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CATEGORICAL ABSTRACT ALGEBRAIC LOGIC: SYNTACTICALLY ALGEBRAIZABLE π-INSTITUTIONS

A b s t r a c t. This paper has a two-fold purpose. On the one hand, it introduces the concept of a syntactically N-algebraizable π-institution, which generalizes in the context of categorical abstract algebraic logic the notion of an algebraizable logic of Blok and Pigozzi. On the other hand, it has the purpose of comparing this important notion with the weaker ones of an N-protoalgebraic and of a syntactically N-equivalential π-institution and with the stronger one of a regularly N-algebraizable π-institution. N-protoalgebraic π-institutions and syntactically N-equivalential π-institutions were previously introduced by the author and abstract in the categorical framework the protoalgebraic logics of Blok and Pigozzi and the equivalential logics of Prucnal and Wroński and of Czelakowski. Regularly N-algebraizable π-institutions are introduced in the present paper taking after work of Czelakowski

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and of Blok and Pigozzi in the sentential logic framework. On the way to defining syntactically N-algebraizable π-institutions, the important notion of an equational π-institution associated with a given quasivariety of N-algebraic systems is also introduced. It is based on the notion of an N-quasivariety imported recently from the theory of Universal Algebra to the categorical level by the author.

1. Introduction

The purpose of this paper is to introduce the concept of a syntactically N-algebraizable π-institution and to compare it to the weaker notions of an N-protoalgebraic π-institution [18] and of a syntactically N-equivalential π-institution [22] and to the stronger notion of a regularly N-algebraizable π-institution, which will also be introduced in the present work. The notion of a syntactically N-algebraizable π-institution adapts to the categorical framework the notion of an algebraizable logic of Blok and Pigozzi [3]. This faithful adaptation has only become possible very recently due to the advances in the theory of varieties and quasivarieties of N-algebraic systems [20, 21, 23, 24]. Those advances were based on the work of Pałasińska and Pigozzi on the theory of varieties and quasivarieties of partially ordered algebras (see [16]) in the framework of abstract algebraic logic.

In this introduction, some concepts and results from abstract algebraic logic that inspired the current developments on the categorical side will be reviewed and, then, an outline of the contents of the present paper will be provided.

Let $\mathcal{L}$ be a fixed algebraic language and Q a quasivariety of $\mathcal{L}$-algebras. The equational deductive system, or 2-deductive system in the terminology of k-deductive systems of [4], associated with the quasivariety Q is the deductive system $\mathcal{S}_{\mathrm{Q}} = \langle \mathcal{L}, \models_{\mathrm{Q}} \rangle$, whose entailment relation

$$\models_{\mathrm{Q}}: \mathcal{P}(\mathrm{Fm}^2_{\mathcal{L}}(V)) \rightarrow \mathrm{Fm}^2_{\mathcal{L}}(V)$$

is the equational entailment relation induced by the class Q of $\mathcal{L}$-algebras, i.e., it is defined, for all $E \cup \{\phi \approx \psi\} \subseteq \mathrm{Fm}^2_{\mathcal{L}}(V)$, by $E \models_{\mathrm{Q}} \phi \approx \psi$ if and only if, for all $\mathbf{A} \in \mathrm{Q}$ and all $\mathcal{L}$-homomorphisms $h : \mathbf{Fm}_{\mathcal{L}}(V) \rightarrow \mathbf{A}$, $h(\epsilon_0) =$

$h(\epsilon_1)$, for all $\epsilon_0 \approx \epsilon_1 \in E$, imply that $h(\phi) = h(\psi)$. The set of theories $\mathrm{Th}(\mathcal{S}_\mathrm{Q})$ of the 2-deductive system $\mathcal{S}_\mathrm{Q}$ coincides with the set $\mathrm{Co}_\mathrm{Q}(\mathbf{Fm}_\mathcal{L}(V))$ of Q-congruences on the formula algebra $\mathbf{Fm}_\mathcal{L}(V)$, i.e., those $\mathcal{L}$-congruences θ on $\mathbf{Fm}_\mathcal{L}(V)$, such that the quotient algebra $\mathbf{Fm}_\mathcal{L}(V)/\theta$ is in Q.

In the seminal monograph [3] Blok and Pigozzi made precise for the first time the notion of an algebraizable logic. Besides the central role that algebraizable logics have played by forming one of the best behaving classes in the Leibniz hierarchy of logics, this monograph is generally acknowledged to be the founding "manifesto" of the theory of abstract algebraic logic. In this subtheory of algebraic logic, previous efforts and case-specific methods and techniques employed for the algebraization of specific logics have been unified and brought under a common general framework. One of the most important concepts on which that of an algebraizable logic is based, is that of an interpretation of the entailment relation of a given deductive system into the equational entailment relation associated with a quasivariety of algebras. Let $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S} \rangle$ be a deductive system and Q a quasivariety of $\mathcal{L}$-algebras. A translation from $\mathcal{S}$ into $\mathcal{S}_\mathrm{Q} = \langle \mathcal{L}, \models_\mathrm{Q} \rangle$ is a set of $\mathcal{L}$-equations $K(x) \approx L(x) = \{\kappa_i(x) \approx \lambda_i(x) : i < n\}$ in one variable x. A translation of $\mathcal{S}$ into $\mathcal{S}_\mathrm{Q}$ is said to be an interpretation if, for all $\Phi \cup \{\phi\} \subseteq \mathrm{Fm}_\mathcal{L}(V)$,

$$\Phi \vdash_\mathcal{S} \phi \quad \text{if and only if} \quad K(\Phi) \approx L(\Phi) \models_\mathrm{Q} K(\phi) \approx L(\phi).$$

Similarly, a translation from $\mathcal{S}_\mathrm{Q}$ into $\mathcal{S}$ is a collection $E = \{\epsilon_j(x, y) : j < m\}$ of $\mathcal{L}$-formulas in two variables x, y. Such a translation is called an interpretation if, for all $\Gamma \approx \Delta \cup \{\phi \approx \psi\} \subseteq \mathrm{Fm}^2_\mathcal{L}(V)$,

$$\Gamma \approx \Delta \models_\mathrm{Q} \phi \approx \psi \quad \text{if and only if} \quad E(\Gamma, \Delta) \vdash_\mathcal{S} E(\phi, \psi).$$

Moreover, the two interpretations $K \approx L$ and E are said to be inverses of one another if and only if, for all $\phi, \psi \in \mathrm{Fm}_\mathcal{L}(V)$,

$$\phi \;{}_\mathcal{S}\!\dashv\vdash_\mathcal{S} E(K(\phi), L(\phi)) \quad \text{and} \quad \phi \approx \psi \;{}_\mathrm{Q}\!=\!\mid\models_\mathrm{Q} K(E(\phi, \psi)) \approx L(E(\phi, \psi)).$$

A deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_\mathcal{S} \rangle$ is said to be algebraizable (in the sense of Blok and Pigozzi) if there exists an invertible interpretation $K \approx L$ from $\mathcal{S}$ into $\mathcal{S}_\mathrm{Q} = \langle \mathcal{L}, \models_\mathrm{Q} \rangle$, for some quasivariety Q of $\mathcal{L}$-algebras. In that case, Q is called an equivalent quasivariety of $\mathcal{S}$, the equations $K \approx L$ are called the defining equations and the formulas E the equivalence formulas of the equivalence witnessing the algebraizability of $\mathcal{S}$. Blok and

Pigozzi showed that, if a deductive system $\mathcal{S}$ is algebraizable, then the quasivariety Q is uniquely determined and they provided a list of very important characterizations of algebraizability, that inspired a large amount of the subsequent research in the field of abstract algebraic logic. Sections 4 and 5 of the present work will be devoted in revisiting the work of [3] and abstracting it to the categorical level to cover logical systems formalized as π-institutions.

One of the intrinsic characterizations of algebraizability provided in Theorem 4.7 of [3] states that a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ is algebraizable if and only if there exist a system E of formulas in two variables and a system $K \approx L$ of equations in a single variable, such that the following conditions hold, for all $\phi, \psi, \chi \in \mathrm{Fm}_{\mathcal{L}}(V)$, $\omega \in \mathcal{L}$, n-ary, and $\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1} \in \mathrm{Fm}_{\mathcal{L}}(V)$:

1. $\vdash_{\mathcal{S}} E(\phi, \phi)$;
2. $E(\phi, \psi) \vdash_{\mathcal{S}} E(\psi, \phi)$;
3. $E(\phi, \psi), E(\psi, \chi) \vdash_{\mathcal{S}} E(\phi, \chi)$;
4. $E(\phi_0, \psi_0), \ldots, E(\phi_{n-1}, \psi_{n-1}) \vdash_{\mathcal{S}} E(\omega(\phi_0, \ldots, \phi_{n-1}), \omega(\psi_0, \ldots, \psi_{n-1}))$;
5. $\phi \;{}_{\mathcal{S}}\!\dashv\vdash_{\mathcal{S}} E(K(\phi), L(\phi))$.

The existence of a system E of formulas in two variables satisfying the first four conditions in this characterization defines the class of equivalential logics, that were first introduced by Prucnal and Wroński [17] and subsequently studied in detail by Czelakowski [7, 8]. In turn, a superclass of the class of equivalential logics consists of the deductive systems $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, that are such that there exists a system E of formulas in two variables, satisfying, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$, the first condition above together with

6. $\phi, E(\phi, \psi) \vdash_{\mathcal{S}} \psi$ (modus ponens)

These logics are called protoalgebraic and were introduced by Blok and Pigozzi in [2]. Classes corresponding to the equivalential deductive systems and to the protoalgebraic deductive systems have already been introduced at the categorical level in [22] and [18], respectively, and the π-institutions belonging to these two classes are called syntactically N-equivalential and N-protoalgebraic, respectively. These two classes of π-institutions will be revisited and comparisons with the class of syntactically N-algebraizable π-institutions will be made in Section 6 of the paper.

Finally, in Corollary 4.8 of [3], Blok and Pigozzi provide a sufficient condition for the algebraizability of a deductive system $\mathcal{S}$. According to this corollary, if there exists a system E of formulas in two variables, such that a deductive system $\mathcal{S}$ satisfies the first four conditions in the characterization above and, also, Condition 6 and, for all $\phi, \psi \in \mathrm{Fm}_{\mathcal{L}}(V)$,

7. $\phi, \psi \vdash_{\mathcal{S}} E(\phi, \psi)$ (G-rule)

then $\mathcal{S}$ is algebraizable with set of equivalence formulas E and set of defining equations $x \approx E(x, x)$. Deductive systems satisfying Conditions (1)-(4),(6) and (7) are called regularly algebraizable and will be at the focus of our investigations in Sections 7 and 8 of the present work.

We turn, now, to an overview of the contents of the present paper.

In Section 2, the reader is reminded of some general notions and results from previous work in categorical abstract algebraic logic that will be useful in better understanding the theory developed in the following sections.

Section 3 starts the main treatment with the definition of the N-free equational π-institution $\mathcal{I}^{N-\mathrm{FEQ}}$ corresponding to a given sentence functor $\mathrm{SEN} : \mathbf{Sign} \rightarrow \mathbf{Set}$, with a category N of natural transformations on SEN. This π-institution captures the essence in the categorical framework of the deductive system that Czelakowski and Pigozzi in [10] have called the *free equational logic*. It is formulated in both a closure system and a proof-theoretic form. After its introduction, it is shown that its theory families exactly coincide with the N-congruence families on the functor SEN in the ordinary sense of categorical abstract algebraic logic.

The study of $\mathcal{I}^{N-\mathrm{FEQ}}$ is followed by the introduction of the π-institution $\mathcal{I}^{\mathtt{Q}}$ corresponding to a given quasivariety $\mathtt{Q}$ of N-algebraic systems. Quasivarieties of N-algebraic systems were introduced in [24]. The notion of a regular or surjectively generated quasivariety is introduced here formally for the first time, but was implicit in the development of [25]. Given an N-quasivariety $\mathtt{Q}$, the π-institution $\mathcal{I}^{\mathtt{Q}}$ corresponds to the notion of an equational deductive system $\mathcal{S}_{\mathrm{Q}}$, associated with a given universal algebraic quasivariety Q, which is termed an *applied equational logic* in [10]. It is shown that the theory families of $\mathcal{I}^{\mathtt{Q}}$ coincide with the N-congruence families $\theta = \{\theta_{\Sigma}\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN, that are such that $\mathrm{SEN}(\Sigma)/\theta_{\Sigma}$ satisfies all N-quasi-identities of $\mathtt{Q}$, for every $\Sigma \in |\mathbf{Sign}|$.

Finally, in the concluding result of Section 3, it is proven that, given a functor SEN, with N a category of natural transformations on SEN, such that SEN^2 is N-rule-based, and a closure system $C \geq C^{N-\mathrm{FEQ}}$ on SEN^2 there always exists an N-quasivariety $\mathtt{Q}$ such that $C = C^{\mathtt{Q}}$ as long as C is finitary.

Section 4 starts with the definition of syntactically N-algebraizable π-institutions. These are the finitary π-institutions whose consequence relations are interpretable into the consequence relations of N-quasivarieties via invertible interpretations. The definition follows very closely the original definition of an algebraizable deductive system of Blok and Pigozzi [3] and captures its essence in the π-institution framework. Section 4 closes with an analog in the categorical framework of Theorem 13 of [11], an improvement of Theorem 2.17 of [3], that provides an axiomatization by N-identities and N-quasi-identities of the equivalent N-quasivariety of a syntactically N-algebraizable π-institution $\mathcal{I}$, based on a given axiomatization of the finitary consequence relation of $\mathcal{I}$ via N-axioms and N-rules of inference.

Section 5 takes after the work of Blok and Pigozzi [3] and has the purpose of showing that, if a π-institution is syntactically N-algebraizable, then its equivalent N-quasivariety is unique. Blok and Pigozzi showed the corresponding result for the deductive systems that they called algebraizable and are now known as the finitary finitely algebraizable deductive systems.

Section 6, on the other hand, is mostly a review section. It connects the notion of syntactic N-algebraizability with those of N-protoalgebraicity and of syntactic N-equivalentiality, that were introduced previously by the author in [18] and [22], respectively. The goal here is to remind the reader of the definitions and to reveal some of the connections between the three notions.

Section 7 deals with regularly algebraizable π-institutions. It starts with the definition of an analog of the well-known G-rule (see rule 7. above) in the context of π-institutions. A π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, has the N-G-rule relative to an N-equivalence system E if, for all $\Sigma \in |\mathbf{Sign}|$, all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, and every theory family T of $\mathcal{I}$, $\phi, \psi \in T_\Sigma$ implies that $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$. The additional complexity in this definition of the G-rule, as compared to the simpler definition in the sentential context, stems from the existence of the signature-changing morphisms in the π-institution framework. However, it is easily seen that, in this framework as well, a π-institution $\mathcal{I}$ has the N-G-rule if and only if, any two Σ-sentences that belong to the Σ-component of a theory family T of $\mathcal{I}$ are indistinguishable modulo the Σ-component of the Leibniz N-congruence system $\Omega^N(T)$ corresponding to the theory family T. In another characterization of the N-G-rule, also inspired by a sentential analog, it is shown that in a syntactically N-equivalential π-institution with an N-equivalence system E, the G-rule holds if and only if, for every $\Sigma \in |\mathbf{Sign}|$, all $\phi \in \mathrm{SEN}(\Sigma)$ and every theory family T of $\mathcal{I}$, $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(t)) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, where t is an arbitrary Σ-theorem of $\mathcal{I}$. Section 7 continues with the introduction of the notion of a systemic π-institution. Roughly speaking, a π-institution $\mathcal{I}$ is systemic if every Σ-theory of $\mathcal{I}$ is the Σ-component of a theory system of $\mathcal{I}$. This notion is shown to be equivalent to the notion of a theory invariant π-institution. A π-institution is said to be theory invariant if, for every $\Sigma \in |\mathbf{Sign}|$ and every Σ-theory T_Σ, $\mathrm{SEN}(f)(T_\Sigma) \subseteq T_\Sigma$, for all $\mathbf{Sign}$-endomorphisms f of Σ. Theorem 16 paves the way for the proof of one of the main results of Section 7 pertaining to regular algebraizability. It asserts, roughly speaking, that an N-equivalence system of a syntactically N-equivalential π-institution provides crucial help in interpreting the equational consequence relation of a certain N-quasivariety into the consequence relation of the π-institution itself. In Theorem 19, it is shown that a finitary, theory invariant and finitely syntactically N-equivalential π-institution $\mathcal{I}$, that

- has the G-rule,
- is such that N contains a constant natural transformation $\top : \mathrm{SEN} \to \mathrm{SEN}$, with $\top_\Sigma(\phi) := \top_\Sigma \in \mathrm{Thm}_\Sigma$, for every $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, and
- whose "associated algebraic counterpart" $\mathcal{I}^{\mathtt{A}^N_{\mathcal{I}}}$ is theory-invariant

is syntactically N-algebraizable. Such π-institutions are termed regularly N-algebraizable.

Section 8 introduces relatively point-regular N-quasivarieties, an analog in the categorical framework of the relatively point-regular quasivarieties of universal algebras. The notion of the assertional π-institution associated with a given pointed N-quasivariety is also introduced in this section.

Moreover, it is shown that for a pointed N-quasivariety $\mathtt{Q}$, the closure system of the assertional π-institution associated with $\mathtt{Q}$ is interpretable into the closure system of the equational π-institution corresponding to $\mathtt{Q}$. In the concluding result of this section, it is proven that, if a π-institution is regularly N-algebraizable, then it is the assertional π-institution of some point-regular N-quasivariety. This is a partial analog of a characterization of regularly algebraizable sentential logics due to Czelakowski and Pigozzi and, independently, also to Blok and Raftery.

The paper ends with Section 9, that revisits the deduction-detachment theorem in the categorical context and uses it to provide an axiomatization of the equivalent N-quasivariety of a regularly N-algebraizable N-rule based π-institution with a finite N-deduction-detachment system.

The interested reader may find an overview of the current state of affairs in the field of abstract algebraic logic in the paper [13], the monograph [12] and the book [9]. For all unexplained categorical notation, any of the standard introductory references in category theory [1, 6, 15] may be consulted.

2. Preliminaries

In this section, some concepts and some results that were introduced previously in the theory of CAAL will be recalled. This exposition of background information will, hopefully, facilitate the reading in the following sections.

Given a category **Sign** and a functor SEN : **Sign** $\to$ **Set** the *clone of all natural transformations on* SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha : \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \to \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \to \mathrm{SEN}$ [18]. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i : i < \beta \rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j : j < \gamma \rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j : j < \gamma \rangle \circ \langle \tau_i : i < \beta \rangle = \langle \sigma_j(\langle \tau_i : i < \beta \rangle) : j < \gamma \rangle.$$

A subcategory N of this category with objects all objects of the form SEN^k for $k < \omega$, and containing all projection morphisms $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}, i <$

$k, k < \omega$, with $p_\Sigma^{k,i} : \mathrm{SEN}(\Sigma)^k \to \mathrm{SEN}(\Sigma)$ given by

$$p_\Sigma^{k,i}(\vec{\phi}) = \phi_i, \quad \text{for all} \quad \vec{\phi} \in \mathrm{SEN}(\Sigma)^k,$$

and such that, for every family $\{\tau_i : \mathrm{SEN}^k \to \mathrm{SEN} : i < l\}$ of natural transformations in N, the sequence $\langle \tau_i : i < l \rangle : \mathrm{SEN}^k \to \mathrm{SEN}^l$ is also in N, is referred to as *a category of natural transformations on* SEN.

Let SEN : **Sign** $\to$ **Set** be a functor and N a category of natural transformations on SEN. In the sequel, the functor of N-terms with variables in an arbitrary set X, that was presented in [24], will also be used. Given a set X, the collection $\mathrm{Te}^N(X)$ of *N-terms in the variables* X is defined recursively as follows:

- $x \in \mathrm{Te}^N(X)$, for all $x \in X$, and
- $\sigma(t_0, \ldots, t_{n-1}) \in \mathrm{Te}^N(X)$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $t_0, \ldots, t_{n-1} \in \mathrm{Te}^N(X)$.

Moreover, given sets X and Y and a mapping $f : X \to Y$, f induces a mapping $\mathrm{Te}^N(f) : \mathrm{Te}^N(X) \to \mathrm{Te}^N(Y)$, defined recursively on the structure of N-terms, by

- $\mathrm{Te}^N(f)(x) = f(x)$, for all $x \in X$, and
- $\mathrm{Te}^N(f)(\sigma(t_0, \ldots, t_{n-1})) = \sigma(\mathrm{Te}^N(f)(t_0), \ldots, \mathrm{Te}^N(f)(t_{n-1}))$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $t_0, \ldots, t_{n-1} \in \mathrm{Te}^N(X)$.

It is not difficult to see that, defined as above, Te^N : **Set** $\to$ **Set** is a functor and that it is equipped with a category N^{t} of natural transformations that is compatible with N. By an *N-term*, we will understand a member of $\mathrm{Te}^N(X)$, for some $X \in |\mathbf{Set}|$.

Given a functor SEN, with N a category of natural transformations on SEN, denote by $\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle : \mathrm{Te}^N \circ \mathrm{SEN} \to \mathrm{SEN}$ the surjective (N^{t}, N)-epimorphic translation, defined by letting, for all $\Sigma \in |\mathbf{Sign}|$,

$$\mu_\Sigma^N : \mathrm{Te}^N(\mathrm{SEN}(\Sigma)) \to \mathrm{SEN}(\Sigma)$$

be given by recursion on the structure of N-terms:

- $\mu_\Sigma^N(\phi) = \phi$, for all $\phi \in \mathrm{SEN}(\Sigma)$, and

- $\mu^N_\Sigma(\sigma(t_0,\ldots,t_{n-1})) = \sigma_\Sigma(\mu^N_\Sigma(t_0),\ldots,\mu^N_\Sigma(t_{n-1}))$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $t_0,\ldots,t_{n-1} \in \mathrm{Te}^N(\mathrm{SEN}(\Sigma))$.

Furthermore, given SEN : **Sign** $\to$ **Set**, with N a category of natural transformations on SEN, an N-term $s(\vec{x})$ in the set of variables X, a $\Sigma \in |\mathbf{Sign}|$ and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^X$, denote by

$$s_\Sigma(\vec{\phi}) := \mu^N_\Sigma(\mathrm{Te}^N(\vec{\phi})(s)).$$

This is the usual operation of substitution of elements of $\mathrm{SEN}(\Sigma)$ for variables. It is obvious that $s_\Sigma(\vec{\phi})$ depends only on the values of the substitution $\vec{\phi}$ on the variables $\vec{x}$ appearing in s.

An N-*equation* is a pair $\langle s,t\rangle$ of N-terms, also denoted by $s \approx t$. An N-*quasiequation* is a nonempty sequence $\langle s_0 \approx t_0,\ldots,s_{n-1} \approx t_{n-1}, u \approx v\rangle$ of N-equations, usually denoted by $s_0 \approx t_0,\ldots,s_{n-1} \approx t_{n-1} \to u \approx v$. The N-equations $s_i \approx t_i, i < n$, are called the *premises* of the N-quasiequation and $u \approx v$ its *conclusion.* N-equations are identified with N-quasiequations with an empty set of premises.

Let SEN : **Sign** $\to$ **Set** be a functor, with N a category of natural transformations on SEN. An N-*algebraic system* is a triple $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, where $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$ is a functor, with N' a category of natural transformations on SEN', and $F' : N \to N'$ a surjective functor that preserves the projections, i.e., such that

- $F'(\mathrm{SEN}^k) = \mathrm{SEN}'^k$, for all $k \in \omega$, and
- $F(p^{k,i}) = p^{k,i}$, for all $k \in \omega, i < k$, where the left $p^{k,i}$ refers to the projection $p^{k,i} : \mathrm{SEN}^k \to \mathrm{SEN}$ onto the i-th coordinate and the right $p^{k,i}$ referes to the projection $p^{k,i} : \mathrm{SEN}'^k \to \mathrm{SEN}'$ onto the i-th coordinate.

The *value of an* N-term $s(\vec{x})$ *at the tuple* $\langle \Sigma, \vec{\phi}\rangle$ *in* the N-algebraic system $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$, denoted by $s_\Sigma(\vec{\phi})$ is defined by recursion on the structure of s by

- $x_\Sigma(\vec{\phi}) = \vec{\phi}(x)$ and
- $\sigma(t_0,\ldots,t_{n-1})_\Sigma(\vec{\phi}) = F'(\sigma)_\Sigma({t_0}_\Sigma(\vec{\phi}),\ldots,{t_{n-1}}_\Sigma(\vec{\phi}))$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $t_0,\ldots,t_{n-1} \in \mathrm{Te}^N(X)$.

The N-algebraic system $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ *satisfies the N-equation* $s \approx t$, denoted

$$\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \models s \approx t,$$

if, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma)^\omega$, $s_\Sigma(\vec{\phi}) = t_\Sigma(\vec{\phi})$. Similarly, the N-algebraic system $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle$ *satisfies the N-quasiequation* $\bigwedge_{i<n} s^i \approx t^i \to s \approx t$, denoted

$$\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \models \bigwedge_{i<n} s^i \approx t^i \to s \approx t,$$

if, for all $\Sigma \in |\mathbf{Sign}'|$ and all $\vec{\phi} \in \mathrm{SEN}'(\Sigma)^\omega$, either $s^i_\Sigma(\vec{\phi}) \neq t^i_\Sigma(\vec{\phi})$, for some $i < n$, or $s_\Sigma(\vec{\phi}) = t_\Sigma(\vec{\phi})$. An *$N$-variety* or a *variety of N-algebraic systems* is a class of N-algebraic systems that is axiomatized by a collection of N-equations. An *N-quasivariety* or a *quasivariety of N-algebraic systems* is a class of N-algebraic systems that is axiomatized by a collection of N-quasiequations. In [24] it has been shown that a class $\mathtt{Q}$ of N-algebraic systems is an N-variety if and only if $\mathbf{HSP}(\mathtt{Q}) = \mathtt{Q}$ and that a class $\mathtt{Q}$ of N-algebraic systems is an N-quasivariety if and only if $\mathbf{SLP}(\mathtt{Q}) = \mathtt{Q}$. The reader is referred to that paper for more details on the closure operators on classes of N-algebraic systems that are involved in this Birkhoff-style and this Mal'cev-style characterizations of N-varieties and N-quasivarieties, respectively. Given an N-quasivariety $\mathtt{Q}$, the *N-core of* $\mathtt{Q}$, denoted by $\mathrm{cor}^N(\mathtt{Q})$, is the subclass of $\mathtt{Q}$ consisting of all those N-algebraic systems $\langle \mathrm{SEN}', \langle N', F' \rangle \rangle \in \mathtt{Q}$, such that, there exists a surjective (N, N')-epimorphic translation $\langle F, \alpha \rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$. $\mathtt{Q}$ is called *N-surjectively generated* or *N-regular* if it is generated as an N-quasivariety by a subclass of its N-core.

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN. Then by an *N-rule of inference*, or, simply, an *N-rule*, of SEN it is understood a member r of the cartesian product $\mathcal{P}(\mathrm{Te}^N(V)) \times \mathrm{Te}^N(V)$. Such a rule is denoted by $r = \langle X, \sigma \rangle$, where $X \subseteq \mathrm{Te}^N(V)$ and $\sigma \in \mathrm{Te}^N(V)$. The *length* of the N-rule $r = \langle X, \sigma \rangle$ is the cardinal number $|r| = |X|^+$. An axiom family $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ on SEN (i.e., $T_\Sigma \subseteq \mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$,) is said to be *closed under the N-rule* $r = \langle X, \sigma \rangle$ if, for all $\Sigma \in |\mathbf{Sign}|$, and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^V$, if $X_\Sigma(\vec{\phi}) \subseteq T_\Sigma$, then $\sigma_\Sigma(\vec{\phi}) \in T_\Sigma$. If T is closed under the rule r, then, we also say that r *preserves* the axiom family T. An N-rule $r = \langle X, \sigma \rangle$ of SEN is an *N-rule* *of the π-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ or *of the closure system* C on SEN if $\sigma_\Sigma(\vec{\phi}) \subseteq C_\Sigma(X_\Sigma(\vec{\phi}))$, for all $\Sigma \in |\mathbf{Sign}|, \vec{\phi} \in \mathrm{SEN}(\Sigma)^V$. If this is the case, then r is said to be *sound for* $\mathcal{I}$ or *for* C. Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor and N a category of natural transformations on SEN. A closure system C on SEN and the corresponding π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ are said to be *N-rule-based* if, for all $\Sigma \in |\mathbf{Sign}|, \Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\Phi)$, there exists an N-rule $\langle X, \sigma\rangle$ of C of length at most $|\Phi|^+$, and $\vec{\psi} \in \mathrm{SEN}(\Sigma)^V$, such that $X_\Sigma(\vec{\psi}) \subseteq \Phi$ and $\sigma_\Sigma(\vec{\psi}) = \phi$, i.e., such that ϕ follows from Φ by an application of $\langle X, \sigma\rangle$. A functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$, with N a category of natural transformations on SEN, is said to be *N-rule based* if every finitary closure system C on SEN is N-rule based. Given a finitary N-rule based π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\Sigma \in |\mathbf{Sign}|$ and $\phi_0, \ldots, \phi_{n-1}, \phi \in \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma(\phi_0, \ldots, \phi_{n-1})$, it is common to denote by $\sigma^{\langle\Sigma,\phi_0\rangle}, \ldots, \sigma^{\langle\Sigma,\phi_{n-1}\rangle}, \sigma^{\langle\Sigma,\phi\rangle} : \mathrm{SEN}^k \to \mathrm{SEN}$ the natural transformations in N, such that

1. $\langle\{\sigma^{\langle\Sigma,\phi_i\rangle} : i < n\}, \sigma^{\langle\Sigma,\phi\rangle}\rangle$ is an N-rule of inference of $\mathcal{I}$ and
2. there exists $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\sigma_\Sigma^{\langle\Sigma,\phi_i\rangle}(\vec{\chi}) = \phi_i, i < n$, and $\sigma_\Sigma^{\langle\Sigma,\phi\rangle}(\vec{\chi}) = \phi$.

3. Equational π-Institutions

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN. Define the triple $\mathcal{I}^{N-\mathrm{FEQ}} = \langle \mathbf{Sign}, \mathrm{SEN}^2, C^{N-\mathrm{FEQ}}\rangle$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $C_\Sigma^{N-\mathrm{FEQ}}$ be the least closure operator, such that

1. $\langle\phi, \phi\rangle \in C_\Sigma^{N-\mathrm{FEQ}}(\emptyset)$, for all $\phi \in \mathrm{SEN}(\Sigma)$,
2. $\langle\psi, \phi\rangle \in C_\Sigma^{N-\mathrm{FEQ}}(\langle\phi, \psi\rangle)$, for all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,
3. $\langle\phi, \chi\rangle \in C_\Sigma^{N-\mathrm{FEQ}}(\langle\phi, \psi\rangle, \langle\psi, \chi\rangle)$, for all $\phi, \psi, \chi \in \mathrm{SEN}(\Sigma)$,
4. $\langle\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})\rangle \in C_\Sigma^{N-\mathrm{FEQ}}(\langle\phi_0, \psi_0\rangle, \ldots, \langle\phi_{n-1}, \psi_{n-1}\rangle)$, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and all $\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$.

A proof-theoretic reformulation of the definition of $C_\Sigma^{N-\mathrm{FEQ}}$ may also be given. Let $E \cup \{\langle\phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$. Say that $\langle\phi, \psi\rangle$ **Σ-directly follows from E in $C^{N-\mathrm{FEQ}}$**, if either

1. $\langle \psi, \phi \rangle \in E$ or
2. $\langle \phi, \chi \rangle, \langle \chi, \psi \rangle \in E$, for some $\chi \in \mathrm{SEN}(\Sigma)$, or
3. there exist $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$, such that $\phi = \sigma_\Sigma(\vec{\phi}), \psi = \sigma_\Sigma(\vec{\psi})$ and $\langle \phi_i, \psi_i \rangle \in E$, for all $i < n$.

Then, a **Σ-proof of $\langle \phi, \psi \rangle$ from E in $C^{N-\mathrm{FEQ}}$** is a finite sequence

$$\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle, \langle \phi_n, \psi_n \rangle = \langle \phi, \psi \rangle,$$

in $\mathrm{SEN}(\Sigma)^2$, such that, for all $k \leq n$, either $\phi_k = \psi_k$ or $\langle \phi_k, \psi_k \rangle$ Σ-directly follows from $\{\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{k-1}, \psi_{k-1} \rangle\}$ in $C^{N-\mathrm{FEQ}}$.

It is not at all difficult to show that, for all $\Sigma \in |\mathbf{Sign}|$ and all $E \cup \{\langle \phi, \psi \rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$, $\langle \phi, \psi \rangle \in C_\Sigma^{N-\mathrm{FEQ}}(E)$ if and only if there exists a Σ-proof of $\langle \phi, \psi \rangle$ from E in $C^{N-\mathrm{FEQ}}$. We sketch the argument which is a standard proof-theoretic argument in mathematical logic. Let $C_\Sigma : \mathcal{P}(\mathrm{SEN}(\Sigma)^2) \to \mathcal{P}(\mathrm{SEN}(\Sigma)^2)$ be defined, for all $E \subseteq \mathrm{SEN}(\Sigma)^2$, by

$$\begin{aligned} C_\Sigma(E) = \{ & \langle \phi, \psi \rangle \in \mathrm{SEN}(\Sigma)^2 : \\ & \text{there exists a } \Sigma\text{-proof of } \langle \phi, \psi \rangle \text{ from } E \text{ in } C^{N-\mathrm{FEQ}}\}. \end{aligned}$$

It can be shown by induction on the length of a Σ-proof that, if $\langle \phi, \psi \rangle \in C_\Sigma(E)$, then $\langle \phi, \psi \rangle \in C_\Sigma^{N-\mathrm{FEQ}}(E)$. On the other hand, it may also be shown that C_Σ is a closure operator on $\mathrm{SEN}(\Sigma)^2$, that satisfies Conditions 1-4 of the definition of $C_\Sigma^{N-\mathrm{FEQ}}$. Thus, by the minimality of $C_\Sigma^{N-\mathrm{FEQ}}$, one obtains that, if $\langle \phi, \psi \rangle \in C_\Sigma^{N-\mathrm{FEQ}}(E)$, then $\langle \phi, \psi \rangle \in C_\Sigma(E)$. Thus, the two closure operators $C_\Sigma^{N-\mathrm{FEQ}}$ and C_Σ on $\mathrm{SEN}(\Sigma)^2$ coincide.

It is shown, next, that $\mathcal{I}^{N-\mathrm{FEQ}}$ is a π-institution and that the theory families of $\mathcal{I}^{N-\mathrm{FEQ}}$ coincide with the N-congruence families on SEN. As a consequence, one also obtains that the theory systems of $\mathcal{I}^{N-\mathrm{FEQ}}$ coincide with the N-congruence systems on SEN.

Proposition 1. *Let* SEN : $\mathbf{Sign} \to \mathbf{Set}$ *be a functor, with N a category of natural transformations on* SEN*. Then*

$$\mathcal{I}^{N-\mathrm{FEQ}} = \langle \mathbf{Sign}, \mathrm{SEN}^2, C^{N-\mathrm{FEQ}} \rangle$$

is a finitary π-institution.

Proof. Since, by definition $C^{N-\mathrm{FEQ}}_\Sigma$ is a closure operator on $\mathrm{SEN}(\Sigma)^2$, it suffices to show that $C^{N-\mathrm{FEQ}}$ is structural. The key observation is that each of the four rules defining $C^{N-\mathrm{FEQ}}$ is structural and, therefore, for all $\Sigma, \Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma, \Sigma')$ and all $E \cup \{\langle\phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$, any Σ-proof of $\langle\phi, \psi\rangle$ from E in $C^{N-\mathrm{FEQ}}$ may be easily converted to a Σ'-proof of $\langle\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)\rangle$ from $\mathrm{SEN}(f)(E)$ in $C^{N-\mathrm{FEQ}}$. In fact, we need only apply $\mathrm{SEN}(f)^2$ to all pairs in the original Σ-proof to get a Σ'-proof. $\square$

The π-institution $\mathcal{I}^{N-\mathrm{FEQ}}$ is called the N**-free equational** π**-institution on** SEN.

Proposition 2. *Let* SEN : $\mathbf{Sign} \to \mathbf{Set}$ *be a functor, with* N *a category of natural transformations on* SEN. *Then* $\mathrm{ThFam}(\mathcal{I}^{N-\mathrm{FEQ}}) = \mathrm{Conf}^N(\mathrm{SEN})$, *where by* $\mathrm{Conf}^N(\mathrm{SEN})$ *is denoted the collection of all* N*-congruence families on* SEN.

Proof. Suppose that $\theta \in \mathrm{ThFam}(\mathcal{I}^{N-\mathrm{FEQ}})$. It suffices to show that, for all $\Sigma \in |\mathbf{Sign}|$, θ_Σ is an N-congruence relation on $\mathrm{SEN}(\Sigma)$. By Properties 1-3 of the definition of $C^{N-\mathrm{FEQ}}_\Sigma$, θ_Σ is an equivalence relation on $\mathrm{SEN}(\Sigma)$. Finally, by Property 4 of the definition of $C^{N-\mathrm{FEQ}}_\Sigma$, we get that θ_Σ is preserved by all natural transformations in N and, therefore, it is in fact an N-congruence relation on $\mathrm{SEN}(\Sigma)$.

Suppose, conversely, that $\theta \in \mathrm{Conf}^N(\mathrm{SEN})$. It suffices to show that $\theta_\Sigma = C^{N-\mathrm{FEQ}}_\Sigma(\theta_\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$. To this end, suppose that $\phi, \psi \in \mathrm{SEN}(\Sigma)$, with $\langle\phi, \psi\rangle \in C^{N-\mathrm{FEQ}}_\Sigma(\theta_\Sigma)$. Using the definition of $C^{N-\mathrm{FEQ}}$ and recursion on the number of steps in a Σ-proof of $\langle\phi, \psi\rangle$ from θ_Σ in $C^{N-\mathrm{FEQ}}$, one of the following must hold:

1. $\phi = \psi$. In this case, since θ_Σ is reflexive, we must have $\langle\phi, \psi\rangle \in \theta_\Sigma$.
2. $\langle\psi, \phi\rangle \in \theta_\Sigma$, whence, since θ_Σ is symmetric, we get that $\langle\phi, \psi\rangle \in \theta_\Sigma$.
3. $\langle\phi, \chi\rangle, \langle\chi, \psi\rangle \in \theta_\Sigma$, for some $\chi \in \mathrm{SEN}(\Sigma)$, whence, since θ_Σ is transitive, we must have $\langle\phi, \psi\rangle \in \theta_\Sigma$.
4. There exist $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$, such that $\phi = \sigma_\Sigma(\vec{\phi})$, $\psi = \sigma_\Sigma(\vec{\psi})$ and $\langle\phi_i, \psi_i\rangle \in \theta_\Sigma$, for all $i < n$. Thus, since θ_Σ is an N-congruence relation on $\mathrm{SEN}(\Sigma)$, we get that $\langle\phi, \psi\rangle = \langle\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi})\rangle \in \theta_\Sigma$.

This concludes the proof that $\theta_\Sigma = C^{N-\mathrm{FEQ}}_\Sigma(\theta_\Sigma)$, showing that $\theta \in \mathrm{ThFam}(\mathcal{I}^{N-\mathrm{FEQ}})$. □

Corollary 3. *Let* SEN : **Sign** → **Set** *be a functor, with* N *a category of natural transformations on* SEN*. Then* $\mathrm{ThSys}(\mathcal{I}^{N-\mathrm{FEQ}}) = \mathrm{Con}^N(\mathrm{SEN})$.

Suppose that SEN : **Sign** → **Set** is a functor and N a category of natural transformations on SEN. Let $\mathtt{Q}$ be a quasivariety of N-algebraic systems. $\mathtt{Q}$ will be said to be **surjectively generated** or **regular** if it is generated by a subclass $\mathtt{K} \subseteq \mathtt{Q}$, such that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, there exists at least one surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$. A class $\mathtt{K}$ of N-algebraic systems having this property will be termed a **surjective class**

Recall from [25], that, given a functor SEN : **Sign** → **Set**, with N a category of natural transformations on SEN, and an N-quasivariety $\mathtt{Q}$, the N-**core** of $\mathtt{Q}$, denoted by $\mathrm{cor}^N(\mathtt{Q})$, is the subclass $\mathtt{K} \subseteq \mathtt{Q}$ of all those N-algebraic systems $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{K}$, that are such that there exists at least one surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \rightarrow^{se} \mathrm{SEN}'$. Regular N-quasivarieties have the following characterization in terms of their core:

Proposition 4. *Let* SEN : **Sign** → **Set***, with* N *a category of natural transformations on* SEN*, be a functor and* $\mathtt{Q}$ *an* N*-quasivariety of* N*-algebraic systems.* $\mathtt{Q}$ *is a regular quasivariety if and only if* $\mathtt{Q} = \mathbf{Q}(\mathrm{cor}^N(\mathtt{Q}))$*, where, for any class* $\mathtt{K}$ *of* N*-algebraic systems, by* $\mathbf{Q}(\mathtt{K})$ *is denoted the* N*-quasivariety generated by* $\mathtt{K}$.

Proof. Suppose that $\mathtt{Q}$ is a regular N-quasivariety. Then, there exists a surjective class $\mathtt{K} \subseteq \mathtt{Q}$, such that $\mathtt{Q} = \mathbf{Q}(\mathtt{K})$. Notice that, since $\mathtt{K}$ is surjective, $\mathtt{K} \subseteq \mathrm{cor}^N(\mathtt{Q})$, whence we obtain that $\mathtt{Q} = \mathbf{Q}(\mathtt{K}) \subseteq \mathbf{Q}(\mathrm{cor}^N(\mathtt{Q})) \subseteq \mathtt{Q}$ and, therefore, $\mathtt{Q} = \mathbf{Q}(\mathrm{cor}^N(\mathtt{Q}))$.

Suppose, conversely, that $\mathtt{Q} = \mathbf{Q}(\mathrm{cor}^N(\mathtt{Q}))$. But, then, since, by definition, $\mathrm{cor}^N(\mathtt{Q})$ is a surjective class, we get that $\mathtt{Q}$ is a regular N-quasivariety. □

Given an N-quasivariety $\mathtt{Q}$, define the triple $\mathcal{I}^{\mathtt{Q}} = \langle \mathbf{Sign}, \mathrm{SEN}^2, C^{\mathtt{Q}}\rangle$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $C^{\mathtt{Q}}_\Sigma$ be the smallest closure operator on $\mathrm{SEN}(\Sigma)^2$ that, in addition to the four rules satisfied by $C^{N-\mathrm{FEQ}}$, it also satisfies,

5. $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in C^{\mathtt{Q}}_\Sigma(\emptyset)$, for all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, for every N-identity $\sigma \approx \tau$ of $\mathtt{Q}$,
6. $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in C^{\mathtt{Q}}_\Sigma(\langle \sigma^0_\Sigma(\vec{\phi}), \tau^0_\Sigma(\vec{\phi})\rangle, \ldots, \langle \sigma^{n-1}_\Sigma(\vec{\phi}), \tau^{n-1}_\Sigma(\vec{\phi})\rangle)$, for all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, for every N-quasi-identity $\bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$ of $\mathtt{Q}$.

Again, we provide a proof-theoretic version of $C^{\mathtt{Q}}$. Given $E \cup \{\langle \phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$, it is said that $\langle \phi, \psi\rangle$ **Σ-directly follows from E in $C^{\mathtt{Q}}$** if either $\langle \phi, \psi\rangle$ Σ-directly follows from E in $C^{N-\mathrm{FEQ}}$ or, there exist

$$\sigma^0, \ldots, \sigma^{n-1}, \tau^0, \ldots, \tau^{n-1}, \sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$$

in N, with $\mathtt{Q} \models \bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$, and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, such that $\phi = \sigma_\Sigma(\vec{\phi}), \psi = \tau_\Sigma(\vec{\phi})$ and $\langle \sigma^i_\Sigma(\vec{\phi}), \tau^i_\Sigma(\vec{\phi})\rangle \in E$, for all $i < n$.

A **Σ-proof of $\langle \phi, \psi\rangle$ from E in $C^{\mathtt{Q}}$** is a finite sequence

$$\langle \phi_0, \psi_0\rangle, \ldots, \langle \phi_{m-1}, \psi_{m-1}\rangle, \langle \phi_m, \psi_m\rangle = \langle \phi, \psi\rangle,$$

in $\mathrm{SEN}(\Sigma)^2$, such that, for all $i \leq m$, either $\phi_i = \psi_i$, or there exist $\sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, with $\mathtt{Q} \models \sigma \approx \tau$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\phi_i = \sigma_\Sigma(\vec{\chi})$ and $\psi_i = \tau_\Sigma(\vec{\chi})$, or $\langle \phi_i, \psi_i\rangle$ Σ-directly follows from $\{\langle \phi_0, \psi_0\rangle, \ldots, \langle \phi_{i-1}, \psi_{i-1}\rangle\}$ in $C^{\mathtt{Q}}$.

It is also not difficult to see that, for all $\Sigma \in |\mathbf{Sign}|$ and all $E \cup \{\langle \phi, \psi\rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$, $\langle \phi, \psi\rangle \in C^{\mathtt{Q}}_\Sigma(E)$ if and only if there exists a Σ-proof of $\langle \phi, \psi\rangle$ from E in $C^{\mathtt{Q}}$. The details of the proof of this statement are very similar to those of the proof of the corresponding statement about $C^{N-\mathrm{FEQ}}$ and will, therefore, be omitted.

Proposition 5. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with* N *a category of natural transformations on* SEN. *If* $\mathtt{Q}$ *is an* N*-quasivariety, then* $\mathcal{I}^{\mathtt{Q}} = \langle \mathbf{Sign}, \mathrm{SEN}^2, C^{\mathtt{Q}}\rangle$ *is a finitary* π*-institution.*

Proof. Very similar to the proof of Proposition 1. □

Now it is not very hard to see that the theory families of $\mathcal{I}^{\mathtt{Q}}$ are exactly the $\mathtt{Q}$-N-congruence families on SEN, i.e., those N-congruence families θ on SEN for which, for every $\Sigma \in |\mathbf{Sign}|$, the quotient $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies all N-quasi-identities of $\mathtt{Q}$. The collection of all such congruence families is denoted by $\mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN})$. Therefore, the theory systems of $\mathcal{I}^{\mathtt{Q}}$ coincide with the $\mathtt{Q}$-N-congruence systems on SEN, i.e., those N-congruence systems θ

on SEN, such that the quotient $\langle \mathrm{SEN}^\theta, \langle N^\theta, F^\theta \rangle\rangle \in \mathtt{Q}$. Similarly with the notation followed for $\mathtt{Q}$-N-congruence families, the collection of all $\mathtt{Q}$-N-congruence systems is denoted by $\mathrm{Con}^N_{\mathtt{Q}}(\mathrm{SEN})$.

Proposition 6. *Let* SEN : **Sign** → **Set** *be a functor, with* N *a category of natural transformations on* SEN*, and* $\mathtt{Q}$ *an* N*-quasivariety. Then* $\mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}}) = \mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN})$ *and, consequently,* $\mathrm{ThSys}(\mathcal{I}^{\mathtt{Q}}) = \mathrm{Con}^N_{\mathtt{Q}}(\mathrm{SEN})$.

Proof. Clearly, we have that

$$\mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}}) \subseteq \mathrm{ThFam}(\mathcal{I}^{N-\mathrm{FEQ}}) = \mathrm{Conf}^N(\mathrm{SEN}),$$

by Proposition 2. So, for left-to-right inclusion, it suffices to show that, if $\theta \in \mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}})$, then, for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies all N-quasi-identities of $\mathtt{Q}$. To this end, let, first, $\sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N be such that $\mathtt{Q} \models \sigma \approx \tau$. Then, for all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in C^{\mathtt{Q}}_\Sigma(\emptyset)$. Thus, since $\theta \in \mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}})$, we must have $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma$. Hence $\sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \tau^\theta_\Sigma(\vec{\phi}/\theta_\Sigma)$, which shows that $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies $\sigma \approx \tau$. Let, next, $\sigma^0, \ldots, \sigma^{n-1}, \tau^0, \ldots, \tau^{n-1}, \sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N be such that $\mathtt{Q} \models \bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$. Then, if $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies $\bigwedge_{i<n} \sigma^i \approx \tau^i$, we have that, for all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, $\sigma^{i\theta}_\Sigma(\vec{\phi}/\theta_\Sigma) = \tau^{i\theta}_\Sigma(\vec{\phi}/\theta_\Sigma), i < n$. Thus, $\langle \sigma^i_\Sigma(\vec{\phi}), \tau^i_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma$, for all $i < n$. Since $\theta \in \mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}})$ and $\mathtt{Q} \models \bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$, we get that $\langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma$. This yields that $\sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \tau^\theta_\Sigma(\vec{\phi}/\theta_\Sigma)$, showing that $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies $\bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$.

For the right-to-left inclusion, notice, again, that

$$\mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN}) \subseteq \mathrm{Conf}^N(\mathrm{SEN}) = \mathrm{ThFam}(\mathcal{I}^{N-\mathrm{FEQ}}),$$

by Proposition 2. Thus, it suffices to show that, if $\theta \in \mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN})$, then the theory family θ of $\mathcal{I}^{N-\mathrm{FEQ}}$ is also closed under the two extra rules of $\mathcal{I}^{\mathtt{Q}}$. To this end, suppose that $\phi, \psi \in \mathrm{SEN}(\Sigma)$, with $\langle \phi, \psi\rangle \in C^{\mathtt{Q}}_\Sigma(\theta_\Sigma)$. Using the definition of $C^{\mathtt{Q}}$ and recursion on the number of steps in a Σ-proof of $\langle \phi, \psi\rangle$ from θ_Σ in $C^{\mathtt{Q}}$, one of the following cases must hold:

1. $\phi = \psi$ or $\langle \psi, \phi\rangle \in \theta_\Sigma$ or $\langle \phi, \chi\rangle, \langle \chi, \psi\rangle \in \theta_\Sigma$, for some $\chi \in \mathrm{SEN}(\Sigma)$, or there exist $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N and $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$, such that $\phi = \sigma_\Sigma(\vec{\phi})$, $\psi = \sigma_\Sigma(\vec{\psi})$ and $\langle \phi_i, \psi_i\rangle \in \theta_\Sigma$, for all $i < n$, and all these cases are handled by the fact that $\mathcal{I}^{\mathtt{Q}}$ is closed under all the rules of $\mathcal{I}^{N-\mathrm{FEQ}}$.

2. There exist $\sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, such that $\mathtt{Q} \models \sigma \approx \tau$, and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, such that $\phi = \sigma_\Sigma(\vec{\phi}), \psi = \tau_\Sigma(\vec{\phi})$. Then, since $\mathtt{Q} \models \sigma \approx \tau$, we have that $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies $\sigma \approx \tau$, i.e., for all $\vec{\psi} \in \mathrm{SEN}(\Sigma)^k$, $\sigma^\theta_\Sigma(\vec{\psi}/\theta_\Sigma) = \tau^\theta_\Sigma(\vec{\psi}/\theta_\Sigma)$, which is equivalent to $\langle \sigma_\Sigma(\vec{\psi}), \tau_\Sigma(\vec{\psi})\rangle \in \theta_\Sigma$. Choosing for $\vec{\psi}$ the specific vector $\vec{\phi}$, we get that $\langle \phi, \psi\rangle = \langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma$.

3. There exist $\sigma^0, \ldots, \sigma^{n-1}, \tau^0, \ldots, \tau^{n-1}, \sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$, such that $\mathtt{Q} \models \bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$, and $\vec{\phi} \in \mathrm{SEN}(\Sigma)^k$, such that
$$\langle \sigma^i_\Sigma(\vec{\phi}), \tau^i_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma, \qquad \text{for all } i < n,$$
and $\phi = \sigma_\Sigma(\vec{\phi}), \psi = \tau_\Sigma(\vec{\phi})$. Then we have that, for all $i < n$, $\sigma^{i\theta}_\Sigma(\vec{\phi}/\theta_\Sigma) = \tau^{i\theta}_\Sigma(\vec{\phi}/\theta_\Sigma)$, whence, since $\mathrm{SEN}(\Sigma)/\theta_\Sigma$ satisfies $\bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$, we get that $\sigma^\theta_\Sigma(\vec{\phi}/\theta_\Sigma) = \tau^\theta_\Sigma(\vec{\phi}/\theta_\Sigma)$, i.e., that $\langle \phi, \psi\rangle = \langle \sigma_\Sigma(\vec{\phi}), \tau_\Sigma(\vec{\phi})\rangle \in \theta_\Sigma$. $\square$

Suppose that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, with N a category of natural transformations on SEN. In the following theorem a characterization is obtained of all closure systems on SEN including the closure system $C^{N-\mathrm{FEQ}}$ that are of the form $C^{\mathtt{Q}}$, for some quasivariety $\mathtt{Q}$ of N-algebraic systems, under the hypothesis that $\mathrm{SEN}^2 : \mathbf{Sign} \to \mathbf{Set}$ is N-rule-based. They are exactly those closure systems that are finitary.

Recall that an N-rule based functor is a functor $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ such that, every finitary closure system C on SEN is N-rule based.

Theorem 7. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$*, with N a category of natural transformations on* SEN^2*, be a functor, such that* SEN^2 *is N-rule-based, and $C \geq C^{N-\mathrm{FEQ}}$ a closure system on* SEN*. Then $C = C^{\mathtt{Q}}$, for some N-quasi-variety $\mathtt{Q}$ if and only if C is finitary.*

Proof. If $C = C^{\mathtt{Q}}$, then, by Proposition 5, we have that C is finitary.

Suppose, conversely, that $C \geq C^{N-\mathrm{FEQ}}$ is a finitary closure system on SEN^2. Let $\mathtt{Q}$ be the N-quasivariety axiomatized by

1. the N-identities $\sigma^{\langle \Sigma, \phi\rangle} \approx \sigma^{\langle \Sigma, \psi\rangle}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\langle \phi, \psi\rangle \in C_\Sigma(\emptyset)$, and

2. the N-quasi-identities $\bigwedge_{i<n} \sigma^{\langle \Sigma, \phi_i\rangle} \approx \sigma^{\langle \Sigma, \psi_i\rangle} \to \sigma^{\langle \Sigma, \phi\rangle} \approx \sigma^{\langle \Sigma, \psi\rangle}$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1}, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such

that

$$\langle \phi, \psi \rangle \in C_\Sigma(\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle).$$

It will be shown that $C_\Sigma = C^{\mathtt{Q}}_\Sigma$, for all $\Sigma \in |\mathbf{Sign}|$. To this end, fix $\Sigma \in |\mathbf{Sign}|$ and suppose, first, that $E \cup \{\langle \phi, \psi \rangle\} \subseteq \mathrm{SEN}(\Sigma)^2$, such that $\langle \phi, \psi \rangle \in C_\Sigma(E)$. Then, there exist, by finitarity, $\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1} \in \mathrm{SEN}(\Sigma)^2$, such that $\langle \phi, \psi \rangle \in C_\Sigma(\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle)$. Hence, since SEN^2 is N-rule-based and C is a finitary closure system on SEN^2, there exist $\sigma^{\langle \Sigma, \phi_i \rangle}, \sigma^{\langle \Sigma, \psi_i \rangle}, \sigma^{\langle \Sigma, \phi \rangle}, \sigma^{\langle \Sigma, \psi \rangle} : \mathrm{SEN}^k \to \mathrm{SEN}, i < n$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that

- $\langle \{\langle \sigma^{\langle \Sigma, \phi_0 \rangle}, \sigma^{\langle \Sigma, \psi_0 \rangle} \rangle, \ldots, \langle \sigma^{\langle \Sigma, \phi_{n-1} \rangle}, \sigma^{\langle \Sigma, \psi_{n-1} \rangle} \rangle\}, \langle \sigma^{\langle \Sigma, \phi \rangle}, \sigma^{\langle \Sigma, \psi \rangle} \rangle \rangle$ is an N-rule of C and
- $\sigma^{\langle \Sigma, \phi_i \rangle}_\Sigma(\vec{\chi}) = \phi_i, \sigma^{\langle \Sigma, \psi_i \rangle}_\Sigma(\vec{\chi}) = \psi_i, \sigma^{\langle \Sigma, \phi \rangle}_\Sigma(\vec{\chi}) = \phi, \sigma^{\langle \Sigma, \psi \rangle}_\Sigma(\vec{\chi}) = \psi$, for all $i < n$.

These two conditions, taken together with the hypothesis that

$$\bigwedge_{i<n} \sigma^{\langle \Sigma, \phi_i \rangle} \approx \sigma^{\langle \Sigma, \psi_i \rangle} \to \sigma^{\langle \Sigma, \phi \rangle} \approx \sigma^{\langle \Sigma, \psi \rangle}$$

is an N-quasi-identity of $\mathtt{Q}$ and the definition of $C^{\mathtt{Q}}$, imply that

$$\langle \sigma^{\langle \Sigma, \phi \rangle}_\Sigma(\vec{\chi}), \sigma^{\langle \Sigma, \psi \rangle}_\Sigma(\vec{\chi}) \rangle \in C^{\mathtt{Q}}_\Sigma(\{\langle \sigma^{\langle \Sigma, \phi_i \rangle}_\Sigma(\vec{\chi}), \sigma^{\langle \Sigma, \psi_i \rangle}_\Sigma(\vec{\chi}) \rangle : i < n\}),$$

i.e., that $\langle \phi, \psi \rangle \in C^{\mathtt{Q}}_\Sigma(\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle)$. Therefore $C \leq C^{\mathtt{Q}}$. Suppose, for the sake of proving the reverse inequality, that

$$\phi_0, \psi_0, \ldots, \phi_{n-1}, \psi_{n-1}, \phi, \psi \in \mathrm{SEN}(\Sigma),$$

such that $\langle \phi, \psi \rangle \in C^{\mathtt{Q}}_\Sigma(\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle)$. Using the definition of $C^{\mathtt{Q}}$ and recursion on the number of steps in a Σ-proof of $\langle \phi, \psi \rangle$ from $\{\langle \phi_i, \psi_i \rangle : i < n\}$ in $C^{\mathtt{Q}}$, we must have one of the following:

- $\langle \phi, \psi \rangle$ follows from $\{\langle \phi_i, \psi_i \rangle : i < n\}$ using one of the N-axioms or N-rules of $\mathcal{I}^{N-\mathrm{FEQ}}$, in which case, since $C \geq C^{N-\mathrm{FEQ}}$, by the hypothesis, we obtain that $\langle \phi, \psi \rangle \in C_\Sigma(\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle)$.
- There exist $\sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, such that $\mathtt{Q} \models \sigma \approx \tau$ and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\phi = \sigma_\Sigma(\vec{\chi})$ and $\psi = \tau_\Sigma(\vec{\chi})$. Therefore, by

the definition of the N-equations holding in $\mathtt{Q}$, there exist $\Sigma' \in |\mathbf{Sign}|$ and $\zeta, \eta \in \mathrm{SEN}(\Sigma')$, such that $\langle \zeta, \eta \rangle \in C_{\Sigma'}(\emptyset)$, causing the existence of the N-axiom $\langle \sigma, \tau \rangle = \langle \sigma^{\langle \Sigma', \zeta \rangle}, \sigma^{\langle \Sigma', \eta \rangle} \rangle$ of C and of $\vec{\chi}' \in \mathrm{SEN}(\Sigma')^k$, such that $\zeta = \sigma_{\Sigma'}(\vec{\chi}')$ and $\eta = \tau_{\Sigma'}(\vec{\chi}')$. Since $\langle \sigma^{\langle \Sigma', \zeta \rangle}, \sigma^{\langle \Sigma', \eta \rangle} \rangle$ is an N-axiom of C, we have that $\langle \phi, \psi \rangle = \langle \sigma_\Sigma(\vec{\chi}), \tau_\Sigma(\vec{\chi}) \rangle = \langle \sigma_\Sigma^{\langle \Sigma', \zeta \rangle}(\vec{\chi}), \sigma_\Sigma^{\langle \Sigma', \eta \rangle}(\vec{\chi}) \rangle \in C_\Sigma(\emptyset)$.

- There exist $\sigma^0, \tau^0, \ldots, \sigma^{n-1}, \tau^{n-1}, \sigma, \tau : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, such that $\mathtt{Q} \models \bigwedge_{i<n} \sigma^i \approx \tau^i \to \sigma \approx \tau$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma)^k$, such that $\phi_i = \sigma_\Sigma^i(\vec{\chi}), \psi_i = \tau_\Sigma^i(\vec{\chi}), i < n$, and $\phi = \sigma_\Sigma(\vec{\chi}), \psi = \tau_\Sigma(\vec{\chi})$. Hence by the definition of the N-quasi-identities of $\mathtt{Q}$, there exist $\Sigma' \in |\mathbf{Sign}|$ and $\zeta_i, \eta_i, \zeta, \eta \in \mathrm{SEN}(\Sigma'), i < n$, such that $\langle \zeta, \eta \rangle \in C_{\Sigma'}(\langle \zeta_0, \eta_0 \rangle, \ldots, \langle \zeta_{n-1}, \eta_{n-1} \rangle)$, causing the existence of the N-rule of inference $\langle \{\sigma^{\langle \Sigma', \zeta_i \rangle} \approx \sigma^{\langle \Sigma', \eta_i \rangle} : i < n\}, \sigma^{\langle \Sigma', \zeta \rangle} \approx \sigma^{\langle \Sigma', \eta \rangle} \rangle = \langle \{\sigma^i \approx \tau^i : i < n\}, \sigma \approx \tau \rangle$ of C and of $\vec{\chi}' \in \mathrm{SEN}(\Sigma')^k$, such that $\sigma_{\Sigma'}^{\langle \Sigma', \zeta_i \rangle}(\vec{\chi}') = \zeta_i, \sigma_{\Sigma'}^{\langle \Sigma', \eta_i \rangle}(\vec{\chi}') = \eta_i$, $i < n$, and $\sigma_{\Sigma'}^{\langle \Sigma', \zeta \rangle}(\vec{\chi}') = \zeta, \sigma_{\Sigma'}^{\langle \Sigma', \eta \rangle}(\vec{\chi}') = \eta$. Thus, we obtain

$$\begin{array}{rcl} \langle \phi, \psi \rangle & = & \langle \sigma_\Sigma(\vec{\chi}), \tau_\Sigma(\vec{\chi}) \rangle \\ & = & \langle \sigma_\Sigma^{\langle \Sigma', \zeta \rangle}(\vec{\chi}), \sigma_\Sigma^{\langle \Sigma', \eta \rangle}(\vec{\chi}) \rangle \\ & \in & C_\Sigma(\{\langle \sigma_\Sigma^{\langle \Sigma', \zeta_i \rangle}(\vec{\chi}), \sigma_\Sigma^{\langle \Sigma', \eta_i \rangle}(\vec{\chi}) \rangle : i < n\}) \\ & = & C_\Sigma(\{\langle \sigma_\Sigma^i(\vec{\chi}), \tau_\Sigma^i(\vec{\chi}) \rangle : i < n\}) \\ & = & C_\Sigma(\{\langle \phi_i, \psi_i \rangle : i < n\}). \end{array}$$

This shows that $C^{\mathtt{Q}} \leq C$ and concludes the proof that C is indeed of the form $C^{\mathtt{Q}}$ for the chosen N-quasivariety $\mathtt{Q}$. □

4. Syntactically Algebraizable π-Institutions

The class of syntactically N-algebraizable π-institutions is in the context of categorical abstract algebraic logic an analog of the class of finitely algebraizable finitary deductive systems, as originally presented by Blok and Pigozzi in [3].

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, be a finitary π-institution. $\mathcal{I}$ is called **syntactically** N-**algebraizable** if there exists an N-quasivariety $\mathtt{Q}$ and a nonempty set

$E(x,y) = \{\epsilon^0(x,y), \ldots, \epsilon^{n-1}(x,y)\}$ of binary natural transformations

$$\epsilon^0, \ldots, \epsilon^{n-1} : \mathrm{SEN}^2 \to \mathrm{SEN}$$

in N and a nonempty set

$$K(x) \approx L(x) = \{\kappa^0(x) \approx \lambda^0(x), \ldots, \kappa^{m-1}(x) \approx \lambda^{m-1}(x)\}$$

of N-equations, with $\kappa^i, \lambda^i : \mathrm{SEN} \to \mathrm{SEN}$ in N, for all $i < m$, such that the following conditions hold, for all $\Sigma \in |\mathbf{Sign}|$, all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$ and all $\Gamma \approx \Delta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$ (usual abstract algebraic logic abbreviations are used in this context):

$$\phi \in C_\Sigma(\Gamma) \quad \text{iff} \quad K_\Sigma(\phi) \approx L_\Sigma(\phi) \subseteq C^{\mathtt{Q}}_\Sigma(K_\Sigma(\Gamma) \approx L_\Sigma(\Gamma)); \tag{1}$$

$$\phi \approx \psi \in C^{\mathtt{Q}}_\Sigma(\Gamma \approx \Delta) \quad \text{iff} \quad E_\Sigma(\phi, \psi) \subseteq C_\Sigma(E_\Sigma(\Gamma, \Delta)); \tag{2}$$

$$C_\Sigma(\phi) = C_\Sigma(E_\Sigma(K_\Sigma(\phi), L_\Sigma(\phi))); \tag{3}$$

$$C^{\mathtt{Q}}_\Sigma(\phi \approx \psi) = C^{\mathtt{Q}}_\Sigma(K_\Sigma(E_\Sigma(\phi, \psi)) \approx L_\Sigma(E_\Sigma(\phi, \psi))). \tag{4}$$

In fact, as is the case for deductive systems, Conditions (1) and (4) imply the remaining two conditions and, similarly, Conditions (2) and (3) also imply the remaining two conditions.

Proposition 8. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a finitary π-institution and* $\mathtt{Q}$ *an N-quasivariety. If E and $K \approx L$ are as above, then Conditions* (1) *and* (4) *hold if and only if Conditions* (2) *and* (3) *hold.*

Proof. Follow the steps in the proof of Corollary 2.9 of [3]. □

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a finitary π-institution and $\mathtt{Q}$ an N-quasivariety. A finite system $K(x) \approx L(x)$, as above, satisfying Condition (1) is said to be an N-**interpretation of $\mathcal{I}$ in $\mathcal{I}^{\mathtt{Q}}$**. Similarly, a finite system $E(x,y)$, as above, satisfying Condition (2) is called an N-**interpretation of $\mathcal{I}^{\mathtt{Q}}$ in $\mathcal{I}$**. If, in addition, the two Conditions (3) and (4) also hold, then $K(x) \approx L(x)$ and $E(x,y)$ are said to be **inverses** of one another.

Thus, using this terminology, the definition of syntactic N-algebraizability may be stated as follows: A finitary π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is syntactically N-algebraizable if and only if there exists an invertible interpretation of $\mathcal{I}$ into $\mathcal{I}^{\mathtt{Q}}$, for some N-quasivariety $\mathtt{Q}$. This N-quasivariety will be shown in Section 5, following the original work of Blok and Pigozzi [3], to be uniquely determined by $\mathcal{I}$ and will be called the **equivalent N-quasivariety of $\mathcal{I}$**.

Still following original terminology established by Blok and Pigozzi, a finitary π-institution $\mathcal{I}$ is called **strongly syntactically N-algebraizable** if it is syntactically N-algebraizable and its equivalent N-quasivariety happens to be an N-variety.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, we define the collection of N-algebraic systems $\mathrm{A}^N(\mathcal{I})$ as follows:

$$\begin{aligned}\mathrm{A}^N(\mathcal{I}) = \{\langle \mathrm{SEN}^\theta, \langle N^\theta, F^\theta\rangle\rangle :\\ \exists\, T \in \mathrm{ThFam}_{\mathcal{I}}^{\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle}(\mathrm{SEN}^\theta)(\Omega^{N^\theta}(T) = \Delta^{\mathrm{SEN}^\theta})\}.\end{aligned}$$

The following lemma characterizes, given a π-institution $\mathcal{I}=\langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, the $\mathrm{A}^N(\mathcal{I})$-$N$-congruence systems on SEN.

Lemma 9. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a finitary π-institution. Then we have*

$$\mathrm{Con}^N_{\mathrm{A}^N(\mathcal{I})}(\mathrm{SEN}) = \{\Omega^N(T) : T \in \mathrm{ThFam}(\mathcal{I})\}.$$

Proof. Let $T \in \mathrm{ThFam}(\mathcal{I})$. To see that $\Omega^N(T) \in \mathrm{Con}^N_{\mathrm{A}^N(\mathcal{I})}(\mathrm{SEN})$, it suffices to show that $\langle \mathrm{SEN}^{\Omega^N(T)}, \langle N^{\Omega^N(T)}, F^{\Omega^N(T)}\rangle\rangle \in \mathrm{A}^N(\mathcal{I})$. But, since $T/\Omega^N(T)$ is a theory family of the $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Omega^N(T)}\rangle$-min $(N, N^{\Omega^N(T)})$-model of $\mathcal{I}$ on $\mathrm{SEN}^{\Omega^N(T)}$, such that $\Omega^{N^{\Omega^N(T)}}(T/\Omega^N(T)) = \Delta^{\mathrm{SEN}^{\Omega^N(T)}}$, we do indeed have $\langle \mathrm{SEN}^{\Omega^N(T)}, \langle N^{\Omega^N(T)}, F^{\Omega^N(T)}\rangle\rangle \in \mathrm{A}^N(\mathcal{I})$.

Suppose, conversely, that $\theta \in \mathrm{Con}^N_{\mathrm{A}^N(\mathcal{I})}(\mathrm{SEN})$. Thus, $\langle \mathrm{SEN}^\theta, \langle N^\theta, F^\theta\rangle\rangle \in \mathrm{A}^N(\mathcal{I})$. Hence, there exists a theory family T of the $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle$-min (N, N^θ)-model of $\mathcal{I}$ on SEN^θ, such that $\Omega^{N^\theta}(T) = \Delta^{\mathrm{SEN}^\theta}$. But, then, we have

$$\begin{aligned}\theta &= (\pi^\theta)^{-1}(\Delta^{\mathrm{SEN}^\theta}) \\ &= (\pi^\theta)^{-1}(\Omega^{N^\theta}(T)) \quad \text{(by hypothesis)} \\ &= \Omega^N((\pi^\theta)^{-1}(T)), \quad \text{(by Lemma 5.26 of [18])}\end{aligned}$$

which, since, by Lemma 4.12 of [18], $(\pi^\theta)^{-1}(T) \in \mathrm{ThFam}(\mathcal{I})$, shows that $\theta \in \{\Omega^N(T) : T \in \mathrm{ThFam}(\mathcal{I})\}$. □

This section concludes with Theorem 10, which is an analog in the π-institution framework of Theorem 13 of [11], which is, in turn, an improvement of Theorem 2.17 of [3]. It provides an axiomatization by N-identities and N-quasi-identities of the equivalent N-quasivariety of a syntactically N-algebraizable π-institution $\mathcal{I}$ based on a given axiomatization of the finitary consequence relation of $\mathcal{I}$ via N-axioms and N-rules of inference.

Theorem 10. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a finitary π-institution, with N a category of natural transformations on* SEN*, whose closure system C is generated by a collection* Ax *of N-axioms and a collection* IR *of N-inference rules. Suppose that $\mathcal{I}$ is syntactically N-algebraizable with equivalent N-quasivariety $\mathtt{Q}$. Let $E(x, y)$ be an N-interpretation of $\mathcal{I}^{\mathtt{Q}} = \langle \mathbf{Sign}, \mathrm{SEN}^2, C^{\mathtt{Q}}\rangle$ in $\mathcal{I}$ with $K(x) \approx L(x)$ its inverse N-interpretation of $\mathcal{I}$ in $\mathcal{I}^{\mathtt{Q}}$. Then, the closure system $C^{\mathtt{Q}}$ of the π-institution $\mathcal{I}^{\mathtt{Q}}$ is generated by the following N-axioms and N-inference rules:*

1. *All rules of $C^{N-\mathrm{FEQ}}$;*
2. *$K(\sigma(\vec{x})) \approx L(\sigma(\vec{x}))$, for all $\sigma(\vec{x}) \in \mathrm{Ax}$;*
3. *$\bigwedge_{i<k} K(\sigma^i(\vec{x})) \approx L(\sigma^i(\vec{x})) \to K(\tau(\vec{x})) \approx L(\tau(\vec{x}))$, for all $\langle\{\sigma^i(\vec{x}) : i < k\}, \tau(\vec{x})\rangle \in \mathrm{IR}$;*
4. *$K(E(x, y)) \approx L(E(x, y)) \to x \approx y$.*

Proof. First, it is shown that each one of the listed N-axioms and N-rules is an N-rule of $\mathcal{I}^{\mathtt{Q}}$. Since, by definition all N-rules of $C^{N-\mathrm{FEQ}}$ are also rules of $C^{\mathtt{Q}}$, this is definitely true for 1. For 2, suppose that $\sigma(\vec{x}) \in \mathrm{Ax}$. This means that, for every $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^l$, $\sigma_\Sigma(\vec{\phi}) \in C_\Sigma(\emptyset)$. Therefore, since $K(x) \approx L(x)$ is an N-interpretation of $\mathcal{I}$ in $\mathcal{I}^{\mathtt{Q}}$, we get that $K_\Sigma(\sigma_\Sigma(\vec{\phi})) \approx L_\Sigma(\sigma_\Sigma(\vec{\phi})) \in C^{\mathtt{Q}}_\Sigma(\emptyset)$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^l$. This means that $K(\sigma(\vec{x})) \approx L(\sigma(\vec{x}))$ is an axiom of $\mathcal{I}^{\mathtt{Q}}$. For 3, we work very similarly as for the proof of 2. Finally, 4 follows from Condition (4) of the definition of syntactic N-algebraizability.

Conversely, it must be shown that every N-identity and N-quasi-identity of $\mathtt{Q}$ is a consequence of N-identities and N-quasi-identities listed in 1-4.

It is not difficult to see that $K(E(x,x)) \approx L(E(x,x))$ are all consequences of 1-4. In fact, since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \approx \phi \in C^{\mathtt{Q}}_\Sigma(\emptyset)$, we have that $E_\Sigma(\phi,\phi) \subseteq C_\Sigma(\emptyset)$. Therefore, there exists a Σ-proof of $E_\Sigma(\phi,\phi)$ in C, using Ax and IR. If $K(x) \approx L(x)$ is applied at each element of the Σ-proof sequence, then a Σ-proof of $K_\Sigma(E_\Sigma(\phi,\phi)) \approx L_\Sigma(E_\Sigma(\phi,\phi))$ is obtained using only N-axioms and N-inference rules of the form 1-4. Since, this holds for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, we have that $K(E(x,x)) \approx L(E(x,x))$ is a consequence of 1-4.

Having the N-identities $K(E(x,x)) \approx L(E(x,x))$ at hand, suppose, now, that $\sigma^0,\ldots,\sigma^{k-1},\tau^0,\ldots,\tau^{k-1},\sigma,\tau : \mathrm{SEN}^l \to \mathrm{SEN}$ are in N, such that $\mathtt{Q} \models \bigwedge_{i<k} \sigma^i \approx \tau^i \to \sigma \approx \tau$. $K(E(x,x)) \approx L(E(x,x))$ is derivable from 1-4. Thus, assuming $\sigma^i \approx \tau^i, i<k$, we get that $K(E(\sigma^i,\tau^i)) \approx L(E(\sigma^i,\tau^i))$, for all $i<k$, based on 1-4. But, since $\mathtt{Q} \models \bigwedge_{i<k} \sigma^i \approx \tau^i \to \sigma \approx \tau$ and E is an N-interpretation of $\mathcal{I}^{\mathtt{Q}}$ in $\mathcal{I}$, we get that $\langle\{E(\sigma^i,\tau^i) : i<k\}, E(\sigma,\tau)\rangle$ is an N-rule of $\mathcal{I}$. Therefore, based on 3, we get that $\mathtt{Q} \models \bigwedge_{i<k} K(E(\sigma^i,\tau^i)) \approx L(E(\sigma^i,\tau^i)) \to K(E(\sigma,\tau)) \approx L(E(\sigma,\tau))$. Hence $K(E(\sigma,\tau)) \approx L(E(\sigma,\tau))$ is derivable from $K(E(\sigma^i,\tau^i)) \approx L(E(\sigma^i,\tau^i)), i<k$, based on 1-4. Now an application of rule 4, gives that $\sigma \approx \tau$ is derivable from $\sigma^i \approx \tau^i, i<k$, based only on 1-4. $\square$

5. Uniqueness of the Equivalent Quasivariety

The goal in this section is to establish that the equivalent N-quasivariety semantics associated to a syntactically N-algebraizable π-institution is uniquely determined. We follow very closely, as expected, Section 2.2.1 of the seminal monograph of Blok and Pigozzi [3], where the analog of this result is shown to hold for the case of finitary and finitely algebraizable deductive systems.

We start, consequently, with pointing out a few simple properties of the N-equivalence formulas $E(x,y)$ that derive from the fact that E is an N-interpretation of the equational consequence relation of $\mathcal{I}^{\mathtt{Q}}$ in the deductive apparatus of $\mathcal{I}$. We first start with the way E reflects the fact that equality is a congruence system on any N-algebraic system.

Lemma 11. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, *with* N *a category of natural transformations on* SEN, *is a syntactically* N*-algebraizable* π*-insti-*

tution. Let $\mathtt{Q}$ *be an equivalent* N*-algebraic semantics for* $\mathcal{I}$ *and* $K \approx L$ *and* E*, respectively, a system of defining* N*-equations and* N*-equivalence formulas witnessing the syntactic* N*-algebraizability of* $\mathcal{I}$*. Then, for all* $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ *in* N*, all* $\Sigma \in |\mathbf{Sign}|$*, and all*

$$\phi, \psi, \chi, \phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}(\Sigma),$$

1. $E_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset)$;
2. $E_\Sigma(\psi, \phi) \subseteq C_\Sigma(E_\Sigma(\phi, \psi))$;
3. $E_\Sigma(\phi, \psi) \subseteq C_\Sigma(E_\Sigma(\phi, \chi) \cup E_\Sigma(\chi, \psi))$;
4. $E_\Sigma(\sigma_\Sigma(\phi_0, \ldots, \phi_{n-1}), \sigma_\Sigma(\psi_0, \ldots, \psi_{n-1})) \subseteq C_\Sigma(\bigcup_{i<n} E_\Sigma(\phi_i, \psi_i))$.

Proof. By the definition of $\mathcal{I}^{\mathtt{Q}}$ (more precisely, the fact that $\mathcal{I}^{\mathtt{Q}}$ is an extension of $\mathcal{I}^{N-\mathrm{FEQ}}$), we conclude that, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi, \chi, \phi_0, \ldots, \phi_{n-1}, \psi_0, \ldots, \psi_{n-1} \in \mathrm{SEN}(\Sigma)$,

1. $\langle \phi, \phi \rangle \in C^{\mathtt{Q}}_\Sigma(\emptyset)$;
2. $\langle \psi, \phi \rangle \in C^{\mathtt{Q}}_\Sigma(\langle \phi, \psi \rangle)$;
3. $\langle \phi, \psi \rangle \in C^{\mathtt{Q}}_\Sigma(\langle \phi, \chi \rangle, \langle \chi, \psi \rangle)$;
4. $\langle \sigma_\Sigma(\phi_0, \ldots, \phi_{n-1}), \sigma_\Sigma(\psi_0, \ldots, \psi_{n-1}) \rangle \in C^{\mathtt{Q}}_\Sigma(\langle \phi_0, \psi_0 \rangle, \ldots, \langle \phi_{n-1}, \psi_{n-1} \rangle)$.

An application of the N-interpretation E to these $C^{\mathtt{Q}}$-relationships gives the four C-relationships postulated in the statement of the lemma. □

Next the analog of Lemma 2.14 of [3], providing a detachment (or modus ponens) property for the N-equivalence formulas, is presented.

Lemma 12. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$*, with* N *a category of natural transformations on* SEN*, is a syntactically* N*-algebraizable* π*-institution. Let* $\mathtt{Q}$ *be an equivalent* N*-algebraic semantics for* $\mathcal{I}$ *and* $K \approx L$ *and* E*, respectively, a system of defining* N*-equations and* N*-equivalence formulas witnessing the syntactic* N*-algebraizability of* $\mathcal{I}$*. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi, \psi \in \mathrm{SEN}(\Sigma)$*,*

$$\psi \in C_\Sigma(\phi, E_\Sigma(\phi, \psi)).$$

Proof. Because of the N-rules defining $C^{\mathtt{Q}}$, we have that $K_\Sigma(\psi) \approx L_\Sigma(\psi) \subseteq C^{\mathtt{Q}}_\Sigma(K_\Sigma(\phi) \approx L_\Sigma(\phi), \phi \approx \psi)$. Now, from Condition (4) of the definition of syntactic N-algebraizability, we get that

$$C^{\mathtt{Q}}_\Sigma(\phi \approx \psi) = C^{\mathtt{Q}}_\Sigma(K_\Sigma(E_\Sigma(\phi, \psi)) \approx L_\Sigma(E_\Sigma(\phi, \psi))),$$

which, combined with the previous condition, yields that

$$K_\Sigma(\psi) \approx L_\Sigma(\psi) \subseteq C^{\mathtt{Q}}_\Sigma(K_\Sigma(\phi) \approx L_\Sigma(\phi), K_\Sigma(E_\Sigma(\phi, \psi)) \approx L_\Sigma(E_\Sigma(\phi, \psi))).$$

Finally, Condition (1) of syntactic N-algebraizability yields that $\psi \in C_\Sigma(\phi, E_\Sigma(\phi, \psi))$. □

Theorem 13 is the promised analog of Theorem 2.15 of [3]. It asserts that any two equivalent N-quasivarieties of a syntactically N-algebraizable π-institution are identical. Therefore, any syntactically N-algebraizable π-institution $\mathcal{I}$ has a unique equivalent N-quasivariety.

Theorem 13. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a syntactically N-algebraizable π-institution, with $\mathtt{Q}$ and $\mathtt{Q}'$ two equivalent N-quasivarieties. Then $\mathtt{Q} = \mathtt{Q}'$.*

Let, moreover, $K \approx L, K' \approx L'$ be defining N-equations for $\mathtt{Q}$ and $\mathtt{Q}'$, respectively, and E, E' N-equivalence formulas for $\mathtt{Q}$ and $\mathtt{Q}'$, respectively. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \subseteq \mathrm{SEN}(\Sigma)$,

$$C_\Sigma(E_\Sigma(\phi, \psi)) = C_\Sigma(E'_\Sigma(\phi, \psi))$$

and

$$C^{\mathtt{Q}}_\Sigma(K_\Sigma(\phi) \approx L_\Sigma(\phi)) = C^{\mathtt{Q}}_\Sigma(K'_\Sigma(\phi) \approx L'_\Sigma(\phi)).$$

Proof. It is shown, first, that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(E_\Sigma(\phi, \psi)) = C_\Sigma(E'_\Sigma(\phi, \psi))$. By Lemma 11, Part 4, since E' consists of natural transformations in N, we get that

$$E_\Sigma(E'_\Sigma(\phi, \phi), E'_\Sigma(\phi, \psi)) \subseteq C_\Sigma(E_\Sigma(\phi, \psi)). \tag{5}$$

But, by Lemma 11, Part 1, $E'_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset)$. Therefore, by Lemma 12,

$$\begin{array}{rcll} E'_\Sigma(\phi, \psi) & \subseteq & C_\Sigma(E'_\Sigma(\phi, \phi), E_\Sigma(E'_\Sigma(\phi, \phi), E'_\Sigma(\phi, \psi))) & \text{(by Lemma 12)} \\ & \subseteq & C_\Sigma(E_\Sigma(E'_\Sigma(\phi, \phi), E'_\Sigma(\phi, \psi))) & \text{(by Lemma 11, Part 1)} \\ & \subseteq & C_\Sigma(E_\Sigma(\phi, \psi)). & \text{(by Condition (5))} \end{array}$$

Now, by symmetry, we have $C_\Sigma(E'_\Sigma(\phi,\psi)) = C_\Sigma(E_\Sigma(\phi,\psi))$.

Hence, for all $\Sigma \in |\mathbf{Sign}|$, $\Gamma \approx \Delta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$, we have that

$$\begin{aligned}
\phi \approx \psi \in C^{\mathtt{Q}}_\Sigma(\Gamma \approx \Delta) \quad &\text{iff} \quad E_\Sigma(\phi,\psi) \subseteq C_\Sigma(E_\Sigma(\Gamma,\Delta)) \quad \text{(by Condition (2))}\\
&\text{iff} \quad E'_\Sigma(\phi,\psi) \subseteq C_\Sigma(E'_\Sigma(\Gamma,\Delta))\\
&\qquad\qquad \text{(by what was shown above)}\\
&\text{iff} \quad \phi \approx \psi \in C^{\mathtt{Q}'}_\Sigma(\Gamma \approx \Delta). \quad \text{(by Condition (2))}
\end{aligned}$$

Thus $C^{\mathtt{Q}} = C^{\mathtt{Q}'}$. This also shows that both $\mathtt{Q}$ and $\mathtt{Q}'$ satisfy exactly the same N-equations and N-quasi-equations and, hence, since they are both N-quasivarieties, that $\mathtt{Q} = \mathtt{Q}'$.

Finally, let us see that, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $C^{\mathtt{Q}}_\Sigma(K_\Sigma(\phi) \approx L_\Sigma(\phi)) = C^{\mathtt{Q}}_\Sigma(K'_\Sigma(\phi) \approx L'_\Sigma(\phi))$. We indeed have

$$\begin{aligned}
C_\Sigma(\phi) = C_\Sigma(\phi) \quad &\text{iff} \quad C_\Sigma(E_\Sigma(K_\Sigma(\phi), L_\Sigma(\phi))) = C_\Sigma(E'_\Sigma(K'_\Sigma(\phi), L'_\Sigma(\phi)))\\
&\qquad\qquad \text{(by Condition (3))}\\
&\text{iff} \quad C_\Sigma(E_\Sigma(K_\Sigma(\phi), L_\Sigma(\phi))) = C_\Sigma(E_\Sigma(K'_\Sigma(\phi), L'_\Sigma(\phi)))\\
&\qquad\qquad \text{(as shown above)}\\
&\text{iff} \quad C^{\mathtt{Q}}_\Sigma(K_\Sigma(\phi) \approx L_\Sigma(\phi)) = C^{\mathtt{Q}}_\Sigma(K'_\Sigma(\phi) \approx L'_\Sigma(\phi))\\
&\qquad\qquad \text{(by Condition (2))}.
\end{aligned}$$

□

As Blok and Pigozzi point out, there exist distinct deductive systems with the same algebraic semantics, showing that the same holds for arbitrary institutions as well. They present a concrete example in 5.2.4 of [3].

6. Syntactic Protoalgebraicity and Equivalentiality

Recall from [18] that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is called N-protoalgebraic if, for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi,\psi \in \mathrm{SEN}(\Sigma)$, $\langle \phi,\psi\rangle \in \Omega^N_\Sigma(T)$ implies that $C_\Sigma(T_\Sigma,\phi) = C_\Sigma(T_\Sigma,\psi)$, i.e., for all theory families T and all $\Sigma \in |\mathbf{Sign}|$, whenever $\phi,\psi \in \mathrm{SEN}(\Sigma)$ are congruent modulo the Leibniz N-congruence system of T, then, they are also Σ-interderivable in $\mathcal{I}$ relative to T.

N-protoalgebraicity was characterized in Lemma 3.8 of [18] as being equivalent to the condition that the N-Leibniz operator is monotone on the theory families of $\mathcal{I}$. That is $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is N-protoalgebraic if and only if, for all $T^1, T^2 \in \mathrm{ThFam}(\mathcal{I})$, if $T^1 \leq T^2$, then $\Omega^N(T^1) \leq \Omega^N(T^2)$.

In Corollary 4.20 of [18] another characterization of N-protoalgebraicity was provided for the special case of π-institutions that are finitary and N-rule based. First, recall that a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, has the family N-correspondence property if, for every functor $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', every surjective (N, N')-epimorphic translation $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, every theory family T of $\mathcal{I}$ and all $\Sigma \in |\mathbf{Sign}|$,

$$\alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\alpha_\Sigma(T_\Sigma))) = C_\Sigma(T_\Sigma \cup \alpha_\Sigma^{-1}(C'_{F(\Sigma)}(\emptyset))),$$

where by $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is denoted the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'.

With this definition in mind, Corollary 4.20 of [18] states that a finitary and N-rule based π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where N is a category of natural transformations on SEN, is N-protoalgebraic if and only if it has the family N-correspondence property.

Note, also, that this property implies that, for all $\mathrm{SEN}' : \mathbf{Sign}' \to \mathbf{Set}$, with N' a category of natural transformations on SEN', and all surjective $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$, if $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN' and $T' \in \mathrm{ThFam}(\mathcal{I}')$, then

$$\alpha^{-1}([T')_{\mathbf{ThFam}(\mathcal{I}')}) = [\alpha^{-1}(T'))_{\mathbf{ThFam}(\mathcal{I})}.$$

Recall, now, from Section 3 of [19] that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, a collection $\Delta = \{\delta^i : i \in I\}$ of natural transformations $\delta^i : \mathrm{SEN}^2 \to \mathrm{SEN}, i \in I$, in N is called an N-implication system or an N-protoequivalence system for $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset) \quad \text{and} \quad \psi \in C_\Sigma(\phi, \Delta_\Sigma(\phi, \psi)),$$

i.e., if and only if $\delta^i(x, x), i \in I$, are N-axioms of $\mathcal{I}$ and $\langle\{x, \Delta(x, y)\}, y\rangle$ is an N-rule of inference of $\mathcal{I}$. The first condition is referred to as Δ-reflexivity and the second as Δ-modus ponens or Δ-detachment.

It has been shown in Proposition 3.2 of [19] that if a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, has an N-implication system, then it is N-protoalgebraic. Let us call those π-institutions, as above, that possess an N-implication system **syntactically N-protoalgebraic**. It has been conjectured that it is not true, in general, that every N-protoalgebraic π-institution is syntactically N-protoalgebraic, i.e., that syntactic N-protoalgebraicity is a properly stronger property than N-protoalgebraicity, which may be termed (in contrast to the syntactic case but also in reference to the use of the N-Leibniz operator in its definition) **semantic N-protoalgebraicity**. In some sense, these two notions coincide when one restricts attention to sentential logics (see, e.g., Theorem 1.1.3 of [9]).

Next, we switch from the presentation of N-protoalgebraic π-institutions and of syntactically N-protoalgebraic π-institutions to the study of (semantically) N-equivalential and of syntactically N-equivalential π-institutions. At the level of deductive systems, equivalential deductive systems were first introduced in [17] and, later, extensively studied by Czelakowski in [7, 8]. At the categorical level an analogous study has been carried out by the author in [22], following both the original work of Czelakowski in [7, 8] and his subsequent exposition of equivalentiality in the context of protoalgebraic logics in Chapter 3 of [9].

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ is a π-institution, with N a category of natural transformations on SEN. A collection $E = \{\epsilon^i : i \in I\}$, with $\epsilon^i : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, for all $i \in I$, is said to be an N-equivalence system for $\mathcal{I}$ if, for all $\sigma : \mathrm{SEN}^n \to \mathrm{SEN}$ in N, $\Sigma \in |\mathbf{Sign}|$, $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$ and all theory families T of $\mathcal{I}$,

1. $E_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset)$;
2. If $(\forall i)(\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)^2(\phi_i, \psi_i)) \subseteq T_{\Sigma'})$, then
$$(\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)^2(\sigma_\Sigma(\vec{\phi}), \sigma_\Sigma(\vec{\psi}))) \subseteq T_{\Sigma'});$$
3. $\phi \in T_\Sigma$ and $(\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)^2(\phi, \psi)) \subseteq T_{\Sigma'})$ imply $\psi \in T_\Sigma$;

where $(\forall f)$ is an abbreviation $(\forall f) := (\forall \Sigma' \in |\mathbf{Sign}|)(\forall f \in \mathbf{Sign}(\Sigma, \Sigma'))$.

In general, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, and any collection E of natural transformations $\mathrm{SEN}^2 \to \mathrm{SEN}$ in N and $T \in \mathrm{ThFam}(\mathcal{I})$, the relation system

$E(T) = \{E_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ on SEN is defined by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$E_\Sigma(T) = \{\langle\phi,\psi\rangle \in \mathrm{SEN}(\Sigma)^2 : (\forall f)(E_{\Sigma'}(\mathrm{SEN}(f)^2(\phi,\psi)) \subseteq T_{\Sigma'})\}. \quad (6)$$

If E is an N-equivalence system for $\mathcal{I}$, then $E(T)$ is in fact an N-congruence system on SEN that is compatible with T, for every theory family T of $\mathcal{I}$. One may infer from this and the fact that, in case $E(T)$ happens to be an N-congruence system on SEN that is compatible with T, then it necessarily coincides with the N-Leibniz congruence system $\Omega^N(T)$, that E is an N-equivalence system for $\mathcal{I}$ if and only if, for every $T \in \mathrm{ThFam}(\mathcal{I})$, $E(T) = \Omega^N(T)$. This characterization of a collection of binary natural transformations in N constituting an N-equivalence system for $\mathcal{I}$ was presented in Theorem 5 of [22].

Proposition 9 of [22] states that, if $\mathcal{I}$ has an N-equivalence system, then it is necessarily N-protoalgebraic.

Two other important characterization theorems for the existence of an N-equivalence system, that are proved in [22] (see Theorems 13 and 15 of [22]) are now revisited to give the reader a better feeling about the properties of the class of π-institutions possessing an N-equivalence system. They both take after corresponding results holding at the deductive system level (see Theorems 3.3.1 and 3.3.3 of [9]).

Let $\mathcal{I} = \langle\mathbf{Sign},\mathrm{SEN},C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Denote by $E^{\max}$ the subcollection of all natural transformations $\epsilon : \mathrm{SEN}^2 \to \mathrm{SEN}$ in N, such that, for all $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$, $\epsilon_\Sigma(\phi,\phi) \in C_\Sigma(\emptyset)$, i.e.,

$$\begin{aligned} E^{\max} = \{&\epsilon : \mathrm{SEN}^2 \to \mathrm{SEN} \text{ in } N :\\ &(\forall\Sigma \in |\mathbf{Sign}|)(\forall\phi \in \mathrm{SEN}(\Sigma))(\epsilon_\Sigma(\phi,\phi) \in C_\Sigma(\emptyset))\}.\end{aligned}$$

Theorem 13 of [22] asserts that the π-institution $\mathcal{I}$ has an N-equivalence system E if and only if $E^{\max}$ is also an N-equivalence system of $\mathcal{I}$. This yields as a corollary that, if $\mathcal{I}$ has an N-equivalence system, then $E^{\max}$ is its largest N-equivalence system. That is also the reason why the notation $E^{\max}$ has been adopted for this collection of binary natural transformations in N.

Theorem 15 of [22], on the other hand, provides a characterization of those π-institutions possessing an N-equivalence system inside the class of N-protoalgebraic π-institutions. The corresponding result for deductive systems is the so-called Herrmann's Test [14].

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on SEN, and E a collection of natural transformations $\mathrm{SEN}^2 \to \mathrm{SEN}$ in N. For all $\Sigma_0 \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma_0)$, define the theory system $E^{\langle \Sigma_0,\phi,\psi\rangle} = \{E_\Sigma^{\langle \Sigma_0,\phi,\psi\rangle}\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$E_\Sigma^{\langle \Sigma_0,\phi,\psi\rangle} = C_\Sigma(\{E_\Sigma(\mathrm{SEN}(f)^2(\phi,\psi)) : f \in \mathbf{Sign}(\Sigma_0,\Sigma)\}).$$

It was shown in Proposition 14 of [22] that $E^{\langle \Sigma_0,\phi,\psi\rangle}$ is in fact a theory system of $\mathcal{I}$, for all $\Sigma_0 \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma_0)$.

Herrmann's Test for π-institutions states that, if $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is an N-protoalgebraic π-institution, and E a collection of natural transformations $\mathrm{SEN}^2 \to \mathrm{SEN}$ in N, then E is an N-equivalence system for $\mathcal{I}$ if and only if, for all $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$E_\Sigma(\phi,\phi) \subseteq C_\Sigma(\emptyset) \quad \text{and} \quad \langle \phi,\psi\rangle \in \Omega_\Sigma^N(E^{\langle \Sigma,\phi,\psi\rangle}).$$

Recall from Theorem 3.3.4 of [9] that a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ is equivalential if and only if the Leibniz operator Ω is monotone and commutes with inverse substitutions on the lattice $\mathrm{Th}\mathcal{S}$ of all theories of $\mathcal{S}$, i.e., if and only if it is protoalgebraic and the Leibniz operator Ω commutes with inverse substitutions on the lattice $\mathrm{Th}\mathcal{S}$ of all theories of $\mathcal{S}$. As was the case with Proposition 3.2 of [19], that gave rise to the distinction between syntactic N-protoalgebraicity and (semantic) N-protoalgebraicity, in Theorem 16 of [22], an analog of only one of the two directions of this characterization of equivalentiality for deductive systems seems to hold for arbitrary π-institutions. Namely, Theorem 16 of [22] asserts that, if $\mathcal{I}$ has an N-equivalence system E, then, the N-Leibniz operator Ω^N is monotone on theory families (i.e., $\mathcal{I}$ is N-protoalgebraic) and, for every (N,N)-logical morphism $\langle F,\alpha\rangle : \mathcal{I} \vdash^{se} \mathcal{I}$, with F surjective,

$$\alpha^{-1}(\Omega^N(T)) = \Omega^N(\alpha^{-1}(T)), \quad \text{for all } T \in \mathrm{ThFam}(\mathcal{I}).$$

This property is viewed in the π-institution framework as an analog of the commutativity of the Leibniz operator with inverse substitutions in the deductive system framework. This result gives in the context of N-equivalentiality another duality, very similar to the one obtained for N-protoalgebraicity. More precisely, because of this result, a π-institution

$\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is said to be (**finitely**) **syntactically** N**-equivalential** if it has a (finite) N-equivalence system and **semantically** N**-equivalential** if it is N-protoalgebraic and, for every (N,N)-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \rangle\!-^{se} \mathcal{I}$, with F surjective,

$$\alpha^{-1}(\Omega^N(T)) = \Omega^N(\alpha^{-1}(T)), \quad \text{for all } T \in \mathrm{ThFam}(\mathcal{I}).$$

7. Regularly Algebraizable π-Institutions

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a syntactically N-equivalential π-institution, with E an N-equivalence system for $\mathcal{I}$. $\mathcal{I}$ is said to satisfy the E-**G-rule** or the **G-rule relative to** E, if, for all $\Sigma \in |\mathbf{Sign}|$, all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, all $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'},$$

or, equivalently, using Definition (6) of the relation system $E(T)$ associated with the theory family T, if, for all $\Sigma \in |\mathbf{Sign}|$, all $\phi, \psi \in \mathrm{SEN}(\Sigma)$ and all $T \in \mathrm{ThFam}(\mathcal{I})$,

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \langle \phi, \psi\rangle \in E_\Sigma(T).$$

Note that the E-G-rule is not an N-inference rule of $\mathcal{I}$, since it cannot be entirely expressed in terms of natural transformations belonging to N. It needs, in addition, a universal quantification over all signature morphisms from Σ. This contrasts with the G-rule in the context of deductive systems.

Recall from Section 3 of [18] that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$ and a theory system T of $\mathcal{I}$, $T^{\langle \Sigma, \Phi\rangle}$ denotes the least theory system T' of $\mathcal{I}$, such that $T \leq T'$ and $\Phi \subseteq T'_\Sigma$. It is given, for all $\Sigma' \in |\mathbf{Sign}|$, by

$$T^{\langle \Sigma, \Phi\rangle}_{\Sigma'} = C_{\Sigma'}(T_{\Sigma'} \cup \{\mathrm{SEN}(f)(\Phi) : f \in \mathbf{Sign}(\Sigma, \Sigma')\}). \tag{7}$$

Also recall that, given a π-institution $\mathcal{I}$, by Thm (or Thm($\mathcal{I}$) if there are many π-institutions considered in the same context) is denoted the theorem system of $\mathcal{I}$, i.e., for all $\Sigma \in |\mathbf{Sign}|$, $\mathrm{Thm}_\Sigma = C_\Sigma(\emptyset)$.

As was pointed out before, it follows from the definition of the N-G-rule that, if $\mathcal{I}$ is syntactically N-equivalential, with an N-equivalent system E, then, for all $\Sigma \in |\mathbf{Sign}|$, all $\phi, \psi \in \mathrm{SEN}(\Sigma)$ and all theory families T of $\mathcal{I}$,

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \begin{array}{l} E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'} \\ \text{for all } \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma'), \end{array}$$

is equivalent, by the definition of $E(T)$, to

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \langle \phi, \psi \rangle \in E_\Sigma(T),$$

which is, in turn, equivalent, by the characterization of N-equivalence systems (Theorem 5 of [22]), to

$$\phi, \psi \in T_\Sigma \quad \text{implies} \quad \langle \phi, \psi \rangle \in \Omega^N_\Sigma(T). \tag{8}$$

Therefore, if E and E' are two N-equivalence systems for the syntactically N-equivalential π-institution $\mathcal{I}$, $\mathcal{I}$ possesses the E-G-rule if and only if it possesses the E'-G-rule. Therefore, we are, in this context as well, free to use the expression "$\mathcal{I}$ has the G-rule" to refer to any of the many possible E-G-rules that may hold in $\mathcal{I}$, as has been a common convention in abstract algebraic logic.

If $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ satisfies the E-G-rule, then, as was shown in Implication (8), for every theory family T of $\mathcal{I}$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, if $\phi, \psi \in T_\Sigma$, then $\langle \phi, \psi \rangle \in \Omega^N_\Sigma(T)$. If, conversely, for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$, $\phi, \psi \in T_\Sigma$ implies that $\langle \phi, \psi \rangle \in \Omega^N_\Sigma(T)$, then, by following the reverse steps in the reasoning proving Implication (8), we can show that the G-rule holds in $\mathcal{I}$. As a consequence, we have that the G-rule holds in a syntactically N-equivalential π-institution if and only if, for every $T \in \mathrm{ThFam}(\mathcal{I})$ and all $\Sigma \in |\mathbf{Sign}|$, T_Σ must be an equivalence class of the equivalence relation $\Omega^N_\Sigma(T)$.

An alternative characterization of the G-rule in a syntactically N-equivalential π-institution $\mathcal{I}$ is provided by the following lemma, which forms an analog for π-institutions of Lemma 27 of [11].

Proposition 14. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on* SEN*, be a syntactically N-equivalential π-institution, with E an N-equivalence system. The E-G-rule holds in $\mathcal{I}$ if and only if,* *for every* $T \in \mathrm{ThFam}(\mathcal{I})$*, all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$*,* $\phi \in T_\Sigma$ *if and only if* $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(t)) \subseteq T_{\Sigma'}$*, for all* $\Sigma' \in |\mathbf{Sign}|$*,* $f \in \mathbf{Sign}(\Sigma, \Sigma')$*, where* t *is an arbitrary* Σ*-theorem of* $\mathcal{I}$*.*

Proof. Suppose, first, that the E-G-rule holds in $\mathcal{I}$ and that $\Sigma \in |\mathbf{Sign}|, \phi \in \mathrm{SEN}(\Sigma)$ and $t \in \mathrm{Thm}_\Sigma$. If $\phi \in T_\Sigma$, then, since $t \in \mathrm{Thm}_\Sigma$, $\phi, t \in T_\Sigma$, whence, by the G-rule, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(t)) \subseteq T_{\Sigma'}.$$

If, on the other hand, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$,

$$E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(t)) \subseteq T_{\Sigma'},$$

then, we have, by the definition of $E(T)$ and the characterization Theorem 5 of [22] of N-equivalence systems, that $\langle \phi, t\rangle \in E_\Sigma(T) = \Omega^N_\Sigma(T)$, whence, using the fact that $t \in \mathrm{Thm}_\Sigma \subseteq T_\Sigma$ and the compatibility of $\Omega^N(T)$ with T, we get that $\phi \in T_\Sigma$.

Suppose, conversely, that, for every $T \in \mathrm{ThFam}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(t)) \in T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and some $t \in \mathrm{Thm}_\Sigma$. Let $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|, \phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\phi, \psi \in T_\Sigma$. Then, by hypothesis, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(t)) \subseteq T_{\Sigma'}$ and $E_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(t)) \subseteq T_{\Sigma'}$. But, since E is an N-equivalence system, $E(T)$ is symmetric and transitive, whence we obtain that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$ $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$. Therefore, the E-G-rule holds in $\mathcal{I}$. □

In the sequel attention will be restricted to a special class of finitary and finitely syntactically N-equivalential π-institutions. To introduce this class let us first call a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ **systemic** if, for all $\Sigma \in |\mathbf{Sign}|$ and all Σ-theories X_Σ of $\mathcal{I}$, there exists a theory system $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, such that $T_\Sigma = X_\Sigma$. Notice that every Σ-theory is trivially the Σ-component of some theory family of $\mathcal{I}$, but it need not necessarily be the Σ-component of a theory system of $\mathcal{I}$. If this happens for all signatures Σ and all Σ-theories, then $\mathcal{I}$ is termed systemic. On the other hand, call a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ **theory invariant** if, for every $\Sigma \in |\mathbf{Sign}|$ and every Σ-theory X_Σ,

$$\mathrm{SEN}(f)(X_\Sigma) \subseteq X_\Sigma, \quad \text{for all } f \in \mathbf{Sign}(\Sigma, \Sigma),$$

i.e., $\mathcal{I}$ is theory invariant if every signature morphism from Σ to itself preserves all Σ-theories, for every $\Sigma \in |\mathbf{Sign}|$.

It will be shown, next, that systemicity and theory invariance are in fact equivalent conditions.

Proposition 15. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution. Then $\mathcal{I}$ is systemic if and only if it is theory invariant.*

Proof. Let $\Sigma_0 \in |\mathbf{Sign}|$ and X_{Σ_0} a Σ_0-theory. Define the collection $T = \{T_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$,

$$T_\Sigma = C_\Sigma(\bigcup\{\mathrm{SEN}(f)(X_{\Sigma_0}) : f \in \mathbf{Sign}(\Sigma_0, \Sigma)\}).$$

Denoting the theorem system of $\mathcal{I}$ by Thm, T is the same collection that was denoted by $\mathrm{Thm}^{\langle \Sigma_0, X_{\Sigma_0}\rangle}$ in Section 3 of [18] and whose definition was recalled in Definition (7). It was shown in Lemma 3.6 of [18] that it is the smallest theory system of $\mathcal{I}$, such that $X_{\Sigma_0} \subseteq T_{\Sigma_0}$. With this in mind, notice that X_{Σ_0} is the Σ_0-component of a theory system if and only if $X_{\Sigma_0} = T_{\Sigma_0}$ if and only if $X_{\Sigma_0} = C_{\Sigma_0}(\bigcup\{\mathrm{SEN}(f)(X_{\Sigma_0}) : f \in \mathbf{Sign}(\Sigma_0, \Sigma_0)\})$ if and only if, for all $f \in \mathbf{Sign}(\Sigma_0, \Sigma_0)$, $\mathrm{SEN}(f)(X_{\Sigma_0}) \subseteq X_{\Sigma_0}$. Therefore $\mathcal{I}$ is systemic if and only if it is theory invariant. $\square$

The following theorem will be very handy in investigating the relationship between syntactically N-equivalential π-institutions satisfying some additional conditions and syntactically N-algebraizable deductive systems. It states that, if a π-institution is finitely syntactically N-equivalential, with E an N-equivalence system for $\mathcal{I}$, then it satisfies a condition similar to Condition (2) of the definition of syntactic N-algebraizability relative to the N-quasivariety $\mathtt{A}^N_{\mathcal{I}} = \mathbf{Q}(\mathrm{A}^N(\mathcal{I}))$ generated by the class $\mathrm{A}^N(\mathcal{I})$.

Theorem 16. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a finitely syntactically N-equivalential π-institution with E an N-equivalence system for $\mathcal{I}$. Let $\mathtt{A}^N_{\mathcal{I}} = \mathbf{Q}(\mathrm{A}^N(\mathcal{I}))$ be the N-quasivariety of N-algebraic systems generated by the class $\mathrm{A}^N(\mathcal{I})$ and assume that the equational π-institution $\mathcal{I}^{\mathtt{A}^N_{\mathcal{I}}}$ is theory invariant. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Gamma \approx \Delta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$,*

$$\phi \approx \psi \in C^{\mathtt{A}^N_{\mathcal{I}}}_\Sigma(\Gamma \approx \Delta) \quad \textit{iff} \quad \Gamma \approx \Delta \in E_\Sigma(T) \Rightarrow \phi \approx \psi \in E_\Sigma(T),$$
$$\textit{for every } T \in \mathrm{ThFam}(\mathcal{I}).$$

Proof. Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a finitely syntactically N-equivalential π-institution, with E an N-equivalence system for $\mathcal{I}$, $\Sigma \in |\mathbf{Sign}|$ and $\Gamma \approx \Delta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$. Then we have that $\phi \approx \psi \notin C_\Sigma^{\mathtt{A}_\mathcal{I}^N}(\Gamma \approx \Delta)$ if and only if, there exists, by definition, $\theta \in \mathrm{ThFam}(\mathcal{I}^{\mathtt{A}_\mathcal{I}^N})$, such that $\Gamma \approx \Delta \subseteq \theta_\Sigma$ but $\phi \approx \psi \notin \theta_\Sigma$. This is equivalent, by Proposition 6, to the existence of $\theta \in \mathrm{Conf}^N_{\mathtt{A}_\mathcal{I}^N}(\mathrm{SEN})$, such that $\Gamma \approx \Delta \subseteq \theta_\Sigma$ but $\phi \approx \psi \notin \theta_\Sigma$, which holds, by Lemma 9 and theory invariance, if and only if, there exists $T \in \mathrm{ThFam}(\mathcal{I})$, such that $\Gamma \approx \Delta \in \Omega^N_\Sigma(T)$ but $\phi \approx \psi \notin \Omega^N_\Sigma(T)$. This, finally, is equivalent, in view of the fact that E is an N-equivalence system for $\mathcal{I}$, to the existence of a theory family T of $\mathcal{I}$, such that $\Gamma \approx \Delta \subseteq E_\Sigma(T)$ but $\phi \approx \psi \notin E_\Sigma(T)$. $\square$

Suppose, now, that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, is a theory invariant and finitely syntactically N-equivalential π-institution, with E a (finite) N-equivalence system for $\mathcal{I}$. Note that, in this case, since, for every $\Sigma \in |\mathbf{Sign}|$ and every Σ-theory T_Σ, T_Σ is the Σ-component of a theory system of $\mathcal{I}$, instead of quantifying over theory families in a specific context, we may quantify, equivalently, only over theory systems. But for a theory system T, the condition $\langle \phi, \psi\rangle \in E_\Sigma(T)$, which is, by definition, equivalent to $E_{\Sigma'}(\mathrm{SEN}(f)(\phi), \mathrm{SEN}(f)(\psi)) \subseteq T_{\Sigma'}$, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma, \Sigma')$, is also equivalent to the much simpler condition $E_\Sigma(\phi, \psi) \subseteq T_\Sigma$, that does not need any quantifications over signatures and signature morphisms.

These observations lead to simplified versions of both Proposition 14 and Theorem 16 in the context of theory invariant π-institutions. We present them here without proof and use them in the formulation of Theorem 19 below.

Proposition 17 (Systemic Version of Proposition 14)**.** *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, be a theory invariant, syntactically N-equivalential π-institution, with E an N-equivalence system. The E-G-rule holds in $\mathcal{I}$ if and only if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(E_\Sigma(\phi, t))$, where t is an arbitrary Σ-theorem of $\mathcal{I}$, i.e., if and only if, for every $T \in \mathrm{ThSys}(\mathcal{I})$, all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $\phi \in T_\Sigma$ if and only if $E_\Sigma(\phi, t) \subseteq T_\Sigma$, where t is an arbitrary Σ-theorem of $\mathcal{I}$.*

Theorem 18 (Systemic Version of Theorem 16). *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is a theory invariant, finitely syntactically* N*-equivalential* π*-institution with* E *an* N*-equivalence system for* $\mathcal{I}$*. Furthermore, assume that the equational* π*-institution* $\mathcal{I}^{\mathtt{A}^N_{\mathcal{I}}}$ *associated with the* N*-quasivariety* $\mathtt{A}^N_{\mathcal{I}} = \mathbf{Q}(\mathrm{A}^N(\mathcal{I}))$ *is theory invariant. Then, for all* $\Sigma \in |\mathbf{Sign}|$ *and all* $\Gamma \approx \Delta \cup \{\phi \approx \psi\} \subseteq \mathrm{SEN}(\Sigma)^2$*,*

$$\phi \approx \psi \in C^{\mathtt{A}^N_{\mathcal{I}}}_{\Sigma}(\Gamma \approx \Delta) \quad \textit{iff} \quad E_\Sigma(\phi, \psi) \subseteq C_\Sigma(E_\Sigma(\Gamma, \Delta)).$$

The following result, an analog of a result first proved as Corollary 4.8 of [3] and then revisited as Theorem 28 of [11], shows that a finitary, theory invariant and finitely syntactically N-equivalential π-institution $\mathcal{I}$, that

- has the G-rule,
- is such that N contains a constant natural transformation $\top : \mathrm{SEN} \to \mathrm{SEN}$, with $\top_\Sigma(\phi) \in \mathrm{Thm}_\Sigma$, for every $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, and
- is such that $\mathcal{I}^{\mathtt{A}^N_{\mathcal{I}}}$ is theory invariant

is syntactically N-algebraizable. In that case, if $\mathtt{Q}$ is the equivalent N-quasivariety of $\mathcal{I}$, then the N-interpretation of $\mathcal{I}$ in $\mathcal{I}^{\mathtt{Q}}$ is given by $\{x \approx \top(x)\}$.

Theorem 19. *Let* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, be a finitary, theory invariant and finitely syntactically* N*-equivalential* π*-institution* $\mathcal{I}$*, that has the G-rule, is such that* N *contains a constant natural transformation* $\top : \mathrm{SEN} \to \mathrm{SEN}$*, with* $\top_\Sigma(\phi) := \top_\Sigma \in \mathrm{Thm}_\Sigma$*, for every* $\Sigma \in |\mathbf{Sign}|$ *and all* $\phi \in \mathrm{SEN}(\Sigma)$ *and such that* $\mathcal{I}^{\mathtt{A}^N_{\mathcal{I}}}$ *is theory invariant. Then,* $\mathcal{I}$ *is syntactically* N*-algebraizable. Moreover, if* $\mathtt{Q}$ *is the equivalent* N*-quasivariety of* $\mathcal{I}$*, then the* N*-interpretation of* $\mathcal{I}$ *in* $\mathcal{I}^{\mathtt{Q}}$ *is given by* $\{x \approx \top(x)\}$*.*

Proof. By Theorem 18, Condition (2) of syntactic N-algebraizability is satisfied with $\mathtt{Q} = \mathtt{A}^N_{\mathcal{I}}$, since $\mathcal{I}$ is assumed to be theory invariant and syntactically N-equivalential, with E an N-equivalence system, and with $\mathcal{I}^{\mathtt{A}^N_{\mathcal{I}}}$ theory invariant. So it suffices to show that Condition (3) is also satisfied.

But, if $K(x) \approx L(x) = \{x \approx \top(x)\}$, then Condition (3) takes the form, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, $C_\Sigma(\phi) = C_\Sigma(E_\Sigma(\phi, \top_\Sigma))$, which is the condition established in Proposition 17 for $\mathcal{I}$, under the presence of the G-rule. $\square$

A finitary, theory invariant and finitely syntactically N-equivalential π-institution $\mathcal{I}$, that has the G-rule, is such that N contains a constant natural transformation $\top : \mathrm{SEN} \to \mathrm{SEN}$, with $\top_\Sigma(\phi) := \top_\Sigma \in \mathrm{Thm}_\Sigma$, for every $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, and such that $\mathcal{I}^{\mathbb{A}^N_\mathcal{I}}$ is theory invariant will be called **regularly** N**-algebraizable**. In that case, the constant natural transformation $\top$ will be called an N**-top**.

Since, by Theorem 19, $\{x \approx \top(x)\}$ is a defining equation for a regularly N-algebraizable π-institution $\mathcal{I}$, Theorem 10 yields immediately the following axiomatization result, which forms an analog in the framework of π-institutions of Theorem 30 of [11].

Theorem 20. *Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, is a finitary π-institution, whose closure system is generated by a collection* Ax *of N-axioms and a collection* IR *of N-inference rules. Assume that $\mathcal{I}$ is regularly N-algebraizable, with a finite N-equivalence system E and N-top $\top$. Then, the unique equivalent N-quasivariety of $\mathcal{I}$ is defined by the following N-identities and N-quasi-identities:*

1. *All rules of* $C^{N-\mathrm{FEQ}}$;
2. $\sigma \approx \top$, *for all* $\sigma(\vec{x}) \in \mathrm{Ax}$;
3. $\bigwedge_{i<k} \sigma^i \approx \top \to \tau \approx \top$, *for all* $\langle\{\sigma^i(\vec{x}) : i < k\}, \tau(\vec{x})\rangle \in \mathrm{IR}$;
4. $E(x, y) \approx \top \to x \approx y$.

8. Relatively Point-Regular Quasivarieties

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN, and $\mathtt{Q}$ an N-quasivariety. If there exists a natural transformation $\sigma : \mathrm{SEN}^k \to \mathrm{SEN}$ in N, such that, for all $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle \in \mathtt{Q}$, $\sigma'_\Sigma(\vec{\phi}) = \sigma'_\Sigma(\vec{\psi})$, for all $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi}, \vec{\psi} \in \mathrm{SEN}'(\Sigma)^k$, then σ is said to be a **constant** N**-term** and $\mathtt{Q}$ is said to be **pointed**. The constant term will be usually denoted by $\top$ and will be called the N**-top**.

A pointed N-quasivariety $\mathtt{Q}$, as above, is called **relatively point-regular** if each $\mathtt{Q}$-N-congruence system θ on SEN is uniquely determined by the family of its $\top$-equivalence classes $\{\top_\Sigma/\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$.

Lemma 21. *Suppose* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *is a functor, with* N *a category of natural transformations on* SEN, *and* $\mathtt{Q}$ *is a relatively point-regular* N*-quasivariety. Then, for every* N*-algebraic system* $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$*, such that there exists at least one surjective* $\langle F, \alpha\rangle : \mathrm{SEN} \to^{se} \mathrm{SEN}'$*, every* $\mathtt{Q}$-N'*-congruence system* θ *on* $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ *is completely determined by the family of its* $\top'$*-equivalence classes.*

Proof. Let θ, θ' be two $\mathtt{Q}$-N'-congruence systems on $\langle \mathrm{SEN}', \langle N', F'\rangle\rangle$ and suppose that, for all $\Sigma \in |\mathbf{Sign}|$, $\top'_{F(\Sigma)}/\theta_{F(\Sigma)} = \top'_{F(\Sigma)}/\theta'_{F(\Sigma)}$. Then

$$\alpha_\Sigma(\top_\Sigma)/\theta_{F(\Sigma)} = \alpha_\Sigma(\top_\Sigma)/\theta'_{F(\Sigma)}.$$

This gives that $\alpha_\Sigma^{-1}(\alpha_\Sigma(\top_\Sigma)/\theta_{F(\Sigma)}) = \alpha_\Sigma^{-1}(\alpha_\Sigma(\top_\Sigma)/\theta'_{F(\Sigma)})$. Therefore, we get that $\top_\Sigma/\alpha_\Sigma^{-1}(\theta_{F(\Sigma)}) = \top_\Sigma/\alpha_\Sigma^{-1}(\theta'_{F(\Sigma)})$. But, it is not difficult to see that, since θ is a $\mathtt{Q}$-N'-congruence system on SEN' and $\langle F, \alpha\rangle$ is surjective, $\alpha^{-1}(\theta)$ is also a $\mathtt{Q}$-N-congruence system on SEN. Therefore, since $\mathtt{Q}$ is relatively point-regular, we must have that $\alpha_\Sigma^{-1}(\theta_{F(\Sigma)}) = \alpha_\Sigma^{-1}(\theta'_{F(\Sigma)})$, for all $\Sigma \in |\mathbf{Sign}|$. This yields that $\alpha_\Sigma(\alpha_\Sigma^{-1}(\theta_{F(\Sigma)})) = \alpha_\Sigma(\alpha_\Sigma^{-1}(\theta'_{F(\Sigma)}))$, for all $\Sigma \in |\mathbf{Sign}|$, which, by the surjectivity of $\langle F, \alpha\rangle$, gives that $\theta_{F(\Sigma)} = \theta'_{F(\Sigma)}$, i.e., that $\theta = \theta'$. Hence θ is uniquely determined by $\{\top'_\Sigma/\theta_\Sigma\}_{\Sigma\in|\mathbf{Sign}'|}$. $\square$

Let $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ be a functor, with N a category of natural transformations on SEN, and $\mathtt{Q}$ a pointed N-quasivariety. Define the triple $\mathcal{I}^{\langle \mathtt{Q},\top\rangle} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\langle \mathtt{Q},\top\rangle}\rangle$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\begin{array}{lll}\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi) & \text{iff} & \mathrm{SEN}(f)(\Phi) \subseteq \top_{\Sigma'}/\theta_{\Sigma'} \text{ implies } \mathrm{SEN}(f)(\phi) \in \top_{\Sigma'}/\theta_{\Sigma'}, \\ & & \text{for all } \theta \in \mathrm{Conf}_{\mathtt{Q}}^N(\mathrm{SEN}), \Sigma' \in |\mathbf{Sign}|, f \in \mathbf{Sign}(\Sigma, \Sigma').\end{array}$$

It will be shown, next, that, thus defined, $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ is a finitary π-institution.

Proposition 22. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with* N *a category of natural transformations on* SEN, *and* $\mathtt{Q}$ *a pointed* N*-quasivariety. Then* $\mathcal{I}^{\langle \mathtt{Q},\top\rangle} = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\langle \mathtt{Q},\top\rangle}\rangle$ *is a* π*-institution.*

Proof. It is very easy to see that, for $\Sigma \in |\mathbf{Sign}|$, the operator $C_\Sigma^{\langle \mathtt{Q},\top\rangle}$ is reflexive and monotone. To see that it is idempotent, suppose $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, such that $\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi))$. Then we have that, for all $\theta \in \mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f)(C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi)) \subseteq \top_{\Sigma'}/\theta_{\Sigma'}$ implies that $\mathrm{SEN}(f)(\phi) \in \top_{\Sigma'}/\theta_{\Sigma'}$. Thus, again by the definition of $C^{\langle \mathtt{Q},\top\rangle}$, we get that, if $\mathrm{SEN}(f)(\Phi) \subseteq \top_{\Sigma'}/\theta_{\Sigma'}$, then $\mathrm{SEN}(f)(C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi)) \subseteq \top_{\Sigma'}/\theta_{\Sigma'}$, whence, by the hypothesis, $\mathrm{SEN}(f)(\phi) \in \top_{\Sigma'}/\theta_{\Sigma'}$. Therefore $\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi)$, showing that $C_\Sigma^{\langle \mathtt{Q},\top\rangle}$ is idempotent. Therefore $C_\Sigma^{\langle \mathtt{Q},\top\rangle}$ is a closure operator on $\mathrm{SEN}(\Sigma)$, for all $\Sigma \in |\mathbf{Sign}|$.

To see that $C^{\langle \mathtt{Q},\top\rangle}$ is a closure system on SEN, it suffices now to show that it is structural. To this end, suppose $\Sigma_1,\Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1,\Sigma_2)$ and $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma_1)$, such that $\phi \in C_{\Sigma_1}^{\langle \mathtt{Q},\top\rangle}(\Phi)$. Thus, for all $\theta \in \mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN})$, all $\Sigma \in |\mathbf{Sign}|$ and all $g \in \mathbf{Sign}(\Sigma_1,\Sigma)$, we have that

$$\begin{tikzcd}
\Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\
& \Sigma &
\end{tikzcd}$$

$\mathrm{SEN}(g)(\Phi) \subseteq \top_\Sigma/\theta_\Sigma$ implies that $\mathrm{SEN}(g)(\phi) \in \top_\Sigma/\theta_\Sigma$. This gives that, for all $\theta \in \mathrm{Conf}^N_{\mathtt{Q}}(\mathrm{SEN})$ and all $h \in \mathbf{Sign}(\Sigma_2,\Sigma)$, $\mathrm{SEN}(hf)(\Phi) \subseteq \top_\Sigma/\theta_\Sigma$ implies that $\mathrm{SEN}(hf)(\phi) \in \top_\Sigma/\theta_\Sigma$, i.e., that, for all $h \in \mathbf{Sign}(\Sigma_2,\Sigma)$, $\mathrm{SEN}(h)(\mathrm{SEN}(f)(\Phi)) \subseteq \top_\Sigma/\theta_\Sigma$ implies that $\mathrm{SEN}(h)(\mathrm{SEN}(f)(\phi)) \in \top_\Sigma/\theta_\Sigma$. Thus, by the definition of $C^{\langle \mathtt{Q},\top\rangle}$, we get that

$$\mathrm{SEN}(f)(\phi) \in C_{\Sigma_2}^{\langle \mathtt{Q},\top\rangle}(\mathrm{SEN}(f)(\Phi))$$

and $C^{\langle \mathtt{Q},\top\rangle}$ is indeed structural, i.e., it forms a closure system on SEN. Thus $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ is a π-institution. □

Next we prove that the closure system of the π-institution $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ and that of the equational π-institution $\mathcal{I}^{\mathtt{Q}}$ are very closely related. Namely, $\{x \approx \top(x)\}$ is an N-interpretation of $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ into $\mathcal{I}^{\mathtt{Q}}$. This will also yield as a corollary the fact that the π-institution $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ is indeed a finitary π-institution.

Proposition 23. *Let* SEN : **Sign** → **Set** *be a functor, with N a category of natural transformations on* SEN*, and* $\mathtt{Q}$ *a pointed N-quasivariety.*

Then, for all $\Sigma \in |\mathbf{Sign}|$ *and all* $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$*,*

$$\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi) \quad \textit{iff} \quad \phi \approx \top_\Sigma \in C_\Sigma^{\mathtt{Q}}(\Phi \approx \top_\Sigma).$$

Proof. Suppose, first, that $\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi)$. Then, for all $\theta \in \mathrm{Conf}_{\mathtt{Q}}^N(\mathrm{SEN})$, all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f)(\Phi) \subseteq \top_{\Sigma'}/\theta_{\Sigma'}$ implies that $\mathrm{SEN}(f)(\phi) \in \top_{\Sigma'}/\theta_{\Sigma'}$. Thus, in particular, we get that $\Phi \subseteq \top_\Sigma/\theta_\Sigma$ implies that $\phi \in \top_\Sigma/\theta_\Sigma$. This is equivalent, by Proposition 6, to, for all $\theta \in \mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}})$, $\Phi \approx \top_\Sigma \subseteq \theta_\Sigma$ implies that $\phi \approx \top_\Sigma \in \theta_\Sigma$, which shows that $\phi \approx \top_\Sigma \in C_\Sigma^{\mathtt{Q}}(\Phi \approx \top_\Sigma)$.

Suppose, conversely, that $\phi \approx \top_\Sigma \in C_\Sigma^{\mathtt{Q}}(\Phi \approx \top_\Sigma)$. Then, since $C^{\mathtt{Q}}$ is a closure system on $\mathrm{SEN}(\Sigma)^2$, we get that, for all $\Sigma' \in |\mathbf{Sign}|$ and all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, $\mathrm{SEN}(f)(\phi) \approx \top_{\Sigma'} \in C_{\Sigma'}^{\mathtt{Q}}(\mathrm{SEN}(f)(\Phi) \approx \top_{\Sigma'})$. Therefore, for all theory families $\theta \in \mathrm{ThFam}(\mathcal{I}^{\mathtt{Q}}) = \mathrm{Conf}_{\mathtt{Q}}^N(\mathrm{SEN})$, we have that $\mathrm{SEN}(f)(\Phi) \approx \top_{\Sigma'} \subseteq \theta_{\Sigma'}$ implies $\mathrm{SEN}(f)(\phi) \approx \top_{\Sigma'} \in \theta_{\Sigma'}$. But this is equivalent to $\mathrm{SEN}(f)(\Phi) \subseteq \top_{\Sigma'}/\theta_{\Sigma'}$ implies $\mathrm{SEN}(f)(\phi) \in \top_{\Sigma'}/\theta_{\Sigma'}$, which, by the definition of $C^{\langle \mathtt{Q},\top\rangle}$, yields that $\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi)$. $\square$

Notice that Proposition 23 yields as a corollary that $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ is a finitary π-institution, a fact that is not obvious by its definition.

Corollary 24. *Let* $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ *be a functor, with* N *a category of natural transformations on* SEN*, and* $\mathtt{Q}$ *a pointed* N*-quasivariety. Then* $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ *is finitary.*

Proof. For all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$, we have $\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Phi)$ if and only if, by Proposition 23, $\phi \approx \top_\Sigma \in C_\Sigma^{\mathtt{Q}}(\Phi \approx \top_\Sigma)$ if and only if, by Proposition 5, $\phi \approx \top_\Sigma \in C_\Sigma^{\mathtt{Q}}(\Psi \approx \top_\Sigma)$, for some finite $\Psi \subseteq \Phi$, if and only if, by Proposition 23, $\phi \in C_\Sigma^{\langle \mathtt{Q},\top\rangle}(\Psi)$, for some finite $\Psi \subseteq \Phi$, showing that $C^{\langle \mathtt{Q},\top\rangle}$ is a finitary closure system on SEN. $\square$

The next result is a partial analog in the context of π-institutions of Theorem 34 of [11], which, in the context or regularly algebraizable deductive systems, asserts that a deductive system is regularly algebraizable if and only if it is the assertional logic of a relatively point-regular quasi-variety. This fact had also been observed independently by Blok and Raftery in [5].

Theorem 25. *If a* π*-institution* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is regularly* N*-algebraizable, then it is of the form* $\mathcal{I}^{\langle \mathtt{Q},\top\rangle}$ *for some relatively point-regular* N*-quasivariety* $\mathtt{Q}$*. In* *fact, if $\mathcal{I}$ is regularly N-algebraizable, then $\mathtt{A}_{\mathcal{I}}^N$ is a relatively point-regular N-quasivariety, with an N-top $\top$, and $\mathcal{I} = \mathcal{I}^{\langle \mathtt{A}_{\mathcal{I}}^N, \top\rangle}$.*

Proof. Suppose that $\mathcal{I}$ is regularly N-algebraizable, with a finite N-equivalence system E and N-top $\top$. Set $\mathtt{Q} := \mathtt{A}_{\mathcal{I}}^N$. By the E-G-rule, we have that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{Thm}_\Sigma$, $E_\Sigma(\phi, \psi) \subseteq T_\Sigma$, which, taking into account the theory invariance of $\mathcal{I}$, shows that $\langle \phi, \psi\rangle \in \Omega_\Sigma^N(T)$, i.e., by Lemma 9 and theory invariance, that $\langle \phi, \psi\rangle \in \theta_\Sigma$, for every $\theta \in \mathrm{Conf}_{\mathtt{Q}}^N(\mathrm{SEN})$. This shows that all Σ-theorems of $\mathcal{I}$ are equivalent in $\mathcal{I}^{\langle \mathtt{Q}, \top\rangle}$ with $\top_\Sigma$. $\mathtt{Q}$ is pointed, with $\top$ its point. By Theorem 19, we get that $\{x \approx \top(x)\}$ is an N-interpretation of $\mathcal{I}$ in $\mathcal{I}^{\mathtt{Q}}$. By proposition 23, $\{x \approx \top(x)\}$ is also an N-interpretation of $\mathcal{I}^{\langle \mathtt{Q}, \top\rangle}$ in $\mathcal{I}^{\mathtt{Q}}$. Therefore $\mathcal{I} = \mathcal{I}^{\langle \mathtt{Q}, \top\rangle}$. It suffices now to show that $\mathtt{Q}$ is relatively point-regular. Suppose, to this end, that $\theta \in \mathrm{Con}_{\mathtt{Q}}^N(\mathrm{SEN})$. If $\theta' \in \mathrm{Con}_{\mathtt{Q}}^N(\mathrm{SEN})$, such that θ' is compatible with $\top/\theta = \{\top_\Sigma/\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$, then $\top/\theta' \leq \top/\theta$. Thus, we get that $\top/\Omega^N(\top/\theta) \leq \top/\theta$. But, on the other hand, θ is compatible with $\top/\theta$, whence $\theta \leq \Omega^N(\top/\theta)$ and, hence $\top/\theta \leq \top/\Omega^N(\top/\theta)$. This shows that $\top/\theta = \top/\Omega^N(\top/\theta)$. By syntactic N-algebraizability, Ω^N is a bijection between the theory systems $\mathrm{ThSys}(\mathcal{I}) = \{\top/\theta : \theta \in \mathrm{Con}_{\mathtt{Q}}^N(\mathrm{SEN})\}$ and those in $\mathrm{Con}_{\mathtt{Q}}^N(\mathrm{SEN})$. Therefore $\Omega^N(\top/\theta) = \theta$, for all $\theta \in \mathrm{Con}_{\mathtt{Q}}^N(\mathrm{SEN})$. Thus, each $\theta \in \mathrm{Con}_{\mathtt{Q}}^N(\mathrm{SEN})$ is uniquely determined by $\top/\theta$. □

9. The Deduction-Detachment Theorem

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, be a π-institution and Δ a set of binary natural transformations in N. Δ is said to be an N-**deduction-detachment system for** $\mathcal{I}$ if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi, \psi\} \in \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\Phi, \phi) \quad \text{iff} \quad \Delta_\Sigma(\phi, \psi) \subseteq C_\Sigma(\Phi).$$

The implication from left-to-right is the N-**deduction theorem relative to** Δ and the converse is the N-**detachment theorem relative to** Δ or the N-**modus ponens**.

A π-institution $\mathcal{I}$ will be said to have the N-**deduction detachment theorem** if it has an N-deduction-detachment system and the N-**uniterm deduction-detachment theorem** if it has a singleton N-deduction-detachment system.

Proposition 26. *Suppose that* $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$*, with* N *a category of natural transformations on* SEN*, is a* π*-institution with an* N*-deduction-detachment system* Δ*. Then* $\mathcal{I}$ *is* N*-protoalgebraic.*

Proof. Since, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi \in \mathrm{SEN}(\Sigma)$, we have that $\phi \in C_\Sigma(\phi)$, we get, by N-deduction, that

$$\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset). \tag{9}$$

Moreover, by N-detachment, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\psi \in C_\Sigma(\phi, \Delta_\Sigma(\phi, \psi)). \tag{10}$$

To show that $\mathcal{I}$ is N-protoalgebraic, suppose that $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$, and $\phi, \psi \in \mathrm{SEN}(\Sigma)$, such that $\langle \phi, \psi\rangle \in \Omega^N_\Sigma(T)$. Then, since $\Omega^N(T)$ is an N-congruence system on SEN and Δ is a collection of natural transformations in N, we get that $\langle \Delta_\Sigma(\phi, \phi), \Delta_\Sigma(\phi, \psi)\rangle \subseteq \Omega^N_\Sigma(T)$. But, by (9), $\Delta_\Sigma(\phi, \phi) \subseteq C_\Sigma(\emptyset) \subseteq T_\Sigma$, whence, by the compatibility of $\Omega^N(T)$ with T, we get that $\Delta_\Sigma(\phi, \psi) \subseteq T_\Sigma$. Now we have, by (10), $\psi \in C_\Sigma(\phi, \Delta_\Sigma(\phi, \psi)) \subseteq C_\Sigma(T_\Sigma, \phi)$. By symmetry, we have that $C_\Sigma(T_\Sigma, \phi) = C_\Sigma(T_\Sigma, \psi)$, which shows that $\mathcal{I}$ is in fact N-protoalgebraic. $\square$

Following [11], we now define collections of Σ-sentences that support an N-deduction-detachment theorem with multiple hypotheses.

To this end, assume that $\mathrm{SEN} : \mathbf{Sign} \to \mathbf{Set}$ is a functor, with N a category of natural transformations on SEN. Suppose that $\Delta(x, y) = \{\delta^i(x, y) : i < n\}$ is a collection of binary natural transformations in N. Define, for every $\Sigma \in |\mathbf{Sign}|$ and all $\vec{\phi} \in \mathrm{SEN}(\Sigma)^m, m \geq 1, \psi \in \mathrm{SEN}(\Sigma)$, the collection $\Delta^*_\Sigma(\vec{\phi}, \psi)$ of $(m+1)$-ary natural transformations in N, by recursion on the value of $m \geq 1$, as follows:

For $m = 1$, we have that

$$\Delta^*_\Sigma(\vec{\phi}, \psi) = \Delta_\Sigma(\phi_0, \psi).$$

For $m > 1$, suppose that $\vec{\phi} = \langle \phi_0, \vec{\phi'}\rangle$, with $\vec{\phi'} = \langle \phi_1, \ldots, \phi_{m-1}\rangle$. Then

$$\Delta^*_\Sigma(\vec{\phi}, \psi) = \bigcup \{\Delta_\Sigma(\phi_0, \chi) : \chi \in \Delta^*_\Sigma(\vec{\phi'}, \psi)\}.$$

The following two lemmas lift to the π-institution framework Lemmas 40 and 41, respectively of [11].

Lemma 27. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a π-institution and Δ a finite system of binary natural transformations in N, such that $\mathcal{I}$ has the N-modus ponens relative to Δ. Then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\phi_0, \ldots, \phi_{m-1}, \psi \in \mathrm{SEN}(\Sigma)$, $\psi \in C_\Sigma(\phi_0, \ldots, \phi_{m-1}, \Delta^*_\Sigma(\vec{\phi}, \psi))$.*

Proof. We work by induction on $m \geq 1$. If $m = 1$, then, the conclusion takes the form $\psi \in C_\Sigma(\phi_0, \Delta_\Sigma(\phi_0, \psi))$, which is simply the N-modus ponens property of Δ. Assume, now, that the conclusion holds for any sequence $\vec{\phi'}$ of length $m-1$ and suppose that $\vec{\phi} = \langle \phi_0, \vec{\phi'}\rangle$, with $\vec{\phi'} = \langle \phi_1, \ldots, \phi_{m-1}\rangle$. Then, by the N-modus ponens relative to Δ, we get that, for all $\chi \in \Delta^*_\Sigma(\vec{\phi'}, \psi)$, we have that $\chi \in C_\Sigma(\phi_0, \Delta_\Sigma(\phi_0, \chi))$, whence, using the induction hypothesis, we get

$$\begin{array}{rcl}
\psi & \in & C_\Sigma(\phi_1, \ldots, \phi_{m-1}, \Delta^*_\Sigma(\vec{\phi'}, \psi)) \\
& \subseteq & C_\Sigma(\phi_1, \ldots, \phi_{m-1}, \bigcup\{C_\Sigma(\phi_0, \Delta_\Sigma(\phi_0, \chi)) : \chi \in \Delta^*_\Sigma(\vec{\phi'}, \psi)\}) \\
& = & C_\Sigma(\phi_0, \ldots, \phi_{m-1}, \bigcup\{\Delta_\Sigma(\phi_0, \chi) : \chi \in \Delta^*_\Sigma(\vec{\phi'}, \psi)\}) \\
& = & C_\Sigma(\phi_0, \ldots, \phi_{m-1}, \Delta^*_\Sigma(\vec{\phi}, \psi)).
\end{array}$$

□

Lemma 28. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *is a π-institution and Δ a finite system of binary natural transformations in N, such that $\mathcal{I}$ has the N-deduction-detachment theorem relative to Δ. Then, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$ and all $\phi_0, \ldots, \phi_{m-1}, \psi \in \mathrm{SEN}(\Sigma)$, such that $\psi \in C_\Sigma(\Phi, \phi_0, \ldots, \phi_{m-1})$, $\Delta^*_\Sigma(\vec{\phi}, \psi) \subseteq C_\Sigma(\Phi)$.*

Proof. The case $m = 1$ follows immediately by the N-deduction theorem of $\mathcal{I}$ relative to Δ. Suppose, again, that the conclusion follows for all sequences $\vec{\phi'}$ of length $m-1$ and assume $\vec{\phi} = \langle \phi_0, \vec{\phi'}\rangle$, with $\vec{\phi'} = \langle \phi_1, \ldots, \phi_{m-1}\rangle$, such that $\psi \in C_\Sigma(\Phi, \phi_0, \ldots, \phi_{m-1})$. Then, by the induction hypothesis, we get that $\Delta^*_\Sigma(\vec{\phi'}, \psi) \subseteq C_\Sigma(\Phi, \phi_0)$, whence by the N-deduction theorem of $\mathcal{I}$ relative to Δ, we obtain that $\bigcup\{\Delta_\Sigma(\phi_0, \chi) : \chi \in \Delta^*_\Sigma(\vec{\phi'}, \psi)\} \subseteq C_\Sigma(\Phi)$. Thus, finally, by the definition of Δ^*, $\Delta^*_\Sigma(\vec{\phi}, \psi) \subseteq C_\Sigma(\Phi)$. □

Theorem 29. *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN, *be a finitary N-rule based π-institution, with a* *finite N-deduction-detachment system Δ. Then $\mathcal{I}$ may be axiomatized via N-axioms and N-inference rules in such a way that*

$$\frac{x, \Delta(x,y)}{y} \tag{11}$$

is its only proper N-rule of inference.

Proof. By hypothesis, since $\mathcal{I}$ is N-rule based, there exists a collection Ax of N-axioms and IR of (proper) N-inference rules that axiomatize $\mathcal{I}$. Consider the new collections Ax$'$ and IR$'$ of N-axioms and N-inference rules constructed from Ax and IR as follows: Ax$'$ consists of the N-axioms in Ax plus all N-axioms of the form $\Delta^*(\vec{\sigma}, \tau)$, for each N-inference rule $\langle\{\sigma^0, \dots, \sigma^{m-1}\}, \tau\rangle$ in IR. IR$'$ consists only of the N-inference rule (11). By Lemma 28, all N-axioms in Ax$'$ are N-axioms of $\mathcal{I}$ and, by hypothesis, (11) is an N-inference rule of $\mathcal{I}$. Let, now, $\mathcal{I}' = \langle \mathbf{Sign}, \mathrm{SEN}, C'\rangle$ be the π-institution axiomatized by Ax$'$ and IR$'$. Then, obviously, by the preceding remarks, $C' \leq C$. On the other hand, since Ax $\subseteq$ Ax$'$ and every N-rule in IR is, by Lemma 27, a derivable N-rule of $\mathcal{I}'$, we get that $C \leq C'$. Therefore $\mathcal{I} = \mathcal{I}'$. □

Theorem 30. *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations, is a regularly N-algebraizable N-rule based π-institution, with a finite N-deduction-detachment system Δ. Let E be a finite N-equivalence system for $\mathcal{I}$ and $\top$ an N-top. Then the unique equivalent N-quasivariety of $\mathcal{I}$ is axiomatized by the N-identities*

$$E(x,x) \approx \top,$$

the two N-quasi-identities

$$x \approx \top \wedge \Delta(x,y) \approx \top \rightarrow y \approx \top, \quad E(x,y) \approx \top \rightarrow x \approx y,$$

and additional N-identities of the form $\sigma \approx \top$, where σ ranges over any fixed set of N-axioms of an axiomatization of $\mathcal{I}$ that has the N-rule (11) *as its only proper N-inference rule.*

Proof. Combine Theorems 20 and 29. □

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