whose n relations are quasiorders. Let $\sim_i = \lesssim_i \cap \gtrsim_i$ for $i = 1, 2, \ldots, n$. An *n-ordered set* $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ is an ordinal structure, such that for all $x, y \in P$ and all $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$,

(1) $x \sim_{i_1} y, \ldots, x \sim_{i_n} y$ imply $x = y$ (*Uniqueness Condition*)

(2) $x \lesssim_{i_1} y, \ldots, x \lesssim_{i_{n-1}} y$ imply $x \gtrsim_{i_n} y$ (*Antiordinal Dependency*)

Each quasiorder $\lesssim_i$ induces an order $\leq_i$ on the set of equivalence classes $P/{\sim_i} = \{[x]_i : x \in P\}, i = 1, 2, \ldots, n$, where $[x]_i = \{y \in P : x \sim_i y\}$.

Let $\mathbf{P} = \langle P, \lesssim_1, \lesssim_2, \ldots, \lesssim_n \rangle$ be an n-ordered set, $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$ and $X_1, X_2, \ldots, X_{n-1} \subseteq P$.

An element $b \in P$ is called a $(j_{n-1}, \ldots, j_1)$-*bound* of $(X_{n-1}, X_{n-2}, \ldots, X_1)$ if $x_i \lesssim_{j_i} b$, for all $x_i \in X_i$ and all $i = 1, \ldots, n-1$. The set of all $(j_{n-1}, \ldots, j_1)$-bounds of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$.

A $(j_{n-1}, \ldots, j_1)$-bound $l \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$ of $(X_{n-1}, \ldots, X_1)$ is called a $(j_{n-1}, \ldots, j_1)$-*limit* of $(X_{n-1}, \ldots, X_1)$ if $l \gtrsim_{j_n} b$, for all $(j_{n-1}, \ldots, j_1)$-bounds $b \in (X_{n-1}, \ldots, X_1)^{(j_{n-1}, \ldots, j_1)}$. The set of all $(j_{n-1}, \ldots, j_1)$-limits of $(X_{n-1}, \ldots, X_1)$ is denoted by $(X_{n-1}, \ldots, X_1)^{\overline{(j_{n-1}, \ldots, j_1)}}$.

Proposition 1.1. *Let $\mathbf{P} = \langle P, \lesssim_1, \ldots, \lesssim_n \rangle$ be an n-ordered set, $X_1, \ldots, X_{n-1} \subseteq P$ and $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. Then there exists at most one $(j_{n-1}, \ldots, j_1)$-limit $\bar{l}$ of $(X_{n-1}, \ldots, X_1)$ satisfying*

(C) *$\bar{l}$ is the largest in $\lesssim_{j_2}$ among the largest limits in $\lesssim_{j_3}$ among $\cdots$ among the largest limits in $\lesssim_{j_{n-1}}$ among the largest limits in $\lesssim_{j_n}$ or, equivalently,*

(C′) *$\bar{l}$ is the smallest in $\lesssim_{j_1}$ among the largest limits in $\lesssim_{j_3}$ among $\cdots$ among the largest limits in $\lesssim_{j_{n-1}}$ among the largest limits in $\lesssim_{j_n}$.*

Proposition 1.1 follows relatively easily by combining the definition of an n-ordered set with those of a bound and of a limit.

If a $(j_{n-1}, \ldots, j_1)$-limit satisfying the statement in Proposition 1.1 exists, it is called the $(j_{n-1}, \ldots, j_1)$-*join* of $(X_{n-1}, \ldots, X_1)$ and will be denoted by $\nabla_{j_{n-1}, \ldots, j_1}(X_{n-1}, \ldots, X_1)$.

A *complete n-lattice* $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ is an n-ordered set in which all $(j_{n-1}, \ldots, j_1)$-joins $\nabla_{j_{n-1}, \ldots, j_1}(X_{n-1}, \ldots, X_1)$ exist, for all $X_1, \ldots, X_{n-1} \subseteq L$ and all $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$. A complete n-lattice is *bounded* by

$$0_{j_n} := \nabla_{j_{n-1}, \ldots, j_1}(L, \ldots, L),$$

where $\{j_1, \ldots, j_n\} = \{1, \ldots, n\}$.

Recall from [10] that, given an n-adic context $(K_1,\dots,K_n,Y)$ and $X_i \subseteq K_{j_i}, i = 1,\dots,n-1$, by $\mathfrak{b}_{j_{n-1},\dots,j_1}(X_{n-1},\dots,X_1)$ is denoted the n-adic concept $(C_1,\dots,C_n)$ with the property that it has the largest j_2-component among all n-adic concepts with the largest j_3-component among those with the largest j_4-component $\cdots$ among all those with the largest j_n-component among those satisfying $X_i \subseteq C_i, i \neq j_n$. In other words, $\mathfrak{b}_{j_{n-1},\dots,j_1}$ is an operator that can be used to form the $(j_{n-1},\dots,j_1)$-join of n-adic concepts under the component-wise quasi-orderings. It is generated by first closing with respect to the j_n-th component, then closing with respect to the j_{n-1}-st component, etc, down to finally closing with respect to the j_1-st component. More precisely, we have

$$\mathfrak{b}_{j_{n-1},\dots,j_1}(X_{n-1},\dots,X_1)_{j_n} = \{x_{j_n} \in K_{j_n} : (x_1,\dots,x_n) \in Y, \text{ for all } x_i \in X_i, i \neq j_n\},$$

and, if, for all $i = k+1,\dots,n$, $Z_{j_i} = \mathfrak{b}_{j_{n-1},\dots,j_1}(X_{n-1},\dots,X_1)_{j_i}$ has already been defined,

$$\mathfrak{b}_{j_{n-1},\dots,j_1}(X_{n-1},\dots,X_1)_{j_k} = \{x_{j_k} \in K_{j_k} : (x_1,\dots,x_n) \in Y, \text{ for all } x_{j_i} \in X_i, i < k, \text{ and } x_{j_i} \in Z_{j_i}, i > k\},$$

for all $k = 1,\dots,n-1$. The ordinal structure of the n-adic concepts of an n-adic formal context forms a complete n-lattice and is the prototypical example that gave rise to the notion of a complete n-lattice.

2. n-closure systems and n-closure operators

Definition 2.1. Let $K_1,K_2,\dots,K_n$ be arbitrary sets. An *n-closure operator* is a mapping C from $\mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_{n-1})$ to $\mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_n)$ such that the following conditions hold:

(1) If $C(X_1,\dots,X_{n-1}) = (A_1,\dots,A_n)$ and $x \in X_i$, then $x \in A_i$ for all $i = 1,\dots,n-1$.
(2) If $C(X_1,\dots,X_{n-1}) = (A_1,\dots,A_n)$, $C(Y_1,\dots,Y_{n-1}) = (B_1,\dots,B_n)$ and $X_i \subseteq Y_i$, for $i = 1,\dots,n-1$, then $B_n \subseteq A_n$.
(3) If $C(X_1,\dots,X_{n-1}) = (A_1,\dots,A_n)$, $C(Y_1,\dots,Y_{n-1}) = (B_1,\dots,B_n)$ and for $k = 1,\dots,n-1$, we have $X_i \subseteq Y_i$, $i \leq k$, and $A_i = B_i$, $i > k$, then $B_k \subseteq A_k$.
(4) If $C(X_1,\dots,X_{n-1}) = (A_1,\dots,A_n)$, $C(Y_1,\dots,Y_{n-1}) = (B_1,\dots,B_n)$ and for $k = 1,\dots,n$, we have $A_i \subseteq B_i$, for $i \neq k$, then $B_k \subseteq A_k$.
(5) If $C(X_1,\dots,X_{n-1}) = (A_1,\dots,A_n)$, then $C(A_1,\dots,A_{n-1}) = (A_1,\dots,A_n)$.

If $C(X_1,\dots,X_{n-1}) = (A_1,\dots,A_n)$, then we adopt the notation

$$C_i(X_1,\dots,X_{n-1}) := A_i, \quad i = 1,\dots,n,$$

with the *warning* that this is just a notational convention and it is *not meant* to imply that C_i is a closure operator in the traditional sense.

A few comments are now given on the conditions in Definition 2.1. First, from the point of view of an ordinary closure operator, the close relationship of Condition 1 with the property of being inflationary and that of Condition 5 with idempotency should be noted. Monotonicity is not given by a single condition in Definition 2.1 because in the ordinary (dyadic) case, i.e., when $n = 2$, it is inferred by Conditions 2 and 4. On the other hand, from the point of view of n-ordered sets, Conditions 4 and 5, taken jointly, imply that the collection of all closed n-tuples form an n-ordered set under component-wise inclusions. This is because Condition 4 reflects the antiordinal dependency law of n-ordered sets and because, in this context, the uniqueness condition holds automatically. Condition 1 provides the bound property of $(n-1,\ldots,1)$-joins in an n-ordered set. In turn, Condition 2 supplies the limit property of $(n-1,\ldots,1)$-joins and Condition 3 yields the remaining properties of an $(n-1,\ldots,1)$-join in the same context.

Given n sets $K_1,\ldots,K_n$ and $A_i, B_i \subseteq K_i$, $i = 1,\ldots,n$, define $\subseteq_i$ for $i = 1,\ldots,n$ by

$$(A_1,\ldots,A_n) \subseteq_i (B_1,\ldots,B_n) \quad \text{iff} \quad A_i \subseteq B_i.$$

Definition 2.2. Let $K_1,\ldots,K_n$ be n sets. An *n-closure system* $\mathcal{L}$ is defined to be a collection of n-tuples of subsets $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$, such that,

(1) $(A_1,\ldots,A_n) \subseteq_i (B_1,\ldots,B_n), i \neq k$, imply $(B_1,\ldots,B_n) \subseteq_k (A_1,\ldots,A_n)$, for all $k = 1,\ldots,n$,

(2) for all $X_i \subseteq K_i, i = 1,\ldots,n-1$, there exists unique $A = (A_1,\ldots,A_n) \in \mathcal{L}$ such that A has the largest second component among all n-tuples in $\mathcal{L}$ with the largest third component among $\cdots$ among all n-tuples with the largest n-th component among all n-tuples $B = (B_1,\ldots,B_n)$ in $\mathcal{L}$ such that $X_i \subseteq B_i, i = 1,\ldots,n-1$.

Using a notation like the one introduced in [9] and adopted in [10], we denote the element $A \in \mathcal{L}$ of Condition 2 in Definition 2.2 by $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)$.

Now, some explanations concerning Definition 2.2 are provided. Consider the relational structure $\mathcal{L} = \langle \mathcal{L}, \subseteq_1,\ldots,\subseteq_n\rangle$. It is easy to see that $\mathcal{L}$ is an ordinal structure, i.e., that $\subseteq_i$ is a quasi-order, for all $i = 1,\ldots,n$. Furthermore, in $\mathcal{L}$ the uniqueness condition holds automatically, since, if $(A_1,\ldots,A_n) \sim_i (B_1,\ldots,B_n)$ for some $i = 1,\ldots,n$, then $A_i \subseteq B_i$ and $B_i \subseteq A_i$, whence $A_i = B_i$. Therefore, if $(A_1,\ldots,A_n) \sim_i (B_1,\ldots,B_n)$ for all $i = 1,\ldots,n$, then $(A_1,\ldots,A_n) = (B_1,\ldots,B_n)$. Finally, Condition 1 in Definition 2.2 requires that $\mathcal{L}$ satisfy the antiordinal dependency law. Therefore, Condition 1 implies that $\mathcal{L}$ is an n-ordered set. Because, by Condition 1, $\mathcal{L}$ is an n-ordered set, it makes sense to ask whether $(n-1,\ldots,1)$-joins exist in $\mathcal{L}$. Condition 2 in Definition 2.2 requires that they do, and, in fact, gives a specific way of finding them. This process is described briefly here. It is not very transparent in the phrasing of Condition 2, but might be clear to readers familiar with polyadic concept analysis or complete n-lattices. Given $\mathcal{A}_i \subseteq \mathcal{L}, i = 1, \ldots, n-1$, the $(n-1, \ldots, 1)$-join of $(\mathcal{A}_{n-1}, \ldots, \mathcal{A}_1)$ is the unique element $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, whose existence is required by Condition 2 of Definition 2.2, for

$$X_i := \bigcup\{A_i : (A_1, \ldots, A_n) \in \mathcal{A}_i\}, \quad \text{for all } i = 1, \ldots, n-1.$$

Another interesting observation that can be made concerning Definition 2.2 is that its second condition is equivalent to a collection of complex interdependent closure conditions on the components of $\mathcal{L}$. These conditions would make the definition look more similar in flavor to the one of ordinary (dyadic) closure systems, but would have been much more complicated to formulate. We glance at this line of thought now and postpone further discussion on the dyadic case until the last section. Condition 2 implies, for instance, that, given $B_i \subseteq K_i, i = 2, \ldots, n-1$, if the collection $\mathcal{A} \subseteq \mathcal{L}$ with

$$\mathcal{A} = \{(A_1, \ldots, A_n) \in \mathcal{A} : A_i = B_i, i = 2, \ldots, n-1\}$$

is nonempty, then it satisfies

$$\left(\bigcap\{A_1 : (A_1, \ldots, A_n) \in \mathcal{A}\}, B_2, \ldots, B_{n-1}, \bigcup\{A_n : (A_1, \ldots, A_n) \in \mathcal{A}\}\right) \in \mathcal{A},$$

since this n-tuple is necessarily the unique n-tuple B in $\mathcal{L}$, such that B has the largest second component among all n-tuples in $\mathcal{L}$ with the largest third component among $\cdots$ among all n-tuples with the largest n-th component among all n-tuples $F = (F_1, \ldots, F_n)$ in $\mathcal{L}$ such that

$$\bigcap\{A_1 : (A_1, \ldots, A_n) \in \mathcal{A}\} \subseteq F_1, \quad B_2 \subseteq F_2, \ldots, B_{n-1} \subseteq F_{n-1}.$$

The following lemma shows how an n-closure system arises from a given n-closure operator. This situation parallels the well-known relationship between ordinary closure operators and closure systems.

Lemma 2.3. *Suppose that $C \colon \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an n-closure operator. Then the collection*

$$\mathcal{L} = \{(A_1, \ldots, A_n) \in \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n) : C(A_1, \ldots, A_{n-1}) = (A_1, \ldots, A_n)\}$$

is an n-closure system.

Proof. First suppose that for $(A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathcal{L}$, we have $(A_1, \ldots, A_n) \subseteq_i (B_1, \ldots, B_n)$ for all $i \neq k$, for some $k = 1, \ldots, n$. Therefore, we obtain, by the definition of $\mathcal{L}$, that $C(A_1, \ldots, A_{n-1}) \subseteq_i C(B_1, \ldots, B_{n-1})$ for all $i \neq k$. Therefore, by Condition 4 of the definition of an n-closure operator, we obtain that $B_k \subseteq A_k$, whence $(B_1, \ldots, B_n) \subseteq_k (A_1, \ldots, A_n)$, and Condition 1 in the definition of an n-closure system holds.

Finally, for Condition 2 of an n-closure system, suppose that $X_i \subseteq K_i, i = 1, \ldots, n-1$. It will be shown that there exists $J \in \mathcal{L}$ such that

$$X_i \subseteq J_i, \quad i = 1, \ldots, n-1, \tag{1}$$

and such that it has the largest second component among all those n-tuples in $\mathcal{L}$ with the largest third component, and so on, among all n-tuples in $\mathcal{L}$ with the largest n-th component among all n-tuples in $\mathcal{L}$ satisfying (1). Because of Condition 5 of the definition of an n-closure operator, it suffices to show that the n-tuple

$$(J_1, \ldots, J_n) := C(X_1, \ldots, X_{n-1}) \in \mathcal{L}$$

satisfies the required property. By Condition 1 of the definition of an n-closure operator, we get that $X_i \subseteq C_i(X_1, \ldots, X_{n-1})$, i.e., that $X_i \subseteq J_i$, for all $i = 1, \ldots, n-1$. Suppose next that $A = (A_1, \ldots, A_n)$ is another n-tuple in $\mathcal{L}$ such that $X_i \subseteq A_i, i = 1, \ldots, n-1$. Then, by Condition 5 in the definition of an n-closure operator, we get that $A = C(A_1, \ldots, A_{n-1})$, whence we obtain, by Condition 2 and the assumption that $X_i \subseteq A_i, i = 1, \ldots, n-1$, that $A_n \subseteq J_n$. This shows that J_n is the largest possible among all candidate A_n's. Next, suppose, in addition, that A is an n-tuple that has the largest n-th component among all n-tuples satisfying (1). We must have $A_n = J_n$. Then, by Conditions 3 and 5 of the definition of an n-closure operator, we get that $A_{n-1} \subseteq J_{n-1}$. We continue down to $n-2, \ldots, 2$ in the same way, finally obtaining that $C(X_1, \ldots, X_{n-1})$ is the unique n-tuple in $\mathcal{L}$ that has the largest second component among all n-tuples with the largest third component, etc., among all those with the largest n-th component among all those satisfying Property (1).

Hence Condition 2 of the definition of an n-closure system is also satisfied and $\mathcal{L}$ is indeed an n-closure system. □

Given an n-closure operator C, the n-closure system $\mathcal{L}$ of Lemma 2.3 will be denoted by $\mathcal{L}_C$ following similar notation adopted in [3] for the dyadic case.

Next, it is shown, again in parallel with the case of ordinary closure systems and closure operators, that an n-closure system gives rise to an n-closure operator. Before stating the relevant lemma, the reader is reminded that the notation $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$ was introduced to denote the unique element in $\mathcal{L}$ provided by Condition 2 in the definition of an n-closure system.

Lemma 2.4. *Suppose that $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an n-closure system. The mapping $C \colon \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$, given by*

$$C(X_1, \ldots, X_{n-1}) = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1),$$

for all $X_i \subseteq K_i, i = 1, \ldots, n-1$, is an n-closure operator.

Proof. Suppose that $X_i \subseteq K_i$ for all $i = 1, \ldots, n-1$, and consider $x \in X_i$ for some $i = 1, \ldots, n-1$. Since, by the definition of $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, we have that $X_i \subseteq \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_i$, we get that $x \in X_i \subseteq C_i(X_1, \ldots, X_n)$ and Condition 1 of the definition of an n-closure operator is satisfied.

Next suppose that $X_i, Y_i \subseteq K_i$ for all $i = 1, \ldots, n-1$, and let $(A_1, \ldots, A_n) = C(X_1, \ldots, X_{n-1})$ and $(B_1, \ldots, B_n) = C(Y_1, \ldots, Y_{n-1})$. If $X_i \subseteq Y_i$ for all $i = 1, \ldots, n-1$, then $\mathfrak{b}_{n-1,\ldots,1}(Y_{n-1}, \ldots, Y_1)$, which is, by definition, an $(n-1, \ldots, 1)$-bound of $(Y_{n-1}, \ldots, Y_1)$, must also be an $(n-1, \ldots, 1)$-bound of $(X_{n-1}, \ldots, X_1)$. Therefore, by the limit property in the definition of $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, we get that

$$B_n = \mathfrak{b}_{n-1,\ldots,1}(Y_{n-1}, \ldots, Y_1)_n \subseteq \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_n = A_n.$$

So Condition 2 of the definition of an n-closure operator is satisfied.

For Condition 3, it must be shown that if, $(A_1, \ldots, A_n) = C(X_1, \ldots, X_{n-1})$, $(B_1, \ldots, B_n) = C(Y_1, \ldots, Y_{n-1})$, and for some $k = 1, \ldots, n-1$, we have that $X_i \subseteq Y_i$ for all $i \leq k$ and $A_i = B_i$ for all $i > k$, then $B_k \subseteq A_k$. Indeed, this must be the case, by the $(n-1, \ldots, 1)$-join property in the definition of $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$ and the fact that $(A_1, \ldots, A_n) = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$.

Condition 4 is a straightforward consequence of Condition 1 in the definition of an n-closure system and of the fact that $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1) \in \mathcal{L}$.

To show that Condition 5 is satisfied, suppose that

$$(A_1, \ldots, A_n) = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$$

and consider the two n-tuples $\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)$ and $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$. Since $X_i \subseteq A_i$ for all $i = 1, \ldots, n-1$, we get, by the limit property of the n-tuple $\mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, that

$$\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)_n \subseteq \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_n = A_n.$$

On the other hand, $(A_1, \ldots, A_n)$ is an n-tuple in $\mathcal{L}$, that is an $(n-1, \ldots, 1)$-bound of $(A_{n-1}, \ldots, A_1)$, whence, by the limit property of $\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)$, we have that $A_n \subseteq \mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)_n$. This proves that

$$\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)_n = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_n.$$

Now, by Condition 3 of the definition of an n-closure operator, that has already been shown to hold, we obtain that

$$\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)_{n-1} \subseteq \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_{n-1} = A_{n-1},$$

whence, once more by the join property of $\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)$, we obtain that

$$\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1)_{n-1} = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)_{n-1}.$$

Work now in the same way down to $n = 2$. This will then conclude the proof that $\mathfrak{b}_{n-1,\ldots,1}(A_{n-1}, \ldots, A_1) = \mathfrak{b}_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$ holds, i.e., that C is idempotent.

All conditions in the definition of an n-closure operator having been demonstrated, we conclude that C is indeed an n-closure operator. $\square$

Given an n-closure system $\mathcal{L}$, the n-closure operator C associated with it, via Lemma 2.4, will be denoted by $C_{\mathcal{L}}$, also following the notation of [3].

The two processes described in Lemmas 2.3 and 2.4 are inverses of each other much in the same way that closure operators and closure systems are inverse constructions in the ordinary (2-dimensional) lattice theory (see, e.g. [3]).

Theorem 2.5. *Let C be an n-closure operator. Then the n-closure operator $C_{\mathcal{L}_C}$ is identical with C. Similarly, given an n-closure system $\mathcal{L}$, the n-closure system $\mathcal{L}_{C_{\mathcal{L}}}$ is identical with $\mathcal{L}$.*

Proof. Suppose, first, that $C\colon \mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_n)$ is an n-closure operator. Pass to the n-closure system $\mathcal{L}_C = \{C(X_1,\dots,X_{n-1}) : X_i \subseteq K_i, i = 1,\dots,n-1\}$ and consider the n-closure operator $C_{\mathcal{L}_C} := \mathfrak{b}_{n-1,\dots,1}$. In the proof of Lemma 2.3 it was established that $C_{\mathcal{L}_C} = C$.

Suppose, similarly, that $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an n-closure system. Pass to the n-closure operator $C_{\mathcal{L}} := \mathfrak{b}_{n-1,\dots,1}$. If $A = (A_1,\dots,A_n) \in \mathcal{L}$, then $A = \mathfrak{b}_{n-1,\dots,1}(A_{n-1},\dots,A_1)$, whence $A \in \mathcal{L}_{C_{\mathcal{L}}}$. On the other hand, if $A \in \mathcal{L}_{C_{\mathcal{L}}}$, then $A = C_{\mathcal{L}}(A_1,\dots,A_{n-1}) := \mathfrak{b}_{n-1,\dots,1}(A_{n-1},\dots,A_1)$, whence $A \in \mathcal{L}$. Therefore $\mathcal{L}=\mathcal{L}_{C_{\mathcal{L}}}$. $\square$

3. Algebraic n-closure systems and n-closure operators

Definition 3.1. An n-closure operator

$$C\colon \mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1)\times\cdots\times\mathcal{P}(K_n)$$

is said to be *algebraic* if, for all $X_i \subseteq K_i, i = 1,\dots,n-1$, we have that

$$C_i(X_1,\dots,X_{n-1}) = \bigcup_{\substack{Y_i \subseteq_\omega X_i,\\ i=1,\dots,n-1}} C_i(Y_1,\dots,Y_{n-1}),$$

for all $i = 1,\dots,n-1$.

In order to define the corresponding notion for an n-closure system, the notion of an i-directed set in an n-ordered space is needed.

Let S be a non-empty subset of an n-ordered set $\mathbf{P} = \langle P, \lesssim_1,\dots,\lesssim_n\rangle$. S is said to be *i-directed* if, for every finite subset $F \subseteq S$, there exists $z \in S$ such that $s \lesssim_i z$ for all $s \in F$. As a consequence, S is *i-directed, for all* $i = 1,\dots,n-1$, if, for every finite subset $F \subseteq S$, there exist $z_1,\dots,z_{n-1} \in S$, such that, for all $s \in F$, $s \lesssim_i z_i$, for all $i = 1,\dots,n-1$.

Definition 3.2. Let $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ be an n-closure system. $\mathcal{L}$ is said to be *algebraic* if, for every collection $\mathcal{A} \subseteq \mathcal{L}$ such that $\mathcal{A}$ is i-directed for all $i = 1, \ldots, n-1$, there exists (a necessarily unique) $B = (B_1, \ldots, B_n) \in \mathcal{L}$ such that

$$B_i = \bigcup\{A_i : (A_1, \ldots, A_n) \in \mathcal{A}\}, \quad \text{for all } i = 1, \ldots, n-1.$$

The next theorem relates algebraic n-closure systems and algebraic n-closure operators. It is an analog of a well-known theorem relating ordinary algebraic closure systems with ordinary algebraic closure operators (see [3], Theorem 3.8).

Theorem 3.3. *Let C be an n-closure operator and $\mathcal{L}$ the associated n-closure system as related by Theorem 2.5. Then the following are equivalent:*

(1) *C is an algebraic n-closure operator.*
(2) *For all directed families $\{X_j^i\}_{j \in J_i} \subseteq \mathcal{P}(K_i), i = 1, \ldots, n-1$, we have that*

$$C_i(\bigcup_{j \in J_1} X_j^1, \ldots, \bigcup_{j \in J_{n-1}} X_j^{n-1}) = \bigcup\{C_i(X_{j_1}^1, \ldots, X_{j_{n-1}}^{n-1}) : j_i \in J_i, i = 1, \ldots, n-1\},$$

for all $i = 1, \ldots, n-1$.
(3) *$\mathcal{L}$ is an algebraic n-closure system.*

Proof. $(1 \to 2)$ Suppose that $C\colon \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an algebraic n-closure operator and consider directed families $\{X_j^i\}_{j \in J_i} \subseteq \mathcal{P}(K_i)$, $i = 1, \ldots, n-1$. Definition 3.1 immediately implies that an algebraic n-closure operator is monotonic in each of the first $n-1$ coordinates, i.e., if $X_i \subseteq Y_i$ for all $i = 1, \ldots, n-1$, then $C_i(X_1, \ldots, X_{n-1}) \subseteq C_i(Y_1, \ldots, Y_{n-1})$ for all $i = 1, \ldots, n-1$. Therefore, we get that

$$\bigcup\{C_i(X_{j_1}^1, \ldots, X_{j_n}^n) : j_i \in J_i, i = 1, \ldots, n-1\} \subseteq C_i(\bigcup_{j \in J_1} X_j^1, \ldots, \bigcup_{j \in J_{n-1}} X_j^{n-1}),$$

for all $i = 1, \ldots, n-1$.

Suppose, conversely, that $x_i \in C_i(\bigcup_{j \in J_1} X_j^1, \ldots, \bigcup_{j \in J_{n-1}} X_j^{n-1})$. Then, since C is algebraic, there exist finite subsets $X_i \subseteq \bigcup_{j \in J_i} X_j^i, i = 1, \ldots, n-1$, such that $x_i \in C_i(X_1, \ldots, X_{n-1})$. But, by hypothesis, $\{X_j^i\}_{j \in J_i}$ is directed, whence there exist $j_i \in J_i$ such that $X_i \subseteq X_{j_i}^i, i = 1, \ldots, n-1$. Therefore, we obtain, again by monotonicity, that $x \in C_i(X_{j_1}^1, \ldots, X_{j_{n-1}}^{n-1})$. Hence, we get

$$C_i(\bigcup_{j \in J_1} X_j^1, \ldots, \bigcup_{j \in J_{n-1}} X_j^{n-1}) \subseteq \bigcup\{C_i(X_{j_1}^1, \ldots, X_{j_{n-1}}^{n-1}) : j_i \in J_i, i = 1, \ldots, n-1\},$$

for all $i = 1, \ldots, n-1$.

$(2 \to 3)$ Suppose now that for all directed families $\{X_j^i\}_{j \in J_i} \subseteq \mathcal{P}(K_i)$, $i = 1, \ldots, n-1$, we have that

$$C_i(\bigcup_{j \in J_1} X_j^1, \ldots, \bigcup_{j \in J_{n-1}} X_j^{n-1}) = \bigcup\{C_i(X_{j_1}^1, \ldots, X_{j_{n-1}}^{n-1}) : j_i \in J_i, i = 1, \ldots, n-1\},$$

for all $i = 1, \ldots, n-1$. To show that $\mathcal{L}$ is an algebraic n-closure system, it suffices to show, in view of Theorem 2.5, that, given $\mathcal{A} \subseteq \mathcal{L}$, such that $\mathcal{A}$ is i-directed, for all $i = 1, \ldots, n-1$,

$$C_i(\bigcup_{A\in\mathcal{A}} A_1, \ldots, \bigcup_{A\in\mathcal{A}} A_{n-1}) = \bigcup_{A\in\mathcal{A}} A_i, \quad \text{for all } i = 1, \ldots, n-1.$$

Since the right-to-left inclusion is obvious, suppose that for some $x \in K_i$, $x \in C_i(\bigcup_{A\in\mathcal{A}} A_1, \ldots, \bigcup_{A\in\mathcal{A}} A_{n-1})$. Then, by the hypothesis,

$$x \in \bigcup\{C_i(A_1^1, \ldots, A_{n-1}^{n-1}) : A^1, \ldots, A^{n-1} \in \mathcal{A}\}.$$

Now, find, by directedness in all components, $A \in \mathcal{A}$, such that $\bigcup_{j=1}^{n-1} A_i^j \subseteq A_i$, for all $i = 1, \ldots, n-1$. Then we have $C_i(A_1^1, \ldots, A_{n-1}^{n-1}) \subseteq C_i(A_1, \ldots, A_{n-1}) = A_i$. This shows that

$$\bigcup\{C_i(A_1^1, \ldots, A_{n-1}^{n-1}) : A^1, \ldots, A^{n-1} \in \mathcal{A}\} \subseteq \bigcup_{A\in\mathcal{A}} A_i,$$

whence $x \in \bigcup_{A\in\mathcal{A}} A_i$, as was to be shown.

$(3 \to 1)$ Suppose now that $\mathcal{L}$ is an algebraic n-closure system. Let $X_i \subseteq K_i, i = 1, \ldots, n-1$. We need to show that

$$C_i(X_1, \ldots, X_{n-1}) = \bigcup_{\substack{Y_i \subseteq_\omega X_i,\\ i=1,\ldots,n-1}} C_i(Y_1, \ldots, Y_{n-1}),$$

for all $i = 1, \ldots, n-1$. Right-to-left inclusion is obvious. For left-to-right inclusion, a similar trick with the one that is used in the dyadic case is employed. Set

$$\mathcal{A} = \{C(Y_1, \ldots, Y_{n-1}) : Y_i \subseteq_\omega X_i : i = 1, \ldots, n-1\}.$$

Then, noting that $\mathcal{A}$ is i-directed for all $i = 1, \ldots, n-1$, we obtain, taking into account the hypothesis, $C_i(X_1, \ldots, X_{n-1}) \subseteq C_i(\bigcup_{A\in\mathcal{A}} A_1, \ldots, \bigcup_{A\in\mathcal{A}} A_{n-1}) = \bigcup_{A\in\mathcal{A}} A_i$. Therefore C is an algebraic n-closure operator. $\square$

4. Complete n-semilattices

In this section abstract algebraic structures, called *complete n-semilattices*, are introduced and they are shown to be related to n-closure operators in the same way as complete lattices are related to closure operators in the dyadic case. Similarly, *algebraic n-semilattices* are introduced and shown to be related to algebraic n-closure operators in a way analogous to the way algebraic lattices are related to algebraic closure operators.

Definition 4.1. An n-ordered set $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ is called a *complete n-semilattice* if, for all $X_1, \ldots, X_{n-1} \subseteq L$, we have that $\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, the $(n-1, \ldots, 1)$-join of $(X_{n-1}, \ldots, X_1)$, exists in $\mathbf{L}$.

From the definition of an n-closure system, it follows directly that an n-closure system forms a complete n-semilattice under component-wise inclusions.

Proposition 4.2. *Suppose $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an n-closure system; then $\langle \mathcal{L}, \subseteq_1, \ldots, \subseteq_n \rangle$ is a complete n-semilattice.*

What is more interesting is that, by analogy with the dyadic case, every complete n-semilattice may be represented as the complete n-semilattice of closed sets of an n-closure system. The representation is analogous to the representation of a complete lattice, via the principal ideals generated by its elements, as the complete lattice of the closed sets of a closure system. This representation is sketched in Exercise 2.9 of [3]. The proof in the n-adic case is analogous to the dyadic one.

Theorem 4.3. *Suppose that $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ is a complete n-semilattice. Then, there exists an n-closure system $\mathcal{L} \subseteq \mathcal{P}(L)^n$, such that $\langle \mathcal{L}, \subseteq_1, \ldots, \subseteq_n \rangle$ is n-order isomorphic to $\mathbf{L}$.*

Proof. Given an element $x \in L$, denote by $(x]_i$ the $\lesssim_i$-order ideal generated by x in $\mathbf{L}$, i.e.,

$$(x]_i = \{y \in L : y \lesssim_i x\}, \quad \text{for all } i = 1, \ldots, n.$$

Consider the mapping $C \colon \mathcal{P}(L)^{n-1} \to \mathcal{P}(L)^n$ given by

$$C(X_1, \ldots, X_{n-1}) = \big((\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)]_1, \ldots, (\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)]_n\big).$$

It will be shown that C is an n-closure operator and that $\mathbf{L} \cong \langle \mathcal{L}_C, \subseteq_1, \ldots, \subseteq_n \rangle$.

First we show that C is an n-closure operator, i.e., that all five conditions of Definition 2.1 are satisfied.

For Condition 1, suppose that $X_1, \ldots, X_{n-1} \subseteq L$, fix $i = 1, \ldots, n$ and let $x \in X_i$. Then, by the bound property of the $(n-1, \ldots, 1)$-join, we have that $x \lesssim_i \nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, whence $x \in (\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)]_i$, i.e., $x \in C_i(X_1, \ldots, X_{n-1})$.

For Condition 2, suppose that $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1} \subseteq L$, with $X_i \subseteq Y_i$, for all $i = 1, \ldots, n-1$. Then, for all $i = 1, \ldots, n-1$, and all $x \in X_i$, we have that $x \in Y_i$, whence, by the bound property of $\nabla_{n-1,\ldots,1}(Y_{n-1}, \ldots, Y_1)$, we obtain that $\nabla_{n-1,\ldots,1}(Y_{n-1}, \ldots, Y_1)$ is an $(n-1, \ldots, 1)$-bound of $(X_{n-1}, \ldots, X_1)$, and, therefore, by the limit property of $\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$, we obtain that $\nabla_{n-1,\ldots,1}(Y_{n-1}, \ldots, Y_1) \lesssim_n \nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)$. This shows that

$$(\nabla_{n-1,\ldots,1}(Y_{n-1}, \ldots, Y_1)]_n \subseteq (\nabla_{n-1,\ldots,1}(X_{n-1}, \ldots, X_1)]_n,$$

i.e., that $C_n(Y_1, \ldots, Y_{n-1}) \subseteq C_n(X_1, \ldots, X_{n-1})$.

For Condition 3, suppose that $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1} \subseteq L$, with $X_i \subseteq Y_i$, for all $i \leq k$ and that $C_i(X_1, \ldots, X_{n-1}) = C_i(Y_1, \ldots, Y_{n-1})$, for all $i > k$. The last condition implies that, for all $i > k$, we have that

$$(\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)]_i = (\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1)]_i,$$

i.e., that $\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1) \sim_i \nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1)$. The first condition and these last conditions show that $\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1)$ is an $(n-1,\ldots,1)$-bound of $(X_{n-1},\ldots,X_1)$, whence, by the join property of the joins, we get that $\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1) \lesssim_k \nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)$. But this yields

$$(\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1)]_k \subseteq (\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)]_k,$$

i.e., that $C_k(Y_1,\ldots,Y_{n-1}) \subseteq C_k(X_1,\ldots,X_{n-1})$.

Condition 4 is easier. Suppose that $X_1,\ldots,X_{n-1},Y_1,\ldots,Y_{n-1} \subseteq L$, with $C_i(X_1,\ldots,X_{n-1}) = C_i(Y_1,\ldots,Y_{n-1})$, for all $i \neq k$. Then, we get that

$$(\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)]_i \subseteq (\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1)]_i,$$

for all $i \neq k$. Thus, we have that

$$\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1) \lesssim_i \nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1),$$

for all $i \neq k$. Thus, by the antiordinal dependency law in $\mathbf{L}$, we obtain that

$$\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1) \lesssim_k \nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1),$$

which yields

$$(\nabla_{n-1,\ldots,1}(Y_{n-1},\ldots,Y_1)]_k \subseteq (\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)]_k,$$

i.e., $C_k(Y_1,\ldots,Y_{n-1}) \subseteq C_k(X_1,\ldots,X_{n-1})$.

Last, for Condition 5, it should be verified that, for all $X_1,\ldots,X_{n-1} \subseteq L$ and all $i = 1,\ldots,n$, we have that

$$\nabla_{n-1,\ldots,1}((\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)]_{n-1},\ldots,(\nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1)]_1) = \nabla_{n-1,\ldots,1}(X_{n-1},\ldots,X_1).$$

This identity is not difficult to verify based on the bound, the limit and join properties of $(n-1,\ldots,1)$-joins in $\mathbf{L}$.

We can now conclude that C is indeed an n-closure operator and, therefore, that $\mathcal{L}_C$ is an n-closure system. It suffices now to show that $\mathcal{L}_C$, ordered by the coordinate-wise inclusions, forms a complete n-semilattice that is isomorphic to $\mathbf{L}$.

To show that $\mathbf{L} \cong \langle \mathcal{L}_C, \subseteq_1,\ldots,\subseteq_n\rangle$, consider the mapping $\phi\colon L \to \mathcal{L}_C$, defined by

$$\phi(x) = ((x]_1,\ldots,(x]_n), \quad \text{for all } x \in L.$$

It is injective, since, if $x, y \in L$, with $\phi(x) = \phi(y)$, then $x \sim_i y$, for all $i = 1,\ldots,n$, and, by the uniqueness condition, $x = y$. It is surjective, since, if $(A_1,\ldots,A_n) \in \mathcal{L}_C$, then $C(A_1, \dots, A_{n-1}) = (A_1, \dots, A_n)$, whence $C_i(A_1, \dots, A_{n-1}) = A_i$. This shows that $A_i = (\nabla_{n-1,\dots,1}(A_{n-1}, \dots, A_1)]_i$, for all $i = 1, \dots, n$, i.e.,

$$(A_1, \dots, A_n) = \phi(\nabla_{n-1,\dots,1}(A_{n-1}, \dots, A_1)).$$

Finally, ϕ is an order isomorphism: $x \lesssim_i y$ iff $(x]_i \subseteq (y]_i$ iff $\phi(x) \subseteq_i \phi(y)$. $\square$

Next, the algebraic case is considered. Compact elements of a complete n-semilattice are introduced first and pave the way for the introduction of algebraic n-semilattices.

Definition 4.4. Let $\mathbf{L} = \langle L, \lesssim_1, \dots, \lesssim_n \rangle$ be a complete n-semilattice. An element $x \in L$ is said to be *compact* if, for all $D \subseteq L$, with D i-directed, for all $i = 1, \dots, n-1$,

$$(\forall i = 1, \dots, n-1)(x \lesssim_i \nabla_{n-1,\dots,1}(D, \dots, D)) \quad \text{implies} \quad (\exists d \in D)(\forall i = 1, \dots, n-1)(x \lesssim_i d).$$

The complete n-semilattice $\mathbf{L}$ is said to be an *algebraic n-semilattice* if every element $x \in L$ can be expressed as the $(n-1, \dots, 1)$-join $\nabla_{n-1,\dots,1}(A, \dots, A)$, where A is a collection of compact elements.

Next, it is shown that, if $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an algebraic n-closure system and $C_{\mathcal{L}}$ is the associated algebraic n-closure operator, then an element $(A_1, \dots, A_n) \in \mathcal{L}$ is compact if and only if it is of the form $C_{\mathcal{L}}(Y_1, \dots, Y_{n-1})$ for some $Y_i \subseteq_\omega K_i, i = 1, \dots, n-1$. As a consequence, we will be able to show that an algebraic n-closure operator gives rise to an algebraic n-semilattice.

Lemma 4.5. *Suppose that $C\colon \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_{n-1}) \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an algebraic n-closure operator. A closed n-tuple $(A_1, \dots, A_n)$ of C is compact in $\langle \mathcal{L}_C, \subseteq_1, \dots, \subseteq_n \rangle$ if and only if there exist $Y_i \subseteq_\omega K_i, i = 1, \dots, n-1$, such that $(A_1, \dots, A_n) = C(Y_1, \dots, Y_{n-1})$.*

Proof. Suppose, first, that $(A_1, \dots, A_n) = C(Y_1, \dots, Y_{n-1})$, for some $Y_i \subseteq_\omega K_i, i = 1, \dots, n-1$. Let $\mathcal{D} \subseteq \mathcal{L}_C$ be a collection of closed subsets that is i-directed for all $i = 1, \dots, n$, and such that

$$(A_1, \dots, A_n) \subseteq_i \nabla_{n-1,\dots,1}(\mathcal{D}, \dots, \mathcal{D}) = C(\bigcup_{D \in \mathcal{D}} D_1, \dots, \bigcup_{D \in \mathcal{D}} D_{n-1})$$

for all $i = 1, \dots, n-1$. Then, we have that

$$C(Y_1, \dots, Y_{n-1}) \subseteq_i C(\bigcup_{D \in \mathcal{D}} D_1, \dots, \bigcup_{D \in \mathcal{D}} D_{n-1})$$

for all $i = 1, \dots, n-1$. Therefore, we obtain, for all $i = 1, \dots, n-1$, that $Y_i \subseteq C_i(Y_1, \dots, Y_{n-1}) = \bigcup_{D \in \mathcal{D}} D_i$, and, since Y_i is finite and $\mathcal{D}$ is i-directed, there exists $D^i \in \mathcal{D}$, such that $Y_i \subseteq D^i_i$. Since, this holds, for all $i = 1, \dots, n-1$, and $\mathcal{D}$ is i-directed, for all $i = 1, \dots, n-1$, we can find $D \in \mathcal{D}$, such that $D^j_i \subseteq D_i$ for all $j = 1, \ldots, n-1$ and all $i = 1, \ldots, n-1$. This shows that $Y_i \subseteq D_i$ for all $i = 1, \ldots, n-1$, whence, by the monotonicity of an algebraic n-closure operator, we obtain that $(A_1, \ldots, A_n) = C(Y_1, \ldots, Y_{n-1}) \subseteq_i D$ for all $i = 1, \ldots, n-1$, and $(A_1, \ldots, A_n)$ is compact in $\mathcal{L}_C$.

If, conversely, $(A_1, \ldots, A_n)$ is compact in $\mathcal{L}_C$, then, since

$$(A_1, \ldots, A_n) = \bigcup \{C_i(Y_1, \ldots, Y_{n-1}) : Y_i \subseteq_\omega A_i, i = 1, \ldots, n-1\},$$

we get that there exists $Y_i \subseteq_\omega A_i, i = 1, \ldots, n-1$, such that $A_i \subseteq C_i(Y_1, \ldots, Y_{n-1})$ for all $i = 1, \ldots, n-1$. But, by monotonicity, we have $C_i(Y_1, \ldots, Y_{n-1}) \subseteq A_i$ for all $i = 1, \ldots, n-1$. Therefore $A_i = C_i(Y_1, \ldots, Y_{n-1})$, for all $i = 1, \ldots, n-1$, and this, finally, yields that $(A_1, \ldots, A_n) = C(Y_1, \ldots, Y_{n-1})$, as was to be shown. □

Proposition 4.6. *If $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ is an algebraic n-closure system, then $\langle \mathcal{L}, \subseteq_1, \ldots, \subseteq_n \rangle$ is an algebraic n-semilattice.*

Proof. By Proposition 4.2, $\langle \mathcal{L}, \subseteq_1, \ldots, \subseteq_n \rangle$ is a complete n-semilattice. By Lemma 4.5, its compact elements are exactly those of the form $C(Y_1, \ldots, Y_{n-1})$, for some $Y_i \subseteq_\omega K_i, i = 1, \ldots, n-1$. Therefore, since for every $(A_1, \ldots, A_n) \in \mathcal{L}$ we have that

$$A_i = \bigcup \{C_i(Y_1, \ldots, Y_{n-1}) : Y_i \subseteq_\omega A_i : i = 1, \ldots, n-1\},$$

we conclude that $\langle \mathcal{L}, \subseteq_1, \ldots, \subseteq_n \rangle$ is indeed an algebraic n-semilattice. □

Unfortunately, we were not able to show that the analogous result to Theorem 4.3 for algebraic n-semilattices holds, i.e., that every algebraic n-semilattice is the complete n-semilattice of the closed sets of an algebraic n-closure operator ordered under component-wise inclusion. We leave this as an open problem for future investigation.

Open Problem. Suppose that $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ is an algebraic n-semilattice. Does there exist an algebraic n-closure system $\mathcal{L}$, such that $\langle \mathcal{L}, \subseteq_1, \ldots, \subseteq_n \rangle$ is n-order isomorphic to $\mathbf{L}$?

5. Brief discussion of the dyadic case

It is shown briefly in this section how the n-closure systems and the n-closure operators of the present work capture the ordinary closure operators and the ordinary closure systems in the dyadic case. In other words, it is shown that the 2-closure systems are the closure systems in the usual sense and the 2-closure operators are the ordinary closure operators and, moreover, that, in that case, Theorem 2.5 reduces to the well-known Theorem 2.21 of [3], establishing a bijective correspondence between closure systems and closure operators.

These facts have been alluded to several times during the development in the previous sections when ordinary closure systems and ordinary closure operators have been referred to as dyadic.

According to the definition, a 2-closure operator is a mapping $C\colon \mathcal{P}(K_1) \to \mathcal{P}(K_1) \times \mathcal{P}(K_2)$, such that

(1) $X_1 \subseteq C_1(X_1)$, for all $X_1 \subseteq K_1$,
(2) if $X_1 \subseteq Y_1$, then $C_2(Y_1) \subseteq C_2(X_1)$, for all $X_1, Y_1 \subseteq K_1$,
(3) if $X_1 \subseteq Y_1$ and $C_2(X_1) = C_2(Y_1)$, then $C_1(Y_1) \subseteq C_1(X_1)$, for all $X_1, Y_1 \subseteq K_1$,
(4) if $C_1(X_1) \subseteq C_1(Y_1)$, then $C_2(Y_1) \subseteq C_2(X_1)$ and, conversely, if $C_2(X_1) \subseteq C_2(Y_1)$, then $C_1(Y_1) \subseteq C_1(X_1)$, for all $X_1, Y_1 \subseteq K_1$, and, finally,
(5) $C(C_1(X_1)) = C(X_1)$, for all $X_1 \subseteq K_1$.

These conditions imply the three conditions that establish that C_1 is a closure operator on K_1. The property of being inflationary is Property 1. Property 5 yields idempotency. Monotonicity is implied by Condition 2 and the second part of Condition 4, taken jointly.

Conversely, every ordinary closure operator $C\colon \mathcal{P}(K) \to \mathcal{P}(K)$ on a set K gives rise to a dyadic closure operator $C'\colon \mathcal{P}(K) \to \mathcal{P}(K) \times \mathcal{P}(K)$ in the present sense if one defines

$$C'_1(X) := C(X), \quad C'_2(X) = C(X)', \quad \text{for all } X \subseteq K,$$

where, by Y' is denoted the complement of Y in K, for all $Y \subseteq K$. It is not difficult to verify (and the reader is invited to do so) that, defined in this way, C' satisfies all five properties of a dyadic closure operator given above.

Consider, on the other hand, the definition of a 2-closure system. A subset $\mathcal{L} \subseteq \mathcal{P}(K_1) \times \mathcal{P}(K_2)$ is a 2-closure system if

(1) $A_1 \subseteq B_1$ implies $B_2 \subseteq A_2$ and, conversely, $A_2 \subseteq B_2$ implies $B_1 \subseteq A_1$ for all $(A_1, A_2), (B_1, B_2) \in \mathcal{L}$, and
(2) for all $X_1 \subseteq K_1$, there exists $(A_1, A_2) \in \mathcal{L}$, such that $X_1 \subseteq A_1$ and (A_1, A_2) is the unique pair in $\mathcal{L}$ with the largest second coordinate among all pairs whose first coordinate contains X_1.

Condition 2 is the usual closure under intersections condition for the closed sets of a closure system. In fact, given Condition 1, it says that, for all $X_1 \in K_1$, there exists a first coordinate A_1 of a closed pair (A_1, A_2), such that $X_1 \subseteq A_1$ and A_1 is the smallest among all B_1 with the same property. So it must be that

$$A_1 = \bigcap\{B_1 : (B_1, B_2) \in \mathcal{L} \text{ and } X_1 \subseteq B_1\}.$$

Since $X \subseteq K_1$ is arbitrary, this shows that the collection of all first components of closed pairs is closed under arbitrary intersections and $\{A_1 : (A_1, A_2) \in \mathcal{L}\}$ is a closure system in the usual sense.

Finally, a few remarks on the algebraic case are in order. Note that, in the dyadic case, the condition $C_1(X_1) = \bigcup_{Y_1 \subseteq_\omega X_1} C_1(Y_1)$ is exactly the condition defining algebraicity for an ordinary (2-dimensional) closure operator. Moreover, the condition $\mathfrak{b}_1(\bigcup_{A_1 \in \mathcal{A}_1} A_1)_1 = \bigcup_{A_1 \in \mathcal{A}_1} A_1$, for every directed collection $\mathcal{A}_1 \subseteq \mathcal{L}$, is equivalent, in the two dimensions, to the condition requiring that the union of a directed family of closed sets of a closure system be a closed set. Therefore it amounts to the algebraicity of a (2-dimensional) closure system. Therefore Theorem 3.3 is an n-dimensional analog of Theorem 3.8 of [3], a well-known result on the correspondence between ordinary algebraic closure systems and ordinary algebraic closure operators (in 2 dimensions).

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