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# Categorical Abstract Algebraic Logic: Ordered Equational Logic and Algebraizable PoVarieties

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**Abstract** A syntactic apparatus is introduced for the study of the algebraic properties of classes of partially ordered algebraic systems (a.k.a. partially ordered functors (pofunctors)). A Birkhoff-style order **HSP** theorem and a Mal'cev-style order **SLP** theorem are proved for partially ordered varieties and partially ordered quasivarieties, respectively, of partially ordered algebraic systems based on this syntactic apparatus. Finally, the notion of a finitely algebraizable partially-ordered quasivariety, in the spirit of Pałasińska and Pigozzi, is introduced and some of the properties of these quasi-povarieties are explored in the categorical framework.

**Key words** varieties · quasi-varieties · order homomorphisms · order isomorphisms · polarities · polarity translations · order translations · algebraic systems · reduced products · subdirect products · subdirect representation theorem · closure operators · Birkhoff's theorem · Mal'cev's theorem · $\pi$-Institutions · protoalgebraic logics · algebraizable logics · protoalgebraic $\pi$-Institutions

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## 1 Introduction

This paper contains the fourth (and final) installment on research concerning an extension of some of the results on partially ordered varieties and quasi-varieties of partially ordered universal algebras obtained by Pałasińska and Pigozzi in the context of abstract algebraic logic and reported in Pigozzi's lecture notes [26]. The original motivation of Pałasińska and Pigozzi was the development of a part of the theory

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of abstract algebraic logic (AAL) suitable for handling logical implication in a way analogous to the way logical equivalence is handled by the well-known (Leibniz and Tarski) operator approach in AAL (see, for instance [2, 3, 6, 8, 20–22, 27] and the surveys [7, 16, 17]). Since, in recent work by the author (see [32–41]), algebraic systems were shown to play a role analogous to that of algebras in the study of logical equivalence in the categorical framework, it is only natural to expect that an approach towards logical implication analogous to that adopted by Pałasińska and Pigozzi in [26] will involve the study of partially ordered algebraic systems or partially ordered functors (pofunctors), as introduced and studied in the preceding three papers of this series [42–44].

Since in both the introduction to [42] and the introduction to [43] a survey of the basic results in [26], that inspired the present work, has been given, in this Introduction, only a brief survey of the results of [42] and [43] will be provided.

In [42] the notion of a polarity $\rho$ for a category of natural transformations $N$ on a given functor SEN is introduced. A functor SEN, with $N$ a category of natural transformations on SEN and $\rho$ a polarity for $N$, may be endowed with different quasi-ordered systems (qosystems) preserving $\rho$, that are termed $\rho$-qosystems. A pair $\langle \mathrm{SEN}, \lesssim\rangle$, where $\lesssim$ is such a $\rho$-qosystem is called a $\rho$-qofunctor (or a $\rho$-quasi-ordered algebraic system). If $\lesssim$ happens to be a partially-ordered system (posystem) then, $\langle \mathrm{SEN}, \lesssim\rangle$ is termed a $\rho$-pofunctor (or a $\rho$-partially ordered algebraic system). An $N$-congruence system $\theta$ on SEN is said to be compatible with a $\rho$-qosystem $\lesssim$ if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi, \phi', \psi' \in \mathrm{SEN}(\Sigma)$, if $\phi\theta_\Sigma\phi'$ and $\psi\theta_\Sigma\psi'$, then $\phi \lesssim_\Sigma \psi$ implies that $\phi' \lesssim_\Sigma \psi'$. Given a $\rho$-qosystem $\lesssim$ on SEN and an $N$-congruence system $\theta$ on SEN, that is compatible with $\lesssim$, one may construct the quotient functor $\mathrm{SEN}^\theta$ and endow it with a $\rho^\theta$-qosystem $\lesssim/\theta$, as is shown in Proposition 4 of [42].

On the other hand, a collection of functors $\mathrm{SEN}^i, i \in I$, are said to have compatible categories of natural transformations $N^i$ on $\mathrm{SEN}^i, i \in I$, if there exists a functor SEN, with $N$ a category of natural transformations on SEN, and surjective functors $F^i : N \to N^i, i \in I$, that preserve all projections. This also implies that the $F^i$'s preserve the arities of the natural transformations involved. Given $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in $N$, by $\sigma^i : (\mathrm{SEN}^i)^k \to \mathrm{SEN}^i$ will be denoted the image of $\sigma$ under $F^i, i \in I$. Moreover, the functors $\mathrm{SEN}^i, i \in I$, are said to have compatible polarities $\rho^i, i \in I$, for the compatible categories of natural transformations $N^i, i \in I$, if corresponding transformations have the same polarity in corresponding argument places. If that is the case, the functors $\mathrm{SEN}^i, i \in I$, are said to be compatible. An $(N, N')$-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$ is said to be an order translation between two pofunctors $\langle \mathrm{SEN}, \lesssim\rangle$ and $\langle \mathrm{SEN}', \lesssim'\rangle$, denoted by $\langle F, \alpha\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to^p \langle \mathrm{SEN}', \lesssim'\rangle$, if it preserves polarities and also preserves the quasi-order systems , i.e., for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \lesssim_\Sigma \psi \quad \text{implies} \quad \alpha_\Sigma(\phi) \lesssim'_{F(\Sigma)} \alpha_\Sigma(\psi).$$

Putting together the notion of a quotient outlined above with the notion of an order translation between pofunctors, analogs of the usual Homomorphism, Isomorphism and Correspondence Theorems of Universal Algebra have been established in [42] for pofunctors (see Theorem 15, Corollary 16 and Theorem 19 of [42]).

The second installment of the work [43] switches gears and studies analogs of the variety and quasi-variety closure operators in the context of classes of compatible pofunctors. It starts with the introduction of the notion of a sub-pofunctor of a given pofunctor, continues with that of a product pofunctor of a collection of pofunctors, goes on with the definition of the order filtered product of a collection of pofunctors and, finally, introduces order direct limits, based on filtered products. One of the basic theorems shows that the operation of taking homomorphic images of sub-pofunctors of products of pofunctors is a closure operator on classes of compatible pofunctors (Theorem 14 of [43]), as is the operation of taking sub-pofunctors of order filtered products of pofunctors (Theorem 20 of [43]). These two operators will be related to the generation of povarieties and quasi-povarieties of pofunctors in the present paper in a way analogous to the way Birkhoff's Theorem and Mal'cev's Theorem, respectively, relate the operators **HSP** and $\mathbf{SP}_{\mathrm{R}}$ to varieties and quasi-varieties of universal algebras.

Another important result, an analog of the well-known Subdirect Representation Theorem of Universal Algebra, that was presented in [44], says that every pofunctor is order isomorphic to an order subdirect product of subdirectly irreducible pofunctors (Theorem 2 of [44]).

To establish the relationship between the closure operators on classes of pofunctors and the generation of povarieties and quasi-povarieties of pofunctors, mentioned above, a syntactic apparatus in which inequations and quasi-inequations may be expressed is developed in the next section, based on the operations provided in the form of the natural transformations in the category $N$. Pofunctors form the natural models of systems of inequations or quasi-inequations in a way directly reflecting the way partially ordered algebras form the models of inequations and quasi-inequations over the same signature in [26]. A class of compatible pofunctors is called a povariety if it is the class of all models of some set of inidentities in this sense and a quasi-povariety if it is the class of all models of a set of quasi-inidentities. The analog of the **HSP** theorem says that a class of pofunctors of this form is a povariety iff it is closed under the operation of taking homomorphic images of sub-pofunctors of product pofunctors of collections of pofunctors in the class. Similarly, the analog of the **SLP** theorem says that a class of pofunctors is a quasi-povariety iff it is closed under taking sub-pofunctors of order direct limits of product pofunctors of collections of pofunctors in the class. The paper continues with the introduction of the notion of a finitely algebraizable quasi-povariety. Roughly speaking, these are the quasi-povarieties for which there exist equations in two variables defining the posystem on every pofunctor in the class. Some of the results shown to hold in the context of algebraizable partially ordered varieties and quasi-varieties of universal algebras in [26] are shown to have analogs in the framework of algebraizable povarieties and quasi-povarieties of pofunctors in this section of the paper. In the final section a few examples are presented illustrating some of the main concepts on which our theory is based. The first example shows how the universal algebraic theory of partially-ordered left-residuated monoids may be perceived as a special case of the abstract categorical setting developed in the paper. It is a representative example that illustrates a general method that may be applied to a wide range of universal algebraic examples if one wishes to treat them as pofunctors. The following two examples are drawn from the two paradigms that led to the development of categorical abstract algebraic logic and, as such, are closer to the spirit of the potential applications at the origin of the present work. The first of these deals with equational logic and uses the presentation of its syntax as it was developed in [29] (resulting in a variant of the variety of the clone algebras of Taylor [28]). The second deals with first-order logic and uses a presentation of its syntax as developed in [30, 31] leading in a natural way to the introduction of a categorical class of pofunctors including all cylindric algebras.

It should be mentioned at this point that a bulk of previous work has paved the way for the development of the theory by Pałasińska and Pigozzi [26]. Sample references include the work of Bloom [4] on varieties of ordered algebras, Mal'cev's work [24, 25] on quasi-varieties of first-order structures, Dellunde and Jansana's [9, 10] and Elgueta's [13, 14] work on first-order structures defined without equality, a special case of which are the structures defined using universal Horn logic without equality, and Dunn's work [11, 12] on gaggle theory. The book on partially ordered algebraic structures by Fuchs [18] should also be mentioned.

Finally, a few general references on concepts used in this paper: for background from category theory the reader is referred to any of [1, 5, 23], for an overview of the current state of affairs in abstract algebraic logic the reader may consult the review article [17], the monograph [16] and the book [7], whereas for more recent developments on the categorical side of the subject the reader may refer to the series of papers [32–41] in the given order.

## 2 Syntax and Semantics

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and $N$ a category of natural transformations on SEN. Given a set $X$, the collection $\mathrm{Te}^N(X)$ of *$N$-terms in the variables $X$* is defined recursively as follows:

- $x \in \mathrm{Te}^N(X)$, for all $x \in X$, and
- $\sigma(t_0, \ldots, t_{n-1}) \in \mathrm{Te}^N(X)$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in $N$ and all $t_0, \ldots, t_{n-1} \in \mathrm{Te}^N(X)$.

Moreover, given sets $X$ and $Y$ and a mapping $f : X \to Y$, $f$ induces a mapping $\mathrm{Te}^N(f) : \mathrm{Te}^N(X) \to \mathrm{Te}^N(Y)$, defined recursively on the structure of $N$-terms, by

- $\mathrm{Te}^N(f)(x) = f(x)$, for all $x \in X$, and
- $\mathrm{Te}^N(f)(\sigma(t_0, \ldots, t_{n-1})) = \sigma(\mathrm{Te}^N(f)(t_0), \ldots, \mathrm{Te}^N(f)(t_{n-1}))$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in $N$ and all $t_0, \ldots, t_{n-1} \in \mathrm{Te}^N(X)$.

It is not difficult to see that, defined as above, $\mathrm{Te}^N : \mathbf{Set} \to \mathbf{Set}$ is a functor and that it is equipped with a category $N'$ of natural transformations that is compatible with $N$. By an $N$-**term**, we will understand a member of $\mathrm{Te}^N(X)$, for some $X \in |\mathbf{Set}|$.

Given a functor SEN, with $N$ a category of natural transformations on SEN, denote by $\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle : \mathrm{Te}^N \circ \mathrm{SEN} \to \mathrm{SEN}$ the surjective $(N', N)$-epimorphic translation, defined by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\mu^N_\Sigma : \mathrm{Te}^N(\mathrm{SEN}(\Sigma)) \to \mathrm{SEN}(\Sigma)$ be given by recursion on the structure of $N$-terms:

- $\mu^N_\Sigma(\phi) = \phi$, for all $\phi \in \mathrm{SEN}(\Sigma)$, and
- $\mu^N_\Sigma(\sigma(t_0, \ldots, t_{n-1})) = \sigma_\Sigma(\mu^N_\Sigma(t_0), \ldots, \mu^N_\Sigma(t_{n-1}))$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in $N$ and all $t_0, \ldots, t_{n-1} \in \mathrm{Te}^N(\mathrm{SEN}(\Sigma))$.

Furthermore, given $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with $N$ a category of natural transformations on SEN, an $N$-term $s(\vec{x})$ in the set of variables $X$, a $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, denote by

$$s_\Sigma(\vec{\phi}) := \mu_\Sigma^N(\mathrm{Te}^N(\vec{\phi})(s)).$$

This is the usual operation of substitution of elements of $\mathrm{SEN}(\Sigma)$ for variables. It is obvious that $s_\Sigma(\vec{\phi})$ depends only on the values of the substitution $\vec{\phi}$ on the variables $\vec{x}$ appearing in $s$.

An $N$-*inequation* is a pair $\langle s, t\rangle$ of $N$-terms, also denoted by $s \preccurlyeq t$. An $N$-*quasi-inequation* is a nonempty sequence $\langle s_0 \preccurlyeq t_0, \ldots, s_{n-1} \preccurlyeq t_{n-1}, u \preccurlyeq v\rangle$ of $N$-inequations, usually denoted by $s_0 \preccurlyeq t_0, \ldots, s_{n-1} \preccurlyeq t_{n-1} \to u \preccurlyeq v$. The $N$-inequations $s_i \preccurlyeq t_i, i < n$, are called the *premises* of the $N$-quasi-inequation and $u \preccurlyeq v$ its *conclusion*. $N$-inequations are identified with $N$-quasi-inequations with an empty set of premises.

Consider a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with $N$ a category of natural transformations on SEN, and $\rho$ a polarity for $N$. A $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ is said to *satisfy* the $N$-inequation $s(\vec{x}) \preccurlyeq t(\vec{x})$ *at* some $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $\Sigma \in |\mathbf{Sign}|$, if $s_\Sigma(\vec{\phi}) \lesssim_\Sigma t_\Sigma(\vec{\phi})$. This is also denoted by $\langle \mathrm{SEN}, \lesssim\rangle \models_\Sigma s(\vec{x}) \preccurlyeq t(\vec{x})[\vec{\phi}]$. Similarly, the $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$ is said to *satisfy* an $N$-quasi-inequation *at* some $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $\Sigma \in |\mathbf{Sign}|$, if it does not satisfy at least one of the premises at $\vec{\phi}$ or it satisfies the conclusion of the $N$-quasi-inequation at $\vec{\phi}$.

An $N$-quasi-inequation $s_0 \preccurlyeq t_0, \ldots, s_{n-1} \preccurlyeq t_{n-1} \to u \preccurlyeq v$ is said to be an $N$-*quasi-inidentity* of the $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$, written

$$\langle \mathrm{SEN}, \lesssim\rangle \models s_0 \preccurlyeq t_0, \ldots, s_{n-1} \preccurlyeq t_{n-1} \to u \preccurlyeq v$$

if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $\langle \mathrm{SEN}, \lesssim\rangle \models_\Sigma s_0 \preccurlyeq t_0, \ldots, s_{n-1} \preccurlyeq t_{n-1} \to u \preccurlyeq v[\vec{\phi}]$. If this is the case, then $\langle \mathrm{SEN}, \lesssim\rangle$ is called a *model* of the $N$-quasi-inequation. If the $N$-quasi-inidentity happens to be an $N$-inequation, then it is called an $N$-*inidentity* of the $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$.

Given a collection Q of $N$-quasi-inequations or $N$-inequations, the class of models of all members of Q is denoted by $\mathbf{Mod}(\mathrm{Q})$.

**Definition 1** Let $\mathtt{Q}$ be a class of compatible pofunctors. $\mathtt{Q}$ is called a $\rho$-*partially ordered quasi-variety* or $\rho$- *quasi-povariety*, if $\mathtt{Q} = \mathbf{Mod}(\mathrm{Q})$ for some set of $N$-quasi-inequations Q. If Q happens to be a set of $N$-inequations, then $\mathtt{Q}$ is said to be a $\rho$-*partially ordered variety* or a $\rho$-*povariety*.

Recall, now, from [43] that, given a class $\mathtt{K}$ of pofunctors and a $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle$, all with compatible categories of natural transformations and compatible polarities, by $\mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{SEN}, \lesssim\rangle)$ is denoted the collection

$$\mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{SEN}, \lesssim\rangle) = \{\lesssim' \in \mathrm{QoSys}_\rho(\langle \mathrm{SEN}, \lesssim\rangle) : \langle \mathrm{SEN}, \lesssim\rangle / \lesssim' \in \mathtt{K}\}.$$

The qosystems in $\mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{SEN}, \lesssim\rangle)$ are referred to as the $\mathtt{K}$-$\rho$-*qosystems of* $\langle \mathrm{SEN}, \lesssim\rangle$.

Let $\mathtt{K}$ be a $\rho$-quasi-povariety and $\langle \mathrm{SEN}, \lesssim \rangle$ a $\rho$-pofunctor, compatible with those in $\mathtt{K}$. Consider a binary relation system $R$ on SEN. By the $\mathtt{K}$-*$\rho$-qosystem of* $\langle \mathrm{SEN}, \lesssim \rangle$ *generated by* $R$, in symbols $\Phi_\rho^{\mathtt{K}}(R)$ is denoted the smallest $\mathtt{K}$-$\rho$-qosystem of $\langle \mathrm{SEN}, \lesssim \rangle$ that includes $R$, if such a $\rho$-qosystem exists, i.e.,

$$\Phi_\rho^{\mathtt{K}}(R) = \bigcap \{ \lesssim' \in \mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{SEN}, \lesssim \rangle) : R \leq \lesssim' \}.$$

As was shown in Proposition 3, Part 1, of [44], if $\mathbf{P}_{\mathrm{SD}}(\mathtt{K}) \subseteq \mathtt{K}$, then $\mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{SEN}, \lesssim \rangle)$ is closed under arbitrary intersections, whence, in that case, $\Phi_\rho^{\mathtt{K}}(R)$ always exists.

## 3 Order HSP Theorem

In this section, we work towards establishing an analog of the well known **HSP** variety theorem of universal algebra in the context of compatible pofunctors. Theorem 4 also abstracts Theorem 3.14 of [26], the Order **HSP** Theorem of Pałasińska and Pigozzi.

**Lemma 2** *Let $\mathtt{K}$ be a class of compatible pofunctors, such that $\mathbf{SP}(\mathtt{K}) \subseteq \mathtt{K}$, and $\langle \mathrm{SEN}, \lesssim \rangle$ a $\rho$-pofunctor in $\mathtt{K}$. Consider the pofunctor $\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle$ and let $\lesssim'$ be the smallest member of $\mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle)$. Then, for all $s(\vec{x})$, $t(\vec{x}) \in \mathrm{Te}^N(X)$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$,*

$$s(\vec{\phi}) \lesssim'_\Sigma t(\vec{\phi}) \quad \textit{implies} \quad s_\Sigma(\vec{\phi}) \lesssim_\Sigma t_\Sigma(\vec{\phi}).$$

*Proof* Let $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$ and suppose that $s(\vec{\phi}) \lesssim'_\Sigma t(\vec{\phi})$. Consider the $(N', N)$-epimorphic translation $\langle I_{\mathbf{Sign}}, \mu^N \rangle : \mathrm{Te}^N \circ \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}$. Then, by the minimality of $\lesssim'$, since

$$\mathrm{OrdKer}(\langle I_{\mathbf{Sign}}, \mu^N \rangle) \in \mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle),$$

we obtain that

$$\begin{array}{rll} s(\vec{\phi}) \lesssim'_\Sigma t(\vec{\phi}) & \text{implies} & s(\vec{\phi}) \ \mathrm{OrdKer}_\Sigma(\langle I_{\mathbf{Sign}}, \mu^N \rangle) \ t(\vec{\phi}) \\ & \text{iff} & \mu_\Sigma^N(s(\vec{\phi})) \lesssim_\Sigma \mu_\Sigma^N(t(\vec{\phi})) \\ & \text{iff} & s_\Sigma(\vec{\phi}) \lesssim_\Sigma t_\Sigma(\vec{\phi}). \end{array}$$

□

**Theorem 3** *Let $\mathtt{K}$ be a class of compatible pofunctors, such that $\mathbf{SP}(\mathtt{K}) \subseteq \mathtt{K}$. Then, for all $s(\vec{x}), t(\vec{x}) \in \mathrm{Te}^N(X)$, $s(\vec{x}) \preccurlyeq t(\vec{x})$ is an $N$-inidentity of $\mathtt{K}$ if and only if, for every functor $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$, with a compatible category $N$ of natural transformations on SEN and compatible polarity $\rho$ for $N$ with those in $\mathtt{K}$, the pofunctor $\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle$, with $\lesssim$ the smallest member of $\mathrm{QoSys}_\rho^{\mathtt{K}}(\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle)$, is such that, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $s(\vec{\phi}) \lesssim_\Sigma t(\vec{\phi})$.*

*Proof* Suppose, first, that $s(\vec{x}) \preccurlyeq t(\vec{x})$ is an $N$-inidentity of $\mathtt{K}$. Then, since $\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle / \sim \in \mathtt{K}$, we have that, for all $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$,

$$\begin{array}{rcl} s(\vec{\phi})/\!\sim_\Sigma & = & s^\sim(\vec{\phi}/\!\sim_\Sigma) \\ & \lesssim_\Sigma/\!\sim_\Sigma & t^\sim(\vec{\phi}/\!\sim_\Sigma) \\ & = & t(\vec{\phi})/\!\sim_\Sigma, \end{array}$$

and, therefore, $s(\vec{\phi}) \lesssim_\Sigma t(\vec{\phi})$.

Suppose, conversely, that, for every functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with a compatible category $N$ of natural transformations on SEN and compatible polarity $\rho$ for $N$, the pofunctor $\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle$, with $\lesssim$ the smallest member of $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle)$, is such that, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $s(\vec{\phi}) \lesssim_\Sigma t(\vec{\phi})$. Then, if $\langle \mathbf{Sign}', \lesssim' \rangle \in \mathtt{K}$, $\Sigma \in |\mathbf{Sign}'|$ and $\vec{\phi} \in \mathrm{SEN}'(\Sigma)^X$, the hypothesis implies the hypothesis of Lemma 2, whence $s_\Sigma(\vec{\phi}) \lesssim'_\Sigma t_\Sigma(\vec{\phi})$. Therefore $s(\vec{x}) \preccurlyeq t(\vec{x})$ is an $N$-inidentity of $\mathtt{K}$. □

Recall from [43] that the operator **HSP** of taking order homomorphic images of order sub-pofunctors of order direct products is a closure operator on classes of compatible pofunctors. In the next theorem, an analog of the Order **HSP** Theorem 3.14 of [26], closed classes of that form are characterized as being exactly the ordered varieties of pofunctors.

**Theorem 4** (Order HSP Theorem) *A class* $\mathtt{K}$ *of compatible pofunctors is an ordered variety if and only if it is closed under the formation of order homomorphic images, order subalgebras and order direct products, i.e., iff* $\mathbf{HSP}(\mathtt{K}) = \mathtt{K}$.

*Proof* Suppose, first, that I is a collection of $N$-inidentities, such that $\mathtt{K} = \mathbf{Mod}(\mathrm{I})$. It suffices to show that $\mathbf{H}(\mathtt{K}) \subseteq \mathtt{K}$, $\mathbf{S}(\mathtt{K}) \subseteq \mathtt{K}$ and $\mathbf{P}(\mathtt{K}) \subseteq \mathtt{K}$.

For $\mathbf{H}(\mathtt{K}) \subseteq \mathtt{K}$, suppose that $\langle \mathrm{SEN}, \lesssim \rangle$, $\langle \mathrm{SEN}', \lesssim' \rangle$ are two pofunctors, such that $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$ and $\langle F, \alpha \rangle : \langle \mathrm{SEN}, \lesssim \rangle \to^p \langle \mathrm{SEN}', \lesssim' \rangle$ is a surjective order translation from $\langle \mathrm{SEN}, \lesssim \rangle$ onto $\langle \mathrm{SEN}', \lesssim' \rangle$. Suppose that $s(\vec{x}), t(\vec{x}) \in \mathrm{Te}^N(X)$, such that $s(\vec{x}) \preccurlyeq t(\vec{x}) \in \mathrm{I}$ and let $\Sigma' \in |\mathbf{Sign}'|$ and $\vec{\psi} \in \mathrm{SEN}'(\Sigma')^X$. Since $\langle F, \alpha \rangle$ is surjective, there exists $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, such that $F(\Sigma) = \Sigma'$ and $\vec{\psi} = \alpha_\Sigma(\vec{\phi})$. Since $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$, we get that $s_\Sigma(\vec{\phi}) \lesssim_\Sigma t_\Sigma(\vec{\phi})$, whence, since $\langle F, \alpha \rangle$ is an order translation, we get that $\alpha_\Sigma(s_\Sigma(\vec{\phi})) \lesssim'_{F(\Sigma)} \alpha_\Sigma(t_\Sigma(\vec{\phi}))$, and, thus, $s_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi})) \lesssim'_{F(\Sigma)} t_{F(\Sigma)}(\alpha_\Sigma(\vec{\phi}))$ and, therefore $s_{\Sigma'}(\vec{\psi}) \lesssim'_{\Sigma'} t_{\Sigma'}(\vec{\psi})$. Therefore $\langle \mathrm{SEN}', \lesssim' \rangle \models s(\vec{x}) \preccurlyeq t(\vec{x})$, showing that $\langle \mathrm{SEN}', \lesssim' \rangle \in \mathtt{K}$.

For $\mathbf{S}(\mathtt{K}) \subseteq \mathtt{K}$, suppose that $\langle \mathrm{SEN}, \lesssim \rangle, \langle \mathrm{SEN}', \lesssim' \rangle$ are pofunctors such that $\langle \mathrm{SEN}', \lesssim' \rangle$ is a $\rho$-subpofunctor of the $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$. Let $s(\vec{x}) \preccurlyeq t(\vec{x}) \in \mathrm{I}$ and consider $\Sigma' \in |\mathbf{Sign}'|, \vec{\psi} \in \mathrm{SEN}'(\Sigma')^X$. Then, since $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$, we have that $s_\Sigma(\vec{\phi}) \lesssim_\Sigma t_\Sigma(\vec{\phi})$, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^X$. But, since $\langle \mathrm{SEN}', \lesssim' \rangle$ is a sub-pofunctor of $\langle \mathrm{SEN}, \lesssim \rangle$, we have that $\Sigma' \in |\mathbf{Sign}'| \subseteq |\mathbf{Sign}|$, $\vec{\psi} \in \mathrm{SEN}'(\Sigma')^X \subseteq \mathrm{SEN}(\Sigma')^X$ and $\lesssim'_{\Sigma'} = \lesssim_{\Sigma'} \cap \mathrm{SEN}'(\Sigma')^2$, which give that $s_{\Sigma'}(\vec{\psi}) \lesssim'_{\Sigma'} t_{\Sigma'}(\vec{\psi})$. Hence $\langle \mathrm{SEN}', \lesssim' \rangle \models s(\vec{x}) \preccurlyeq t(\vec{x})$ and $\langle \mathrm{SEN}', \lesssim' \rangle \in \mathtt{K}$.

Finally, for $\mathbf{P}(\mathtt{K}) \subseteq \mathtt{K}$, suppose that $\langle \mathrm{SEN}^i, \lesssim^i \rangle \in \mathtt{K}$, for all $i \in I$. Consider $s(\vec{x}) \preccurlyeq t(\vec{x}) \in \mathrm{I}$ and let $\Sigma_i \in |\mathbf{Sign}^i|, \vec{\phi}^i \in \mathrm{SEN}^i(\Sigma_i)^X$, for all $i \in I$. Then, we have $s_{\Sigma_i}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} t_{\Sigma_i}(\vec{\phi}^i)$, for all $i \in I$. This yields immediately $\prod_{i \in I} s_{\Sigma_i}(\vec{\phi}^i) \prod_{i \in I} \lesssim^i_{\Sigma_i} \prod_{i \in I} t_{\Sigma_i}(\vec{\phi}^i)$, and, therefore, $s_{\prod_{i\in I}\Sigma_i}(\vec{\phi})\prod_{i\in I}\precsim^i_{\prod_{i\in I}\Sigma_i}t_{\prod_{i\in I}\Sigma_i}(\vec{\phi})$, where, $\vec{\phi}=\langle\vec{\phi}(x):x\in X\rangle$, with $\vec{\phi}(x)=\langle\vec{\phi}^i(x):i\in I\rangle$, for all $x\in X$. This shows that $\prod_{i\in I}\langle\mathrm{SEN}^i,\precsim^i\rangle\models s(\vec{x})\preccurlyeq t(\vec{x})$, and, hence, $\prod_{i\in I}\langle\mathrm{SEN}^i,\precsim^i\rangle\in\mathtt{K}$.

Suppose, conversely, that $\mathbf{HSP}(\mathtt{K})\subseteq\mathtt{K}$. Consider the collection I of all $N$-inidentities that are satisfied by all pofunctors in $\mathtt{K}$. Then, obviously $\mathtt{K}\subseteq\mathbf{Mod}(\mathrm{I})$ and it suffices to show the reverse inclusion, i.e., that $\mathbf{Mod}(\mathrm{I})\subseteq\mathtt{K}$. So, suppose that $\langle\mathrm{SEN},\precsim\rangle\in\mathbf{Mod}(\mathrm{I})$. Consider the surjective order translation $\langle I_{\mathbf{Sign}},\mu^N\rangle:\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle\to^p\langle\mathrm{SEN},\precsim\rangle$. Then, let $\precsim'$ be the smallest member of $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle)$. It exists, by Proposition 3 of [44], since $\mathbf{SP}(\mathtt{K})\subseteq\mathtt{K}$. By Theorem 3, we obtain that, for all $\Sigma\in|\mathbf{Sign}|$, $s(\vec{\phi}),t(\vec{\phi})\in\mathrm{Te}^N(\mathrm{SEN}(\Sigma))$, $s(\vec{\phi})\precsim'_\Sigma t(\vec{\phi})$ implies $s_\Sigma(\vec{\phi})\precsim_\Sigma t_\Sigma(\vec{\phi})$. Therefore, $\precsim'\leq\mathrm{OrdKer}(\langle I_{\mathbf{Sign}},\mu^N\rangle)$, which, using the Order Homomorphism Theorem, yields the existence of

$$\langle G,\beta\rangle:\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/\precsim'\to\langle\mathrm{SEN},\precsim\rangle,$$

such that $\langle I_{\mathbf{Sign}},\mu^N\rangle=\langle G,\beta\rangle\circ\langle I_{\mathbf{Sign}},\pi^{\sim'}\rangle$.

$$\begin{tikzcd}
\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle \arrow[rr, "\langle I_{\mathbf{Sign}},\pi^{\sim'}\rangle"] \arrow[dr, "\langle I_{\mathbf{Sign}},\mu^N\rangle"'] & & \langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/{\sim'} \arrow[dl, "\langle G,\beta\rangle"] \\
& \langle\mathrm{SEN},\precsim\rangle &
\end{tikzcd}$$

Note that $\langle G,\beta\rangle$ is also a surjective order translation, whence $\langle\mathrm{SEN},\precsim\rangle$ is a homomorphic image of $\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/{\sim'}$. This, combined with the fact that $\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/{\sim'}\in\mathtt{K}$, gives $\langle\mathrm{SEN},\precsim\rangle\in\mathbf{H}(\mathtt{K})\subseteq\mathtt{K}$. □

## 4 Order SLP Theorem

Let $\mathtt{K}$ be a class of pofunctors, such that $\mathbf{SP}(\mathtt{K})\subseteq\mathtt{K}$ and $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$ a functor with $N$ a category of natural transformations on SEN and $\rho$ a polarity for $N$, both compatible with those of $\mathtt{K}$. In the next lemma, the smallest member $\precsim$ of $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle\mathrm{Te}^N\circ\mathrm{SEN},\Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle)$ containing a specific relation system induced by a finite collection of inequations is characterized. This characterization will play a crucial role in the proof of the main theorem of the section, the Order **SLP** Theorem, an abstraction of the corresponding Theorem 3.17 of [26].

**Lemma 5** *Let $\mathtt{K}$ be a class of pofunctors, such that $\mathbf{SP}(\mathtt{K})\subseteq\mathtt{K}$, and $s_0(\vec{x})\preccurlyeq t_0(\vec{x}),\ldots,s_{n-1}(\vec{x})\preccurlyeq t_{n-1}(\vec{x})$ a finite set of $N$-inequations in the variables $X$. Given a functor $\mathrm{SEN}:\mathbf{Sign}\to\mathbf{Set}$, with compatible category of natural transformations $N$ and compatible polarity $\rho$ for $N$ with those in $\mathtt{K}$, consider the relation system $R=\{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\mathrm{Te}^N\circ\mathrm{SEN}$, defined, for all $\Sigma\in|\mathbf{Sign}|$, by*

$$R_\Sigma=\{\langle s_i(\vec{\phi}),t_i(\vec{\phi})\rangle:\vec{\phi}\in\mathrm{SEN}(\Sigma)^X,i<n\},$$

*and denote by* $\Phi^{\mathbb{K}}_{\rho}(R)$ *the smallest member* $\lesssim$ *of* $\mathrm{QoSys}^{\mathbb{K}}_{\rho}(\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle)$ *such that* $s_i(\vec{\phi}) \lesssim_\Sigma t_i(\vec{\phi})$, *for all* $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$ *and* $i < n$, *i.e., such that* $R \leq \lesssim$. *For any* $N$*-inequation* $u(\vec{x}) \preccurlyeq v(\vec{x})$, *the* $N$*-quasi-inequation*

$$s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x})$$

*is an* $N$*-quasi-inidentity of* $\mathbb{K}$ *if and only if, for every functor* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $u(\vec{\phi})\ \Phi^{\mathbb{K}}_{\rho}(R)_\Sigma\ v(\vec{\phi})$.

*Proof* Let $\lesssim := \Phi^{\mathbb{K}}_{\rho}(R)$. Then, by its definition, we have that, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $s_i(\vec{\phi})/{\sim_\Sigma} \lesssim_\Sigma/{\sim_\Sigma}\ t_i(\vec{\phi})/{\sim_\Sigma}$, for all $i < n$, whence, it follows that

$$s_i^{\sim}(\vec{\phi}/{\sim_\Sigma}) \lesssim_\Sigma/{\sim_\Sigma}\ t_i^{\sim}(\vec{\phi}/{\sim_\Sigma}), \quad i < n. \tag{1}$$

Now, for the left-to-right implication, suppose that

$$s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x})$$

is an $N$-quasi-inidentity of $\mathbb{K}$. Then, since $\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^{\mathrm{N}} \circ \mathrm{SEN}} \rangle/{\lesssim} \in \mathbb{K}$, we obtain, by Condition (1), $u^{\sim}(\vec{\phi}/{\sim_\Sigma}) \lesssim_\Sigma/{\sim_\Sigma}\ v^{\sim}(\vec{\phi}/{\sim_\Sigma})$, whence $u(\vec{\phi})/{\sim_\Sigma} \lesssim_\Sigma/{\sim_\Sigma}\ v(\vec{\phi})/{\sim_\Sigma}$, which yields $u(\vec{\phi}) \lesssim_\Sigma v(\vec{\phi})$.

For the right-to-left implication, suppose that for each functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, $u(\vec{\phi})\ \Phi^{\mathbb{K}}_{\rho}(R)_\Sigma\ v(\vec{\phi})$. Let $\langle \mathrm{SEN}, \lesssim \rangle \in \mathbb{K}$, $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, such that $s_{i_\Sigma}(\vec{\phi}) \lesssim_\Sigma t_{i_\Sigma}(\vec{\phi})$, for all $i < n$. The qosystem $\mathrm{OrdKer}(\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle)$ is a member of $\mathrm{QoSys}^{\mathbb{K}}_{\rho}(\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N \circ \mathrm{SEN}} \rangle)$ and $R \leq \mathrm{OrdKer}(\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle)$, whence, by the minimality of $\Phi^{\mathbb{K}}_{\rho}(R)$, we get that $\Phi^{\mathbb{K}}_{\rho}(R) \leq \mathrm{OrdKer}(\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle)$, and, hence, by the hypothesis, $u(\vec{\phi})\mathrm{OrdKer}(\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle)_\Sigma v(\vec{\phi})$, which yields that $u_\Sigma(\vec{\phi}) \lesssim_\Sigma v_\Sigma(\vec{\phi})$. Therefore the $N$-quasi-inequation $s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x})$ is an $N$-quasi-inidentity of $\mathbb{K}$. □

The next lemma, which will also be used in the proof of the Order **SLP** Theorem, states that the quotient of a given pofunctor by the union of a collection of upward directed qosystems is isomorphic to the order direct limit of the quotients of the pofunctor by each of the qosystems in the directed collection by the system of the natural surjective order translations from one quotient onto another, formed by a larger qosystem. It forms an analog of Lemma 3.16 of [26] in the context of pofunctors.

Recall the notation and terminology established for order direct limits of pofunctors in Section 2 of [43].

**Lemma 6** *Let* $\langle \mathrm{SEN}, \lesssim \rangle$ *be a* $\rho$*-pofunctor and* $\mathcal{K}$ *a collection of* $\rho$*-qosystems of* $\langle \mathrm{SEN}, \lesssim \rangle$ *that is upward directed by signature-wise inclusion so that* $\bigcup \mathcal{K}$ *is also a* $\rho$*-qosystem. Then* $\langle \mathrm{SEN}, \lesssim \rangle / \bigcup \mathcal{K}$ *is isomorphic to the order direct limit of the system of* $\rho$*-pofunctors* $\{\langle \mathrm{SEN}, \lesssim \rangle/{\lesssim'} : {\lesssim'} \in \mathcal{K}\}$ *by the system of surjective order translations*

$$\begin{aligned} \mathcal{F} = \langle \langle F^{\lesssim', \lesssim''}, \alpha^{\lesssim', \lesssim''} \rangle &: \langle \mathrm{SEN}, \lesssim \rangle/{\lesssim'} \\ &\to^p \langle \mathrm{SEN}, \lesssim \rangle/{\lesssim''} : {\lesssim'}, {\lesssim''} \in \mathcal{K}, {\lesssim'} \leq {\lesssim''} \rangle, \end{aligned}$$

*where, for all* $\lesssim', \lesssim'' \in \mathcal{K}$, $F^{\lesssim',\lesssim''} = \mathrm{I}_{\mathbf{Sign}}$ *and, for all* $\Sigma \in |\mathbf{Sign}|$, $\phi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_{\Sigma}^{\lesssim',\lesssim''}(\phi/{\sim'_{\Sigma}}) = \phi/{\sim''_{\Sigma}}.$$

*Proof* Given $\lesssim' \in \mathcal{K}$, let $[\lesssim') := \{\lesssim'' \in \mathcal{K} : \lesssim' \leq \lesssim''\}$. Now, set

$$\mathcal{D} = \{\mathcal{G} \subseteq \mathcal{K} : (\exists \lesssim)([\lesssim) \subseteq \mathcal{G})\}$$

and define $\langle G, \beta\rangle : \langle \mathrm{SEN}, \lesssim\rangle \to \lim^{\mathcal{F}}_{\lesssim'\in\mathcal{K}}\langle \mathrm{SEN}, \lesssim\rangle/{\sim'}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, by

$$\beta_{\Sigma}(\phi) = \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \phi/{\sim'_{\Sigma}}.$$

$\langle G, \beta\rangle$ is an order translation. To show this, it is first shown that it is $(N, \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} N^{\sim'})$-epimorphic and, then, that it preserves corresponding posystems.

We do have, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$, $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \ldots, \phi_{n-1} \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}
&\beta_{\Sigma}\,(\sigma_{\Sigma}(\phi_0, \ldots, \phi_{n-1}))\\
&\quad = \textstyle\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \sigma_{\Sigma}(\phi_0, \ldots, \phi_{n-1})/{\sim'_{\Sigma}}\\
&\quad = \textstyle\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \sigma^{\sim'}_{\Sigma}(\phi_0/{\sim'_{\Sigma}}, \ldots, \phi_{n-1}/{\sim'_{\Sigma}})\\
&\quad = \textstyle\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \sigma^{\sim'}{}_{\prod_{\lesssim'\in\mathcal{K}}\Sigma}\left(\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \phi_0/{\sim'_{\Sigma}}, \ldots, \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \phi_{n-1}/{\sim'_{\Sigma}}\right)\\
&\quad = \textstyle\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \sigma^{\sim'}{}_{\prod_{\lesssim'\in\mathcal{K}}\Sigma}\big(\beta_{\Sigma}(\phi_0), \ldots, \beta_{\Sigma}(\phi_{n-1})\big)
\end{aligned}$$

Hence $\langle G, \beta\rangle$ is $(N, \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} N^{\sim'})$-epimorphic.

Moreover, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}
\phi \lesssim_{\Sigma} \psi \ &\text{implies}\ (\forall \lesssim' \in \mathcal{K})(\phi/{\sim'_{\Sigma}} \lesssim'_{\Sigma}/{\sim'_{\Sigma}}\ \psi/{\sim'_{\Sigma}})\\
&\text{iff}\quad \textstyle\prod_{\lesssim'\in\mathcal{K}} \phi/{\sim'_{\Sigma}}\ \prod_{\lesssim'\in\mathcal{K}} \lesssim'_{\Sigma}/{\sim'_{\Sigma}}\ \prod_{\lesssim'\in\mathcal{K}} \psi/{\sim'_{\Sigma}}\\
&\text{implies}\ \textstyle\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \phi/{\sim'_{\Sigma}}\ \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \lesssim'/{\sim'}_{\prod_{\lesssim'\in\mathcal{K}}\Sigma}\ \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \psi/{\sim'_{\Sigma}}\\
&\text{iff}\quad \beta_{\Sigma}(\phi)\ \textstyle\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \lesssim'/{\sim'}_{\prod_{\lesssim'\in\mathcal{K}}\Sigma}\ \beta_{\Sigma}(\psi),
\end{aligned}$$

whence $\langle G, \beta\rangle$ is an order translation.

Next, it is shown that $\langle G, \beta\rangle$ is a surjective order translation. Recall that, for all $\Sigma \in |\mathbf{Sign}|$, every element of $[\lim^{\mathcal{F}}_{\lesssim'\in\mathcal{K}}\langle \mathrm{SEN}, \lesssim\rangle/{\sim'}]_{\prod_{\lesssim'\in\mathcal{K}}\Sigma}$ is of the form $\prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \phi_{\lesssim'}/{\sim'_{\Sigma}}$, where, there exists $\lesssim'' \in \mathcal{K}$, such that $\alpha_{\Sigma}^{\lesssim'',\lesssim'}(\phi_{\lesssim''}/{\sim''_{\Sigma}}) = \phi_{\lesssim'}/{\sim'_{\Sigma}}$, for all $\lesssim' \in \mathcal{K}$, with $\lesssim'' \leq \lesssim'$. Set $\psi := \phi_{\lesssim''}$. Then, for all $\lesssim' \geq \lesssim''$, we have $\phi_{\lesssim'}/{\sim'_{\Sigma}} = \alpha_{\Sigma}^{\lesssim'',\lesssim'}(\phi_{\lesssim''}/{\sim''_{\Sigma}}) = \psi/{\sim'_{\Sigma}}$ and, therefore, $\prod_{\lesssim'\in\mathcal{K}} \psi/{\sim'_{\Sigma}} \equiv^{\mathcal{D}}_{\prod_{\lesssim'\in\mathcal{K}}\Sigma} \prod_{\lesssim'\in\mathcal{K}} \phi_{\lesssim'}/{\sim'_{\Sigma}}$, i.e., $\beta_{\Sigma}(\psi) = \prod^{\mathcal{D}}_{\lesssim'\in\mathcal{K}} \phi_{\lesssim'}/{\sim'_{\Sigma}}$, and $\langle G, \beta\rangle$ is indeed surjective.

Finally, the result will follow from the Order Isomorphism Theorem (Corollary 16 of [42]), if it is shown that

$$\mathrm{OrdKer}(\langle G, \beta\rangle) = \bigcup \mathcal{K}.$$

In fact, we have, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}
&\beta_\Sigma(\phi) \textstyle\prod^{\mathcal{D}}_{\lesssim' \in \mathcal{K}} \lesssim'/{\sim'}_{\prod_{\lesssim' \in \mathcal{K}} \Sigma}\, \beta_\Sigma(\psi)\\
&\quad \text{iff} \quad \textstyle\prod^{\mathcal{D}}_{\lesssim' \in \mathcal{K}} \phi/{\sim'_\Sigma} \prod^{\mathcal{D}}_{\lesssim' \in \mathcal{K}} \lesssim'/{\sim'}_{\prod_{\lesssim' \in \mathcal{K}} \Sigma} \prod^{\mathcal{D}}_{\lesssim' \in \mathcal{K}} \psi/{\sim'_\Sigma}\\
&\quad \text{iff} \quad (\exists \lesssim'')(\forall \lesssim' \geq \lesssim'')(\phi/{\sim'_\Sigma} \lesssim'_\Sigma/{\sim'_\Sigma}\, \psi/{\sim'_\Sigma})\\
&\quad \text{iff} \quad (\exists \lesssim'')(\forall \lesssim' \geq \lesssim'')(\phi \lesssim'_\Sigma \psi)\\
&\quad \text{iff} \quad \phi\, (\textstyle\bigcup \mathcal{K})_\Sigma\, \psi. \qquad \square
\end{aligned}$$

Having Lemmas 5 and 6 at hand, we proceed now with stating and proving the Order **SLP** Theorem, the main theorem of this section, stating that a class of compatible pofunctors is a quasi-povariety if and only if it is closed under the formation of subpofunctors, order direct limits and order direct products. The Order **SLP** Theorem abstracts the Order **SLP** Theorem for partially ordered universal algebras of Pałasińska and Pigozzi (Theorem 3.14 of [26]).

**Theorem 7** (Order SLP Theorem) *A class* $\mathtt{K}$ *of compatible pofunctors is a quasi-po-variety iff it is closed under the formation of subpofunctors, order direct limits and direct products, i.e., iff* $\mathbf{SLP}(\mathtt{K}) = \mathtt{K}$.

*Proof* Suppose, first, that Q is a collection of $N$-quasi-inidentities, such that $\mathtt{K} = \mathbf{Mod}(\mathrm{Q})$. It suffices to show that $\mathbf{S}(\mathtt{K}) \subseteq \mathtt{K}$ and $\mathbf{P}_{\mathrm{R}}(\mathtt{K}) \subseteq \mathtt{K}$, since order direct products are special cases of order reduced products and order direct limits are subpofunctors of order reduced products.

For $\mathbf{S}(\mathtt{K}) \subseteq \mathtt{K}$, suppose that $\langle \mathrm{SEN}, \lesssim\rangle, \langle \mathrm{SEN}', \lesssim'\rangle$ are pofunctors such that $\langle \mathrm{SEN}', \lesssim'\rangle$ is a $\rho$-subpofunctor of the $\rho$-pofunctor $\langle \mathrm{SEN}, \lesssim\rangle \in \mathtt{K}$. Let

$$s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x}) \in \mathrm{Q}$$

and consider $\Sigma' \in |\mathbf{Sign}'|$, $\vec{\psi} \in \mathrm{SEN}'(\Sigma')^X$, such that $s_{i_{\Sigma'}}(\vec{\psi}) \lesssim'_{\Sigma'} t_{i_{\Sigma'}}(\vec{\psi})$, for all $i < n$. But, since $\langle \mathrm{SEN}, \lesssim\rangle \in \mathtt{K}$, we have that $s_{i_\Sigma}(\vec{\phi}) \lesssim_\Sigma t_{i_\Sigma}(\vec{\phi})$, for all $i < n$, imply that $u_\Sigma(\vec{\phi}) \lesssim_\Sigma v_\Sigma(\vec{\phi})$, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$. But, since $\langle \mathrm{SEN}', \lesssim'\rangle$ is a subpofunctor of $\langle \mathrm{SEN}, \lesssim\rangle$, we have that $\Sigma' \in |\mathbf{Sign}'| \subseteq |\mathbf{Sign}|$ and $\vec{\psi} \in \mathrm{SEN}'(\Sigma')^X \subseteq \mathrm{SEN}(\Sigma')^X$ and $\lesssim'_{\Sigma'} = \lesssim_{\Sigma'} \cap \mathrm{SEN}'(\Sigma')^2$, which give that $u_{\Sigma'}(\vec{\psi}) \lesssim'_{\Sigma'} v_{\Sigma'}(\vec{\psi})$. Hence $\langle \mathrm{SEN}', \lesssim'\rangle \models s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x})$ and $\langle \mathrm{SEN}', \lesssim'\rangle \in \mathtt{K}$.

For $\mathbf{P}_{\mathrm{R}}(\mathtt{K}) \subseteq \mathtt{K}$, suppose that $\langle \mathrm{SEN}^i, \lesssim^i\rangle \in \mathtt{K}$, for all $i \in I$, and let $F$ be a proper filter over $I$. Consider $s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x}) \in \mathrm{Q}$. We have, for all $\Sigma_i \in |\mathbf{Sign}^i|$, $\vec{\phi}^i \in \mathrm{SEN}^i(\Sigma_i)^X$, $i \in I$, $s_{j_{\Sigma_i}}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} t_{j_{\Sigma_i}}(\vec{\phi}^i)$, for all $j < n$, imply that $u_{\Sigma_i}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} v_{\Sigma_i}(\vec{\phi}^i)$. Now, consider $\Sigma_i \in |\mathbf{Sign}^i|$, $\vec{\phi}^i \in \mathrm{SEN}^i(\Sigma_i)^X$, $i \in I$, such that

$$s_{j_{\prod_{i \in I} \Sigma_i}}\left(\prod^F_{i \in I} \vec{\phi}^i\right) \prod^F_{i \in I} \lesssim^i_{\prod_{i \in I} \Sigma_i} t_{j_{\prod_{i \in I} \Sigma_i}}\left(\prod^F_{i \in I} \vec{\phi}^i\right), \quad \text{for all} \quad j < n.$$

This is equivalent to

$$s_{j_{\prod_{i \in I} \Sigma_i}}\left(\prod_{i \in I} \vec{\phi}^i\right)/{\equiv^F_{\prod_{i \in I} \Sigma_i}} \prod^F_{i \in I} \lesssim^i_{\prod_{i \in I} \Sigma_i} t_{j_{\prod_{i \in I} \Sigma_i}}\left(\prod_{i \in I} \vec{\phi}^i\right)/{\equiv^F_{\prod_{i \in I} \Sigma_i}},$$

for all $j < n$. Therefore $\{i \in I : s_{j_{\Sigma_i}}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} t_{j_{\Sigma_i}}(\vec{\phi}^i)\} \in F$, for all $j < n$, which yields that $\bigcap_{j<n}\{i \in I : s_{j_{\Sigma_i}}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} t_{j_{\Sigma_i}}(\vec{\phi}^i)\} \in F$. But note that, by our hypothesis,

$$\left\{i \in I : u_{\Sigma_i}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} v_{\Sigma_i}(\vec{\phi}^i)\right\} \supseteq \bigcap_{j<n}\left\{i \in I : s_{j_{\Sigma_i}}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} t_{j_{\Sigma_i}}(\vec{\phi}^i)\right\},$$

and, therefore $\{i \in I : u_{\Sigma_i}(\vec{\phi}^i) \lesssim^i_{\Sigma_i} v_{\Sigma_i}(\vec{\phi}^i)\} \in F$, which proves that

$$u_{\prod_{i\in I}\Sigma_i}\left(\prod_{i\in I}^F \vec{\phi}^i\right) \prod_{i\in I}^F \lesssim^i_{\prod_{i\in I}\Sigma_i} v_{\prod_{i\in I}\Sigma_i}\left(\prod_{i\in I}^F \vec{\phi}^i\right).$$

Therefore $\prod_{i\in I}^F\langle \mathrm{SEN}^i, \lesssim^i\rangle \models s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x})$ and $\prod_{i\in I}^F\langle \mathrm{SEN}^i, \lesssim^i\rangle \in \mathtt{K}$.

Suppose, conversely, that $\mathbf{SLP}(\mathtt{K}) = \mathtt{K}$. Consider the collection Q of all $N$-quasi-inidentities that are satisfied by all pofunctors in $\mathtt{K}$. Then, obviously $\mathtt{K} \subseteq \mathbf{Mod}(\mathrm{Q})$ and it suffices to show the reverse inclusion, i.e., that $\mathbf{Mod}(\mathrm{Q}) \subseteq \mathtt{K}$. So, suppose that $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{Mod}(\mathrm{Q})$. Consider the surjective order translation $\langle I_{\mathbf{Sign}}, \mu^N\rangle : \langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle \to \langle \mathrm{SEN}, \lesssim\rangle$. Let $\lesssim' = \mathrm{OrdKer}(\langle I_{\mathbf{Sign}}, \mu^N\rangle)$ and consider the relation system $R = \{R_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ on $\mathrm{Te}^N \circ \mathrm{SEN}$, defined, for all $\Sigma \in |\mathbf{Sign}|$, by

$$R_\Sigma = \{\langle s_i(\vec{\phi}), t_i(\vec{\phi})\rangle : \vec{\phi} \in \mathrm{SEN}(\Sigma)^X, i < n\},$$

where $s_{i_\Sigma}(\vec{\phi}) \lesssim'_\Sigma t_{i_\Sigma}(\vec{\phi})$, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$ and all $i < n$. Let $u(\vec{x}), v(\vec{x})$ be such that $u(\vec{\phi})\ \Phi^{\mathtt{K}}_\rho(R)_\Sigma\ v(\vec{\phi})$, for all functors SEN, all $\Sigma \in |\mathbf{Sign}|$, all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$ and all collections $R$ induced by $\langle s_i(\vec{x}), t_i(\vec{x})\rangle$, $i < n$, on SEN, as defined above. Then, by Lemma 5,

$$s_0(\vec{x}) \preccurlyeq t_0(\vec{x}), \ldots, s_{n-1}(\vec{x}) \preccurlyeq t_{n-1}(\vec{x}) \to u(\vec{x}) \preccurlyeq v(\vec{x})$$

is an $N$-quasi-inidentity of $\mathtt{K}$. But, by hypothesis, it is also an $N$-quasi-inidentity of $\langle \mathrm{SEN}, \lesssim\rangle$. Hence, since $R \leq \lesssim'$, we have, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, and all $i < n$, $s_i(\vec{\phi}) \lesssim'_\Sigma t_i(\vec{\phi})$ giving $s_{i_\Sigma}(\vec{\phi}) \lesssim_\Sigma t_{i_\Sigma}(\vec{\phi})$, whence $u_\Sigma(\vec{\phi}) \lesssim_\Sigma v_\Sigma(\vec{\phi})$ and, therefore, $u(\vec{\phi}) \lesssim'_\Sigma v(\vec{\phi})$. Thus, $\Phi^{\mathtt{K}}_\rho(R) \leq \lesssim'$, for all $R \leq \lesssim'$ induced by a finite set of inequations, denoted $R \leq_{\mathrm{f}} \lesssim'$. Thus, $\lesssim' = \bigcup_{R\leq_{\mathrm{f}}\lesssim'}\Phi^{\mathtt{K}}_\rho(R)$. Since, in addition, the collection $\{\Phi^{\mathtt{K}}_\rho(R) : R \leq_{\mathrm{f}} \lesssim'\}$ is upward directed by signature-wise inclusions, we conclude, by Lemma 6, that $\langle \mathrm{SEN}, \lesssim\rangle = \langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/\lesssim'$ is the order direct limit of $\langle\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/\Phi^{\mathtt{K}}_\rho(R) : R \leq_{\mathrm{f}} \lesssim'\rangle$. Therefore, since $\langle \mathrm{Te}^N \circ \mathrm{SEN}, \Delta^{\mathrm{Te}^N\circ\mathrm{SEN}}\rangle/\Phi^{\mathtt{K}}_\rho(R) \in \mathtt{K}$, for all $R \leq_{\mathrm{f}} \lesssim'$, we obtain that $\langle \mathrm{SEN}, \lesssim\rangle \in \mathbf{L}(\mathtt{K}) = \mathtt{K}$. □

This result immediately implies

**Corollary 8** *A class* $\mathtt{K}$ *of compatible pofunctors is a quasi-povariety iff* $\mathbf{SP}_{\mathrm{R}}(\mathtt{K}) = \mathtt{K}$ *iff* $\mathbf{SPP}_{\mathrm{U}}(\mathtt{K}) = \mathtt{K}$.

Recall that both $\mathbf{SP}_{\mathrm{R}}$ and $\mathbf{SPP}_{\mathrm{U}}$ were shown to be closure operators on classes of compatible pofunctors in Theorem 18 of [43]. Thus, both Theorem 7 and Corollary 8 serve in characterizing by logical, rather than algebraic, means the classes of compatible pofunctors that these two closure operators generate.

## 5 Algebraizable PoVarieties

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, $N$ a category of natural transformations on SEN and $\rho$ a polarity for $N$. Given a class $\mathtt{K}$ of pofunctors, compatible with SEN, denote by

$$\mathrm{Alg}(\mathtt{K}) = \{\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle : \langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}\}.$$

**Lemma 9** *If* $\mathtt{K}$ *is a quasi-povariety, then* $\mathrm{Alg}(\mathtt{K})$ *is a quasi-variety.*

*Proof* If $\mathtt{K}$ is closed under subpofunctors and order filtered products, then it is not very difficult to verify that $\mathrm{Alg}(\mathtt{K})$ is also closed under subfunctors and filtered products. Hence it is a quasi-variety on its own right. □

In what follows, the notation SEN will sometimes be used to denote, except for the functor SEN itself, also the pofunctor $\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle$.

Consider a quasivariety $\mathtt{Q}$ of functors (with compatible categories of natural transformations) and SEN a functor with compatible category $N$ of natural transformations with those in $\mathtt{Q}$ but not necessarily in $\mathtt{Q}$. Those $N$-congruence systems $\theta$ on SEN, such that $\mathrm{SEN}^\theta \in \mathtt{Q}$ are called $\mathtt{Q}$-*$N$-congruence systems*. Define

$$\mathrm{Con}^N_{\mathtt{Q}}(\mathrm{SEN}) = \{\theta \in \mathrm{Con}^N(\mathrm{SEN}) : \mathrm{SEN}^\theta \in \mathtt{Q}\}.$$

It is next shown that, given a quasi-povariety $\mathtt{K}$ and a functor SEN, with compatible category of natural transformations and compatible polarity with those of the pofunctors in $\mathtt{K}$, the passage from $\lesssim$ to $\sim$ is a surjective mapping from $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle)$ onto $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$. This forms an analog of Proposition 4.1 of [26] for quasi-povarieties of pofunctors.

**Proposition 10** *Suppose that* $\mathtt{K}$ *is a quasi-povariety and* SEN *a functor compatible with those in* $\mathtt{K}$*. Then the mapping* $\lesssim \mapsto \sim$ *is a surjective mapping from* $\mathrm{QoSys}^{\mathtt{K}}_\rho$ $(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle)$ *onto* $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$.

*Proof* First, let $\lesssim \in \mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle)$. This means, by the definition of $\mathrm{QoSys}^{\mathtt{K}}_\rho$ $(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle)$, that $\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle/{\sim} \in \mathtt{K}$. And this, in turn, means, by taking into account the definition of $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$, that $\sim \in \mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$. Thus, $\lesssim \mapsto \sim$ is a mapping from $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle)$ into $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$. To finish the proof, it remains to show that $\lesssim \mapsto \sim$ maps $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle)$ *onto* $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$.

To this end, suppose that $\theta \in \mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$. Hence, by the definition of $\mathrm{Alg}(\mathtt{K})$-$N$-congruence systems, $\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}} \rangle/\theta \in \mathrm{Alg}(\mathtt{K})$. Thus, by the definition of $\mathrm{Alg}(\mathtt{K})$, there exists a $\rho^\theta$-posystem

$$\lesssim \in \mathrm{PoSys}_{\rho^\theta}(\mathrm{SEN}^\theta),$$

such that $\langle \mathrm{SEN}^\theta, \lesssim\rangle \in \mathtt{K}$. Let $\lesssim' := (\pi^\theta)^{-1}(\lesssim)$, where $\langle I_{\mathbf{Sign}}, \pi^\theta\rangle : \mathrm{SEN} \to \mathrm{SEN}^\theta$ is the natural projection translation. We have, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}\phi \sim'_\Sigma \psi \text{ iff } & \phi\ (\pi^\theta_\Sigma)^{-1}(\lesssim_\Sigma) \cap (\pi^\theta_\Sigma)^{-1}(\gtrsim_\Sigma)\ \psi \\ \text{iff } & \phi\ (\pi^\theta_\Sigma)^{-1}(\sim_\Sigma)\ \psi \\ \text{iff } & \phi\ (\pi^\theta_\Sigma)^{-1}(\Delta^{\mathrm{SEN}^\theta}_\Sigma)\ \psi \\ \text{iff } & \phi\ \theta_\Sigma\ \psi,\end{aligned}$$

i.e., $\sim' = \theta$ and, therefore, $\lesssim'/\sim' = \lesssim'/\theta = \lesssim$. This yields that $\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}}\rangle/\sim' = \langle \mathrm{SEN}/\sim', \lesssim'/\sim'\rangle = \langle \mathrm{SEN}^\theta, \lesssim\rangle \in \mathtt{K}$. Therefore $\lesssim' \in \mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}}\rangle)$, which, taking into account that $\sim' = \theta$, yields the surjectivity of $\lesssim \mapsto \sim$ as a mapping from $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}}\rangle)$ onto $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$. □

Next, following Definition 4.2 of [26], the key concept of an $N$-finitely algebraizable quasi-povariety is introduced.

**Definition 11** Consider a functor SEN, with $N$ a category of natural transformations on SEN and $\rho$ a polarity for $N$. Let $\langle \mathrm{SEN}, \lesssim\rangle$ be a $\rho$-pofunctor and $S(x, y) \approx T(x, y) = \{s^i(x, y) \approx t^i(x, y) : i < n\}$ a finite set of equations in two variables, where $s^i, t^i : \mathrm{SEN}^2 \to \mathrm{SEN}$ are natural transformations in $N$. $\lesssim$ is said to be *$N$-definable by* $S(x, y) \approx T(x, y)$ if, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\phi \lesssim_\Sigma \psi \quad \text{iff} \quad S_\Sigma(\phi, \psi) = T_\Sigma(\phi, \psi),$$

where $S_\Sigma(\phi, \psi) = T_\Sigma(\phi, \psi)$ abbreviates the condition

$$(\forall i < n)(s^i_\Sigma(\phi, \psi) = t^i_\Sigma(\phi, \psi)).$$

A quasi-povariety $\mathtt{K}$ is *$N$-finitely algebraizable* if there exists a finite set of $N$-equations $S(x, y) \approx T(x, y)$, as above, such that, for all $\langle \mathrm{SEN}', \lesssim'\rangle \in \mathtt{K}$, $S' \approx T'$ defines $\lesssim'$, where by $S' \approx T'$ are denoted the $N'$-equations that correspond to the $N$-equations in $S \approx T$ via the compatibility property. In that case $S \approx T$ is called a *defining set of $N$-equations* for $\mathtt{K}$.

The following proposition reveals a tie between a defining set of $N$-equations in an $N$-finitely algebraizable quasi-povariety $\mathtt{K}$ and the collection of $\mathtt{K}$-$\rho$-qosystems of members of $\mathtt{K}$. It forms an analog of Proposition 4.3 of [26].

**Proposition 12** *Let $\mathtt{K}$ be an $N$-finitely algebraizable quasi-povariety with defining $N$-equations $S \approx T$. Then, for all $\langle \mathrm{SEN}, \lesssim\rangle \in \mathtt{K}$, all $\lesssim' \in \mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,*

$$\phi \lesssim'_\Sigma \psi \quad \textit{iff} \quad S_\Sigma(\phi, \psi) \sim'_\Sigma T_\Sigma(\phi, \psi).$$

*Proof* Suppose $\langle \mathrm{SEN}, \lesssim\rangle \in \mathtt{K}$, $\lesssim' \in \mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \lesssim\rangle)$, $\Sigma \in |\mathbf{Sign}|$ and $\phi, \psi \in \mathrm{SEN}(\Sigma)$. We have

$$\begin{aligned}\phi \lesssim'_\Sigma \psi \text{ iff } & \phi/\sim'_\Sigma \lesssim'_\Sigma/\sim'_\Sigma \psi/\sim'_\Sigma \\ \text{iff } & S^{\sim'}_\Sigma(\phi/\sim'_\Sigma, \psi/\sim'_\Sigma) = T^{\sim'}_\Sigma(\phi/\sim'_\Sigma, \psi/\sim'_\Sigma) \\ \text{iff } & S_\Sigma(\phi, \psi)/\sim'_\Sigma = T_\Sigma(\phi, \psi)/\sim'_\Sigma \\ \text{iff } & S_\Sigma(\phi, \psi) \sim'_\Sigma T_\Sigma(\phi, \psi).\end{aligned}$$

□

Let $\mathrm{K}$ be an $N$-finitely algebraizable quasi-povariety with defining $N$-equations $S \approx T$. For every $\langle \mathrm{SEN}, \lesssim \rangle \in \mathrm{K}$, all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, we have that

$$\phi = \psi \quad \text{iff} \quad \phi \lesssim_\Sigma \psi \text{ and } \psi \lesssim_\Sigma \phi.$$

Therefore, we obtain, by Definition 11, that

$$\phi = \psi \quad \text{iff} \quad S_\Sigma(\phi, \psi) = T_\Sigma(\phi, \psi) \text{ and } S_\Sigma(\psi, \phi) = T_\Sigma(\psi, \phi).$$

Hence, if $\mathrm{K}$ is $N$-finitely algebraizable with defining $N$-equations $S \approx T$, then $S(x, x) \approx T(x, x)$ are $N$-identities of $\mathrm{Alg}(\mathrm{K})$ and the $N$-quasi-equation $S(x, y) \approx T(x, y), S(y, x) \approx T(y, x) \to x \approx y$ is an $N$-quasi-identity of $\mathrm{Alg}(\mathrm{K})$.

Corollary 13, that follows, shows that, in case $\mathrm{K}$ is an $N$-finitely algebraizable povariery, the mapping $\lesssim' \mapsto \sim'$ of Proposition 10 is an injection between $\mathrm{QoSys}^{\mathrm{K}}_\rho(\langle \mathrm{SEN}, \lesssim \rangle)$ and $\mathrm{Con}^N_{\mathrm{Alg}(\mathrm{K})}(\mathrm{SEN})$, for every pofunctor $\langle \mathrm{SEN}, \lesssim \rangle \in \mathrm{K}$.

**Corollary 13** *Let $\mathrm{K}$ be an $N$-finitely algebraizable quasi-povariety.*

1. *For all* $\mathrm{SEN} \in \mathrm{Alg}(\mathrm{K})$, *there exists unique $\rho$-posystem $\lesssim$ of* SEN, *such that* $\langle \mathrm{SEN}, \lesssim \rangle \in \mathrm{K}$.
2. *The mapping $\lesssim' \mapsto \sim'$ from* $\mathrm{QoSys}^{\mathrm{K}}_\rho(\langle \mathrm{SEN}, \lesssim \rangle)$ *to* $\mathrm{Con}^N_{\mathrm{Alg}(\mathrm{K})}(\mathrm{SEN})$ *is injective, for every pofunctor* $\langle \mathrm{SEN}, \lesssim \rangle \in \mathrm{K}$.

*Proof* For Part 1, note that, since, by hypothesis, $\mathrm{K}$ is an $N$-finitely algebraizable quasi-povariety, there exists a finite set $S(x, y) \approx T(x, y)$ of defining $N$-equations for $\mathrm{K}$. Hence, if $\langle \mathrm{SEN}, \lesssim \rangle, \langle \mathrm{SEN}, \lesssim' \rangle \in \mathrm{K}$, then, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi \lesssim_\Sigma \psi$ iff $S_\Sigma(\phi, \psi) = T_\Sigma(\phi, \psi)$ iff $\phi \lesssim'_\Sigma \psi$. Therefore $\lesssim = \lesssim'$.

For Part 2, suppose that $\lesssim', \lesssim'' \in \mathrm{QoSys}^{\mathrm{K}}_\rho(\langle \mathrm{SEN}, \lesssim \rangle)$, such that $\sim' = \sim''$. Then, we have $\langle \mathrm{SEN}, \lesssim \rangle / {\sim'} = \langle \mathrm{SEN}, \lesssim \rangle / {\sim''}$, whence $\langle \mathrm{SEN}/{\sim'}, \lesssim'/{\sim'} \rangle = \langle \mathrm{SEN}/{\sim''}, \lesssim''/{\sim''} \rangle$. Thus, since $\sim' = \sim''$, we get that

$$\langle \mathrm{SEN}/{\sim'}, \lesssim'/{\sim'} \rangle = \langle \mathrm{SEN}/{\sim'}, \lesssim''/{\sim'} \rangle.$$

But both $\langle \mathrm{SEN}/{\sim'}, \lesssim'/{\sim'} \rangle$ and $\langle \mathrm{SEN}/{\sim'}, \lesssim''/{\sim'} \rangle$ are in $\mathrm{K}$, whence $\mathrm{SEN}/{\sim'} \in \mathrm{Alg}(\mathrm{K})$ and, therefore, by Part 1, $\lesssim'/{\sim'} = \lesssim''/{\sim'}$, which yields that $\lesssim' = \lesssim''$. Thus, the mapping $\lesssim' \mapsto \sim'$ from the collection $\mathrm{QoSys}^{\mathrm{K}}_\rho(\langle \mathrm{SEN}, \lesssim \rangle)$ to $\mathrm{Con}^N_{\mathrm{Alg}(\mathrm{K})}(\mathrm{SEN})$ is injective. □

If an $N$-finitely algebraizable quasi-povariety $\mathrm{K}$ has a known axiomatization in terms of a collection I of inidentities and a collection Q of quasi-inidentities, then there is a way of discovering a collection of identities and quasi-identities that axiomatize the quasivariety $\mathrm{Alg}(\mathrm{K})$, given by Lemma 9. The identities and quasi-identities, presented below, form a straightforward translation in the present context of the ones given by Pałasińska and Pigozzi in the context of finitely algebraizable quasi-povarieties of universal poalgebras (see Proposition 4.6 of [26]).

**Proposition 14** *Let $\mathrm{K}$ be an $N$-finitely algebraizable quasi-povariety. If $S \approx T$ is a defining set of $N$-equations for $\mathrm{K}$, then the quasi-variety* $\mathrm{Alg}(\mathrm{K})$ *is defined by the* *following identities and quasi-identities, where* I *and* Q *are, respectively, any set of* $N$*-inidentities and* $N$*-quasi-inidentities defining* $\mathtt{K}$:

$$\begin{aligned}
S(x, x) &\approx T(x, x) && (2)\\
S(x, y) &\approx T(x, y), S(y, z) \approx T(y, z) \to S(x, z) \approx T(x, z) && (3)\\
S(x, y) &\approx T(x, y) \to S(\sigma(\vec{z}_{<i}, x, \vec{z}_{>i}), \sigma(\vec{z}_{<i}, y, \vec{z}_{>i}))\\
&\approx T(\sigma(\vec{z}_{<i}, x, \vec{z}_{>i}), \sigma(\vec{z}_{<i}, y, \vec{z}_{>i})),\\
\sigma : \mathrm{SEN}^k &\to \mathrm{SEN} \textit{ in } N, \rho(\sigma, i) = +. && (4)\\
S(x, y) &\approx T(x, y) \to S(\sigma(\vec{z}_{<i}, y, \vec{z}_{>i}), \sigma(\vec{z}_{<i}, x, \vec{z}_{>i}))\\
&\approx T(\sigma(\vec{z}_{<i}, y, \vec{z}_{>i}), \sigma(\vec{z}_{<i}, x, \vec{z}_{>i})),\\
\sigma : \mathrm{SEN}^k &\to \mathrm{SEN} \textit{ in } N, \rho(\sigma, i) = -. && (5)\\
S(x, y) &\approx T(x, y), S(y, x) \approx T(y, x) \to x \approx y && (6)\\
S(s, t) &\approx T(s, t), s \preccurlyeq t \in \mathrm{I} && (7)\\
S(s_0, t_0) &\approx T(s_0, t_0), \ldots, S(s_{n-1}, t_{n-1})\\
&\approx T(s_{n-1}, t_{n-1}) \to S(u, v) \approx T(u, v),\\
s_0 &\preccurlyeq t_0, \ldots, s_{n-1} \preccurlyeq t_{n-1} \to u \preccurlyeq v \in \mathrm{Q}. && (8)
\end{aligned}$$

*Proof* Let E be the collection of $N$-equations and $N$-quasi Eqs. 2–8. It is not difficult to see from the discussion following Proposition 12 that every SEN $\in \mathrm{Alg}(\mathtt{K})$ does satisfy every $N$-equation and $N$-quasi-equation in the list. Thus $\mathrm{Alg}(\mathtt{K}) \subseteq \mathbf{Mod}(\mathrm{E})$. To see that the reverse inclusion also holds, let SEN $\in \mathbf{Mod}(\mathrm{E})$. Define $\lesssim = \{\lesssim_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, for all $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, by

$$\phi \lesssim_\Sigma \psi \quad \text{iff} \quad S_\Sigma(\phi, \psi) = T_\Sigma(\phi, \psi).$$

Then $\lesssim$ is a $\rho$-qosystem of SEN, by Eqs. 2–5. It is a posystem, by Eq. 6, and $\langle \mathrm{SEN}, \lesssim \rangle \in \mathbf{Mod}(\mathtt{K})$, by Eqs. 7 and 8. Therefore SEN $\in \mathrm{Alg}(\mathtt{K})$. □

Finally, some properties of $N$-finitely algebraizable quasi-povarieties are given, detailing the connection between the lattices $\mathbf{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \lesssim \rangle)$ to $\mathbf{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$, for an arbitrary $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$. This result forms a partial analog of Theorem 4.8 of [26]. In the context of partially ordered quasi-povarieties of universal algebras, the analogs of all three parts of Theorem 15 are shown to be equivalent to each other. This, however, does not seem to be the case in this, more general, setting.

**Theorem 15** *Let* $\mathtt{K}$ *be a quasi-povariety. The following are related by* $1 \to 2 \leftrightarrow 3$:

1. $\mathtt{K}$ *is* $N$*-finitely algebraizable.*
2. *For all* $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$, $\lesssim' \mapsto \sim'$ *is an injective map from the set* $\mathrm{QoSys}^{\mathtt{K}}_\rho(\langle \mathrm{SEN}, \lesssim \rangle)$ *to* $\mathrm{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$.
3. *For all* $\langle \mathrm{SEN}, \lesssim \rangle \in \mathtt{K}$, $\lesssim' \mapsto \sim'$ *is an isomorphism from the lattice* $\mathbf{QoSys}^{\mathtt{K}}_\rho$ $(\langle \mathrm{SEN}, \lesssim \rangle)$ *to* $\mathbf{Con}^N_{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$.

*Proof*

$1 \to 2$ By Corollary 13.

$2 \to 3$ Assume that $\lesssim' \mapsto \sim'$ is an injective map from $\mathrm{QoSys}^{\mathtt{K}}_{\rho}(\langle \mathrm{SEN}, \lesssim\rangle)$ to $\mathrm{Con}^{\mathrm{Alg}(\mathtt{K})}(\mathrm{SEN})$. It is also an order-preserving mapping. To show that it is surjective, observe that

$$\mathrm{QoSys}^{\mathtt{K}}_{\rho}(\langle \mathrm{SEN}, \lesssim\rangle) = \mathrm{QoSys}^{\mathtt{K}}_{\rho}(\langle \mathrm{SEN}, \Delta^{\mathrm{SEN}}\rangle),$$

because, by the hypothesis 2, $\lesssim$ is the unique $\rho$-posystem $\lesssim'$ on SEN, such that $\langle \mathrm{SEN}, \lesssim'\rangle \in \mathtt{K}$, and use Proposition 10.

$3 \to 2$ Obvious. □

## 6 Examples

In this section a few examples are presented to help illustrate some of the concepts of the theory that was developed in the previous sections. First, it is shown how all ordered varieties and quasi-varieties of universal algebras may be viewed as special cases of $\rho$-quasi-povarieties and $\rho$-povarieties, respectively, in the sense of this paper. Then, two examples of special interest to categorical abstract algebraic logic (CAAL), that of equational logic and of first-order logic without terms, that were the paradigms on which the initial development of CAAL was based, are revisited in the present context. In a slightly different vein, despite the fact that, as it was mentioned before the statement of Theorem 15, it does not seem likely that the implication $(2 \leftrightarrow 3) \to 1$ of Theorem 15 holds in general, we were, unfortunately, unable to provide a counterexample proving this at the present time. As a result, this question has to be left open for future investigation.

### 6.1 Povarieties and Quasi-Povarieties of Universal Algebras

Both the **HSP** and the **SLP** Theorem of Pałasińska and Pigozzi, Theorem 3.14 and Theorem 3.17 of [26], are special cases of Theorem 7 and Theorem 4, respectively. This may be easily seen if one identifies ordinary partially ordered algebras with pofunctors over a trivial signature category. The sentence functor maps the single signature object to the domain of the partially ordered algebra and the category $N$ of natural transformations consists of the entire clone of natural transformations generated by the basic operations of the algebra. Polarities are the polarities inherited by those attached to the different arguments of the basic operations under the composition compatibility property (see Section 2 of [42]). Since taking order homomorphic images, order subpofunctors, order direct products and order direct limits of a pofunctor or a collection of pofunctors with trivial signature categories results to pofunctors with the same property, both classes coincide with the corresponding classes generated by the corresponding operators in the universal algebraic framework.

Thus, one may provide as concrete examples in the present framework some appropriately modified examples from [26]. For instance, consider the partially-ordered left-residuated monoids (POLRMs), defined in [26] as those structures $\mathbf{A} = \langle\langle A, \cdot, \Rightarrow, 1\rangle, \leq\rangle$, where

- $\langle A, \cdot, 1\rangle$ is a monoid,
- $\langle\langle A, \cdot\rangle, \leq\rangle$ is a poalgebra and
- For all $a, b \in A$, $(\forall z \in A)(z \cdot a \leq b \quad \text{iff} \quad z \leq a \Rightarrow b)$.

It was shown in Proposition 2.2 of [26] that every POLRM is a $\rho$-poalgebra with polarity $\rho$ on $\cdot, \Rightarrow$ satisfying $\rho(\cdot, 0) = \rho(\cdot, 1) = \rho(\Rightarrow, 1) = +$ and $\rho(\Rightarrow, 0) = -$. Furthermore, it was shown in Proposition 3.4 of [26] that POLRMs form a $\rho$-povariety defined by the logical quasi-inidentities

$$x \preccurlyeq x$$
$$x \preccurlyeq y, y \preccurlyeq z \to x \preccurlyeq z$$
$$x_1 \preccurlyeq x_2, y_1 \preccurlyeq y_2 \to x_1 \cdot y_1 \preccurlyeq x_2 \cdot y_2$$
$$x_2 \preccurlyeq x_1, y_1 \preccurlyeq y_2 \to x_1 \Rightarrow y_1 \preccurlyeq x_2 \Rightarrow y_2$$

and the extra-logical inidentities

$$(x \cdot y) \cdot z \preccurlyeq\succcurlyeq x \cdot (y \cdot z)$$
$$1 \cdot x \preccurlyeq\succcurlyeq x$$
$$x \cdot 1 \preccurlyeq\succcurlyeq x$$
$$x \cdot (x \Rightarrow y) \preccurlyeq y$$
$$y \preccurlyeq x \Rightarrow x \cdot y$$

It was also remarked in [26] that partially-ordered groups, i.e., structures of the form $\langle\langle G, \cdot, ^{-1}, e\rangle, \leq\rangle$, where

- $\langle G, \cdot, ^{-1}, e\rangle$ is a group and
- $\leq$ is a partial ordering on $G$ with respect to which $\cdot$ is monotone in both arguments

may be viewed as POLRMs $\langle\langle G, \cdot, \Rightarrow, e\rangle, \leq\rangle$, where $x \Rightarrow y := x^{-1} \cdot y$. In this sense partially-ordered groups form a $\rho$-subpovariety of the class of POLRMs. The two partially-ordered groups (seeing as POLRMs) $\langle\langle \mathbf{Z}, +, -, 0\rangle, \leq\rangle$ and $\langle\langle \mathbf{Z}, +, -, 0\rangle, \Delta_{\mathbf{Z}}\rangle$, where $\mathbf{Z}$ is the set of integers and $\leq$ its natural ordering, show, via an application of Corollary 13, that the $\rho$-povariety of partially ordered groups is not algebraizable, which immediately implies, since it is a $\rho$-subpovariety of POLRMs, that the $\rho$-povariety of POLRMs is not algebraizable either.

### 6.2 Equational Logic

One of the paradigmatic examples that motivated the development of both the theory of institutions and the theory of categorical abstract algebraic logic was that of equational logic. One may consider pofunctors with underlying sentence functor the functor with domain the category of all algebraic signatures and mapping each signature to the collection of all terms over that signature. Based on the main result of [29], pofunctors of this form would correspond to ordered substitution algebras or ordered clone algebras, with the term substitution algebra used to suggest a variant of the algebras of Feldman [15] and the term clone algebra a variant of the algebras of Taylor [28]. We provide some more details here, based on the work presented in [29].

Recall from [29] that a chain set A is a family of sets $A = \{A_k : k \in \omega\}$, such that $A_k \subseteq A_{k+1}$, for every $k \in \omega$. A chain set morphism $f : A \to B$ is a family of set maps $f = \{f_k : A_k \to B_k : k \in \omega\}$, such that the following diagram commutes, for every $k \in \omega$,

$$\begin{tikzcd} A_{k+1} \arrow[r, "f_{k+1}"] & B_{k+1} \\ A_k \arrow[u, "i"] \arrow[r, "f_k"'] & B_k \arrow[u, "i"'] \end{tikzcd}$$

where by $i : A_k \hookrightarrow A_{k+1}$ and $i : B_k \hookrightarrow B_{k+1}$ are denoted the inclusion maps.

Given two chain set morphisms $f : A \to B$ and $g : B \to C$ we define their composite $gf : A \to C$ to be the collection of maps $gf = \{g_k f_k : A_k \to C_k : k \in \omega\}$. With this composition the collection of chain sets with chain set morphisms between them forms a category, the category of chain sets, which is denoted by **CSet**.

The chain set of $X$-terms $\mathrm{Tm}_X(V) = \{\mathrm{Tm}_X(V)_k : k \in \omega\} \in |\mathbf{CSet}|$ is defined by letting $\mathrm{Tm}_X(V)_k$ be the smallest set with

– $v_i \in \mathrm{Tm}_X(V)_k, i < k$,
– $x(t_0, \ldots, t_{n-1}) \in \mathrm{Tm}_X(V)_k$, for all $n \in \omega, x \in X_n - X_{n-1}, t_0, \ldots, t_{n-1} \in \mathrm{Tm}_X(V)_k$.

Given $X, Y \in |\mathbf{CSet}|$ and $f : X \to \mathrm{Tm}_Y(V) \in \mathrm{Mor}(\mathbf{CSet})$ the chain set morphism $f^* : \mathrm{Tm}_X(V) \to \mathrm{Tm}_Y(V)$, with $f^*_k : \mathrm{Tm}_X(V)_k \to \mathrm{Tm}_Y(V)_k$, for every $k \in \omega$, is defined by recursion on the structure of $X$-terms in the usual way.

The notation $f : X \rightharpoondown Y$ is used to denote a **CSet**-map $f : X \to \mathrm{Tm}_Y(V)$. Given two such maps $f : X \rightharpoondown Y$ and $g : Y \rightharpoondown Z$, their composition $g \circ f : X \rightharpoondown Z$ is defined to be the **CSet**-map

$$g \circ f = g^* f.$$

If, for every $X \in |\mathbf{CSet}|$, $j_X : X \rightharpoondown X$, is defined by letting $j_{X_k} : X_k \to \mathrm{Tm}_X(V)_k$ be given by

$$j_{X_k}(x) = x(v_0, \ldots, v_{k-1}), \text{ for all } x \in X_k - X_{k-1},$$

then **Sign**, having collection of objects $|\mathbf{CSet}|$ and collections of morphisms

$$\mathbf{Sign}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{CSet}(X, \mathrm{Tm}_Y(V))\},$$

for all $X, Y \in |\mathbf{CSet}|$, with composition $\circ$ and $X$-identity $j_X$, is a category.

The sentence functor $\mathrm{SEN} : \mathbf{CSet} \to \mathbf{Set}$ is defined at the object level, for every $X \in |\mathbf{Sign}|$, by setting

$$\mathrm{SEN}(X) = \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)_k.$$

and at the morphism level, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$, by letting $\mathrm{SEN}(f) : \mathrm{SEN}(X) \to \mathrm{SEN}(Y)$ be given, for all $t \in \bigcup_{k=0}^{\infty} \mathrm{Tm}_X(V)_k$,

$$\mathrm{SEN}(f)(t) = f_k^*(t), \quad \text{if} \quad t \in \mathrm{Tm}_X(V)_k.$$

$\mathrm{SEN}(f)$ is well-defined, because, if $t \in \mathrm{Tm}_X(V)_k \cap \mathrm{Tm}_X(V)_l$, then $f_k^*(t) = f_l^*(t)$, by the definition of a **CSet**-morphism. Birkhoff's equational logic is now obtained by considering the category of natural transformations $N$ on SEN corresponding to the entire clone of operations generated by the basic operations of clone algebras, as introduced and studied in [29], and by imposing positive polarities in all argument places, i.e., the logical quasi-inidentities

$$\begin{array}{l} x \preccurlyeq x \\ x \preccurlyeq y, y \preccurlyeq z \to x \preccurlyeq z \\ x \preccurlyeq x', y_0 \preccurlyeq y_0', \ldots, y_{n-1} \preccurlyeq y_{n-1}' \\ \quad \to C_n(x, y_0, \ldots, y_{n-1}) \preccurlyeq C_n(x', y_0', \ldots, y_{n-1}'), \text{for all } n \in \omega, \end{array}$$

and the extra-logical quasi-inidentity

$$x \preccurlyeq y \to y \preccurlyeq x.$$

These quasi-inidentities ensure that the $\rho$-qosystem on any member of this $\rho$-quasi-povariety is an $N$-congruence relation on a substitution algebra whose elements correspond to algebraic operations with fixed arities. Therefore, the operations of these algebras are divided into equivalence classes that respect the substitutivity property of terms for variables into term operations. The reader is invited to compare the $\rho$-quasi-povariety briefly described here with the presentation of the variety of algebras over a fixed signature as a quasi-povariety given in Example 3.6 of [26]. The logical axioms of monotonicity for each of the fundamental operations in the fixed signature in [26] are here replaced by a logical axiom scheme stipulating monotonicity of the substitution operations.

## 6.3 First-order Logic Without Terms

In this section $\subseteq_{\mathrm{f}}$ will denote the finite subset relation and $\mathcal{P}_{\mathrm{f}}$ the finite powerset operator. Recall from [31] that a hierarchy of sets or, simply, an h-set A is a family of sets $A = \{A_P : P \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, such that $A_P \cap A_Q = A_{P \cap Q}$, for every $P, Q \subset_{\mathrm{f}} \omega$. By a morphism of h-sets or,simply, an h-set morphism $f : A \to B$, is meant a family of set maps $f = \{f_P : A_P \to B_P : P \in \mathcal{P}_{\mathrm{f}}(\omega)\}$, such that the following diagram commutes, for every $P \subseteq Q \subset_{\mathrm{f}} \omega$,

$$\begin{tikzcd} A_Q \arrow[r, "f_Q"] & B_Q \\ A_P \arrow[u, "i"] \arrow[r, "f_P"'] & B_P \arrow[u, "i"'] \end{tikzcd}$$

where by $i : A_P \hookrightarrow A_Q$ and $i : B_P \hookrightarrow B_Q$ we denote the inclusion maps, i.e., $f_Q \restriction_{A_P} = f_P$, for all $P \subseteq Q \subset_{\mathrm{f}} \omega$. Given two chain set morphisms $f : A \to B$ and

$g : B \to C$, their composite $gf : A \to C$ is a collection of maps $gf = \{g_P f_P : A_P \to C_P : P \in \mathcal{P}_{\mathrm{f}}(\omega)\}$. With this composition the collection of h-sets with h-set morphisms between them forms a category, the category of h-sets, which is denoted by **HSet**.

In the sequel, by $\mathcal{L}$ will be denoted the set of symbols $\{\neg, \wedge\} \cup \{\exists_k : k \in \omega\}$, which will be used as connectives and quantifiers, respectively, in the construction of the formulas below. Given a set $X$, by $\overline{X}$ will be denoted an isomorphic copy of $X$ constructed in some canonical way. $\overline{x}$ will denote the copy of $x \in X$ in the set $\overline{X}$.

The h-set of $X$-formulas $\mathrm{Fm}_{\mathcal{L}}(X) = \{\mathrm{Fm}_{\mathcal{L}}(X)_P : P \in \mathcal{P}_{\mathrm{f}}(\omega)\} \in |\mathbf{HSet}|$ is defined by letting $\mathrm{Fm}_{\mathcal{L}}(X)_P$ be the smallest set with

- $\mathrm{v}_i \approx \mathrm{v}_j \in \mathrm{Fm}_{\mathcal{L}}(X)_P$, for all $i, j \in P$,
- $\overline{x} \in \mathrm{Fm}_{\mathcal{L}}(X)_P$, for every $x \in X_P$,
- $\neg\phi, \phi_1 \wedge \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_P$, for all $\phi, \phi_1, \phi_2 \in \mathrm{Fm}_{\mathcal{L}}(X)_P$,
- $\exists_k \phi \in \mathrm{Fm}_{\mathcal{L}}(X)_P$, for every $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_{P\cup\{k\}}$.

Moreover, given two h-sets $X$ and $Y$, any h-set morphism $f$ from $X$ into the h-set $\mathrm{Fm}_{\mathcal{L}}(Y)$ may be extended to an h-set morphism $f^*$ from $\mathrm{Fm}_{\mathcal{L}}(X)$ into $\mathrm{Fm}_{\mathcal{L}}(Y)$ by recursion on the structure of $X$-formulas in the ordinary way. We write $f : X \rightharpoondown Y$ to denote an **HSet**-morphism $f : X \to \mathrm{Fm}_{\mathcal{L}}(Y)$. Given two such maps $f : X \rightharpoondown Y$, $g : Y \rightharpoondown Z$, their composition $g \circ f : X \rightharpoondown Z$ is defined to be the **HSet**-morphism $g \circ f = g^* f$. If, for all $X \in |\mathbf{HSet}|$, $j_X : X \rightharpoondown X$ is given by $j_{X_P} : X_P \to \mathrm{Fm}_{\mathcal{L}}(X)_P$, with $j_{X_P}(x) = \overline{x}$, for all $x \in X_P$, then **Sign**, with collection of objects $|\mathbf{HSet}|$ and collection of morphisms $\mathbf{Sign}(X, Y) = \{f : X \rightharpoondown Y : f \in \mathbf{HSet}(X, \mathrm{Fm}_{\mathcal{L}}(Y))\}$, for all $X, Y \in |\mathbf{HSet}|$, with composition $\circ$ and $X$-identity $j_X$, is a category.

The sentence functor is defined at the object level, for every $X \in |\mathbf{Sign}|$, by setting $\mathrm{SEN}(X) = \mathrm{Fm}_{\mathcal{L}}(X)_\emptyset$ and at the morphism level, given $f : X \rightharpoondown Y \in \mathrm{Mor}(\mathbf{Sign})$, by letting $\mathrm{SEN}(f) : \mathrm{SEN}(X) \to \mathrm{SEN}(Y)$ be given, for all $\phi \in \mathrm{Fm}_{\mathcal{L}}(X)_\emptyset$, by $\mathrm{SEN}(f)(\phi) = f^*_\emptyset(\phi)$.

By a result presented in [30], **Sign** is the Kleisli category of an algebraic theory in monoid form that corresponds to the variety of all algebras over the signature of cylindric algebras. The different sets that form the relational signatures become then generators of corresponding absolutely free algebras. We may generate a quasi-povariety of pofunctors that contains all pofunctors corresponding to cylindric algebras by imposing the appropriate logical and extra-logical quasi-inidentities over the cylindric signatures. The logical ones are

$$x \preccurlyeq x$$

$$x \preccurlyeq y, y \preccurlyeq z \to x \preccurlyeq z$$

$$x_1 \preccurlyeq y_1, x_2 \preccurlyeq y_2 \to x_1 + x_2 \preccurlyeq y_1 + y_2$$

$$x_1 \preccurlyeq y_1, x_2 \preccurlyeq y_2 \to x_1 \cdot x_2 \preccurlyeq y_1 \cdot y_2$$

$$x \preccurlyeq y \to -y \preccurlyeq -x$$

and the extra-logical ones consist of

$$x \preccurlyeq y \to y \preccurlyeq x$$

together with all inidentities that form axioms of cylindric algebras (see, e.g., Section 3 of [30] and also [19]).

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