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Categorical Abstract Algebraic Logic: Algebraic Semantics for π-Institutions

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Various aspects of the work of Blok and Rebagliato on the algebraic semantics for deductive systems are studied in the context of logics formalized as π-institutions. Three kinds of semantics are surveyed: institution, matrix (system) and algebraic (system) semantics, corresponding, respectively, to the generalized matrix, matrix and algebraic semantics of the theory of sentential logics. After some connections between matrix and algebraic semantics are revealed, it is shown that every (finitary) N-rule based extension of an N-rule based π-institution possessing an algebraic semantics also possesses an algebraic semantics. This result abstracts one of the main theorems of Blok and Rebagliato. An attempt at a Blok-Rebagliato-style characterization of those π-institutions with a mono-unary category of natural transformations on their sentence functors having an algebraic semantics is also made. Finally, a necessary condition for a π-institution to possess an algebraic semantics is provided.

1 Introduction

Blok and Pigozzi [2] introduced the notion of an algebraizable logic to formalize the close connection that exists between some finitary deductive systems and corresponding quasi-varieties of algebras, such as, e.g., classical propositional calculus and the variety of Boolean algebras. The algebraic counterpart of such a deductive system was termed an *equivalent algebraic semantics*. Roughly speaking, a class of algebras of the same similarity type as that of the deductive system is an equivalent algebraic semantics of the deductive system if there exist mutually inverse interpretations from the consequence relation of the deductive system to the equational consequence induced by the class of algebras and vice-versa. In subsequent work, Blok and Rebagliato [3] studied the property of a deductive system having an *algebraic semantics*, which may not be equivalent. This only requires the existence of an interpretation from the consequence $\vdash$ of the deductive system to the equational consequence $\models_K$ of the algebraic semantics K, i.e., of a finite set of equations $\delta(p) \approx \varepsilon(p) = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ in one variable p, such that, for all sets of formulas $\Phi \cup \{\psi\}$,

$$\Phi \vdash \psi \text{ iff } \delta(\Phi) \approx \varepsilon(\Phi) \models_K \delta(\psi) \approx \varepsilon(\psi),$$

where $\delta(\Phi) \approx \varepsilon(\Phi) = \{\delta_i(\varphi) \approx \varepsilon_i(\varphi) : i < n, \varphi \in \Phi\}$ and $\delta(\psi) \approx \varepsilon(\psi) = \{\delta_i(\psi) \approx \varepsilon_i(\psi) : i < n\}$. In both cases an algebraic completeness theorem is obtained but in the case of an algebraic semantics that is not equivalent the connection between the metalogical properties of the deductive system and the algebraic properties of the corresponding class of algebras is not as tight as when the semantics is equivalent. Moreover, whereas an equivalent algebraic semantics, whenever it exists, is essentially unique (cf. [2, Section 2.2.1]), the same does not hold for an algebraic semantics in general, as is illustrated in [3, Section 2.1].

The question as to whether the property of possessing an algebraic semantics is preserved under extensions was already raised in [2]. There, it was mistakenly claimed that the property is not preserved on passing from a

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deductive system to an extension. In [10], on the other hand, Rebagliato and Verdú showed that any *axiomatic* extension of a deductive system possessing an algebraic semantics itself possesses an algebraic semantics. Blok and Rebagliato prove in [3, Theorem 2.15] that *every* extension of a deductive system that possesses an algebraic semantics also possesses an algebraic semantics. In this paper this result is further abstracted to cover logics formalized as π-institutions. Namely, it is shown that every (finitary) N-rule based extension of a (finitary) N-rule based π-institution possessing a $\sigma \approx \tau$-algebraic semantics has itself a $\sigma \approx \tau$-algebraic semantics.

Among the most interesting conclusive results of [3] is the characterization of those mono-unary deductive systems that possess an algebraic semantics. Despite its relatively limited applicability, the interest of this result lies on the fact that it is the only result of its kind known at present about algebraic semantics and, also, that, possibly, by a more careful analysis of its content and proof method, it might pave the way for more general characterization results along similar lines. Furthermore, it has motivated the introduction in the present work of the concept of a mono-unary category of natural transformations on a sentence functor, which abstracts the notion of a clone of operations generated by a single unary operation. It is shown in Theorem 7.4 that, whereas a condition similar to that used in [3] proves to be necessary for a π-institution with a mono-unary category of natural transformations on its sentence functor to possess an algebraic semantics, it does not seem to be sufficient. Additional technical conditions on the sentence functor are needed to ensure sufficiency. The exact characterization of those functors for which [3, Theorem 2.20] may be carried virtually unchanged to the categorical level is left open.

In [3, Theorem 2.16] and [3, Proposition 2.17], it is shown that a necessary condition for a deductive system to possess an algebraic semantics with defining equations $\delta(p) \approx \varepsilon(p) = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ in the single variable p is that, for all $i < n$, the pair $\langle \delta_i(p), \varepsilon_i(p)\rangle$ belongs to the Leibniz congruence of every theory T that includes p. Corollary 8.3 generalizes this result to the categorical level. Finally, in Theorem 9.3, a sufficient condition is given for a π-institution to have an algebraic semantics. This theorem abstracts [3, Theorem 3.3]. Blok and Rebagliato provide various refinements of this result in [3, Theorems 3.6 & 3.1].

The reader in encouraged to consult [5–7] for a relatively up-to-date overview of the field of abstract algebraic logic as well as either of [1, 4, 8] for all unexplained categorical terminology and notation.

2 Institution, Matrix and Algebraic Semantics

Let $\mathbf{Sign}$ be a category and SEN : $\mathbf{Sign} \to \mathbf{Set}$ a functor. The *clone of all natural transformations on* SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma\rangle \circ \langle \tau_i : i < \beta\rangle = \langle \sigma_j(\langle \tau_i : i < \beta\rangle) : j < \gamma\rangle.$$

A subcategory N of this category containing *all* objects of the form SEN^k for $k < \omega$, and all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$, $i < k, k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\varphi}) = \varphi_i, \quad \text{for all} \quad \vec{\varphi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l\rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as a *category of natural transformations on* SEN.

Let SEN : $\mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor, with N a category of natural transformations on SEN. Two set-valued functors $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ and $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$, with N' and N'' categories of natural transformations on SEN' and SEN'', respectively, will be said to be *similar* if there exist surjective functors $F' : N \to N'$ and $F'' : N \to N''$, that preserve all projection natural transformations. Here, a surjective functor is one both of whose object and morphism parts are surjective. In that case, given $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, we use the notation σ' and σ'' to denote the natural transformations $F'(\sigma) : \mathrm{SEN}'^k \to \mathrm{SEN}'$ and $F''(\sigma) : \mathrm{SEN}''^k \to \mathrm{SEN}''$, respectively. Similarity is intended to capture in the categorical framework of set-valued functors with designated categories of natural transformations on them (also known as algebraic systems) the same concept as that of similar algebras in the context of universal algebra, i.e., algebras over the same algebraic signature.

Let SEN : $\mathbf{Sign} \to \mathbf{Set}$, SEN$'$: $\mathbf{Sign}' \to \mathbf{Set}$, with N and N' categories of natural transformations on SEN and SEN$'$, respectively, be similar set-valued functors. An (N, N')-*epimorphic translation* $\langle F, \alpha\rangle$: SEN $\to$ SEN$'$ consists of a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation α : SEN $\to$ SEN$' \circ F$, such that

$$\alpha_\Sigma(\sigma_\Sigma(\varphi_0, \ldots, \varphi_{n-1})) = \sigma'_{F(\Sigma)}(\alpha_\Sigma(\varphi_0), \ldots, \alpha_\Sigma(\varphi_{n-1})), \tag{1}$$

for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi_0, \ldots, \varphi_{n-1} \in \mathrm{SEN}(\Sigma)$. If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are two π-institutions, with N, N' categories of natural transformations on SEN, SEN$'$, respectively, then an (N, N')-*epimorphic translation* $\langle F, \alpha\rangle : \mathcal{I} \to \mathcal{I}'$ is an (N, N')-epimorphic translation $\langle F, \alpha\rangle$: SEN $\to$ SEN$'$.

Recall from [12] the concept of an (N, N')-model of a given π-institution $\mathcal{I}$. Namely, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, a π-institution $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', D\rangle$, with N' a category of natural transformations on SEN$'$, is said to be an (N, N')-*model* of $\mathcal{I}$ if there exists an (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathcal{I} \to \mathcal{I}'$, such that, for every $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \quad \text{implies} \quad \alpha_\Sigma(\varphi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

In this case, the translation is said to be an (N, N')-*logical morphism* (an (N, N')-*epimorphic semi-interpretation*) and is denoted by $\langle F, \alpha\rangle : \mathcal{I} \rangle\!\!- \mathcal{I}'$.

An alternative equivalent way of viewing models is to look at the closure system $D^{\langle F,\alpha\rangle}$ on SEN that is induced by the (N, N')-epimorphic translation $\langle F, \alpha\rangle$: SEN $\to$ SEN$'$ and the π-institution $\mathcal{I}'$. This is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in D^{\langle F,\alpha\rangle}_\Sigma(\Phi) \quad \text{iff} \quad \alpha_\Sigma(\varphi) \in D_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

It is, perhaps, a tedious but not very difficult exercise to show that the collection $D^{\langle F,\alpha\rangle} = \{D^{\langle F,\alpha\rangle}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ is indeed a closure system on SEN. Then $\mathcal{I}'$ is an (N, N')-model of $\mathcal{I}$ via $\langle F, \alpha\rangle$: SEN $\to$ SEN$'$ if and only if $C \leq D^{\langle F,\alpha\rangle}$, i.e., if and only if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $\varphi \in C_\Sigma(\Phi)$ implies $\varphi \in D^{\langle F,\alpha\rangle}_\Sigma(\Phi)$. In a similar way, given an indexed collection $\mathfrak{I}$ of π-institutions $\mathcal{I}^i = \langle \mathbf{Sign}^i, \mathrm{SEN}^i, D^i\rangle$, $i \in I$, with N^i a category of natural transformations on SEN^i, and (N, N^i)-epimorphic translations $\langle F^i, \alpha^i\rangle$: SEN $\to$ SEN^i, $i \in I$, one may define the closure system $D^{\mathfrak{I}}$ on SEN induced by $\mathfrak{I}$. In fact, in that case, $D^{\mathfrak{I}} = \bigcap_{i\in I}(D^i)^{\langle F^i,\alpha^i\rangle}$, i.e., for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \subseteq \mathrm{SEN}(\Sigma)$, $D^{\mathfrak{I}}_\Sigma(\Phi) = \bigcap_{i\in I}(D^i)^{\langle F^i,\alpha^i\rangle}_\Sigma(\Phi)$. If $\mathcal{I}^i$ is an (N, N^i)-model of $\mathcal{I}$ via $\langle F^i, \alpha^i\rangle$, for all $i \in I$, then $\mathfrak{I} = \{\langle F^i, \alpha^i\rangle : \mathcal{I} \rangle\!\!- \mathcal{I}^i, i \in I\}$ is said to be an (*institution*) *semantics* of $\mathcal{I}$ if, in addition, $C = D^{\mathfrak{I}}$.

For the remainder of this study, the notion of an N-matrix system for a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, borrowed from [16], will also be needed. We briefly review it here.

Given a functor SEN : $\mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, an N-*matrix system* $M = \langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ *for* SEN consists of

(1) a functor SEN$'$: $\mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN$'$;
(2) an (N, N')-epimorphic translation $\langle F, \alpha\rangle$: SEN $\to$ SEN$'$;
(3) an axiom family T' of SEN$'$, i.e., a collection $T' = \{T'_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$, with $T'_\Sigma \subseteq \mathrm{SEN}'(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}'|$.

Given a class $\mathfrak{M}$ of matrix systems for SEN, $\mathfrak{M}$ induces a closure system $C^{\mathfrak{M}}$ on SEN, defined, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\begin{aligned}\varphi \in C^{\mathfrak{M}}_\Sigma(\Phi) \quad &\text{iff for all} \quad \langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \in \mathfrak{M}, \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'),\\ &\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \quad \text{implies} \quad \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}.\end{aligned}$$

The last implication will be usually abbreviated in the form

$$(\forall f)\big(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F(\Sigma')} \Rightarrow \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F(\Sigma')}\big).$$

In [16, Lemma 20] it was shown that, given a functor SEN : $\mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, and a class $\mathfrak{M}$ of matrix systems for SEN, $C^{\mathfrak{M}}$ is a closure system on SEN. As a consequence, the structure $\mathcal{I}^{\mathfrak{M}} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathfrak{M}} \rangle$ is a π-institution.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, an N-matrix system $M = \langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ for SEN is said to be an *N-matrix system for $\mathcal{I}$* if $C \leq C^M := C^{\{M\}}$. A class $\mathfrak{M}$ of N-matrix systems for a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is said to be *strongly adequate for $\mathcal{I}$* or an *N-matrix (system) semantics of $\mathcal{I}$* if $C = C^{\mathfrak{M}}$.

Consider, again, a functor SEN : $\mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN. An *N-algebraic system for* SEN is a pair $\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle$, where SEN' is a functor, with N' a category of natural transformations on SEN', and $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ is an (N, N')-epimorphic translation.

Given a class $\mathfrak{F} = \{\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle : i \in I\}$ of N-algebraic systems for SEN, define the $|\mathbf{Sign}|$-indexed collection $\{C^{\mathfrak{F}}_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$C^{\mathfrak{F}}_\Sigma : \mathcal{P}\left(\mathrm{SEN}(\Sigma)^2\right) \to \mathcal{P}\left(\mathrm{SEN}(\Sigma)^2\right)$$

be given, for all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$, by

$$\begin{aligned}
&\varphi \approx \psi \in C^{\mathfrak{F}}_\Sigma(E) \quad \text{iff} \quad (\forall i \in I)\big(\forall \Sigma' \in |\mathbf{Sign}|\big)(\forall f \in \mathbf{Sign}(\Sigma, \Sigma'))\\
&\quad \big(\big(\forall e^0 \approx e^1 \in E\big)\big(\alpha^i_{\Sigma'}\big(\mathrm{SEN}(f)\big(e^0\big)\big) = \alpha^i_{\Sigma'}\big(\mathrm{SEN}(f)\big(e^1\big)\big)\big)\\
&\quad \Longrightarrow \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi))).
\end{aligned}$$

Sometimes the condition on the right-hand side of the equivalence above will be abbreviated to

$$(\forall i)(\forall f)\big(\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \Rightarrow \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi))\big).$$

We use the notation C^A instead of $C^{\{A\}}$ in case $\mathfrak{F} = \{A\}$. It is now shown that $C^{\mathfrak{F}}$, as defined above, is a closure system on the functor $\mathrm{SEN}^2 : \mathbf{Sign} \to \mathbf{Set}$.

Lemma 2.1 *Given a functor* SEN : $\mathbf{Sign} \to \mathbf{Set}$*, with N a category of natural transformations on* SEN*, and a class $\mathfrak{F} = \{\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle : i \in I\}$ of N-algebraic systems for* SEN*, $C^{\mathfrak{F}}$ is a closure system on* SEN^2.

Proof. Suppose that $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}^2(\Sigma)$ and $\varphi \approx \psi \in E$. Then we have, for all $i \in I$, $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')}$ implies that $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi))$. Therefore $\varphi \approx \psi \in C^{\mathfrak{F}}_\Sigma(E)$ and $C^{\mathfrak{F}}$ is inflationary.

If $\Sigma \in |\mathbf{Sign}|$ and $E \subseteq E' \subseteq \mathrm{SEN}^2(\Sigma)$, then, if $\varphi \approx \psi \in C^{\mathfrak{F}}_\Sigma(E)$, we have, for all $i \in I$ and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \quad \text{implies} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi)).$$

Therefore, for all $i \in I$ and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(E')) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \quad \text{implies} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')},$$

which gives $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi))$ and, hence, $\varphi \approx \psi \in C^{\mathfrak{F}}_\Sigma(E')$. Thus, $C^{\mathfrak{F}}$ is also monotone.

Suppose that $\Sigma \in |\mathbf{Sign}|$, $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}^2(\Sigma)$, such that $\varphi \approx \psi \in C^{\mathfrak{F}}_\Sigma\big(C^{\mathfrak{F}}_\Sigma(E)\big)$. This means that, for all $i \in I$ and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha^i_{\Sigma'}\big(\mathrm{SEN}(f)\big(C^{\mathfrak{F}}_\Sigma(E)\big)\big) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \quad \text{implies} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi)).$$

But then, for all $i \in I$, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')}$ implies, by the definition of $C^{\mathfrak{F}}$, that $\alpha^i_{\Sigma'}\big(\mathrm{SEN}(f)\big(C^{\mathfrak{F}}_\Sigma(E)\big)\big) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')}$, which yields that $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(f)(\psi))$, whence $\varphi \approx \psi \in C^{\mathfrak{F}}_\Sigma(E)$ and $C^{\mathfrak{F}}$ is idempotent.

Finally, suppose $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}^2(\Sigma_1)$, such that $\varphi \approx \psi \in C^{\mathfrak{F}}_{\Sigma_1}(E)$. Thus, for all $i \in I$, and all $\Sigma' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$,

$$\begin{tikzcd}
\Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\
& \Sigma' &
\end{tikzcd}$$

$$\alpha^i_{\Sigma'}(\mathrm{SEN}(g)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \quad \text{implies} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(g)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(g)(\psi)).$$

Hence, we have, for all $i \in I$ and all $\Sigma' \in |\mathbf{Sign}|$, $h \in \mathbf{Sign}(\Sigma_2, \Sigma')$,

$$\begin{aligned}
&\alpha^i_{\Sigma'}(\mathrm{SEN}(h)(\mathrm{SEN}(f)(E))) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \\
&\quad \text{iff} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(hf)(E)) \subseteq \Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')} \\
&\quad \text{implies} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(hf)(\varphi)) = \alpha^i_{\Sigma'}(\mathrm{SEN}(hf)(\psi)) \\
&\quad \text{iff} \quad \alpha^i_{\Sigma'}(\mathrm{SEN}(h)(\mathrm{SEN}(f)(\varphi))) = \alpha^i_{\Sigma'}(\mathrm{SEN}(h)(\mathrm{SEN}(f)(\psi)))
\end{aligned}$$

which yields that $\mathrm{SEN}^2(f)(\varphi \approx \psi) \in C^{\mathfrak{F}}_{\Sigma_2}(\mathrm{SEN}(f)(E))$, and, hence, $C^{\mathfrak{F}}$ is also structural. □

As a consequence of Lemma 2.1, it follows that the structure $\mathcal{I}^{\mathfrak{F}} = \langle \mathbf{Sign}, \mathrm{SEN}^2, C^{\mathfrak{F}} \rangle$ is a π-institution.

Suppose, next, that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a π-institution, with N a category of natural transformations on SEN, and that $\sigma \approx \tau$ is a collection of pairs of natural transformations $\mathrm{SEN} \to \mathrm{SEN}$ in N. We shall use the term *N-translation* to refer to such a collection $\sigma \approx \tau$. Define, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$,

$$[\sigma \approx \tau]_\Sigma(\varphi) := \sigma_\Sigma(\varphi) \approx \tau_\Sigma(\varphi).$$

Moreover, given a class $\mathfrak{F} = \{\langle \mathrm{SEN}^i, \langle F^i, \alpha^i \rangle \rangle : i \in I\}$ of N-algebraic systems for SEN, define $C^{\mathfrak{F},\sigma\approx\tau} = \{C^{\mathfrak{F},\sigma\approx\tau}_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C^{\mathfrak{F},\sigma\approx\tau}_\Sigma(\Phi) \quad \text{iff} \quad (\forall A \in \mathfrak{F})([\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)).$$

It is shown in the next lemma that $C^{\mathfrak{F},\sigma\approx\tau}$ is a closure system on SEN.

Lemma 2.2 *Given a functor* SEN : $\mathbf{Sign} \to \mathbf{Set}$*, with N a category of natural transformations on* SEN*, an N-translation $\sigma \approx \tau$ and a class $\mathfrak{F} = \{\langle \mathrm{SEN}^i, \langle F^i, \alpha^i \rangle \rangle : i \in I\}$ of N-algebraic systems for* SEN*, $C^{\mathfrak{F},\sigma\approx\tau}$ is a closure system on* SEN.

Proof. The fact that, for all $\Sigma \in |\mathbf{Sign}|$, $C^{\mathfrak{F},\sigma\approx\tau}_\Sigma$ is inflationary, monotone and idempotent follows from Lemma 2.1. To see that $C^{\mathfrak{F},\sigma\approx\tau}$ is structural, note that, for all $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$ and all $\varphi \in \mathrm{SEN}(\Sigma_1)$, $\mathrm{SEN}(f)([\sigma \approx \tau]_{\Sigma_1}(\varphi)) = [\sigma \approx \tau]_{\Sigma_2}(\mathrm{SEN}(f)(\varphi))$ and, then, also use Lemma 2.1. □

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution, with N a category of natural transformations on SEN, and $\sigma \approx \tau$ an N-translation. The N-algebraic system $A = \langle \mathrm{SEN}', \langle F, \alpha \rangle \rangle$ for SEN is said to be a *$\sigma \approx \tau$-algebraic* (*system*) *model* of $\mathcal{I}$ if $C \leq C^{A,\sigma\approx\tau} := C^{\{A\},\sigma\approx\tau}$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \quad \text{implies} \quad \varphi \in C^{A,\sigma\approx\tau}_\Sigma(\Phi).$$

By $\mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ will be denoted the class of all $\sigma \approx \tau$-algebraic models of $\mathcal{I}$. A class $\mathfrak{F} = \{\langle \mathrm{SEN}^i, \langle F^i, \alpha^i \rangle \rangle : i \in I\}$ of $\sigma \approx \tau$-algebraic models of a π-institution $\mathcal{I}$ is a *$\sigma \approx \tau$-algebraic semantics of $\mathcal{I}$* if $C = C^{\mathfrak{F},\sigma\approx\tau}$.

3 Basic Preservation Properties

In this section, the task of proving some preservation properties for $\sigma \approx \tau$-algebraic models of a π-institution $\mathcal{I}$ is undertaken. Namely, in Proposition 3.1 it will be shown that a $\sigma \approx \tau$-algebraic model is preserved both in the forward and in the backward directions by (N', N'')-epimorphic translations, whose natural transformation components are injective. It will also be shown in Proposition 3.2 that $\sigma \approx \tau$-models of a π-institution $\mathcal{I}$ are preserved under the formation of products.

Proposition 3.1 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*, and $\sigma \approx \tau$ an N-translation. Let, also, $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, $\mathrm{SEN}'' : \mathbf{Sign}'' \to \mathbf{Set}$ be functors, with N', N'' categories of natural transformations on $\mathrm{SEN}', \mathrm{SEN}''$, respectively, $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$ an (N, N')-epimorphic translation and $\langle G, \beta\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ an (N', N'')-epimorphic translation, such that $\beta_\Sigma : \mathrm{SEN}'(\Sigma) \to \mathrm{SEN}''(G(\Sigma))$ is injective, for all $\Sigma \in |\mathbf{Sign}'|$.*

$$\mathrm{SEN} \xrightarrow{\langle F,\alpha\rangle} \mathrm{SEN}' \xrightarrow{\langle G,\beta\rangle} \mathrm{SEN}''$$

Then $A = \langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle$ is a $\sigma \approx \tau$-algebraic model of $\mathcal{I}$ if and only if $B = \big\langle \mathrm{SEN}'', \langle GF, \beta_F\alpha\rangle\big\rangle$ is a $\sigma \approx \tau$-algebraic model of $\mathcal{I}$.

Proof. For the implication from left to right, suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $\varphi \in C_\Sigma(\Phi)$. This implies, by the hypothesis, that

$$(2) \qquad [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)).$$

Now, in order to show that also $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^B_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$, suppose that $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that

$$\beta_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\Phi)))) \subseteq \Delta^{\mathrm{SEN}''}_{G(F(\Sigma'))}.$$

By the injectivity of $\beta_{F(\Sigma')}$ we now get that $\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\Phi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}$. Thus, by (2), we have that $\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}$, which yields that $\beta_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi)))) \subseteq \Delta^{\mathrm{SEN}''}_{G(F(\Sigma'))}$. This shows that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^B_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$.

For the right to left implication, suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$. Then, by the hypothesis, we obtain that

$$(3) \qquad [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^B_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)).$$

To show that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$, suppose that $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, such that $\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\Phi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}$. Then, clearly, we obtain $\beta_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\Phi)))) \subseteq \Delta^{\mathrm{SEN}''}_{G(F(\Sigma'))}$. Therefore, using (3), we get $\beta_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi)))) \subseteq \Delta^{\mathrm{SEN}''}_{G(F(\Sigma'))}$, which, by the injectivity of $\beta_{F(\Sigma')}$, yields that $\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}$. This proves that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$. □

We next turn to preservation under products. Recall from [11] that, given an indexed collection $\mathrm{SEN}^i : \mathbf{Sign}^i \to \mathbf{Set}$, $i \in I$, of set-valued functors, with N^i a category of natural transformations on SEN^i, together with a collection $\langle F^i, \alpha^i\rangle : \mathrm{SEN} \to \mathrm{SEN}^i$ of (N, N^i)-epimorphic translations, one may define the product functor $\prod_{i\in I} \mathrm{SEN}^i : \prod_{i\in I} \mathbf{Sign}^i \to \mathbf{Set}$ by setting, for all $\Sigma_i \in |\mathbf{Sign}^i|, i \in I$,

$$\prod_{i\in I} \mathrm{SEN}^i(\langle \Sigma_i : i \in I\rangle) = \prod_{i\in I} \mathrm{SEN}^i(\Sigma_i),$$

and, similarly for morphisms. Moreover, the N^i together with the (N, N^i)-epimorphic property of the $\langle F^i, \alpha^i\rangle$, $i \in I$, induce a category of natural transformations $\prod_{i\in I} N^i$ on $\prod_{i\in I} \mathrm{SEN}^i$ and an $(N, \prod_{i\in I} N^i)$-epimorphic translation $\prod_{i\in I}\langle F^i,\alpha^i\rangle := \langle\prod_{i\in I}F^i,\prod_{i\in I}\alpha^i\rangle : \mathrm{SEN}\to\prod_{i\in I}\mathrm{SEN}^i$. This is defined, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi\in\mathrm{SEN}(\Sigma)$, by

$$\prod_{i\in I}\alpha^i_\Sigma(\varphi) = \langle\alpha^i_\Sigma(\varphi) : i\in I\rangle .$$

The following proposition takes into account these definitions to formalize the closure property of algebraic models of a π-institution under the formation of products.

Proposition 3.2 *Suppose that $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*, and $\sigma\approx\tau$ an N-translation. Let, also, $\mathrm{SEN}^i : \mathbf{Sign}^i\to\mathbf{Set}, i\in I$, be a collection of functors, with N^i a category of natural transformations on $\mathrm{SEN}^i, i\in I$, and $\langle F^i,\alpha^i\rangle : \mathrm{SEN}\to\mathrm{SEN}^i$ an (N,N^i)-epimorphic translation, for all $i\in I$. If $A^i = \langle\mathrm{SEN}^i,\langle F^i,\alpha^i\rangle\rangle$ are $\sigma\approx\tau$-algebraic models of $\mathcal{I}$, then $\prod A^i = \langle\prod_{i\in I}\mathrm{SEN}^i,\prod_{i\in I}\langle F^i,\alpha^i\rangle\rangle$ is also a $\sigma\approx\tau$-algebraic model of $\mathcal{I}$.*

Proof. Suppose that $A^i = \langle\mathrm{SEN}^i,\langle F^i,\alpha^i\rangle\rangle$ are $\sigma\approx\tau$-algebraic models of $\mathcal{I}$. To show that $\prod A^i$ is also a $\sigma\approx\tau$-algebraic model of $\mathcal{I}$, let $\Sigma\in|\mathbf{Sign}|$, $\Phi\cup\{\varphi\}\subseteq\mathrm{SEN}(\Sigma)$, such that $\varphi\in C_\Sigma(\Phi)$.

$$\begin{tikzcd}
\mathrm{SEN}^i & \prod_{i\in I}\mathrm{SEN}^i \arrow[l, "\langle P^i,\pi^i\rangle"'] \\
\mathrm{SEN} \arrow[u, "\langle F^i,\alpha^i\rangle"] \arrow[ur, "\prod_{i\in I}\langle F^i,\alpha^i\rangle"']
\end{tikzcd}$$

Then, by the hypothesis,

$$[\sigma\approx\tau]_\Sigma(\varphi)\subseteq C^{A^i}_\Sigma([\sigma\approx\tau]_\Sigma(\Phi)),\quad\text{for all}\quad i\in I. \tag{4}$$

Assume, now, that $\Sigma'\in|\mathbf{Sign}|$, $f\in\mathbf{Sign}(\Sigma,\Sigma')$ and let

$$\prod_{i\in I}\alpha^i_{\Sigma'}(\mathrm{SEN}(f)([\sigma\approx\tau]_\Sigma(\Phi)))\subseteq\Delta^{\prod_{i\in I}\mathrm{SEN}^i}_{\prod_{i\in I}F^i(\Sigma')}.$$

This gives, unfolding all product definitions, $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)([\sigma\approx\tau]_\Sigma(\Phi)))\subseteq\Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')}$, for all $i\in I$. Therefore, by (4), we obtain that $\alpha^i_{\Sigma'}(\mathrm{SEN}(f)([\sigma\approx\tau]_\Sigma(\varphi)))\subseteq\Delta^{\mathrm{SEN}^i}_{F^i(\Sigma')}$, for all $i\in I$. But this, also by the definition of the product, implies that $\prod_{i\in I}\alpha^i_{\Sigma'}(\mathrm{SEN}(f)([\sigma\approx\tau]_\Sigma(\varphi)))\subseteq\Delta^{\prod_{i\in I}\mathrm{SEN}^i}_{\prod_{i\in I}F^i(\Sigma')}$, which yields that $[\sigma\approx\tau]_\Sigma(\varphi)\subseteq C^{\prod A^i}_\Sigma([\sigma\approx\tau]_\Sigma(\Phi))$. Therefore, $\prod A^i$ is indeed a $\sigma\approx\tau$-algebraic model of $\mathcal{I}$. □

4 Matrix Semantics and Algebraic Semantics

Let $\mathrm{SEN} : \mathbf{Sign}\to\mathbf{Set}$ be a functor, with N a category of natural transformations on SEN, and $\sigma\approx\tau$ an N-translation. Consider, also, a functor $\mathrm{SEN}' : \mathbf{Sign}'\to\mathbf{Set}$, with N' a category of natural transformations on SEN', and an (N,N')-epimorphic translation $\langle F,\alpha\rangle : \mathrm{SEN}\to\mathrm{SEN}'$. Denote by $T'^{\sigma\approx\tau}$ the axiom family on SEN' given, for all $\Sigma\in|\mathbf{Sign}'|$, by

$$T'^{\sigma\approx\tau}_\Sigma = \{\varphi\in\mathrm{SEN}'(\Sigma) : \sigma'_\Sigma(\varphi) = \tau'_\Sigma(\varphi)\},$$

where by σ' and τ' are denoted the natural transformations on SEN' corresponding to σ and τ, respectively, via the (N,N')-epimorphic property (cf. Equation (1)). The axiom family $T'^{\sigma\approx\tau}$ corresponds in this context to the subset $F^\tau_{\mathbf{A}} = \{a\in A : \delta^{\mathbf{A}}_i(a) = \varepsilon^{\mathbf{A}}_i(a), i<n\}$ of the carrier A of an algebra $\mathbf{A}$, defined via a translation $\tau = \{\delta_i(p)\approx\varepsilon_i(p) : i<n\}$ in [3, p. 161].

In the following lemma it is shown that the closure system C^M induced by the N-matrix system $M = \langle\langle\mathrm{SEN}',\langle F,\alpha\rangle\rangle, T'^{\sigma\approx\tau}\rangle$ on SEN is interpreted into the closure system C^A induced by the N-algebraic system $A = \langle\mathrm{SEN}',\langle F,\alpha\rangle\rangle$ on SEN^2 via the N-translation $\sigma\approx\tau$.

Lemma 4.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, $A = \big\langle \mathrm{SEN}', \langle F, \alpha\rangle\big\rangle$ an N-algebraic system for* SEN*, and $\sigma \approx \tau$ an N-translation. Then $C^{\langle A, T'^{\sigma\approx\tau}\rangle} = C^{A,\sigma\approx\tau}$.*

Proof. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$. Then, we have

$$\begin{aligned}
\varphi \in C^{A,\sigma\approx\tau}_\Sigma(\Phi) \text{ iff } & [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)) \\
\text{iff } & (\forall f)\Big(\alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\Phi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')} \Rightarrow \alpha_{\Sigma'}(\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}\Big) \\
\text{iff } & (\forall f)\Big(\alpha_{\Sigma'}([\sigma \approx \tau]_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')} \Rightarrow \alpha_{\Sigma'}([\sigma \approx \tau]_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}\Big) \\
\text{iff } & (\forall f)\Big([\sigma' \approx \tau']_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')} \Rightarrow \\
& \quad [\sigma' \approx \tau']_{F(\Sigma')}(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \subseteq \Delta^{\mathrm{SEN}'}_{F(\Sigma')}\Big) \\
\text{iff } & (\forall f)\Big(\alpha_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'^{\sigma\approx\tau}_{F(\Sigma')} \Rightarrow \alpha_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'^{\sigma\approx\tau}_{F(\Sigma')}\Big) \\
\text{iff } & \varphi \in C^{\langle A, T'^{\sigma\approx\tau}\rangle}_\Sigma(\Phi).
\end{aligned}$$ □

Thus, the following analog of [2, Theorem 2.4], [3, Theorem 2.3], holds:

Theorem 4.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, $A = \big\langle \mathrm{SEN}', \langle F, \alpha\rangle\big\rangle$ an N-algebraic system for* SEN*, and $\sigma \approx \tau$ an N-translation. Then A is a $\sigma \approx \tau$-algebraic model of $\mathcal{I}$ if and only if $\langle A, T'^{\sigma\approx\tau}\rangle$ is an N-matrix system for $\mathcal{I}$.*

Proof. Follows directly from Lemma 4.1. □

The following result relates a $\sigma \approx \tau$-algebraic semantics of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, with the corresponding N-matrix system semantics obtained from it as in the statement of Theorem 4.2.

Theorem 4.3 *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*, $\mathfrak{F} = \big\{A^i = \big\langle \mathrm{SEN}^i, \big\langle F^i, \alpha^i\big\rangle\big\rangle : i \in I\big\}$ a collection of N-algebraic systems for* SEN*, and $\sigma \approx \tau$ an N-translation. Then $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$ if and only if $\mathfrak{M} = \{\langle A^i, (T^i)^{\sigma\approx\tau}\rangle : i \in I\}$ is an N-matrix system semantics for $\mathcal{I}$.*

Proof. Taking into account Lemma 4.1,

$$C^{\mathfrak{M}} = \bigcap_{i\in I} C^{\langle A^i, (T^i)^{\sigma\approx\tau}\rangle} = \bigcap_{i\in I} C^{A^i,\sigma\approx\tau} = C^{\mathfrak{F},\sigma\approx\tau}.$$

Thus, $C = C^{\mathfrak{M}}$ if and only if $C = C^{\mathfrak{F},\sigma\approx\tau}$, which gives the required equivalence. □

Recall, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and an N-translation $\sigma \approx \tau$, the definition of the class $\mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ of all $\sigma \approx \tau$-algebraic models of $\mathcal{I}$. This definition gives immediately the following proposition, forming an analog of [3, Proposition 2.8].

Proposition 4.4 *If a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, has a $\sigma \approx \tau$-algebraic semantics for an N-translation $\sigma \approx \tau$, then $\mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ is its largest $\sigma \approx \tau$-algebraic semantics.*

Proof. Suppose that $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics of $\mathcal{I}$. Then, by the definition of $\mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$, $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$. Moreover, by the definition of a $\sigma \approx \tau$-algebraic semantics, $C = C^{\mathfrak{F},\sigma\approx\tau}$. Therefore, we obtain $C \leq C^{\mathfrak{F}(\mathcal{I},\sigma\approx\tau),\sigma\approx\tau} \leq C^{\mathfrak{F},\sigma\approx\tau} = C$. Thus, $\mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ is also a $\sigma \approx \tau$-algebraic semantics of $\mathcal{I}$ and, hence, the largest one. □

5 The Class of $\sigma \approx \tau$-Algebraic Models

Recall that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is *finitary* if, for all $\Sigma \in |\mathbf{Sign}|$, C_Σ is a finitary closure operator on $\mathrm{SEN}(\Sigma)$, i.e., if, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$, there exists a finite $\Phi' \subseteq \Phi$, such that $\varphi \in C_\Sigma(\Phi')$. On the other hand, if N is a category of natural transformations on SEN, an N*-rule* is a tuple $\left\langle \left\{\sigma^0, \ldots, \sigma^{n-1}\right\}, \sigma^n \right\rangle$, where $\sigma^i : \mathrm{SEN}^k \to \mathrm{SEN}$ is a natural transformation in N, for all $i \leq n$ (the arity k is arbitrary but fixed for all n). The N-rule is said to be an N*-rule of* $\mathcal{I}$, if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\sigma^n_\Sigma(\vec{\chi}) \in C_\Sigma\left(\sigma^0_\Sigma(\vec{\chi}), \ldots, \sigma^{n-1}_\Sigma(\vec{\chi})\right)$. A *finitary* π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is called N*-rule based* if, for every $\Sigma \in |\mathbf{Sign}|$ and every finite $\{\varphi_0, \ldots, \varphi_n\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi_n \in C_\Sigma(\varphi_0, \ldots, \varphi_{n-1})$, there exists an N-rule $\left\langle \left\{\sigma^0, \ldots, \sigma^{n-1}\right\}, \sigma^n \right\rangle$ of $\mathcal{I}$ and a $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^i_\Sigma(\vec{\chi}) = \varphi_i$, for all $i \leq n$. It was shown in Theorem 3.5 of [17] that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is an N-rule based π-institution, then $C = C^R$, the closure operator induced by a set R of N-rules (in a way similar to the ordinary proof-theoretic one). The set R is then said to *axiomatize* the closure system C.

To study equational entailments on sentence functors of the form SEN^2, induced by classes $\mathfrak{F}$ of N-algebraic systems, we introduce a similar notion of an equational inference rule. Namely, given a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, an *equational* N*-rule* is a tuple $\left\langle \left\{\sigma^0 \approx \tau^0, \ldots, \sigma^{n-1} \approx \tau^{n-1}\right\}, \sigma^n \approx \tau^n \right\rangle$, where $\sigma^i, \tau^i : \mathrm{SEN}^k \to \mathrm{SEN}$ are natural transformations in N, for all $i \leq n$ (as before, the arity k is arbitrary but fixed for all n). An equational N-rule $\left\langle \left\{\sigma^0 \approx \tau^0, \ldots, \sigma^{n-1} \approx \tau^{n-1}\right\}, \sigma^n \approx \tau^n \right\rangle$ is a *rule of* the closure system C on SEN^2 and the corresponding π-institution $\langle \mathbf{Sign}, \mathrm{SEN}^2, C \rangle$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$,

$$\sigma^n_\Sigma(\vec{\chi}) \approx \tau^n_\Sigma(\vec{\chi}) \in C_\Sigma\left(\sigma^0_\Sigma(\vec{\chi}) \approx \tau^0_\Sigma(\vec{\chi}), \ldots, \sigma^{n-1}_\Sigma(\vec{\chi}) \approx \tau^{n-1}_\Sigma(\vec{\chi})\right).$$

The equational N-rule is said to be a *rule of* the N-algebraic system $A = \left\langle \mathrm{SEN}', \langle F, \alpha \rangle \right\rangle$ or to *hold in* A if, for all $\Sigma \in |\mathbf{Sign}|$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\alpha_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\vec{\chi})\right)\right) = \alpha_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\vec{\chi})\right)\right)$, for all $i < n$, implies $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma^n_\Sigma(\vec{\chi}))) = \alpha_{\Sigma'}(\mathrm{SEN}(f)(\tau^n_\Sigma(\vec{\chi})))$.

It is not very difficult to show that the definitions of an equational N-rule holding in an N-algebraic system $A = \langle \mathrm{SEN}', \langle F, \alpha \rangle \rangle$ and being a rule of the closure system C^A on SEN^2 induced by A coincide:

Lemma 5.1 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with* N *a category of natural transformations on* SEN*, and* $A = \langle \mathrm{SEN}', \langle F, \alpha \rangle \rangle$ *an* N*-algebraic system. An equational* N*-rule* $\left\langle \left\{\sigma^0 \approx \tau^0, \ldots, \sigma^{n-1} \approx \tau^{n-1}\right\}, \sigma^n \approx \tau^n \right\rangle$ *holds in* A *iff it is a rule of* $\mathcal{I}^A = \left\langle \mathbf{Sign}, \mathrm{SEN}^2, C^A \right\rangle$.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN, R a collection of equational N-rules, $\Sigma \in |\mathbf{Sign}|$ and $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$. We say that $\varphi \approx \psi$ *follows from* E *via* R if there exists a rule $\left\langle \left\{\sigma^0 \approx \tau^0, \ldots, \sigma^{n-1} \approx \tau^{n-1}\right\}, \sigma^n \approx \tau^n \right\rangle$ in R and a $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma^i_\Sigma(\vec{\chi}) \approx \tau^i_\Sigma(\vec{\chi}) \in E$, for all $i < n$, and $\sigma^n_\Sigma(\vec{\chi}) = \varphi$, $\tau^n_\Sigma(\vec{\chi}) = \psi$. On the other hand, a *proof of* $\varphi \approx \psi$ *from hypotheses* E *via* R is, as usual a sequence $\varphi_0 \approx \psi_0, \ldots, \varphi_{m-1} \approx \psi_{m-1}, \varphi_m \approx \psi_m$, in $\mathrm{SEN}(\Sigma)^2$, such that, for every $i \leq m$, $\varphi_i \approx \psi_i$ is either in E or follows from $\{\varphi_0 \approx \psi_0, \ldots, \varphi_{i-1} \approx \psi_{i-1}\}$ via R. If there exists a proof of $\varphi \approx \psi$ from hypotheses E via R, we also say that $\psi \approx \psi$ is R*-derivable from* E.

Assume, next, that N is a category of natural transformations on SEN. Given a collection R of equational N-rules, we shall denote by $C^R = \left\{C^R_\Sigma\right\}_{\Sigma \in |\mathbf{Sign}|}$ the closure system on SEN^2 *axiomatized by* R, which is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$, $\varphi \approx \psi \in C^R_\Sigma(E)$ iff there exists an R-proof of $\varphi \approx \psi$ from premises E.

Lemma 5.2 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with* N *a category of natural transformations on* SEN*, and* R *a collection of equational* N*-rules. Then* $C^R : \mathcal{P}\mathrm{SEN}^2 \to \mathcal{P}\mathrm{SEN}^2$ *is a closure system on* SEN^2.

Proof. If $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$, such that $\varphi \approx \psi \in E$, then, clearly, $\varphi \approx \psi \in C^R_\Sigma(E)$, since $\varphi \approx \psi$ is a proof of $\varphi \approx \psi$ from premises E. Thus C^R_Σ is inflationary, for all $\Sigma \in |\mathbf{Sign}|$.

That C^R_Σ is monotone follows from the fact that, if $E, F \subseteq \mathrm{SEN}(\Sigma)^2$, such that $E \subseteq F$, any R-proof of a Σ-equation from premises in E is also an R-proof of the same equation from premises in F.

To show that C^R_Σ is transitive, suppose that $E \cup \{\varphi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$, with $\varphi \approx \psi \in C^R_\Sigma\left(C^R_\Sigma(E)\right)$. Then, there exists an R-proof of $\varphi \approx \psi$ from premises $C^R_\Sigma(E)$, say

$$\varphi_0 \approx \psi_0, \varphi_1 \approx \psi_1, \ldots, \varphi_{n-1} \approx \psi_{n-1}, \varphi \approx \psi.$$

We show by induction on i that there exists an R-proof of $\varphi_i \approx \psi_i$ from E.

(a) If $i = 0$, then $\varphi_0 \approx \psi_0$ can either be an instance of an axiom $\sigma \approx \tau$ in R or an element of $C^R_\Sigma(E)$.
  (i) If $\varphi_0 \approx \psi_0$ is an instance of an axiom, then clearly $\varphi_0 \approx \psi_0$ is an R-proof of $\varphi_0 \approx \psi_0$ from E.
  (ii) If $\varphi_0 \approx \psi_0 \in C^R_\Sigma(E)$, there is nothing to prove.

(b) Assume, as the induction hypothesis, that, for all $k < i \leq n$, there exists an R-proof of $\varphi_k \approx \psi_k$ from E. Consider the derivation of $\varphi_i \approx \psi_i$. In that derivation, $\varphi_i \approx \psi_i$ is an instance of an axiom $\sigma \approx \tau$ in R or is in $C^R_\Sigma(E)$ or follows by an application of an R-rule $\left\langle \left\{\sigma^0 \approx \tau^0, \ldots, \sigma^{q-1} \approx \tau^{q-1}\right\}, \sigma \approx \tau\right\rangle$ on previous equations $\varphi_{j_0} \approx \psi_{j_0}, \ldots, \varphi_{j_{q-1}} \approx \psi_{j_{q-1}}$ of the R-proof. The first two cases are handled in exactly the same way as were cases (a)(i) and (a)(ii) above. For the last case, note that, by the induction hypothesis, there exist R-proofs of each of $\varphi_{j_l} \approx \psi_{j_l}$, $l < q$, from premises E. Juxtaposing all these proofs and adding as the last equation of the sequence $\varphi_i \approx \psi_i$ yields an R-proof of $\varphi_i \approx \psi_i$ from premises E. This yields the conclusion.

Finally, it suffices to prove structurality, i.e., that, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$ and $f \in \mathbf{Sign}(\Sigma, \Sigma')$, if $\varphi \approx \psi \in C^R_\Sigma(E)$, then $\mathrm{SEN}(f)(\varphi) \approx \mathrm{SEN}(f)(\psi) \in C^R_{\Sigma'}(\mathrm{SEN}(f)^2(E))$. This is easy, since, if $\varphi \approx \psi \in C^R_\Sigma(E)$, there exists an R-proof of $\varphi \approx \psi$ from premises E. By applying $\mathrm{SEN}(f)^2$ to all Σ-equations in the proof, we get an R-proof of $\mathrm{SEN}(f)(\varphi) \approx \mathrm{SEN}(f)(\psi)$ from premises $\mathrm{SEN}(f)^2(E)$, whence the conclusion follows. □

Suppose that the π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is N-rule based and $\sigma \approx \tau$ an N-translation. The following lemma asserts that all axioms and all rules of inference that hold in the π-institution $\mathcal{I}$ induce corresponding rules that hold in the closure system $C^{\mathfrak{F}}$ on SEN^2, where $\mathfrak{F}$ is any class of $\sigma \approx \tau$-algebraic models of $\mathcal{I}$.

Lemma 5.3 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN, *$\sigma \approx \tau = \left\{\sigma^j \approx \tau^j : j \in J\right\}$ a finite N-translation and $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$. If $\mathcal{I}$ is axiomatized by a set* Ax *of axioms and a set* IR *of rules of inference, in the sense of* [17]*, then the axioms $\sigma^j(\rho) \approx \tau^j(\rho), j \in J, \rho \in \mathrm{Ax}$ and the rules $\dfrac{\sigma^j(\rho^i) \approx \tau^j(\rho^i) : j \in J, i < n}{\sigma^j(\rho^n) \approx \tau^j(\rho^n)}$, $j \in J$, $\dfrac{\rho^i : i < n}{\rho^n} \in \mathrm{IR}$, hold in $\mathcal{I}^{\mathfrak{F}} = \left\langle \mathbf{Sign}, \mathrm{SEN}^2, C^{\mathfrak{F}}\right\rangle$.*

Proof. Let $\rho \in \mathrm{Ax}$. This means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\rho_\Sigma(\vec{\chi}) \in C_\Sigma(\varnothing)$. Thus, since $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$, for all $\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle \in \mathfrak{F}$, we get $\alpha_{\Sigma'}\big(\mathrm{SEN}(f)\big(\sigma^j_\Sigma(\rho_\Sigma(\vec{\chi}))\big)\big) = \alpha_{\Sigma'}\big(\mathrm{SEN}(f)\big(\tau^j_\Sigma(\rho_\Sigma(\vec{\chi}))\big)\big)$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and $j \in J$. Thus, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\sigma^j_\Sigma(\rho_\Sigma(\vec{\chi})) \approx \tau^j_\Sigma(\rho_\Sigma(\vec{\chi})) \in C^{\mathfrak{F}}_\Sigma(\varnothing)$, which shows that $\sigma^j(\rho) \approx \tau^j(\rho)$ is an axiom of $\mathcal{I}^{\mathfrak{F}}$.

Suppose, next that $\dfrac{\rho^i : i < n}{\rho^n} \in \mathrm{IR}$. This means that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\rho^n_\Sigma(\vec{\chi}) \in C_\Sigma(\{\rho^i_\Sigma(\vec{\chi}) : i < n\})$. Thus, since $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$, we obtain that, for all $\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle \in \mathfrak{F}$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, if $\alpha_{\Sigma'}\big(\mathrm{SEN}(f)\big(\sigma^j_\Sigma(\rho^i_\Sigma(\vec{\chi}))\big)\big) = \alpha_{\Sigma'}\big(\mathrm{SEN}(f)\big(\tau^j_\Sigma(\rho^i_\Sigma(\vec{\chi}))\big)\big)$, for all $i < n$ and all $j \in J$, then $\alpha_{\Sigma'}\big(\mathrm{SEN}(f)\big(\sigma^j_\Sigma(\rho^n_\Sigma(\vec{\chi}))\big)\big) = \alpha_{\Sigma'}\big(\mathrm{SEN}(f)\big(\tau^j_\Sigma(\rho^n_\Sigma(\vec{\chi}))\big)\big)$, for all $j \in J$. Thus, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\sigma^j_\Sigma\big(\rho^n_\Sigma(\vec{\chi})\big) \approx \tau^j_\Sigma(\rho^n_\Sigma(\vec{\chi})) \in C^{\mathfrak{F}}_\Sigma\big(\big\{\sigma^j_\Sigma\big(\rho^i_\Sigma(\vec{\chi})\big) \approx \tau^j_\Sigma\big(\rho^i_\Sigma(\vec{\chi})\big) : i < n, j \in J\big\}\big)$, for all $j \in J$, whence

$$\frac{\sigma^j(\rho^i) \approx \tau^j(\rho^i) : j \in J, i < n}{\sigma^j(\rho^n) \approx \tau^j(\rho^n)}$$

is a rule of $\mathcal{I}^{\mathfrak{F}}$, for all $j \in J$. □

The next lemma asserts, roughly speaking, that the translation of every valid consequence of a π-institution $\mathcal{I}$ via the equations $\sigma \approx \tau$ holds in every N-algebraic system satisfying the translates under $\sigma \approx \tau$ of all the axioms and all the rules of inference of $\mathcal{I}$.

Lemma 5.4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN, *and $\sigma \approx \tau = \left\{\sigma^j \approx \tau^j : j \in J\right\}$ a finite N-translation. Assume that $\mathcal{I}$ is axiomatized by a set* Ax *of axioms and a set* IR *of rules of inference, in the sense of* [17]*, and that the axioms $\sigma^j(\rho) \approx \tau^j(\rho), j \in J, \rho \in \mathrm{Ax}$*

and the rules $\dfrac{\sigma^j(\rho^i) \approx \tau^j(\rho^i) : j \in J, i < n}{\sigma^j(\rho^n) \approx \tau^j(\rho^n)}$, $j \in J$, *and* $\dfrac{\rho^i : i < n}{\rho^n} \in \mathrm{IR}$ *hold in an* N*-algebraic system* $A = \langle \mathrm{SEN}', \langle F, \alpha \rangle\rangle$*. Then, for every* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$*, such that* $\varphi \in C_\Sigma(\Phi)$*,* $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$.

Proof. Since $\varphi \in C_\Sigma(\Phi)$ and $\mathcal{I}$ is axiomatized by the set Ax of axioms and the set IR of rules of inference, there exists a proof of φ from premises Φ, i.e., a sequence $\varphi_0, \ldots, \varphi_{m-1}, \varphi_m = \varphi$, such that, for all $i \leq m$, $\varphi_i = \rho_\Sigma(\vec{\chi})$, for some axiom ρ and some $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, or $\varphi_i \in \Phi$, or $\varphi_i = \rho^n_\Sigma(\vec{\chi})$, for some rule of inference $\dfrac{\rho^0 \quad \cdots \quad \rho^{n-1}}{\rho^n}$ and some $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\{\rho^0_\Sigma(\vec{\chi}), \ldots, \rho^{n-1}_\Sigma(\vec{\chi})\} \subseteq \{\varphi_0, \ldots, \varphi_{i-1}\}$. We show, by induction on $i \leq m$, that, for all $j \in J$, $[\sigma^j \approx \tau^j]_\Sigma(\varphi_i) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$.

Base Case: If $\varphi_0 = \rho_\Sigma(\vec{\chi})$, for some $\rho \in \mathrm{Ax}$ and some $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, then, since $\sigma^j(\rho) \approx \tau^j(\rho)$, $j \in J$, hold in A, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\alpha_{\Sigma'}(\mathrm{SEN}(f)(\sigma^j_\Sigma(\rho_\Sigma(\vec{\chi})))) = \alpha_{\Sigma'}(\mathrm{SEN}(f)(\tau^j_\Sigma(\rho_\Sigma(\vec{\chi}))))$, whence $[\sigma^j \approx \tau^j]_\Sigma(\varphi_0) \subseteq C^A_\Sigma(\varnothing) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$. If, on the other hand, $\varphi_0 \in \Phi$, then the same conclusion follows from the reflexivity of C^A.

Induction step: Assume that, for all $j \in J$ and all $i < p \leq m$, $[\sigma^j \approx \tau^j]_\Sigma(\varphi_i) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$. If $\varphi_p = \rho_\Sigma(\vec{\chi})$, for some $\rho \in \mathrm{Ax}$ and some $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, or $\varphi_p \in \Phi$, then the conclusion follows exactly as in the base case. So assume that $\varphi_p = \rho^n_\Sigma(\vec{\chi})$, for some rule of inference $\dfrac{\rho^0 \quad \cdots \quad \rho^{n-1}}{\rho^n}$ and some $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\{\rho^0_\Sigma(\vec{\chi}), \ldots, \rho^{n-1}_\Sigma(\vec{\chi})\} \subseteq \{\varphi_0, \ldots, \varphi_{p-1}\}$. Then, by the induction hypothesis

$$[\sigma^j \approx \tau^j]_\Sigma(\rho^l_\Sigma(\vec{\chi})) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)), j \in J, l < n,$$

and, since $\dfrac{\sigma^j(\rho^i) \approx \tau^j(\rho^i) : j \in J, i < n}{\sigma^j(\rho^n) \approx \tau^j(\rho^n)}$ holds in A,

$$[\sigma^j \approx \tau^j]_\Sigma(\rho^n_\Sigma(\vec{\chi})) \subseteq C^A_\Sigma(\{[\sigma^j \approx \tau^j]_\Sigma(\rho^l_\Sigma(\vec{\chi})) : j \in J, l < n\}).$$

Therefore, by the transitivity of C^A, we get that

$$\begin{aligned}[\sigma^j \approx \tau^j]_\Sigma(\rho^n_\Sigma(\vec{\chi})) &\subseteq C^A_\Sigma(\{[\sigma^j \approx \tau^j]_\Sigma(\rho^l_\Sigma(\vec{\chi})) : j \in J, l < n\}) \\ &\subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)),\end{aligned}$$

i.e.,

$$[\sigma^j \approx \tau^j]_\Sigma(\varphi_p) \subseteq C^A_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)).$$ □

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN be an N-rule based π-institution axiomatized by a set Ax of axioms and a set IR of rules of inference and define $R := R(\mathrm{Ax}, \mathrm{IR})$ to be the set consisting of the equational N-rules

(Identity) $\iota \approx \iota$

(Axiom) $\sigma^j(\rho) \approx \tau^j(\rho), j \in J, \rho \in \mathrm{Ax}$

and

(Symmetry) $\dfrac{p^{2,0} \approx p^{2,1}}{p^{2,1} \approx p^{2,0}}$

(Transitivity) $\dfrac{p^{3,0} \approx p^{3,1} \qquad p^{3,1} \approx p^{3,2}}{p^{3,0} \approx p^{3,2}}$

(Substitution) $\dfrac{\eta \approx \theta \qquad \eta^0 \approx \theta^0 \qquad \ldots \qquad \eta^{n-1} \approx \theta^{n-1}}{\eta(\eta^0, \ldots, \eta^{n-1}) \approx \theta(\theta^0, \ldots, \theta^{n-1})}$

$$\text{(Inference Rule)} \quad \frac{\sigma^j(\rho^i) \approx \tau^j(\rho^i) : j \in J, i < n}{\sigma^j(\rho^n) \approx \tau^j(\rho^n)} \, j \in J, \ \frac{\rho^i : i < n}{\rho^n} \in \mathrm{IR}.$$

For an arbitrary but fixed signature Σ_0 and a subset $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, we set, for all $\Sigma \in |\mathbf{Sign}|$,

$$[\sigma \approx \tau]^*_{\Sigma_0,\Sigma}(\Phi_0) := \bigcup_{f \in \mathbf{Sign}(\Sigma_0,\Sigma)} [\sigma \approx \tau]_\Sigma(\mathrm{SEN}(f)(\Phi_0)).$$

Moreover, we define on SEN the relation family $\theta^{\langle \Sigma_0,\Phi_0\rangle} = \left\{\theta^{\langle \Sigma_0,\Phi_0\rangle}_\Sigma\right\}_{\Sigma \in |\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\theta^{\langle \Sigma_0,\Phi_0\rangle}_\Sigma = \left\{\langle \varphi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : \langle \varphi,\psi\rangle \text{ is } R\text{-derivable from } [\sigma \approx \tau]^*_{\Sigma_0,\Sigma}(\Phi_0)\right\},$$

the $R = R(\mathrm{Ax},\mathrm{IR})$ referring to the set of equational N-rules defined above.

Lemma 5.5 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-rule based π-institution axiomatized by a set* Ax *of axioms and a set* IR *of rules of inference, $\sigma \approx \tau$ an N-translation and $R = R(\mathrm{Ax},\mathrm{IR})$. Then, for all $\Sigma_0 \in |\mathbf{Sign}|$ and all $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, $\theta^{\langle \Sigma_0,\Phi_0\rangle}$ is an N-congruence system on* SEN.

P r o o f. For all $\Sigma \in |\mathbf{Sign}|$, $\theta^{\langle \Sigma_0,\Phi_0\rangle}_\Sigma$ is, by definition, closed under all R-rules. Thus, by closure under (Identity), (Symmetry) and (Transitivity), it is an equivalence relation on $\mathrm{SEN}(\Sigma)$. Moreover, by closure under (Substitution), it is an N-congruence relation on $\mathrm{SEN}(\Sigma)$. Thus, it suffices to show that $\theta^{\langle \Sigma_0,\Phi_0\rangle}$ satisfies also the system property, i.e., it is invariant under all signature morphisms. To this end, suppose $\Sigma,\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma,\Sigma')$ and $\varphi,\psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \varphi,\psi\rangle \in \theta^{\langle \Sigma_0,\Phi_0\rangle}_\Sigma$. Thus, there exists an R-proof of $\varphi \approx \psi$ from premises $[\sigma \approx \tau]^*_{\Sigma_0,\Sigma}(\Phi_0)$. Applying to both sides of all Σ-equations in the proof $\mathrm{SEN}(f)$, we get an R-proof of $\mathrm{SEN}(f)(\varphi) \approx \mathrm{SEN}(f)(\psi)$ from premises in $\mathrm{SEN}(f)^2([\sigma \approx \tau]^*_{\Sigma_0,\Sigma}(\Phi_0))$. Now, it suffices to notice that

$$\begin{aligned}
\mathrm{SEN}(f)^2([\sigma \approx \tau]^*_{\Sigma_0,\Sigma}(\Phi_0)) &= \mathrm{SEN}(f)^2\left(\bigcup_{g \in \mathbf{Sign}(\Sigma_0,\Sigma)} [\sigma \approx \tau]_\Sigma(\mathrm{SEN}(g)(\Phi_0))\right)\\
&= \bigcup_{g \in \mathbf{Sign}(\Sigma_0,\Sigma)} ([\sigma \approx \tau]_{\Sigma'}(\mathrm{SEN}(fg)(\Phi_0)))\\
&\subseteq \bigcup_{h \in \mathbf{Sign}(\Sigma_0,\Sigma')} [\sigma \approx \tau]_{\Sigma'}(\mathrm{SEN}(h)(\Phi_0))\\
&= [\sigma \approx \tau]^*_{\Sigma_0,\Sigma'}(\Phi_0).
\end{aligned}$$

Thus, $\langle \mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi)\rangle \in \theta^{\langle \Sigma_0,\Phi_0\rangle}_{\Sigma'}$, showing that the system property holds. □

Because of Lemma 5.5, we may consider, for all $\Sigma_0 \in |\mathbf{Sign}|$ and all $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, the quotient N-algebraic system $\mathcal{A}^{\langle \Sigma_0,\Phi_0\rangle} = \left\langle \mathrm{SEN}/\theta^{\langle \Sigma_0,\Phi_0\rangle}, \left\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\theta^{\langle \Sigma_0,\Phi_0\rangle}}\right\rangle\right\rangle$. To simplify notation, we shall be denoting $\pi^{\theta^{\langle \Sigma_0,\Phi_0\rangle}}$ simply by $\pi^{\langle \Sigma_0,\Phi_0\rangle}$.

Lemma 5.6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-rule based π-institution axiomatized by a set* Ax *of axioms and a set* IR *of rules of inference, $\sigma \approx \tau$ an N-translation and $R = R(\mathrm{Ax},\mathrm{IR})$. Then, for all $\Sigma_0 \in |\mathbf{Sign}|$ and all $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, $\mathcal{A}^{\langle \Sigma_0,\Phi_0\rangle} \in \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$.*

P r o o f. We must show that, given $\Sigma_0 \in |\mathbf{Sign}|$ and $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, the N-algebraic system $\mathcal{A}^{\langle \Sigma_0,\Phi_0\rangle}$ is a $\sigma \approx \tau$-algebraic system model of $\mathcal{I}$, i.e., that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C_\Sigma(\Phi)$, we have that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^{\mathcal{A}^{\langle \Sigma_0,\Phi_0\rangle}}_\Sigma([\sigma \approx \tau]_\Sigma(\Phi))$. Thus, by Lemma 5.4, it suffices to show that all equational N-rules in R hold in $\mathrm{SEN}/\theta^{\langle \Sigma_0,\Phi_0\rangle}$. But this holds since, by definition, $\theta^{\langle \Sigma_0,\Phi_0\rangle}$ is closed under those rules. □

Next, define $\mathfrak{F}^* = \{A^{\langle\Sigma_0,\Phi_0\rangle} : \Sigma_0 \in |\mathbf{Sign}|, \Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)\}$. By Lemma 5.6, $\mathfrak{F}^* \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$. In the next lemma, we work to show that all inferences on SEN induced via $\sigma \approx \tau$ by the closure system $C^{\mathfrak{F}^*}$ on SEN^2 generated by the class $\mathfrak{F}^*$ are also induced via $\sigma \approx \tau$ by the closure C^R on SEN^2 axiomatized by R, i.e., that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}^*}([\sigma \approx \tau]_\Sigma(\Phi)) \quad \text{implies} \quad [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^R([\sigma \approx \tau]_\Sigma(\Phi)).$$

Lemma 5.7 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-rule based π-institution axiomatized by a set* Ax *of axioms and a set* IR *of rules of inference, $\sigma \approx \tau$ an N-translation and $R = R(\mathrm{Ax}, \mathrm{IR})$. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,*

$$[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}^*}([\sigma \approx \tau]_\Sigma(\Phi)) \quad \textit{implies} \quad [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^R([\sigma \approx \tau]_\Sigma(\Phi)).$$

P r o o f. Suppose that $\Sigma \in |\mathbf{Sign}|$ and $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}^*}([\sigma \approx \tau]_\Sigma(\Phi))$. Thus, for every $\Sigma_0 \in |\mathbf{Sign}|$ and all $\Phi_0 \subseteq \mathrm{SEN}(\Sigma_0)$, we have that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{A^{\langle\Sigma_0,\Phi_0\rangle}}([\sigma \approx \tau]_\Sigma(\Phi))$. In particular, we obtain that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{A^{\langle\Sigma,\Phi\rangle}}([\sigma \approx \tau]_\Sigma(\Phi))$. This means that, for every $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\Phi)) \subseteq \theta_{\Sigma'}^{\langle\Sigma,\Phi\rangle} \quad \text{implies} \quad \mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi)) \subseteq \theta_{\Sigma'}^{\langle\Sigma,\Phi\rangle}. \tag{5}$$

But, since $\theta_{\Sigma'}^{\langle\Sigma,\Phi\rangle}$ consists, by definition, of all equations that are R-provable from hypothesis $[\sigma \approx \tau]^*_{\Sigma,\Sigma'}(\Phi)$, the hypothesis of the displayed Implication (5) always holds. Therefore, we have that for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)([\sigma \approx \tau]_\Sigma(\varphi)) \subseteq \theta_{\Sigma'}^{\langle\Sigma,\Phi\rangle}$. Therefore, we get that $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq \theta_{\Sigma}^{\langle\Sigma,\Phi\rangle}$, i.e., that the equations $[\sigma \approx \tau]_\Sigma(\varphi)$ are R-provable from hypotheses $[\sigma \approx \tau]^*_{\Sigma,\Sigma}(\Phi)$. Thus, by the definition of C^R, $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^R([\sigma \approx \tau]_\Sigma(\Phi))$. □

Now we can prove the following proposition, that forms an analog of [3, Corollary 2.10].

Proposition 5.8 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*, $\sigma \approx \tau = \{\sigma^j \approx \tau^j : j \in J\}$ a finite N-translation and $\mathfrak{F} = \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$. If $\mathcal{I}$ is axiomatized by a set* Ax *of axioms and a set* IR *of rules of inference in the sense of* [17] *and $R = R(\mathrm{Ax}, \mathrm{IR})$ then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,*

$$[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi)) \textit{ iff } [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^R([\sigma \approx \tau]_\Sigma(\Phi)).$$

P r o o f. The right-to-left implication follows from Lemma 5.3 with $\mathfrak{F} = \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$. For the left-to-right implication, we have that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$ implies, by Lemma 5.6, $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}^*}([\sigma \approx \tau]_\Sigma(\Phi))$, and, therefore, by Lemma 5.7, $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^R([\sigma \approx \tau]_\Sigma(\Phi))$. □

6 Algebraic Semantics and Extensions

Recall that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, a *theory family* $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of $\mathcal{I}$ is a collection of Σ-theories T_Σ, $\Sigma \in |\mathbf{Sign}|$, i.e., $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, such that $C_\Sigma(T_\Sigma) = T_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$. By $\mathrm{ThFam}(\mathcal{I})$ is denoted the collection of all theory families of the π-institution $\mathcal{I}$.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Suppose that $\sigma \approx \tau$ is an N-translation and $\mathfrak{F} = \{\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle : i \in I\}$ a collection of N-algebraic systems for SEN. Define the mapping $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I}^{\mathfrak{F}})$ by setting, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}|$,

$$(\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}(T) = C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(T_\Sigma)).$$

This mapping forms an analog in the π-institution framework of the mapping $\tau_{\mathcal{S},\mathrm{K}} : \mathrm{Th}\mathcal{S} \to \mathrm{ThK}$ of [3, Definition 2.11] for deductive systems. It does obey the following lemma, which forms an analog in this context of [3, Lemma 2.12]. Roughly speaking, deductively closing in the domain and then mapping via $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}}$ is the same as mapping first and then closing in the codomain. For its exact formulation, the following notation borrowed from [15] will be used: Given a theory family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, a signature $\Sigma_0 \in |\mathbf{Sign}|$ and $\Phi \subseteq \mathrm{SEN}(\Sigma_0)$, let $T^{[\langle \Sigma_0, \Phi\rangle]} = \left\{T_\Sigma^{[\langle \Sigma_0, \Phi\rangle]}\right\}_{\Sigma \in |\mathbf{Sign}|}$ be the theory family of $\mathcal{I}$ defined by

$$T_\Sigma^{[\langle \Sigma_0, \Phi\rangle]} = \begin{cases} C_\Sigma(T_\Sigma \cup \Phi) & \text{if} \quad \Sigma = \Sigma_0, \\ T_\Sigma & \text{otherwise.} \end{cases}$$

Recall, also, that $\mathrm{Thm} = \{C_\Sigma(\varnothing)\}_{\Sigma \in |\mathbf{Sign}|}$ denotes the theorem family (which is actually a theory system, i.e., invariant under signature morphisms) of $\mathcal{I}$.

Lemma 6.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, $\sigma \approx \tau$ an N-translation and $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ a class of $\sigma \approx \tau$-algebraic models of $\mathcal{I}$. Then, for every $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$,*

$$(\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}\big(\mathrm{Thm}^{[\langle \Sigma, \Phi\rangle]}\big) = C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi)).$$

P r o o f. Since, by definition, for all $\Sigma \in |\mathbf{Sign}|, \Phi \subseteq \mathrm{SEN}(\Sigma)$ and all $T \in \mathrm{ThFam}(\mathcal{I})$, we have $(\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}(T) = C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(T_\Sigma))$, it suffices to show that

$$C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(C_\Sigma(\Phi))) = C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi)).$$

Since $\Phi \subseteq C_\Sigma(\Phi)$, we have $[\sigma \approx \tau]_\Sigma(\Phi) \subseteq [\sigma \approx \tau]_\Sigma(C_\Sigma(\Phi))$ and, therefore, $C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi)) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(C_\Sigma(\Phi)))$. For the reverse inclusion, let $\zeta \approx \eta \in C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(C_\Sigma(\Phi)))$. Now, since $\mathfrak{F}$ is a class of $\sigma \approx \tau$-algebraic models of $\mathcal{I}$, we have that, for every $\varphi \in C_\Sigma(\Phi)$, $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$, whence $[\sigma \approx \tau]_\Sigma(C_\Sigma(\Phi)) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$. This yields $\zeta \approx \eta \in C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(C_\Sigma(\Phi))) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$. □

The next theorem, an analog of [3, Theorem 2.13], provides a characterization of a class of $\sigma \approx \tau$-algebraic models of a given π-institution as being a $\sigma \approx \tau$-algebraic semantics in terms of the injectivity of the mapping $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}}$.

Theorem 6.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, $\sigma \approx \tau$ an N-translation and $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ a class of $\sigma \approx \tau$-algebraic models of $\mathcal{I}$. Then $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$ if and only if $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I}^{\mathfrak{F}})$ is injective.*

P r o o f. Suppose, first, that $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$. Consider $T, T' \in \mathrm{ThFam}(\mathcal{I})$, such that, for every $\Sigma \in |\mathbf{Sign}|$, $(\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}(T) = (\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}(T')$. This gives, by the definition of $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}}$, that

$$C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(T_\Sigma)) = C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(T'_\Sigma)),$$

whence, since $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$, we get that $C_\Sigma(T_\Sigma) = C_\Sigma(T'_\Sigma)$, i.e., that $T_\Sigma = T'_\Sigma$. Since this holds for all $\Sigma \in |\mathbf{Sign}|$, $T = T'$ and $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}}$ is indeed injective.

Suppose, conversely, that $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}}$ is injective. It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$ implies that $\varphi \in C_\Sigma(\Phi)$. If $[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$, then $C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi \cup \{\varphi\})) = C_\Sigma^{\mathfrak{F}}([\sigma \approx \tau]_\Sigma(\Phi))$, whence, by Lemma 6.1, we get that

$$(\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}\big(\mathrm{Thm}^{[\langle \Sigma, \Phi \cup \{\varphi\}\rangle]}\big) = (\sigma \approx \tau)_\Sigma^{\mathcal{I},\mathfrak{F}}\big(\mathrm{Thm}^{[\langle \Sigma, \Phi\rangle]}\big).$$

Thus, by the injectivity of $(\sigma \approx \tau)^{\mathcal{I},\mathfrak{F}}$, we obtain that $\mathrm{Thm}_\Sigma^{[\langle \Sigma, \Phi \cup \{\varphi\}\rangle]} = \mathrm{Thm}_\Sigma^{[\langle \Sigma, \Phi\rangle]}$, i.e., $C_\Sigma(\Phi \cup \{\varphi\}) = C_\Sigma(\Phi)$, showing that $\varphi \in C_\Sigma(\Phi)$. □

A π-institution $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ is said to be an *extension* of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ if, for every $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\varphi \in C_\Sigma(\Phi) \quad \text{implies} \quad \varphi \in C'_\Sigma(\Phi).$$

Equivalently, $\mathcal{I}'$ is an extension of $\mathcal{I}$ if and only if $C \leq C'$ if and only if $\mathrm{ThFam}(\mathcal{I}') \subseteq \mathrm{ThFam}(\mathcal{I})$.

Theorem 6.3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be an N-rule based π-institution, $\sigma \approx \tau$ an N-translation and $\mathfrak{F} \subseteq \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$. Suppose that $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ is an N-rule-based extension of $\mathcal{I}$ and $\mathfrak{F}' = \mathfrak{F}(\mathcal{I}', \sigma \approx \tau)$. Then*

$$(\sigma \approx \tau)^{\mathcal{I}', \mathfrak{F}'} = (\sigma \approx \tau)^{\mathcal{I}, \mathfrak{F}} \restriction_{\mathrm{ThFam}(\mathcal{I}')} .$$

P r o o f. Notice that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, we have

$$\begin{aligned}
[\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)) \text{ iff } \quad & \varphi \in C_\Sigma(\Phi) \\
& \text{(since } \mathfrak{F} \text{ is a } \sigma \approx \tau\text{-algebraic semantics of } \mathcal{I}) \\
\text{implies} \quad & \varphi \in C'_\Sigma(\Phi) \text{(since } C \leq C') \\
\text{implies} \quad & [\sigma \approx \tau]_\Sigma(\varphi) \subseteq C^{\mathfrak{F}'}_\Sigma([\sigma \approx \tau]_\Sigma(\Phi)) \\
& \text{(since } \mathfrak{F}' \text{ is a class of } \sigma \approx \tau\text{-models of } \mathcal{I}').
\end{aligned}$$

Therefore, for all $\Sigma \in |\mathbf{Sign}|$ and all $T' \in \mathrm{ThFam}(\mathcal{I}')$, we get

$$(\sigma \approx \tau)^{\mathcal{I}, \mathfrak{F}}_\Sigma(T') = C^{\mathfrak{F}}_\Sigma\left([\sigma \approx \tau]_\Sigma\left(T'_\Sigma\right)\right) \subseteq C^{\mathfrak{F}'}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma)) = (\sigma \approx \tau)^{\mathcal{I}', \mathfrak{F}'}_\Sigma(T').$$

For the reverse inclusion, it suffices to show that $C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$ is a $C^{\mathfrak{F}'}$-theory, since, then, as $C^{\mathfrak{F}'}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$ is by definition the least $C^{\mathfrak{F}'}$-theory containing $[\sigma \approx \tau]_\Sigma(T'_\Sigma)$, we shall have $C^{\mathfrak{F}'}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma)) \subseteq C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$. To see that this is the case, we use Proposition 5.8, taking advantage of the hypothesis that $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ is an N-rule-based extension of $\mathcal{I}$.

Assume, first, that ρ is an axiom of C'. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, $\rho_\Sigma(\vec{\chi}) \in C'_\Sigma(\varnothing)$, whence, we obtain that $\rho_\Sigma(\vec{\chi}) \in T'_\Sigma$. Thus, $[\sigma \approx \tau]_\Sigma(\rho_\Sigma(\vec{\chi})) \subseteq [\sigma \approx \tau]_\Sigma(T'_\Sigma)$ and, therefore, $[\sigma \approx \tau]_\Sigma(\rho_\Sigma(\vec{\chi})) \subseteq C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$. Thus, $C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$ contains all instances of the axioms $\sigma(\rho) \approx \tau(\rho)$, for all axioms ρ of C'.

Assume, next that $\dfrac{\rho^0 \quad \ldots \quad \rho^{n-1}}{\rho^n}$ is a rule of inference of C' and let $\Sigma \in |\mathbf{Sign}|$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $[\sigma \approx \tau]_\Sigma(\rho^i_\Sigma(\vec{\chi})) \in C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$, for all $i < n$. Since $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$, we get that $\rho^i_\Sigma(\vec{\chi}) \in C_\Sigma(T'_\Sigma) = T'_\Sigma$, for all $i < n$. Hence, since $\dfrac{\rho^0 \quad \ldots \quad \rho^{n-1}}{\rho^n}$ is a rule of inference of C' and $T' \in \mathrm{ThFam}(\mathcal{I}')$, $\rho^n_\Sigma(\vec{\chi}) \in T'_\Sigma$. This yields that $[\sigma \approx \tau]_\Sigma(\rho^n_\Sigma(\vec{\chi})) \in [\sigma \approx \tau]_\Sigma(T'_\Sigma) \subseteq C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$. Thus, $C^{\mathfrak{F}}_\Sigma([\sigma \approx \tau]_\Sigma(T'_\Sigma))$ is closed under all rules of inference of $C^{\mathfrak{F}'}$ and is, thus, a $C^{\mathfrak{F}'}$-theory, as was to be shown. □

Theorem 6.4 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is an N-rule based π-institution and $\sigma \approx \tau$ an N-translation. If $\mathcal{I}$ has a $\sigma \approx \tau$-algebraic semantics, then so does any N-rule based extension $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ of $\mathcal{I}$.*

P r o o f. Suppose that $\mathcal{I}$ has a $\sigma \approx \tau$-algebraic semantics. Let $\mathcal{I}'$ be an N-rule-based extension of $\mathcal{I}$ and $\mathfrak{F}' = \mathfrak{F}(\mathcal{I}', \sigma \approx \tau)$. By Proposition 4.4, $\mathfrak{F} = \mathfrak{F}(\mathcal{I}, \sigma \approx \tau)$ is a $\sigma \approx \tau$-algebraic semantics of $\mathcal{I}$, whence, by Theorem 6.2, the mapping $(\sigma \approx \tau)^{\mathcal{I}, \mathfrak{F}} : \mathrm{ThFam}(\mathcal{I}) \to \mathrm{ThFam}(\mathcal{I}^{\mathfrak{F}})$ is injective. By Theorem 6.3, the mapping $(\sigma \approx \tau)^{\mathcal{I}', \mathfrak{F}'} : \mathrm{ThFam}(\mathcal{I}') \to \mathrm{ThFam}(\mathcal{I}^{\mathfrak{F}'})$ is also injective. Therefore, by Theorem 6.2, $\mathcal{I}'$ also has a $\sigma \approx \tau$-algebraic semantics, namely $\mathfrak{F}'$. □

7 A Blok-Rebagliato Style Theorem

In [3, Theorem 2.20], Blok and Rebagliato prove that a necessary and sufficient condition for a mono-unary deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S}\rangle$, with $\mathcal{L} = \{f\}$, f a unary operation, to have an algebraic semantics is that $p \vdash_\mathcal{S} f(p)$. Moreover, in that case, $p \approx f(p)$ is the defining equation. The usefulness of this result for proving the existence or non-existence of algebraic semantics in various examples has been demonstrated in [9]. In this section, an analog of this result is proven. Because f is used in the present context to denote signature morphisms and, in general, we opt for lowercase Greek letters (σ, τ, etc.) for natural transformations in N, we shall denote the unary natural transformation that corresponds to the unary operation f of Blok and Rebagliato by μ. First, it is shown that, if a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with a mono-unary category of natural transformations N on SEN, generated by $\mu : \mathrm{SEN} \to \mathrm{SEN}$, has an algebraic semantics, then, for every $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\mu_\Sigma(\varphi) \in C_\Sigma(\varphi)$, i.e., the N-rule of inference $\langle\{\iota\}, \mu\rangle$ is a rule of $\mathcal{I}$. The converse, however, does not seem to hold without further qualifications. Namely, to be able to apply the proof method of [3, Theorem 2.20] to ensure that an N-rule based π-institution $\mathcal{I}$, as above, with $\langle\{\iota\}, \mu\rangle$ an N-rule of inference of $\mathcal{I}$, has a universal $\iota \approx \mu$-algebraic semantics, several additional technical conditions must be imposed on $\mathcal{I}$. These are spelled out in detail in the second part of Theorem 7.4, the main theorem of this section.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a set-valued functor and N a category of natural transformations on SEN. N is said to be *mono-unary* if it is generated as a category of natural transformations by a unary natural transformation $\mu : \mathrm{SEN} \to \mathrm{SEN}$.

Lemma 7.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution with N a mono-unary category of natural transformations on* SEN*, generated by $\mu : \mathrm{SEN} \to \mathrm{SEN}$. If, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\mu_\Sigma(\varphi) \in C_\Sigma(\varphi)$, then $\mu^n_\Sigma(\chi) \notin C_\Sigma(\Phi)$ implies that, for all $i < n$, $\mu^i_\Sigma(\chi) \notin \Phi$, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\chi\} \subseteq \mathrm{SEN}(\Sigma)$.*

Proof. Suppose that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\mu_\Sigma(\varphi) \in C_\Sigma(\varphi)$. Using this relation multiple times, we get

$$\begin{aligned}\mu^n_\Sigma(\chi) &= \mu_\Sigma\big(\mu^{n-1}_\Sigma(\chi)\big)\\ &\in C_\Sigma\big(\mu^{n-1}_\Sigma(\chi)\big)\\ &\subseteq C_\Sigma\big(\mu^{n-2}_\Sigma(\chi)\big)\\ &\subseteq \cdots\\ &\subseteq C_\Sigma\big(\mu^{i}_\Sigma(\chi)\big).\end{aligned}$$

Thus, if $\mu^i_\Sigma(\chi) \in \Phi$, then $\mu^n_\Sigma(\chi) \in C_\Sigma(\Phi)$. □

In the sequel, the symbol ★ will be used to denote a trivial category. When this is done, the unique object of that category will also be denoted by ★. Hopefully, the context will make clear in which sense ★ is used and this overloading will not cause any confusion.

Lemma 7.2 *Let ★ be the trivial category, $n \in \omega$ and $\mathrm{SEN}' : \bigstar \to \mathbf{Set}$ the set-valued functor defined by $\mathrm{SEN}'(\bigstar) = \{0, 1, \ldots, n+1\}$. Then $\mu' : \mathrm{SEN}' \to \mathrm{SEN}'$ defined, for all $i < n+2$, by*

$$\mu'_\bigstar(i) = \begin{cases} i+1 & \text{if} \quad i < n+1, \\ n+1 & \text{if} \quad i = n+1, \end{cases} \tag{6}$$

is a natural transformation on SEN′*, which generates a mono-unary category of natural transformations on* SEN′.

Proof. The proof relies on the trivial fact that the following rectangle commutes:

$$\begin{tikzcd} \mathrm{SEN}'(\bigstar) \arrow[r, "\mu'_\bigstar"] \arrow[d, "\mathrm{SEN}'(i_\bigstar)"'] & \mathrm{SEN}'(\bigstar) \arrow[d, "\mathrm{SEN}'(i_\bigstar)"] \\ \mathrm{SEN}'(\bigstar) \arrow[r, "\mu'_\bigstar"'] & \mathrm{SEN}'(\bigstar) \end{tikzcd}$$

where $i_\bigstar : \bigstar \to \bigstar$ is the identity arrow on the single object ★ of the category ★. □

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ and $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ be two set-valued functors, with N, N' similar categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$ respectively, $F : \mathbf{Sign} \to \mathbf{Sign}'$ be a functor and $\Sigma_0 \in |\mathbf{Sign}|$. A function $\alpha_{\Sigma_0} : \mathrm{SEN}(\Sigma_0) \to \mathrm{SEN}'(F(\Sigma_0))$ is called (N, N')-*admissible* if, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $\vec{\varphi} \in \mathrm{SEN}(\Sigma_0)^n$,

$$\begin{array}{ccc} \mathrm{SEN}(\Sigma_0)^n & \xrightarrow{\alpha^n_{\Sigma_0}} & \mathrm{SEN}'(F(\Sigma_0))^n \\ {\scriptstyle \sigma_{\Sigma_0}}\downarrow & & \downarrow{\scriptstyle \sigma'_{F(\Sigma_0)}} \\ \mathrm{SEN}(\Sigma_0) & \xrightarrow[\alpha_{\Sigma_0}]{} & \mathrm{SEN}'(F(\Sigma_0)) \end{array}$$

$$\alpha_{\Sigma_0}(\sigma_{\Sigma_0}(\vec{\varphi})) = \sigma'_{F(\Sigma_0)}\big(\alpha^n_{\Sigma_0}(\vec{\varphi})\big).$$

An (N, N')-admissible function $\alpha_{\Sigma_0} : \mathrm{SEN}(\Sigma_0) \to \mathrm{SEN}'(F(\Sigma_0))$ is said to be (N, N')-**extendable** if it can be extended to an (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$.

The (N, N')-extendability of functions of the form $\alpha_{\Sigma_0} : \mathrm{SEN}(\Sigma_0) \to \mathrm{SEN}'(F(\Sigma_0))$ seems to be a necessary assumption for some of our subsequent work in this section since, in the theory of π-institutions, a crucial role is played by (N, N')-translations and there is no a priori guarantee that a conveniently defined function $\alpha_{\Sigma_0} : \mathrm{SEN}(\Sigma_0) \to \mathrm{SEN}'(F(\Sigma_0))$, for a given fixed $\Sigma_0 \in |\mathbf{Sign}|$, should be (N, N')-admissible and, if it is, that it should be (N, N')-extendable.

As an example consider the discrete category $\mathbf{C}$ with set of objects $|\mathbf{C}| = \{\{0, 1\}, \{0, a, 1\}\}$ and the discrete category $\mathbf{D}$ with object $\{0, 1\}$. Assume that the sentence functors $C : \mathbf{C} \to \mathbf{Set}$ and $D : \mathbf{D} \to \mathbf{Set}$ are defined as the identities (the reason for presenting $\mathbf{C}$ and $\mathbf{D}$ as subcategories of $\mathbf{Set}$). Moreover, assume that N_C and N_D are the mono-unary categories of natural transformations on C, D, respectively, generated by $\neg : C \to C$ and $\sim: D \to D$, that are defined by stipulating that $\neg_{\{0,1\}}, \sim_{\{0,1\}} : \{0, 1\} \to \{0, 1\}$ is the Boolean negation and $\neg_{\{0,a,1\}} : \{0, a, 1\} \to \{0, a, 1\}$ maps $0, a$ and 1 to $1, a$ and 0, respectively.

Clearly, the mapping $\alpha_{\{0,a,1\}} : \{0, a, 1\} \to \{0, 1\}$, given by $0 \mapsto 0, a \mapsto 0, 1 \mapsto 1$ is not (N_C, N_D)-admissible. In fact, there exists no (N_C, N_D)-admissible mapping $\alpha_{\{0,a,1\}} : \{0, a, 1\} \to \{0, 1\}$. It just suffices to observe that this would require that

$$\alpha_{\{0,a,1\}}(a) =\sim_{\{0,1\}} \big(\alpha_{\{0,a,1\}}(a)\big),$$

which is impossible, since $\sim_{\{0,1\}}$ was defined as Boolean negation on $\{0, 1\}$. On the other hand, the mapping $\alpha_{\{0,1\}} : \{0, 1\} \to \{0, 1\}$ defined as the identity mapping on the set $\{0, 1\}$ is clearly (N_C, N_D)-admissible. It is not, however, (N_C, N_D)-extendable, since, as was just observed, there is no way of extending it to an (N_C, N_D)-epimorphic translation $\langle F, \alpha\rangle : C \to D$.

Lemma 7.3 *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a set-valued functor, with* N *a mono-unary category of natural transformations on* SEN*, generated by* $\mu : \mathrm{SEN} \to \mathrm{SEN}$. *Let, also* SEN' *and* N' *be as in Lemma 7.2 and* $\star : \mathbf{Sign} \to \bigstar$ *be the constant functor, such that* $\star(\Sigma) = \bigstar$*, for all* $\Sigma \in |\mathbf{Sign}|$. *Suppose that there exist* $\Sigma_0 \in |\mathbf{Sign}|$, $\chi \in \mathrm{SEN}(\Sigma_0)$*, such that, if* $\mathcal{X} = \{\mu^k_{\Sigma_0}(\chi) : k \in \omega\}$*, then*

- $\mu^k_{\Sigma_0}(\chi) \neq \mu^l_{\Sigma_0}(\chi)$*, for all* $k, l \in \omega$*, with* $k \neq l$;
- *if* $\mu_{\Sigma_0}(\varphi) \in \mathcal{X}$*, then* $\varphi \in \mathcal{X}$*, for all* $\varphi \in \mathrm{SEN}(\Sigma_0)$.

Define $\alpha_{\Sigma_0} : \mathrm{SEN}(\Sigma_0) \to \mathrm{SEN}'(\bigstar)$ *by setting*

- $\alpha_{\Sigma_0}(\psi) = k$*, if* $\psi = \mu^k_{\Sigma_0}(\chi)$*, for some* $k \leq n$ *and* $\alpha_{\Sigma_0}(\psi) = n + 1$*, if* $\psi = \mu^k_{\Sigma_0}(\chi)$*, for some* $k > n$;
- $\alpha_{\Sigma_0}(\psi) = n + 1$*, if* $\psi \neq \mu^k_{\Sigma_0}(\chi)$*, for all* $k \in \omega$.

Then α_{Σ_0} *is well-defined and* (N, N')*-admissible.*

P r o o f. That α_{Σ_0} is well-defined follows from the postulated conditions on $\chi \in \mathrm{SEN}(\Sigma_0)$. For the admissibility condition, it suffices to show that, for all $\varphi \in \mathrm{SEN}(\Sigma_0)$, $\alpha_{\Sigma_0}(\mu_{\Sigma_0}(\varphi)) = \mu'_\bigstar(\alpha_{\Sigma_0}(\varphi))$. We distinguish three cases:

(1) If $\varphi = \mu^k_{\Sigma_0}(\chi)$, for some $k \leq n - 1$, we get

$$\alpha_{\Sigma_0}\big(\mu_{\Sigma_0}\big(\mu^k_{\Sigma_0}(\chi)\big)\big) = \alpha_{\Sigma_0}\big(\mu^{k+1}_{\Sigma_0}(\chi)\big) = k + 1 = \mu'_\bigstar(k) = \mu'_\bigstar\big(\alpha_{\Sigma_0}\big(\mu^k_{\Sigma_0}(\chi)\big)\big).$$

(2) If $\varphi = \mu^k_{\Sigma_0}(\chi)$, for some $k \geq n$, then $\alpha_{\Sigma_0}\left(\mu_{\Sigma_0}\left(\mu^k_{\Sigma_0}(\chi)\right)\right) = \alpha_{\Sigma_0}\left(\mu^{k+1}_{\Sigma_0}(\chi)\right) = n+1 = \left\{\begin{array}{l}\mu'_\bigstar(n+1)\\ \mu'_\bigstar(n)\end{array}\right\} = \mu'_\bigstar(\alpha_{\Sigma_0}(\mu^k_{\Sigma_0}(\chi)))$.

(3) The case in which $\varphi \neq \mu^k_{\Sigma_0}(\chi)$, for all $k \in \omega$, may be handled similarly and depends crucially on the postulated conditions on $\chi \in \mathrm{SEN}(\Sigma_0)$. □

The (N, N')-admissible mappings of the form of Lemma 7.3 will be called $\langle\Sigma_0, \chi\rangle$*-generated*.

Theorem 7.4 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a mono-unary category of natural transformations on* SEN*, generated by $\mu : \mathrm{SEN} \to \mathrm{SEN}$.*

(1) *If $\mathcal{I}$ has an algebraic semantics then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\mu_\Sigma(\varphi) \in C_\Sigma(\varphi)$.*

(2) *If $\mathcal{I}$ is N-rule-based, such that $\langle\{\iota\}, \mu\rangle$ is an N-rule of $\mathcal{I}$ and, for all $\Sigma_0 \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma_0)$, there exists $\chi \in \mathrm{SEN}(\Sigma_0)$, with $\varphi \in \mathcal{X} := \left\{\mu^k_{\Sigma_0}(\chi) : k \in \omega\right\}$, such that*
   - *$\mu^k_{\Sigma_0}(\chi) \neq \mu^l_{\Sigma_0}(\chi)$, for all $k, l \in \omega$, with $k \neq l$;*
   - *if $\mu_{\Sigma_0}(\psi) \in \mathcal{X}$, then $\psi \in \mathcal{X}$, for all $\psi \in \mathrm{SEN}(\Sigma_0)$;*
   - *the $\langle\Sigma_0, \chi\rangle$-generated mapping is (N, N')-extendable*;

then $\mathcal{I}$ has an $\iota \approx \mu$-algebraic semantics.

Proof.

1. Suppose that $\mathfrak{F}$ is an N-algebraic semantics of $\mathcal{I}$ with defining equations $\sigma \approx \tau = \{\sigma^i \approx \tau^i : i < n\}$. Since N is mono-unary, there exist $m_i, n_i < \omega$, such that $\sigma^i = \mu^{m_i}$ and $\tau^i = \mu^{n_i}$. Thus, we get that $\sigma \circ \mu = \mu \circ \sigma$ and, similarly, $\tau \circ \mu = \mu \circ \tau$. Hence, the equation $\sigma \circ \mu \approx \tau \circ \mu$ coincides with $\mu \circ \sigma \approx \mu \circ \tau$. But, then, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, $\sigma_\Sigma(\mu_\Sigma(\varphi)) \approx \tau_\Sigma(\mu_\Sigma(\varphi)) \subseteq C^{\mathfrak{F}}_\Sigma(\sigma_\Sigma(\varphi) \approx \tau_\Sigma(\varphi))$, we get, by the hypothesis, that $\mu_\Sigma(\varphi) \in C_\Sigma(\varphi)$.
2. By Theorem 6.4, it suffices to show that the π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, whose only N-rule of inference is $\langle\{\iota\}, \mu\rangle$, has an $\iota \approx \mu$-algebraic semantics.
   Let $\mathfrak{F}$ consist of all N-algebraic systems $\left\langle \mathrm{SEN}', \langle F, \alpha\rangle\right\rangle$ and $\sigma \approx \tau$ be $\{\iota \approx \mu\}$. Clearly, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, $\mu_\Sigma(\varphi) \approx \mu_\Sigma(\mu_\Sigma(\varphi)) \in C^{\mathfrak{F}}_\Sigma(\varphi \approx \mu_\Sigma(\varphi))$. Therefore $C \leq C^{\mathfrak{F}, \sigma\approx\tau}$. Thus, it suffices to show that $C^{\mathfrak{F}, \sigma\approx\tau} \leq C$, or, by contraposition, that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \notin C_\Sigma(\Phi)$, there exists $A = \left\langle \mathrm{SEN}', \langle F, \alpha\rangle\right\rangle \in \mathfrak{F}$, such that $\varphi \approx \mu_\Sigma(\varphi) \notin C^A_\Sigma([\iota \approx \mu]_\Sigma(\Phi))$.
   We use virtually the same construction as in the proof of [3, Theorem 2.20] but suitably lifted to a functorial context. This is the point where the additional technical conditions of the hypothesis come into play. Suppose $\Sigma \in |\mathbf{Sign}|$, $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \notin C_\Sigma(\Phi)$. By hypothesis, there exists $\chi \in \mathrm{SEN}(\Sigma)$, satisfying the itemized conditions in the statement of the theorem, and $n \in \omega$, such that $\varphi = \mu^n_\Sigma(\chi)$. Then, by Lemma 7.1, for all $i \leq n$, we have that $\mu^i_\Sigma(\chi) \notin \Phi$. Taking into account Lemma 7.2, we may consider the functor $\mathrm{SEN}' : \bigstar \to \mathbf{Set}$, such that $\mathrm{SEN}'(\bigstar) = \{0, 1, \ldots, n+1\}$, and the mono-unary category N' of natural transformations on SEN' generated by the natural transformation $\mu' : \mathrm{SEN}' \to \mathrm{SEN}'$ of Definition (6). By the hypothesis and Lemma 7.3, there exists an (N, N')-epimorphic translation $\langle\star, \alpha\rangle : \mathrm{SEN} \to \mathrm{SEN}'$, such that $\alpha_\Sigma(\chi) = 0$ and $\alpha_\Sigma(\psi) = n+1$, for all $\psi \neq \mu^k_\Sigma(\chi)$, for any $k = 0, 1, \ldots, n-1$. Set $A = \left\langle \mathrm{SEN}', \langle\star, \alpha\rangle\right\rangle$. Then, for every $\psi \in \Phi$, $\alpha_\Sigma(\mu_\Sigma(\psi)) = \alpha_\Sigma(\psi) = n+1$, whereas $\alpha_\Sigma(\varphi) = n \neq n+1 = \alpha_\Sigma(\mu_\Sigma(\varphi))$. This shows that $\varphi \approx \mu_\Sigma(\varphi) \notin C^A_\Sigma([\iota \approx \mu]_\Sigma(\Phi))$. □

As an application to Theorem 7.4, consider the discrete category $\mathbf{Sign}$ with set of objects $|\mathbf{Sign}| = \omega = \{0, 1, 2, \ldots\}$. Define the functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ by

$$\mathrm{SEN}(n) = n \times \omega = \{(i, m) : i < n, m \in \omega\},$$

for all $n \in \omega$, i.e., $\mathrm{SEN}(n)$ consists of n countable sequences $(i, 0), (i, 1), \ldots$, $i < n$. For all $n \in \omega$ and all $X \subseteq \mathrm{SEN}(n)$, define

$$C_n(X) = \uparrow X := \{(i, m) \in n \times \omega : (\exists x \in \omega)((i, x) \in X \wedge x \leq m)\},$$

i.e., $C_n(X)$ is the upset generated by X, if one perceives of SEN(n) as the disjoint union of n linear orders $(i,0),(i,1),\ldots$, $i<n$. Next, consider N, the mono-unary category of natural transformations on SEN generated by μ : SEN $\to$ SEN defined, for all $n\in\omega$, by

$$\mu_n((i,m)) = (i,m+1), \quad \text{for all} \quad (i,m)\in \mathrm{SEN}(n).$$

Observe that $\mu_n((i,m)) \in C_n((i,m))$, for all $n\in\omega$ and all $(i,m)\in \mathrm{SEN}(n)$, which, by Part 1 of Theorem 7.4, is necessary for $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ to possess an algebraic semantics. Moreover, $\mathcal{I}$ is N-rule based, $\langle\{\iota\},\mu\rangle$ is an N-rule of $\mathcal{I}$ and, given $n\in\omega$, $(i,m)\in \mathrm{SEN}(n)$, the element $(i,0)\in \mathrm{SEN}(n)$ is such that $(i,m)\in \mathcal{X} := \{\mu_n^k((i,0)) : k\in\omega\}$ and this $\mathcal{X}$ satisfies all the conditions listed in Part 2 of Theorem 7.4. Therefore, by Theorem 7.4, $\mathcal{I}$ does have an $\iota\approx\mu$-algebraic semantics.

In Theorem 7.4, sufficient conditions on the π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with a mono-unary category of natural transformations N on SEN, generated by μ : SEN $\to$ SEN, were given that guarantee the existence of an algebraic semantics. An *open problem* is to find necessary and sufficient conditions on SEN so that $\mathcal{I}$ has an algebraic semantics if and only if it is N-rule based, with $\langle\{\iota\},\mu\rangle$ an N-rule of inference.

8 A Necessary Condition

Theorem 8.1, that follows, is an analog in the present setting of [3, Theorem 2.16]. It provides a property that will be used to establish the main necessary condition for the existence of a $\sigma\approx\tau$-algebraic semantics in Proposition 8.2. Roughly speaking, it says that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and a $\sigma\approx\tau$-algebraic semantics $\mathfrak{F}$, for all $\xi : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, $f\in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in \mathrm{SEN}(\Sigma')^{n-1}$, the two sentences

$$\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right) \quad \text{and} \quad \xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\vec{\chi}\right) \tag{7}$$

are interderivable in $\mathcal{I}$ modulo SEN$(f)(\varphi)$, for every $\varphi\in \mathrm{SEN}(\Sigma)$.

Note that in (7), we have followed a common convention in categorical abstract algebraic logic by which the expressions in (7) are shorthands for the more cumbersome expressions: for all $\vec{\chi}\in \mathrm{SEN}(\Sigma')^n$ and all $j<n$,

$$\begin{aligned}&\xi_{\Sigma'}\left(\chi_0,\ldots,\chi_{j-1},\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\chi_{j+1},\ldots,\chi_{n-1}\right)\\ &\quad\text{and}\quad \xi_{\Sigma'}\left(\chi_0,\ldots,\chi_{j-1},\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\chi_{j+1},\ldots,\chi_{n-1}\right).\end{aligned}$$

Thus, SEN$(f)\left(\sigma^i_\Sigma(\varphi)\right)$ and SEN$(f)\left(\tau^i_\Sigma(\varphi)\right)$ may appear as arguments in any position of $\xi_{\Sigma'}$ and not just the first, as long as they both appear in the same position.

Theorem 8.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, having a $\sigma\approx\tau$-algebraic semantics $\mathfrak{F}$, with $\sigma\approx\tau = \{\sigma^i\approx\tau^i : i\in I\}$. Then, for all $\Sigma\in|\mathbf{Sign}|$ and all $\varphi\in \mathrm{SEN}(\Sigma)$, we have that, for all $\xi : \mathrm{SEN}^n\to \mathrm{SEN}$ in N, all $\Sigma'\in|\mathbf{Sign}|$, $f\in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in \mathrm{SEN}(\Sigma')^{n-1}$,*

$$\begin{aligned}&\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\vec{\chi}\right) \in C_{\Sigma'}\left(\left\{\mathrm{SEN}(f)(\varphi),\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right\}\right)\\ &\quad\text{and}\quad \xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right) \in C_{\Sigma'}\left(\left\{\mathrm{SEN}(f)(\varphi),\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right\}\right).\end{aligned} \tag{8}$$

Proof. Suppose that $\mathfrak{F}$ is a $\sigma\approx\tau$-algebraic semantics of $\mathcal{I}$. Then, we have, for all $\xi : \mathrm{SEN}^n\to \mathrm{SEN}$ in N, all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, $f\in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in \mathrm{SEN}(\Sigma')^{n-1}$,

$$\begin{aligned}&\sigma_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right) \approx \tau_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right) \subseteq\\ &\quad C^{\mathfrak{F}}_{\Sigma'}(\mathrm{SEN}(f)(\sigma_\Sigma(\varphi)) \approx \mathrm{SEN}(f)(\tau_\Sigma(\varphi))\cup\\ &\quad \sigma_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right) \approx \tau_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right)).\end{aligned}$$

Then, for all $\xi : \mathrm{SEN}^n\to \mathrm{SEN}$ in N, all $\Sigma,\Sigma'\in|\mathbf{Sign}|$, $f\in \mathbf{Sign}(\Sigma,\Sigma')$ and all $\vec{\chi}\in \mathrm{SEN}(\Sigma')^{n-1}$,

$$\begin{aligned}&\sigma_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)(\tau^i_\Sigma(\varphi)),\vec{\chi}\right)\right) \approx \tau_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right) \subseteq\\ &\quad C^{\mathfrak{F}}_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \approx \tau_{\Sigma'}(\mathrm{SEN}(f)(\varphi))\cup\\ &\quad \sigma_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right) \approx \tau_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right),\vec{\chi}\right)\right)).\end{aligned}$$

Thus, by the hypothesis, since $\mathfrak{F}$ is a $\sigma \approx \tau$-algebraic semantics of $\mathcal{I}$, we obtain that for all $\xi : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$\xi_{\Sigma'}\big(\mathrm{SEN}(f)\big(\tau^i_\Sigma(\varphi)\big), \vec{\chi}\big) \in C_{\Sigma'}\big(\big\{\mathrm{SEN}(f)(\varphi), \xi_{\Sigma'}\big(\mathrm{SEN}(f)\big(\sigma^i_\Sigma(\varphi)\big), \vec{\chi}\big)\big\}\big).$$

The second of the conditions in (8) follows by symmetry. □

The main result of the subsection, a necessary condition for the existence of a $\sigma \approx \tau$-algebraic semantics, is a partial analog of [3, Proposition 2.17]. It says that the two conditions in (8) imply that $\big\langle \sigma^i_\Sigma(\varphi), \tau^i_\Sigma(\varphi)\big\rangle \in \Omega^N_\Sigma(T)$, for all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}|, \varphi \in \mathrm{SEN}(\Sigma)$, such that $\mathrm{SEN}(f)(\varphi) \in T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$.

Proposition 8.2 *If a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with a category N of natural transformations on* SEN*, satisfies Conditions* (8)*, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi \in \mathrm{SEN}(\Sigma)$, all $\xi : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$, then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, such that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\varphi) \in T_{\Sigma'}$, $\sigma_\Sigma(\varphi) \approx \tau_\Sigma(\varphi) \subseteq \Omega^N_\Sigma(T)$.*

Proof. Suppose that $T \in \mathrm{ThFam}(\mathcal{I})$ and $\Sigma \in |\mathbf{Sign}|, \varphi \in \mathrm{SEN}(\Sigma)$, such that $\mathrm{SEN}(f)(\varphi) \in T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$. Let $\xi : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$, such that $\xi_{\Sigma'}(\mathrm{SEN}(f)(\sigma^i_\Sigma(\varphi)), \vec{\chi}) \in T_{\Sigma'}$. Then, we have

$$\begin{aligned}
\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right), \vec{\chi}\right) &\in C_{\Sigma'}\left(\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right), \vec{\chi}\right), \mathrm{SEN}(f)(\varphi)\right)\\
&\subseteq C_{\Sigma'}(T_{\Sigma'})\\
&= T_{\Sigma'}.
\end{aligned}$$

Hence, by symmetry, we obtain that

$$\xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\sigma^i_\Sigma(\varphi)\right), \vec{\chi}\right) \in T_{\Sigma'} \quad \text{iff} \quad \xi_{\Sigma'}\left(\mathrm{SEN}(f)\left(\tau^i_\Sigma(\varphi)\right), \vec{\chi}\right) \in T_{\Sigma'}.$$

By [15, Proposition 2.3], we get $\big\langle \sigma^i_\Sigma(\varphi), \tau^i_\Sigma(\varphi)\big\rangle \in \Omega^N_\Sigma(T)$. □

Theorem 8.1 and Proposition 8.2 yield immediately the following necessary condition for the existence of a $\sigma \approx \tau$-algebraic semantics for a π-institution $\mathcal{I}$.

Corollary 8.3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, having a $\sigma \approx \tau$-algebraic semantics $\mathfrak{F}$. Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$, such that for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\mathrm{SEN}(f)(\varphi) \in T_{\Sigma'}$, $\sigma_\Sigma(\varphi) \approx \tau_\Sigma(\varphi) \subseteq \Omega^N_\Sigma(T)$.*

In [3], Blok and Rebagliato use the special case of Corollary 8.3, applicable to deductive systems, to show that the deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S}\rangle$ whose language $\mathcal{L}$ consists of one binary connective $\to$ and which is axiomatically defined by the single axiom $p \to p$ and has Modus Ponens as its only rule of inference does not have an algebraic semantics despite the fact that it is protoalgebraic with $\{p \to q\}$ as its system of implication formulas. (Cf. [3, Theorem 2.19] for the details.)

9 A Sufficient Condition

In this final section, our aim is to prove a sufficient condition for the existence of an algebraic semantics for a given π-institution. The intended condition forms an analog in the π-institution framework of the one established in [3, Theorem 3.3].

We first formulate an analog of [3, Proposition 1.2], which states, roughly speaking, that given two logical matrices and a surjective homomorphism between them that preserves the filters, the two logical entailments induced on the formula algebra by each of the two logical matrices coincide. [3, Proposition 1.2] is used to prove a key lemma, [3, Lemma 3.2], that is, in turn, used to prove [3, Theorem 3.3]. Similarly, in the present context, Proposition 9.1 will be used to prove a key Lemma 9.2, which will then be used to prove Theorem 9.3, the analog of [3, Theorem 3.3].

Proposition 9.1 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN. *Let $M' = \langle\langle \mathrm{SEN}', \langle F', \alpha'\rangle\rangle, T'\rangle$ and $M'' = \langle\langle \mathrm{SEN}'', \langle F'', \alpha''\rangle\rangle, T''\rangle$ be N-matrix systems for* SEN *and $\langle G, \beta\rangle : \mathrm{SEN}' \to \mathrm{SEN}''$ a surjective (N', N'')-epimorphic translation, such that the following triangle commutes*

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\langle F', \alpha'\rangle"'] \arrow[dr, "\langle F'', \alpha''\rangle"] & \\ \mathrm{SEN}' \arrow[rr, "\langle G, \beta\rangle"'] & & \mathrm{SEN}'' \end{tikzcd}$$

and $\beta^{-1}(T'') = T'$. Then $C^{M'} = C^{M''}$.

P r o o f. Let $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C^{M'}_\Sigma(\Phi)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F'(\Sigma')}$ implies $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F'(\Sigma')}$. Therefore, since $\beta^{-1}(T'') = T'$, we obtain that, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\beta_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq T''_{G(F'(\Sigma'))} \quad \text{implies} \quad \beta_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \in T''_{G(F'(\Sigma'))}.$$

By hypothesis, this yields that, for all $\Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T''_{F''(\Sigma')} \quad \text{implies} \quad \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T''_{F''(\Sigma')}.$$

Therefore, we obtain that $\varphi \in C^{M''}_\Sigma(\Phi)$. Thus, $C^{M'} \leq C^{M''}$.

For the sake of proving the converse, suppose that $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\varphi \in C^{M''}_\Sigma(\Phi)$. Then, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have that

$$\alpha''_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T''_{F''(\Sigma')} \quad \text{implies} \quad \alpha''_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T''_{F''(\Sigma')}.$$

Thus, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, we have that

$$\beta_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi))) \subseteq T''_{F''(\Sigma')} \quad \text{implies} \quad \beta_{F'(\Sigma')}(\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\varphi))) \in T''_{F''(\Sigma')}.$$

Therefore, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq \beta^{-1}_{F'(\Sigma')}\big(T''_{F''(\Sigma')}\big) \quad \text{implies} \quad \alpha'_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in \beta^{-1}_{F'(\Sigma')}\big(T''_{F''(\Sigma')}\big)$$

and, thus, by the hypothesis, $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\Phi)) \subseteq T'_{F'(\Sigma')}$ implies $\alpha'_{\Sigma'}(\mathrm{SEN}(f)(\varphi)) \in T'_{F'(\Sigma')}$. Hence $\varphi \in C^{M'}_\Sigma(\Phi)$. Thus, $C^{M''} \leq C^{M'}$, as was to be shown. □

Proposition 9.1 will now be used to prove Lemma 9.2, an analog of [3, Lemma 3.2] for π-institutions. Note the close similarities of parts of the hypothesis of Lemma 9.2 with the hypothesis of Proposition 9.1.

Lemma 9.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Suppose that $\mathfrak{M} = \{\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle : i \in I\}$ is an N-matrix system semantics for $\mathcal{I}$ and $\sigma \approx \tau$ an N-translation. Suppose, further, that, for all $i \in I$, there exist an N-matrix system $\langle\langle \mathrm{SEN}'^i, \langle F'^i, \alpha'^i\rangle\rangle, T'^i\rangle$ and a surjective (N'^i, N^i)-epimorphic translation $\langle G^i, \beta^i\rangle : \mathrm{SEN}'^i \to \mathrm{SEN}^i$, such that*

– *the following diagram commutes*

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\langle F'^i, \alpha'^i\rangle"'] \arrow[dr, "\langle F^i, \alpha^i\rangle"] & \\ \mathrm{SEN}'^i \arrow[rr, "\langle G^i, \beta^i\rangle"'] & & \mathrm{SEN}^i \end{tikzcd}$$

– $(\beta^i)^{-1}(T^i) = T'^i$ *and*

– $T'^i_\Sigma = \{\varphi \in \mathrm{SEN}'^i(\Sigma) : \sigma'^i_\Sigma(\varphi) = \tau'^i_\Sigma(\varphi)\}$, *for all* $\Sigma \in |\mathbf{Sign}'^i|$.

Set $\mathfrak{M}' = \{\langle\langle \mathrm{SEN}'^i, \langle F'^i, \alpha'^i\rangle\rangle, T'^i\rangle : i \in I\}$. *Then, the collection* $\mathfrak{F}' = \{\langle \mathrm{SEN}'^i, \langle F'^i, \alpha'^i\rangle\rangle : i \in I\rangle\}$ *is a* $\sigma \approx \tau$*-algebraic semantics for* $\mathcal{I}$.

Proof. By Proposition 9.1, we have $C^{\{\langle\langle \mathrm{SEN}'^i, \langle F'^i, \alpha'^i\rangle\rangle, T'^i\rangle\}} = C^{\{\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle\}}$, for all $i \in I$. Moreover, by hypothesis, $\mathfrak{M}$ is an N-matrix system semantics for $\mathcal{I}$. Therefore, $\mathfrak{M}'$ is also an N-matrix system semantics for $\mathcal{I}$. But, then, again by the hypothesis, using Theorem 4.3, we obtain that $\mathfrak{F}'$ is a $\sigma \approx \tau$-algebraic semantics for $\mathcal{I}$. □

The ground has now been laid for proving Theorem 9.3, the promised analog of [3, Theorem 3.3], providing a sufficient condition for the existence of a $\sigma \approx \tau$-algebraic semantics for a given π-institution $\mathcal{I}$. We note that in [3], two refinements of this condition have been provided in the framework of deductive systems. The interested reader should consult [3, Theorems 3.6 & 3.1]. Some applications of these and related results are also given in [3, Section 3].

Theorem 9.3 *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ *be a* π*-institution, with* N *a category of natural transformations on* SEN. *Assume that* $\mathfrak{M} = \{\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle : i \in I\}$ *is an* N*-matrix semantics of* $\mathcal{I}$*, such that, for all* $i \in I$*,* $\Sigma, \Sigma' \in |\mathbf{Sign}^i|$*,* $f \in \mathbf{Sign}^i(\Sigma, \Sigma')$ *and* $\varphi \in \mathrm{SEN}^i(\Sigma)$*,* $\varphi \notin T^i_\Sigma$ *implies* $\mathrm{SEN}^i(f)(\varphi) \notin T^i_{\Sigma'}$*. Let, also,* $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ *be a natural transformation in* N*. If, for all* $i \in I$*, all* $\Sigma \in |\mathbf{Sign}^i|$ *and all* $\varphi \in \mathrm{SEN}^i(\Sigma)$*,* $\sigma^i_\Sigma(\varphi, \ldots, \varphi) = \varphi$ *(*$\sigma^i : (\mathrm{SEN}^i)^n \to \mathrm{SEN}^i$ *the natural transformation on* SEN^i *corresponding to* σ*), then* $\mathcal{I}$ *has a* $\sigma(\vec{\iota}) \approx \iota$*-algebraic semantics, where* $\iota : \mathrm{SEN} \to \mathrm{SEN}$ *is the identity (projection) function and* $\vec{\iota} := \langle \iota, \iota, \ldots, \iota\rangle : \mathrm{SEN} \to \mathrm{SEN}^n$.

Proof. Assume that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on SEN, that $\mathfrak{M} = \{\langle\langle \mathrm{SEN}^i, \langle F^i, \alpha^i\rangle\rangle, T^i\rangle : i \in I\}$ is an N-matrix semantics of $\mathcal{I}$, satisfying the condition of the statement, and that $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ is a natural transformation in N, such that for all $i \in I$, all $\Sigma \in |\mathbf{Sign}^i|$ and all $\varphi \in \mathrm{SEN}^i(\Sigma)$, $\sigma^i_\Sigma(\varphi, \ldots, \varphi) = \varphi$.

Fix an $i \in I$. Let $\mathrm{SEN}'^i : \mathbf{Sign}^i \to \mathbf{Set}$ be defined as follows: For all $\Sigma \in |\mathbf{Sign}^i|$,

$$\mathrm{SEN}'^i(\Sigma) = \mathrm{SEN}^i(\Sigma) \cup \{\varphi' : \varphi \in \mathrm{SEN}^i(\Sigma) \backslash T^i_\Sigma\},$$

i.e., $\mathrm{SEN}'^i(\Sigma)$ consists of $\mathrm{SEN}^i(\Sigma)$ plus an additional copy of the complement in $\mathrm{SEN}^i(\Sigma)$ of T^i_Σ. We pay attention so that, if $\varphi, \psi \in \mathrm{SEN}^i(\Sigma) \backslash T^i_\Sigma$, we have $\varphi', \psi' \notin \mathrm{SEN}^i(\Sigma)$ and also that, if $\varphi \neq \psi$, then $\varphi' \neq \psi'$. For all $\Sigma, \Sigma' \in |\mathbf{Sign}^i|$, $f \in \mathbf{Sign}^i(\Sigma, \Sigma')$,

$$\mathrm{SEN}'^i(f)(\chi) = \begin{cases} \mathrm{SEN}^i(f)(\chi) & \text{if} \quad \chi \in \mathrm{SEN}^i(\Sigma), \\ \mathrm{SEN}^i(f)(\varphi)' & \text{if} \quad \chi = \varphi', \quad \text{for some} \quad \varphi \in \mathrm{SEN}^i(\Sigma) \backslash T^i_\Sigma. \end{cases}$$

Because of the hypothesis, this mapping is well-defined. Moreover, it is not difficult to verify that $\mathrm{SEN}'^i : \mathbf{Sign}^i \to \mathbf{Set}$, defined as above on objects and morphisms, is a functor.

For every $\tau^i : (\mathrm{SEN}^i)^k \to \mathrm{SEN}^i$ in N^i, with $\tau^i \neq \sigma^i$, define $\tau'^i : (\mathrm{SEN}'^i)^k \to \mathrm{SEN}'^i$, by setting, for all $\Sigma \in |\mathbf{Sign}^i|$, $\vec{\varphi}'' \in \mathrm{SEN}'^i(\Sigma)^k$,

$$\tau'^i_\Sigma(\vec{\varphi}'') = \tau^i_\Sigma(\vec{\varphi}),$$

where $\varphi_i = \varphi''_i$, if $\varphi''_i \in \mathrm{SEN}^i(\Sigma)$ and $\varphi_i = \psi_i$, if $\varphi''_i = \psi'_i$, for some $\psi_i \in \mathrm{SEN}^i(\Sigma) \backslash T^i_\Sigma$.

Finally, let $\sigma'^i : (\mathrm{SEN}'^i)^n \to \mathrm{SEN}'^i$ be given, for all $\vec{\varphi}'' \in \mathrm{SEN}'^i(\Sigma)^n$, by

$$\sigma'^i_\Sigma(\vec{\varphi}'') = \begin{cases} \psi' & \text{if} \quad \varphi''_j = \psi \in \mathrm{SEN}^i(\Sigma) \backslash T^i_\Sigma, \quad \text{for all} \quad j < n, \\ \sigma^i_\Sigma(\vec{\varphi}) & \text{otherwise,} \end{cases}$$

where $\vec{\varphi}$ is defined from $\vec{\varphi}''$ as in the previous case.

Each $\tau'^i : (\mathrm{SEN}'^i)^k \to \mathrm{SEN}'^i$, τ in N, is a natural transformation on SEN'^i. Consider the category N'^i of natural transformations that is generated by the collection of all natural transformations of the form τ'^i, with τ in N. Notice, first, that this category consists, in general, of more natural transformations than just those of the form σ'^i, for some σ in N. For instance, the projection natural transformations $p^{k,l}$ give rise to natural transformations $\left(p^{k,l}\right)'^i$, which may not be projection natural transformations in the category N'^i because of their postulated action on the newly introduced elements that are "copies" of "old" elements outside the matrix system filter. Notice, however, that the mapping $\tau^i \mapsto \tau'^i$, τ in N, preserves compositions, i.e., $\rho'^i \circ \vec{\tau}'^i = \left(\rho^i \circ \vec{\tau}^i\right)'$: To see this, we must consider various cases (using notation introduced above). The easiest is the case, when none of $\rho, \vec{\tau}$ is σ. In that case, for all $\Sigma \in |\mathbf{Sign}^i|$ and all $\vec{\varphi}'' \in \mathrm{SEN}'^i(\Sigma)^k$,

$$\rho'^i_\Sigma\left(\vec{\tau}'^i_\Sigma\left(\vec{\varphi}''\right)\right) = \rho'^i_\Sigma\left(\vec{\tau}^i_\Sigma\left(\vec{\varphi}\right)\right) = \rho^i_\Sigma\left(\vec{\tau}^i_\Sigma\left(\vec{\varphi}\right)\right) = \left(\rho^i_\Sigma \circ \vec{\tau}^i_\Sigma\right)\left(\vec{\varphi}\right) = \left(\rho^i_\Sigma \circ \vec{\tau}^i_\Sigma\right)'\left(\vec{\varphi}''\right).$$

Another (perhaps the most interesting) case is when ρ and all τ's are equal to σ and $\varphi''_j = \psi \in \mathrm{SEN}^i(\Sigma)\backslash T^i_\Sigma$, for all $j < n$. Then, we have

$$\sigma'^i_\Sigma\left(\vec{\sigma}'^i_\Sigma(\vec{\varphi}'')\right) = \sigma'^i_\Sigma(\psi', \ldots, \psi') = \sigma'^i_\Sigma(\psi, \ldots, \psi) = (\sigma^i_\Sigma)'(\vec{\psi}'') = \left(\sigma^i_\Sigma \circ \vec{\sigma}^i_\Sigma\right)'(\vec{\varphi}'').$$

With these observations at hand, let us now show that N'^i is a category of natural transformations similar to N (and, thus, also to N^i). Let $\bar{N}$ denote the free category generated by the natural transformations in N taken as formal names for arrows, the projections acting as transformation names on equal standing with other arrow names, i.e., the projections of the generated category $\bar{N}$ will be different than the arrows denoted by those labeled by the projections of N. It is clear that mapping projections to projections and generators of $\bar{N}$ to the corresponding transformations in N, we obtain a surjective functor from $\bar{N}$ onto N. On the other hand, notice that the mapping $\bar{N} \to N'^i$ defined as the identity on projections and as sending each τ in N to τ'^i is well-defined, since, for all $\rho, \vec{\tau}$,

$$\rho'^i \circ \vec{\tau}'^i = (\rho^i \circ \vec{\tau}^i)' = (\rho \circ \vec{\tau})'^i,$$

the last equality being valid from the hypothesis that N and N^i are similar under the postulated correspondence. Now, the similarity of N with N'^i follows from the fact that N'^i is generated by the images of this mapping, and, hence, this mapping is surjective.

Next, define, for all $i \in I$, $\langle F'^i, \alpha'^i\rangle : \mathrm{SEN} \to \mathrm{SEN}'^i$ by setting $F'^i = F^i$ and $\alpha'^i_\Sigma(\varphi) = \alpha^i_\Sigma(\varphi)$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$. Define, also, $\langle G^i, \beta^i\rangle : \mathrm{SEN}'^i \to \mathrm{SEN}^i$, by setting $G^i = \mathrm{I}_{\mathbf{Sign}^i}$, the identity functor on $\mathbf{Sign}^i$, and, for all $\Sigma \in |\mathbf{Sign}^i|, \psi \in \mathrm{SEN}'^i(\Sigma)$,

$$\beta^i_\Sigma(\psi) = \begin{cases} \psi & \text{if} \quad \psi \in \mathrm{SEN}^i(\Sigma), \\ \varphi & \text{if} \quad \psi = \varphi', \quad \text{for some} \quad \varphi \in \mathrm{SEN}^i(\Sigma)\backslash T^i_\Sigma \end{cases}$$

Thus defined, $\langle G^i, \beta^i\rangle : \mathrm{SEN}'^i \to \mathrm{SEN}^i$ is a surjective (N'^i, N^i)-epimorphic translation, such that $\beta^{-1}(T^i) = T^i$ and

$$T^i_\Sigma = \left\{\psi \in \mathrm{SEN}'^i(\Sigma) : \sigma'^i_\Sigma(\psi, \ldots, \psi) = \psi\right\}, \quad \text{for all} \quad \Sigma \in \left|\mathbf{Sign}^i\right|.$$

Indeed, if $\psi \in T^i_\Sigma$, then $\sigma'^i_\Sigma(\psi, \ldots, \psi) = \sigma^i_\Sigma(\beta_\Sigma(\psi), \ldots, \beta_\Sigma(\psi)) = \beta_\Sigma(\psi) = \psi$. On the other hand, if $\psi \notin T^i_\Sigma$, then either $\psi \in \mathrm{SEN}^i(\Sigma)$ or $\psi = \chi'$, for some $\chi \in \mathrm{SEN}^i(\Sigma)\backslash T^i_\Sigma$.

- If $\psi \in \mathrm{SEN}^i(\Sigma)$, then $\psi \in \mathrm{SEN}^i(\Sigma)\backslash T^i_\Sigma$, whence $\sigma^i_\Sigma(\psi, \ldots, \psi) = \psi' \notin \mathrm{SEN}^i(\Sigma)$ and, therefore, $\sigma'^i_\Sigma(\psi, \ldots, \psi) \neq \psi$.
- If $\psi = \chi'$, for some $\chi \in \mathrm{SEN}^i(\Sigma)$, then $\sigma'^i_\Sigma(\psi, \ldots, \psi) = \sigma'^i_\Sigma(\chi', \ldots, \chi') = \sigma^i_\Sigma(\chi, \ldots, \chi) = \chi \in \mathrm{SEN}^i(\Sigma)$, whence $\sigma'^i_\Sigma(\psi, \ldots, \psi) \neq \psi$.

Thus, all conditions in the hypothesis of Lemma 9.2 are satisfied and, therefore, $\mathfrak{F} = \left\{\left\langle \mathrm{SEN}'^i, \left\langle F'^i, \alpha'^i\right\rangle\right\rangle : i \in I\right\}$ is a $\sigma(\vec{\iota}) \approx \iota$-algebraic semantics of $\mathcal{I}$, as was to be shown. □

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