Order **19:** 295–304, 2002.

Polyadic Concept Analysis ⋆

GEORGE VOUTSADAKIS
School of Mathematics and Computer Science, Lake Superior State University, 650 W. Easterday Avenue, Sault Sainte Marie, MI 49783, U.S.A. e-mail: gvoutsad@lssu.edu

(Received: 19 March 2002; in final form: 22 August 2002)

Abstract. The framework and the basic results of Wille on triadic concept analysis, including his Basic Theorem of Triadic Concept Analysis, are here generalized to n-dimensional formal contexts.

Mathematics Subject Classifications (2000): Primary: 06A23; secondary: 62-07.

Key words: complete lattices, formal concept analysis, formal contexts, triadic concept analysis, triordered sets.

1. Introduction

In [4], Ganter and Wille introduce a framework for the analysis of data via lattice-theoretic techniques. The basic notion is that of a *formal context* which consists of two sets, a set of *objects* and a set of *attributes*, and a binary relation between objects and attributes. The Galois connection naturally arising from this binary relation is considered and a formal analysis of the lattice structures of the lattices of the closed sets of this connection, so-called *formal concepts*, is carried out. Many constructions are provided which are inspired by the use of the framework for data analysis.

This setting has been generalized in [3] to cover the case of three sets, the sets of *objects*, *attributes* and *conditions*, and a ternary relation between them that form a *triadic context*. In a way similar to the binary case, one may consider the *triadic concepts* of this triadic context. Out of combining two of the objects, attributes and conditions in various ways, several binary formal contexts also arise. The relations of these with the triadic concepts is studied and a construction for obtaining triadic concepts out of the dyadic formal concepts of these binary contexts, induced by a given triadic context, is also given.

In the same paper, the notion of a *complete trilattice* is introduced as a poset-theoretic construct and it is shown that the posets of the triadic concepts of a triadic context form naturally a complete trilattice in this abstract sense. The Basic Theorem of Triadic Concept Analysis gives necessary and sufficient conditions for a complete trilattice to be representable as the concept trilattice of a triadic context.

⋆ This research was partially supported by a grant from the U.S. Department of Defense.

In this paper, the framework, the constructs and the results of Wille [3] are generalized to n-adic contexts. For $n = 2$ and 3, the formal contexts and the triadic contexts, respectively, are obtained and the results specialize exactly to the known results from formal concept analysis on dyadic and triadic contexts, respectively.

2. *n*-Adic Concepts

Fix a natural number $n \geq 2$ and denote by $\mathbf{n}$ the set $\mathbf{n} = \{1, 2, \ldots, n\}$.

An *n-adic context* is an $(n+1)$-tuple $\mathbb{K} = (K_1, K_2, \ldots, K_n, Y)$, where $K_1, \ldots, K_n$ are sets and Y an n-ary relation between $K_1, \ldots, K_n$.

An n-adic context gives rise to (Stirling number of the second kind) $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$ many k-adic contexts, $2 \leq k \leq n$, which are in 1-1 correspondence with the possible ways of placing n distinguishable objects to k indistinguishable boxes with at least one object placed in each box (see [2], section 8.2). Such placements will be called *partitions* in the sequel. The *k-adic context corresponding* to the partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is denoted by $\mathbb{K}^{(\pi)} = (\prod_{i\in\pi_1} K_i, \ldots, \prod_{i\in\pi_k} K_i, Y^{(\pi)})$ with

$$(a^{(1)}, \ldots, a^{(k)}) \in Y^{(\pi)} \quad \text{iff } (a_1, \ldots, a_n) \in Y,$$
$$\text{for all } a^{(j)} \in \prod_{i\in\pi_j} K_i,\ 1 \leq j \leq k,$$

where $a_i = a_i^{(j)}$, for all $1 \leq i \leq n$, $1 \leq j \leq k$, such that $i \in \pi_j$.

In what follows, given a subset $I \subseteq \mathbf{n}$, denote by $I' = \mathbf{n} - I$ and, given sets $K_i, i \in I$, and elements $a_i \in K_i, i \in I$, use the notation $a_I = \langle a_i : i \in I\rangle$ and $K_I = \langle K_i : i \in I\rangle$. By slightly abusing notation, when no confusion is likely to occur, the notation i and i' will be used in place of $\{i\}$ and $\{i\}'$, respectively.

Now suppose that $I \subseteq \mathbf{n}$ and $A_i \subseteq K_i$, for all $i \in I$. Define the *$|I'|$-adic context determined by* the A_i's, denoted $\mathbb{K}_{A_I}^{I'} = (K_{I'}, Y_{A_I}^{I'})$, such that, for all $a_{I'} \in K_{I'}$,

$$a_{I'} \in Y_{A_I}^{I'} \quad \text{iff} \quad \text{for all } a_I \in A_I,\ a_{\mathbf{n}} \in Y.$$

With every dyadic context corresponding to a binary partition $\pi = (\pi_1, \pi_2)$ there are associated *derivation operators* $Z \mapsto Z^{(\pi)}$ and with every 2-index subset $I = \{i, j\}$ and subsets $A_k \subseteq K_k, k \in I'$, derivation operators $Z \mapsto Z^{(i,j,A_{I'})}$. These derivations are defined by considering the dyadic concepts $\mathbb{K}^{(\pi)}$ and $\mathbb{K}_{A_{I'}}^{I}$, respectively, that were defined above, and taking their derivation operators as defined in dyadic concept analysis. Many more derivation operations are obtained in the n-adic case ($n \geq 3$), by combining these two kinds. For instance, if $\pi = (\pi_1, \pi_2)$, with $i \in \pi_1$ and $\pi_1 - \{i\} \neq \emptyset$, then $Z \mapsto Z^{(\pi_1 - \{i\}, \pi_2, A_i)}$ is a derivation that results by considering the dyadic context $\mathbb{K}_{A_i}^{i'(\pi_1-\{i\},\pi_2)}$. Also note that the two constructions above allow us, for any pair of nonempty index sets I, J, with $I \cap J = \emptyset$, $(I \cup J)' \neq \emptyset$, and subsets $A_k \subseteq K_k, k \in (I \cup J)'$, to define derivation operators

$Z \mapsto Z^{(I,J,A_{(I\cup J)'})}$. These are binary derivation operators on the dyadic context $(\mathbb{K}^{I\cup J}_{A_{(I\cup J)'}})^{(I,J)}$. In this case, when an $|I|$-tuple or a $|J|$-tuple of sets appears in place of the place-holder Z, it will be taken to denote the direct product of the sets in the tuple.

An *n-adic concept* of $\mathbb{K} = (K_1, \ldots, K_n, Y)$ is an n-tuple $(A_1, \ldots, A_n)$ with $A_i \subseteq K_i$, $i = 1, 2, \ldots, n$, and $A_i = A_{i'}^{(i,i')}$, for all $i = 1, \ldots, n$.

PROPOSITION 1. *The n-adic concepts of an n-adic context* $(K_1, \ldots, K_n, Y)$ *are exactly the maximal n-tuples* $(A_1, \ldots, A_n)$ *in* $\mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ *with* $A_1 \times \cdots \times A_n \subseteq Y$ *with respect to component-wise set inclusion.*

Proof. For $A_i \subseteq B_i \subseteq K_i$, $1 \leq i \leq n$, $B_1 \times \cdots \times B_n \subseteq Y$ implies that $B_i \subseteq A_{i'}^{(i,i')}$, for all $i = 1, \ldots, n$, which proves the assertion. □

The collection $\mathfrak{C}(\mathbb{K})$ of all n-adic concepts of the n-adic context $\mathbb{K} = (K_1, \ldots, K_n, Y)$ is quasi-ordered by the quasi-orders $\lesssim_i$, $1 \leq i \leq n$, defined by

$$(A_1, \ldots, A_n) \lesssim_i (B_1, \ldots, B_n) \quad \text{iff } A_i \subseteq B_i.$$

By $\sim_i$, $1 \leq i \leq n$, are denoted the induced equivalence relations, defined by

$$(A_1, \ldots, A_n) \sim_i (B_1, \ldots, B_n) \quad \text{iff } A_i = B_i, \quad i = 1, 2, \ldots, n,$$

and by $[(A_1, \ldots, A_n)]_i$ the equivalence class of $\sim_i$ represented by the concept $(A_1, \ldots, A_n)$. Then $\lesssim_i$ induces a partial ordering $\leq_i$ on $\mathfrak{C}(\mathbb{K})/{\sim_i}$.

PROPOSITION 2. *Suppose that* $i \in \mathbf{n}$ *and that, for all* $j \neq i$, $(A_1, \ldots, A_n) \lesssim_j (B_1, \ldots, B_n)$. *Then* $(A_1, \ldots, A_n) \gtrsim_i (B_1, \ldots, B_n)$, *for all n-adic concepts* $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$ *of* $\mathbb{K}$. *Furthermore,* $\bigcap_{j\neq i} \sim_j = \Delta_{\mathfrak{C}(\mathbb{K})}$.

Proof. $(A_1, \ldots, A_n) \lesssim_j (B_1, \ldots, B_n)$ means $A_j \subseteq B_j$, $j \neq i$. Hence $A_{i'} \subseteq B_{i'}$ with component-wise inclusion, whence $A_i = A_{i'}^{(i,i')} \supseteq B_{i'}^{(i,i')} = B_i$. Thus $(A_1, \ldots, A_n) \gtrsim_i (B_1, \ldots, B_n)$. For the second statement $A_i = A_{i'}^{(i,i')}$, whence, if $A_j = B_j$, for all $j \neq i$, then $A_i = B_i$ and, therefore, $\bigcap_{j\neq i} \sim_j = \Delta_{\mathfrak{C}(\mathbb{K})}$. □

PROPOSITION 3. *Let* $\{j_1, \ldots, j_n\} = \mathbf{n}$ *and* $X_i \subseteq K_i$, $i \neq j_n$. *Define*

$$A_{j_n} = X_{j_{n-1}}^{(j_n, j_{n-1}, X_{\{j_n, j_{n-1}\}'})}, \tag{1}$$

$$A_{j_{n-1}} = A_{j_n}^{(j_n, j_{n-1}, X_{\{j_n, j_{n-1}\}'})}, \tag{2}$$

$$A_{j_{n-2}} = A_{\{j_n, j_{n-1}\}}^{(\{j_n, j_{n-1}\}, j_{n-2}, X_{\{j_n, j_{n-1}, j_{n-2}\}'})}, \tag{3}$$

$$\vdots$$

$$A_{j_k} = A_{\{j_n, j_{n-1}, \ldots, j_{k+1}\}}^{(\{j_n, j_{n-1}, \ldots, j_{k+1}\}, j_k, X_{\{j_1, \ldots, j_{k-1}\}})} \tag{4}$$

$$\vdots$$

$$A_{j_1} = A_{\{j_n, \ldots, j_2\}}^{(\{j_n, \ldots, j_2\}, j_1)}. \tag{5}$$

Then $(A_1, \dots, A_n)$ is the n-adic concept $\mathfrak{b}_{j_{n-1},\dots,j_1}(X_{j_n'})$ with the property that it has the largest j_2-component among all n-adic concepts $(B_1, \dots, B_n)$ with the largest j_3-component among those with the largest j_4-component, …, among all those with the largest j_n-component, satisfying $X_i \subseteq B_i, i \neq j_n$. Thus, if $(C_1, \dots, C_n)$ is a concept, then $\mathfrak{b}_{j_{n-1},\dots,j_1}(C_{j_n'}) = (C_1, \dots, C_n)$.

Proof. Assume, for notational simplicity and without loss of generality, that $j_k = k$, for all $k = 1, \dots, n$. From (1) and (2), $X_{n-1} \subseteq A_{n-1}$. From (3) then, $X_{n-2} \subseteq A_{n-2}$, and so on, until, finally, from (5), $X_1 \subseteq A_1$.

We show, first, that $(A_1, \dots, A_n)$ is an n-adic concept. We have $A_1 = A_{\{2,\dots,n\}}^{(\{2,\dots,n\},1)}$, by (5). Then

$$A_2 \subseteq A_{\{3,\dots,n\}}^{(\{3,\dots,n\},2,A_{\{2,\dots,n\}}^{(\{2,\dots,n\},1)})} = A_{\{3,\dots,n\}}^{(\{3,\dots,n\},2,A_1)} \subseteq A_{\{3,\dots,n\}}^{(\{3,\dots,n\},2,X_1)} = A_2,$$

whence

$$A_2 = A_{\{3,\dots,n\}}^{(\{3,\dots,n\},2,A_1)} = A_{\{1,3,\dots,n\}}^{(\{1,3,\dots,n\},2)}.$$

Suppose now that

$$3 \leq k \leq n-1 \quad \text{and} \quad A_j = A_{j'}^{(j',j)}, \quad \text{for all } j < k.$$

Then

$$\begin{aligned} A_k \;&\subseteq\; A_{\{k+1,\dots,n\}}^{(\{k+1,\dots,n\},k,A_{1'}^{(1',1)},\dots,A_{(k-1)'}^{((k-1)',k-1)})} = A_{\{k+1,\dots,n\}}^{(\{k+1,\dots,n\},k,A_{\{1,\dots,k-1\}})} \\ &\subseteq\; A_{\{k+1,\dots,n\}}^{(\{k+1,\dots,n\},k,X_{\{1,\dots,k-1\}})} = A_k. \end{aligned}$$

Thus $A_k = A_{\{k+1,\dots,n\}}^{(\{k+1,\dots,n\},k,A_{\{1,\dots,k-1\}})} = A_{k'}^{(k',k)}$ and similarly for A_n.

Suppose, next, that $(B_1, \dots, B_n) \in \mathfrak{C}(\mathbb{K})$, with $X_i \subseteq B_i, i < n$. Then

$$B_n = B_{n'}^{(n,n')} = B_{n-1}^{(n,n-1,B_{\{n,n-1\}'})} \subseteq X_{n-1}^{(n,n-1,X_{\{n,n-1\}'})} = A_n,$$

so $B_n \subseteq A_n$. Set $B_n = A_n$. Then

$$B_{n-1} = B_{(n-1)'}^{((n-1)',n-1)} = B_n^{(n,n-1,B_{\{n,n-1\}'})} \subseteq A_n^{(n,n-1,X_{\{n,n-1\}'})} = A_{n-1}.$$

Thus, $B_{n-1} \subseteq A_{n-1}$. Set $B_{n-1} = A_{n-1}$. Then we get as above that $B_{n-2} \subseteq A_{n-2}$. We continue in a similar way up to $B_3 \subseteq A_3$ and set $B_3 = A_3$. Then we get $B_2 \subseteq A_2$, which gives, finally, that $A_1 \subseteq B_1$, since both $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-adic concepts.

For the last statement, since $C_{\{2,\dots,n-1\}}^{(\{2,\dots,n-1\},n,A_1)} = C_n$, the first statement forces $\mathfrak{b}_{n-1,\dots,1}(C_{n'})_n = C_n$. Now we may proceed to $n-1$ and work downwards as before. □

Define, for $\{j_1, \dots, j_n\} = \mathbf{n}$, the $(j_{n-1}, \dots, j_1)$-*join* of $n-1$ sets $\mathfrak{X}_i, i \neq j_n$, of n-adic concepts of $\mathbb{K}$ by

$$\nabla_{j_{n-1},\dots,j_1}(\mathfrak{X}_{j_n'}) = \mathfrak{b}_{j_{n-1},\dots,j_1}\left(\left\langle \bigcup \{A_i : A_{\{1,\dots,n\}} \in \mathfrak{X}_i\} : i \neq j_n \right\rangle\right). \tag{6}$$

3. Complete *n*-Lattices

An *n-ordered set* is a relational structure $\mathbf{S} = \langle S, \lesssim_1, \ldots, \lesssim_n \rangle$ for which $\lesssim_i$, $1 \le i \le n$, is a quasi-order on S, such that $\bigcap_{j \neq i} \lesssim_j \subseteq \gtrsim_i$, for all $i = 1, \ldots, n$, and, if $\sim_i = \lesssim_i \cap \gtrsim_i$, then $\bigcap_{i=1}^n \sim_i = \Delta_S$.

It immediately follows that $\bigcap_{j \neq i} \sim_j = \Delta_S$, for all $i = 1, \ldots, n$. In fact $\bigcap_{j \neq i} \sim_j = \bigcap_{j \neq i} \lesssim_j \cap \bigcap_{j \neq i} \gtrsim_j \subseteq \gtrsim_i \cap \lesssim_i = \sim_i$. Thus $\bigcap_{j \neq i} \sim_i = \bigcap_{i=1}^n \sim_i = \Delta_S$.

For $x \in S$, let $[x]_i = \{y \in S : x \sim_i y\}$. Denote by $\le_i$ the partial ordering induced by the quasi-ordering $\lesssim_i$ on $S/\sim_i$. For $\{j_1, \ldots, j_n\} = \mathbf{n}$, $X_i \subseteq S$, $i \neq j_n$, an element $u \in S$ is a $(j_{n-1}, \ldots, j_1)$*-bound* of $X_{j_n'}$ if $u \gtrsim_i x_i$, for all $x_i \in X_i$, $i \neq j_n$. A $(j_{n-1}, \ldots, j_1)$-bound is a $(j_{n-1}, \ldots, j_1)$*-limit* of $X_{j_n'}$ if $u \gtrsim_{j_n} v$, for all $(j_{n-1}, \ldots, j_1)$-bounds v of $X_{j_n'}$.

PROPOSITION 4. *Let* $\mathbf{S} = \langle S, \lesssim_1, \ldots, \lesssim_n \rangle$ *be an n-ordered set,* $\{j_1, \ldots, j_n\} = \mathbf{n}$ *and* $X_i \subseteq S$, $i \neq j_n$. *Then there exists at most one* $(j_{n-1}, \ldots, j_1)$*-limit u of* $X_{j_n'}$ *that is the largest in* $\lesssim_{j_2}$ *among the largest limits in* $\lesssim_{j_3}$ *among ... among the largest limits in* $\lesssim_{j_{n-1}}$ *among the largest limits in* $\lesssim_{j_n}$. *u is called the* $(j_{n-1}, \ldots, j_1)$-join *of* $X_{j_n'}$ *and denoted by* $\nabla_{j_{n-1},\ldots,j_1} X_{j_n'}$.

Before proving Proposition 4, observe that the given condition $\phi(u)$ which the unique element u in the statement must satisfy can be formally written using restricted quantification over $(j_{n-1}, \ldots, j_1)$-limits of $X_{j_n'}$ as follows

$$\begin{aligned}\phi(x) \;=\; \bigwedge_{k=2}^{n-1} [\forall x_k (\forall x_{k+1} (\ldots \forall x_{n-2} (\forall x_{n-1} (x_{n-2} \gtrsim_{j_{n-1}} x_{n-1}) \\ \Rightarrow x_{n-3} \gtrsim_{j_{n-2}} x_{n-2}) \ldots \Rightarrow x_k \gtrsim_{j_{k+1}} x_{k+1}) \Rightarrow x \gtrsim_{j_k} x_k)].\end{aligned}$$

Proof of Proposition 4. Suppose there are two u_1, u_2 $(j_{n-1}, \ldots, j_1)$-limits of $X_{j_n'}$ satisfying the conditions of the hypothesis. Then, since u_1, u_2 are $(j_{n-1}, \ldots, j_1)$-limits of $X_{j_n'}$, $u_1 \sim_{j_n} u_2$. But then, by the hypothesis, $u_1 \gtrsim_{j_{n-1}} u_2$ and $u_2 \gtrsim_{j_{n-1}} u_1$, i.e., $u_1 \sim_{j_{n-1}} u_2$. Similarly, we get, in sequence, $u_1 \sim_{j_{n-2}} u_2, \ldots, u_1 \sim_{j_2} u_2$. But $\bigcap_{j \neq j_1} \sim_j = \Delta_S$, whence $u_1 = u_2$. □

A *complete n-lattice* is an n-ordered set $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ in which the $(j_{n-1}, \ldots, j_1)$-joins $\nabla_{j_{n-1},\ldots,j_1} X_{j_n'}$ exist for all $\{j_1, \ldots, j_n\} = \mathbf{n}$ and all $(n-1)$-tuples $X_{j_n'}$ of subsets of S.

By Propositions 2 and 3, the relational structure $\mathcal{C}(\mathbb{K})$ derived from an n-adic context $\mathbb{K}$ is an n-ordered set in which $\nabla_{j_{n-1},\ldots,j_1}(\mathfrak{X}_{j_n'}) = \mathfrak{b}_{j_{n-1},\ldots,j_1}(\langle \bigcup \{A_i : A_{\{1,\ldots,n\}} \in \mathfrak{X}_i\} : i \neq j_n \rangle)$ is always the $(j_{n-1}, \ldots, j_1)$-join of $\mathfrak{X}_{j_n'}$. Thus, $\mathcal{C}(\mathbb{K})$ is a complete n-lattice.

Next, the three-dimensional examples of Wille [3] are generalized to n dimensions.

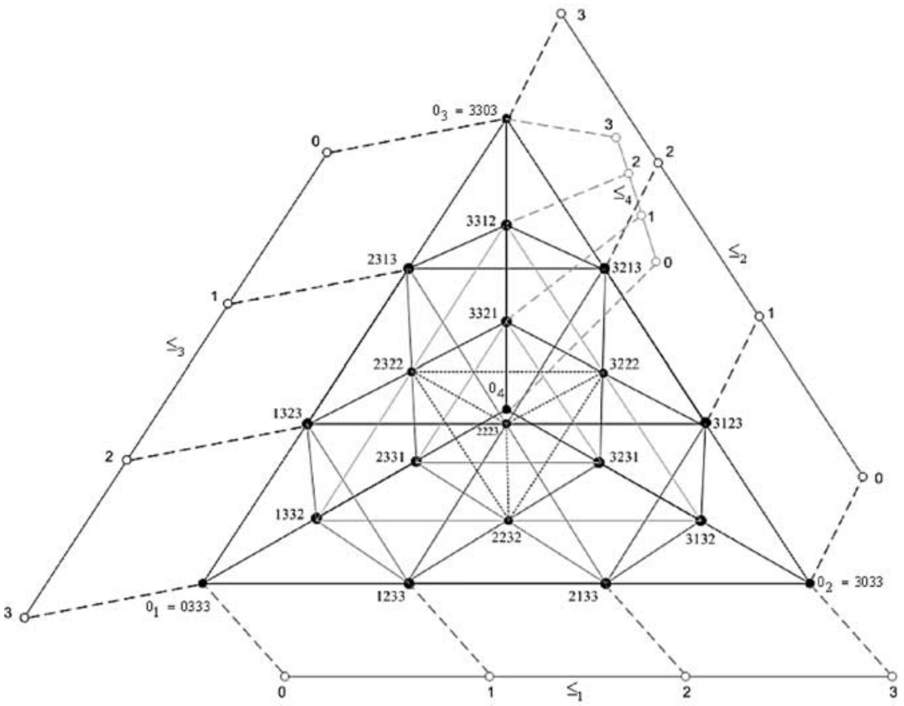

Figure 1. The equilateral 4-chain $\mathbf{4C}_3$.

A *complete n-chain* is a complete n-lattice $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$, such that $\langle L/{\sim_i}, \leq_i \rangle$, $1 \leq i \leq n$, is a complete chain. An example is the *equilateral n-chain* $\mathbf{nC}_k = \langle nC_k, \lesssim_1, \ldots, \lesssim_n \rangle$ with

$$\begin{aligned} &nC_k = \{(x_1, \ldots, x_n) \in \{0, \ldots, k\}^n : x_1 + \cdots + x_n = (n-1)k\} \\ &\text{and} \quad (x_1, \ldots, x_n) \lesssim_i (y_1, \ldots, y_n) \end{aligned}$$

if and only if $x_i \leq y_i$, $i = 1, \ldots, n$. $\mathbf{nC}_k$ is isomorphic to $\mathcal{C}(\mathbb{K}_k^{nc})$ with

$$\mathbb{K}_k^{nc} = (\{1, \ldots, k\}, \ldots, \{1, \ldots, k\}, Y_k^{nc}) \quad \text{and} \quad (x_1, \ldots, x_n) \in Y_k^{nc}$$

if and only if $x_1 + \cdots + x_n \leq (n-1)k$.

A diagram for $\mathbf{3C}_5$ is given in [3, p. 153]. For $\mathbf{4C}_3$ we get the tetrahedron of Figure 1.

A *complete Boolean n-lattice* is a complete n-lattice $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$, such that $\langle L/{\sim_i}, \leq_i \rangle$, $1 \leq i \leq n$, is a complete Boolean lattice. An example is provided by the subsets of a set S. Let $\mathbf{B}_n(S) = \langle B_n(S), \lesssim_1, \ldots, \lesssim_n \rangle$, with

$$\begin{aligned} B_n(S) \;=\; \Big\{ &(X_1, \ldots, X_n) \in \mathcal{P}(S)^n : X_1 \cap \cdots \cap X_n = \emptyset \\ &\text{and} \quad \bigcup_{j \neq i} X_j = S, \quad \text{for all } i \in \mathbf{n} \Big\} \end{aligned}$$

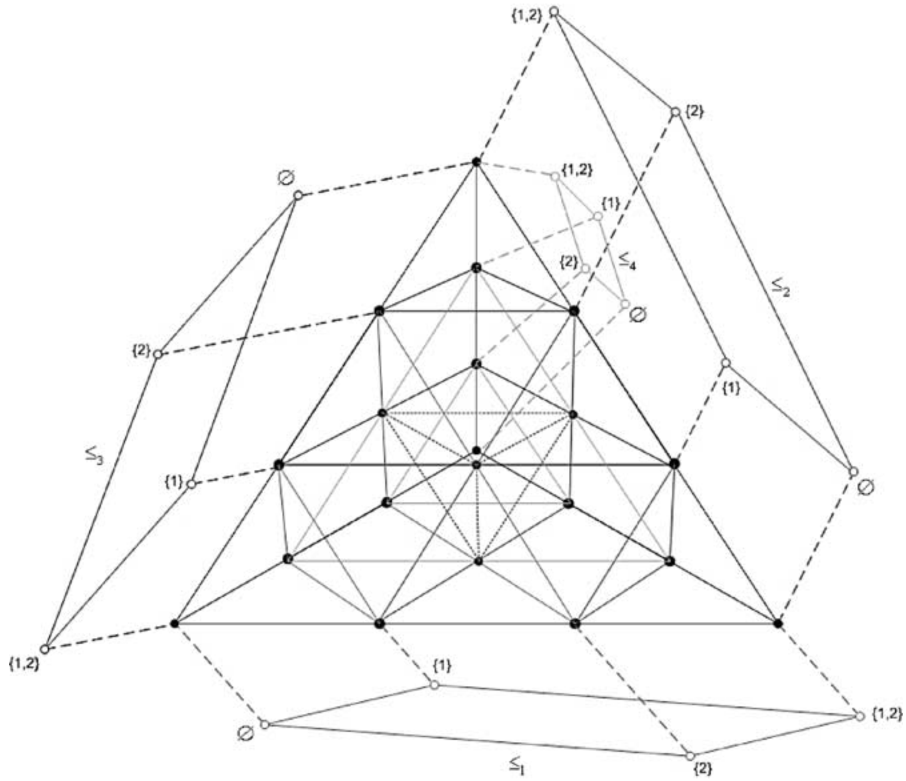

Figure 2. The complete Boolean 4-lattice $\mathbf{B}_4(\{1, 2\})$.

and $(X_1, \dots, X_n) \lesssim_i (Y_1, \dots, Y_n)$ if and only if $X_i \subseteq Y_i$, for all $i = 1, \dots, n$. The elements of $\mathbf{B}_n(S)$ are the n-adic concepts of the n-adic context $\mathbb{K}_S^{nb} = \langle S, \dots, S, Y_S^{nb}\rangle$ with $Y_S^{nb} = S^n - \{(x, \dots, x) : x \in S\}$. $\mathbf{B}_3(\{1, 2\})$ is given in [3, p. 154]. Figure 2 depicts $\mathbf{B}_4(\{1, 2\})$.

An n-tuple $(X_1, \dots, X_n)$ of subsets of an n-ordered set is said to be *joined* if there exists an element u with $u \gtrsim_i x_i$, for all $x_i \in X_i$, $1 \leq i \leq n$, i.e., u is a $(j_{n-1}, \dots, j_1)$-bound of $X_{j_n'}$, for all $\{j_1, \dots, j_n\} = \mathbf{n}$. An n-tuple $(x_1, \dots, x_n)$ of elements of an n-ordered set is *joined* if $(\{x_1\}, \dots, \{x_n\})$ is joined.

PROPOSITION 5. *Let $X_1, X_2, \dots, X_n$ be subsets of a complete n-lattice. Then $(X_1, \dots, X_n)$ is joined if and only if $(x_1, \dots, x_n)$ is joined, for all $x_i \in X_i$, $1 \leq i \leq n$. In particular, if $(X_1, \dots, X_n)$ is joined, then $\nabla_{1,2,\dots,n-1} X_{n'} \gtrsim_i x_i$, for all $x_i \in X_i$, $1 \leq i \leq n$.*

Proof. If $(X_1, \dots, X_n)$ is joined, then it is obvious that $(x_1, \dots, x_n)$ is joined, for all $x_i \in X_i$, $1 \leq i \leq n$.

Suppose, conversely, that $(x_1, \dots, x_n)$ is joined, for all $x_i \in X_i$, $1 \leq i \leq n$. Let u be such that $u \gtrsim_i x_i$, $1 \leq i \leq n$. Then $u \lesssim_1 \nabla_{2,\dots,n}(x_2, \dots, x_n)$ and, thus, $x_1 \lesssim_1 \nabla_{2,\dots,n}(x_2 \dots, x_n)$. Hence $\nabla_{2,\dots,n}(x_2, \dots, x_n)$ is a $(1, 3, \dots, n)$-bound for $X_1, x_3, \ldots, x_n$. Since $\nabla_{1,3,\ldots,n}(X_1, x_3, \ldots, x_n)$ is a $(1, 3, \ldots, n)$-limit of $X_1, x_3, \ldots, x_n$, we get

$$\nabla_{2,3,\ldots,n}(x_2, \ldots, x_n) \lesssim_2 \nabla_{1,3,\ldots,n}(X_1, x_3, \ldots, x_n).$$

Now $\nabla_{1,3,\ldots,n}(X_1, x_3, \ldots, x_n)$ is a $(1, 2, 4 \ldots, n)$-bound of $X_1, X_2, x_4, \ldots, x_n$, whence $\nabla_{1,2,4,\ldots,n}(X_1, X_2, x_4, \ldots, x_n)$ being a $(1, 2, 4, \ldots, n)$-limit, we get

$$\nabla_{1,3,\ldots,n}(X_1, x_3, \ldots, x_n) \lesssim_3 \nabla_{1,2,4,\ldots,n}(X_1, X_2, x_4, \ldots, x_n).$$

Continue until

$$\nabla_{1,\ldots,n-2,n}(X_1, \ldots, X_{n-2}, x_n) \lesssim_n \nabla_{1,\ldots,n-1}(X_1, \ldots, X_{n-1}).$$

But then

$$x_n \lesssim_n \nabla_{1,\ldots,n-2,n}(X_1, \ldots, X_{n-2}, x_n) \lesssim_n \nabla_{1,\ldots,n-1}(X_1, \ldots, X_{n-1})$$

and the assertion follows. □

4. The Basic Theorem of *n*-adic Concept Analysis

The n-adic concepts of an n-adic context $\mathbb{K} = (K_1, \ldots, K_n, Y)$ form a complete n-lattice with respect to component-wise defined quasi-orders. $\mathfrak{C}(\mathbb{K})$ is the *concept n-lattice* of the n-adic context $\mathbb{K}$. The Basic Theorem of Triadic Concept Analysis of Wille [3] states that every complete 3-lattice is isomorphic to a concept 3-lattice of a suitable 3-adic context. In this section the analog of the Basic Theorem of Triadic Concept Analysis is proved for n-adic contexts. Namely, it is shown that every complete n-lattice is isomorphic to the concept n-lattice of a suitable n-adic context.

Let $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n \rangle$ be a complete n-lattice. The set of all order filters of $\langle L, \lesssim_i \rangle$ is denoted by $\mathcal{F}_i(\mathbf{L})$, $1 \leq i \leq n$, where an *order filter* of $\langle L, \lesssim_i \rangle$ is a subset $F \subseteq L$ such that $x \in F$ and $x \lesssim_i y$ imply $y \in F$. A *principal filter* of $\langle L, \lesssim_i \rangle$ is defined by $[x)_i = \{y \in L : x \lesssim_i y\}$. An $\mathfrak{X} \subseteq \mathcal{F}_i(\mathbf{L})$ is called *i-dense* with respect to $\mathbf{L}$ if each principal filter of $\langle L, \lesssim_i \rangle$ is the intersection of some order filters from $\mathfrak{X}$. The principal filter generated by the n-adic concept $(A_1, \ldots, A_n)$ in $(\mathfrak{C}(\mathbb{K}), \lesssim_i)$ equals

$$\bigcap_{a_i \in A_i} \{(B_1, \ldots, B_n) \in \mathfrak{C}(\mathbb{K}) : a_i \in B_i\} \in \mathcal{F}_i(\mathfrak{C}(\mathbb{K}))$$

and, therefore, if

$$\kappa_i(a_i) = \{(B_1, \ldots, B_n) \in \mathfrak{C}(\mathbb{K}) : a_i \in B_i\}, \quad a_i \in K_i,$$

then $\kappa_i(K_i)$ is an i-dense set of order filters of $(\mathfrak{C}(\mathbb{K}), \lesssim_i)$, $1 \leq i \leq n$.

The proof of the basic theorem of n-adic concept analysis follows mutatis mutandis the proof of the Basic Theorem of Triadic Concept Analysis. It is only included here for the sake of completeness.

THEOREM 6. *Let* $\mathbb{K} = (K_1, \ldots, K_n, Y)$ *be an* n*-adic context. Then* $\mathfrak{C}(\mathbb{K})$ *is a complete* n*-lattice for which the* $(j_{n-1}, \ldots, j_1)$*-joins* ($\{j_1, \ldots, j_n\} = \mathbf{n}$) *are described by*

$$\nabla_{j_{n-1},\ldots,j_1}\mathfrak{X}_{j'_n} = \mathfrak{b}_{j_{n-1},\ldots,j_1}\Big(\Big\langle\bigcup\{A_i : (A_1, \ldots, A_n) \in \mathfrak{X}_i\} : i \neq j_n\Big\rangle\Big).$$

In general, a complete n*-lattice* $\mathbf{L} = \langle L, \lesssim_1, \ldots, \lesssim_n\rangle$ *is isomorphic to* $\mathfrak{C}(\mathbb{K})$ *if and only if there exist mappings* $\tilde{\kappa}_i$: $K_i \to \mathcal{F}_i(\mathbf{L})$, $1 \leq i \leq n$, *such that* $\tilde{\kappa}_i(K_i)$ *is* i*-dense with respect to* $\mathbf{L}$ *and* $A_1 \times \cdots \times A_n \subseteq Y$ *if and only if* $\bigcap_{i=1}^n \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i) \neq \emptyset$, *for all* $A_i \subseteq K_i, 1 \leq i \leq n$. *In particular* $\mathbf{L} \cong \mathfrak{C}(L, \ldots, L, Y_{\mathbf{L}})$, *with* $Y_{\mathbf{L}} = \{(x_1, \ldots, x_n) \in L^n : (x_1, \ldots, x_n)$ *is joined.*$\}$.

Proof. The first assertion follows from Proposition 3.

Let ϕ: $\mathfrak{C}(\mathbb{K}) \to \mathbf{L}$ be a complete n-lattice isomorphism. For $i \in \mathbf{n}$, define $\tilde{\kappa}_i(a_i) = \phi(\kappa_i(a_i))$, for all $a_i \in K_i$. Since $\kappa_i(K_i)$ is i-dense with respect to $\mathfrak{C}(\mathbb{K})$, $\tilde{\kappa}_i(K_i)$ is i-dense with respect to $\mathbf{L}$. Furthermore, $A_1 \times \cdots \times A_n \subseteq Y$ if and only if $\bigcap_{i=1}^n \bigcap_{a_i \in A_i} \kappa_i(a_i) \neq \emptyset$ if and only if $\bigcap_{i=1}^n \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i) \neq \emptyset$.

Conversely, let $\tilde{\kappa}_i$: $K_i \to \mathcal{F}_i(\mathbf{L})$, $1 \leq i \leq n$, be maps satisfying the hypothesis. Let ψ: $L \to \mathcal{P}(K_1) \times \cdots \times \mathcal{P}(K_n)$ be given by $\psi(x) = (A_1^x, \ldots, A_n^x)$ with $A_i^x = \{a_i \in K_i : x \in \tilde{\kappa}_i(a_i)\}$, $1 \leq i \leq n$. Since

$$[x)_1 \cap \cdots \cap [x)_n = \{x\} \quad \text{and} \quad [x)_i = \bigcap_{a_i \in A_i^x} \tilde{\kappa}_i(a_i)$$

by i-density, we get $\bigcap_{i=1}^n \bigcap_{a_i \in A_i^x} \tilde{\kappa}_i(a_i) = \{x\}$. Thus, by the second property, $A_1^x \times \cdots \times A_n^x \subseteq Y$. Now let $\hat{A}_n^x = (A_1^x \times \cdots \times A_{n-1}^x)^{(n',n)}$. Then $A_1^x \times \cdots \times A_{n-1}^x \times \hat{A}_n^x \subseteq Y$, whence

$$\bigcap_{a_1 \in A_1^x} \tilde{\kappa}_1(a_1) \cap \cdots \cap \bigcap_{a_{n-1} \in A_{n-1}^x} \tilde{\kappa}_{n-1}(a_{n-1}) \cap \bigcap_{a_n \in \hat{A}_n^x} \tilde{\kappa}_n(a_n) \neq \emptyset.$$

Since $A_n^x \subseteq \hat{A}_n^x$, we get that

$$\bigcap_{i=1}^{n-1} \bigcap_{a_i \in A_i^x} \tilde{\kappa}_i(a_i) \cap \bigcap_{a_n \in \hat{A}_n^x} \tilde{\kappa}_n(a_n) = \{x\},$$

whence $\hat{A}_n^x = A_n^x$ and, similarly, for $i = 1, 2, \ldots, n-1$. Thus $\psi(x) \in \mathfrak{C}(\mathbb{K})$. ψ preserves $\lesssim_1, \ldots, \lesssim_n$. Now, if $(A_1, \ldots, A_n) \in \mathfrak{C}(\mathbb{K})$, consider $x \in \bigcap_{i=1}^n \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i)$. Then $(A_1, \ldots, A_n) = \psi(x)$, i.e., ψ is surjective. But, as before, $\bigcap_{i=1}^n \bigcap_{a_i \in A_i} \tilde{\kappa}_i(a_i) = \{x\}$, whence ψ is also injective. ψ^{-1} also preserves $\lesssim_1, \ldots, \lesssim_n$. Thus ψ is an isomorphism.

To show that $\mathbf{L} \cong \mathfrak{C}(L, \ldots, L, Y_{\mathbf{L}})$, define $\tilde{\kappa}_i$: $L \to \mathcal{F}_i(\mathbf{L})$ by

$$\tilde{\kappa}_i(x) = [x)_i, \quad 1 \leq i \leq n, \ x \in L.$$

Then $\tilde{\kappa}_i(K_i)$ is i-dense with respect to $\mathbf{L}$. Let $A_1 \times \cdots \times A_n \subseteq Y_{\mathbf{L}}$, with $A_1, \ldots, A_n \subseteq L$. Then, by Proposition 5, $(A_1, \ldots, A_n)$ is joined. Now the second condition guaranteeing $\mathbf{L} \cong \mathfrak{C}(L, \ldots, L, Y_{\mathbf{L}})$ is satisfied. □

Finally, it is not difficult to show, following [3], that the following analogs of propositions 6 and 7 of [3] hold for n-adic contexts.

PROPOSITION 7. *Let* $(P, \leq)$ *be a poset with smallest element* 0 *and greatest element* 1 *and*

$$Y = \{(x_1, \ldots, x_n) \in P^n : 0 \neq x_1 \leq x_2 = x_3 = \cdots = x_n\}.$$

Then $(P, \leq) \cong (\mathfrak{C}(P, \ldots, P, Y)/\sim_1, \leq_1)$.

PROPOSITION 8. *Let* $(L_i, \leq_i)$, $1 \leq i \leq n-1$, *be complete lattices and*

$$\begin{aligned} Y \;=\; & \{(x_1, \ldots, x_{n-1}, (y_1, \ldots, y_{n-1})) \in \\ & L_1 \times \cdots \times L_{n-1} \times (L_1 \times \cdots \times L_{n-1}) : 0 \neq x_i \leq y_i, 1 \leq i \leq n-1\}. \end{aligned}$$

Then

$$\begin{aligned} (L_i, \leq_i) \;\cong\; & (\mathfrak{C}(L_1, \ldots, L_{n-1}, (L_1 \times \cdots \times L_{n-1}), Y)/\sim_i, \leq_i), \\ & 1 \leq i \leq n-1. \end{aligned}$$

5. An Open Problem

It would be very interesting to investigate whether n-lattices may obtained from complete n-lattices in the same way as trilattices are obtained from complete trilattices in [1] and discover equations that characterize n-lattices analogous to the equations of Biedermann ([1], theorems 3.1 and 4.1).

Acknowledgements

The author wishes to thank Dr Abdallah Taha for his help in preparing the figures. Also, thanks to Professor Rudolf Wille for sharing some of his thoughts and intuitions on formal concept analysis while at New Mexico State University. Finally, thanks to the two anonymous referees for useful comments, corrections and suggestions.

References

1. Biedermann, K.: An equational theory for trilattices, *Algebra Universalis* **42** (1999), 253–268.
2. Brualdi, R. A.: *Introductory Combinatorics,* 3rd edn, Prentice-Hall, New Jersey, 1999.
3. Wille, R.: The basic theorem of triadic concept analysis, *Order* **12** (1995), 149–158.
4. Ganter, B. and Wille, R.: *Formal Concept Analysis, Mathematical Foundations*, Springer-Verlag, Berlin, 1999.