Business and Life Calculus

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LSSU Math 112

1. Functions
   - Inequalities and Lines
   - Exponents
   - Linear and Quadratic Functions
   - Polynomial, Rational, Piece-wise and Composite Functions

Subsection 1

Inequalities and Lines

Inequalities

- $a < b$ means "a is less than b";
- $a \leq b$ means "a is less than or equal to b";
- $a > b$ means "a is greater than b";
- $a \geq b$ means "a is greater than or equal to b";
- Example: Which of the following statements are true and which are false?

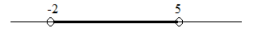
Inequality	Truth Value
$-3 < 2$	✓
$-5 < -9$	✠
$1 \leq 1$	✓
$2.2 \geq -1.7$	✓

- A **double inequality** $a < x < b$ means "x is between a and b", i.e., *both* $a < x$ and $x < b$ hold;
- Example: $-2 < x < 5$ means that x lies between -2 and 5 on the real line.

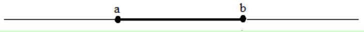
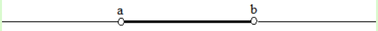
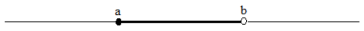
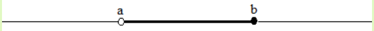
Sets and Intervals

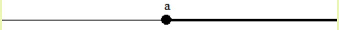
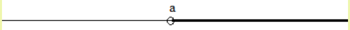
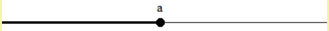
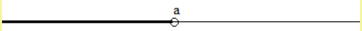
- The notation
  $$\{x : x > 3\}$$
  means "the set of all x, such that x is greater than 3";
- Similarly,
  $$\{x : -2 < x < 5\}$$
  means "the set of all x, such that x is between -2 and 5";
- These sets may also be expressed in **interval notation**;
  - The first set above is $(3, \infty)$;

  - And the second set is $(-2, 5)$;



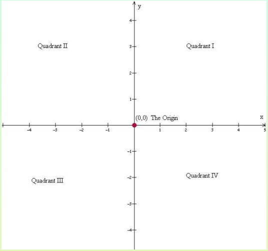
Finite and Infinite Intervals

Set Notation	Interval Notation	Graph
$\{x : a \leq x \leq b\}$	$[a, b]$	
$\{x : a < x < b\}$	(a, b)	
$\{x : a \leq x < b\}$	$[a, b)$	
$\{x : a < x \leq b\}$	$(a, b]$	

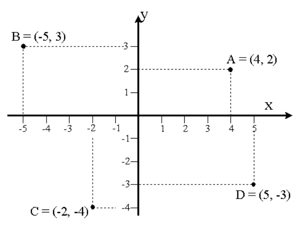
Set Notation	Interval Notation	Graph
$\{x : x \geq a\}$	$[a, \infty)$	
$\{x : x > a\}$	(a, ∞)	
$\{x : x \leq a\}$	$(-\infty, a]$	
$\{x : x < a\}$	$(-\infty, a)$	

Cartesian Plane

- The **Cartesian plane** is defined by
  - the x-**axis**;
  - the y-**axis**;
  - a **unit** of measurement, determining the x- and the y-coordinates of points on the plane;



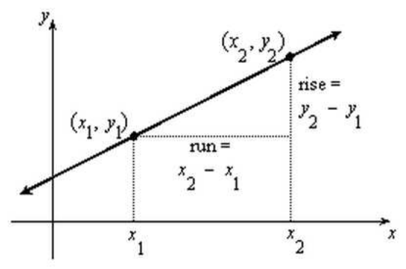
- Example: Some points and their coordinates:



Slopes

- If a line ℓ passes through two points (x_1, y_1) and (x_2, y_2), then we define its **slope** m by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1};$$



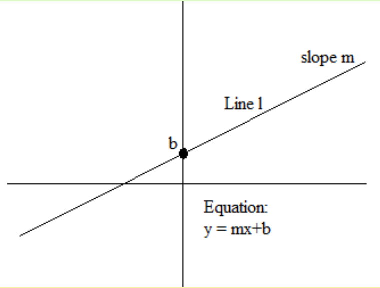
- A horizontal line has slope $m = 0$;
- A vertical line has slope undefined;
- Example: Find the slope of the line passing through $(-2, 3)$ and $(18, -12)$;

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-12 - 3}{18 - (-2)} = \frac{-15}{20} = -\frac{3}{4};$$

Equations of Lines: The Slope-Intercept Form

- If a line ℓ has slope m and y-intercept $(0, b)$, then its equation is

$$y = mx + b;$$



Example: Find an equation of the line passing through $(0, 4)$ and $(2, 0)$;

We first compute the slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{2-0} = -2$;

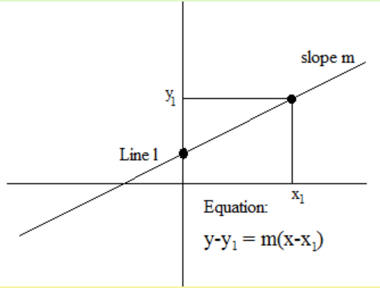
Then, we use the slope-intercept form with $b = 4$;

$$y = -2x + 4.$$

Equations of Lines: The Point-Slope Form

- If a line ℓ has slope m and passes through the point (x_1, y_1), then its equation is

$$y - y_1 = m(x - x_1);$$



Example: Find an equation of the line passing through $(4, 1)$ and $(7, -2)$;

We first compute the slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 1}{7 - 4} = -1$;

Then, we use the point-slope form with $(x_1, y_1) = (4, 1)$;

$$y - 1 = (-1)(x - 4)$$

or $y = -x + 5$.

General Linear Equation

- The **general form** of an equation of a line is
$$ax + by = c,$$
with a, b, c real constants, such that a, b are not both zero;
- Example: If a line ℓ passes through $(-2, 10)$ and $(1, -2)$, find an equation for ℓ in the general form;
First, compute the slope:
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 10}{1 - (-2)} = \frac{-12}{3} = -4;$$
Now use the point-slope form:
$$\begin{aligned} y - (-2) &= -4(x - 1) \\ \Rightarrow \quad y + 2 &= -4x + 4 \\ \Rightarrow \quad 4x + y &= 2. \end{aligned}$$

General Linear Equation: Another Example

- Example: If a line ℓ has equation $2x + 3y = 12$, what is slope m and what is its y-intercept b?

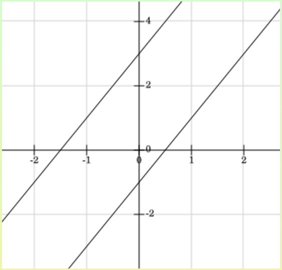
  Solve for y to transform into the slope-intercept form:

$$\begin{aligned} & 2x + 3y = 12 \\ \Rightarrow \quad & 3y = -2x + 12 \\ \Rightarrow \quad & y = -\frac{2}{3}x + 4; \end{aligned}$$

  Thus, the line has slope $m = -\frac{2}{3}$ and y-intercept $b = 4$.

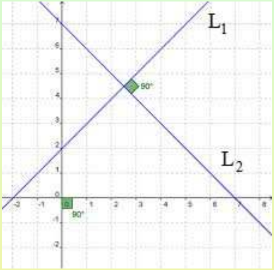
Parallel and Perpendicular Lines

Two lines L_1 and L_2 are **parallel**, written $L_1 \parallel L_2$, if they have no points in common;



If L_1 has slope m_1 and L_2 has slope m_2, we have $L_1 \parallel L_2$ if and only if $m_1 = m_2$;

Two lines L_1 and L_2 are **perpendicular**, written $L_1 \perp L_2$, if they intersect at a right (90°) angle;



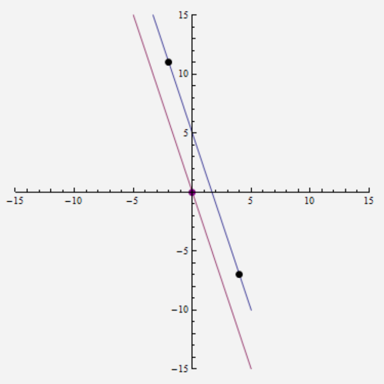
If L_1 has slope m_1 and L_2 has slope m_2, we have $L_1 \perp L_2$ if and only if $m_1 = -\frac{1}{m_2}$.

Example I

Find the slope of the line ℓ that passes through the origin and that is parallel to the line ℓ' passing through the points $(-2, 11)$ and $(4, -7)$.

Line ℓ' has slope

$$\begin{aligned} m' &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-7 - 11}{4 - (-2)} \\ &= \frac{-18}{6} = -3; \end{aligned}$$



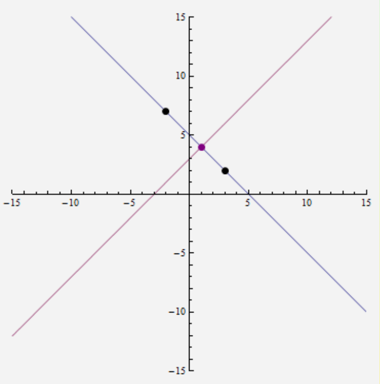
Since $\ell \parallel \ell'$, we must have $m = m' = -3$;

Example II

Find an equation for the line ℓ that passes through the point $(1, 4)$ and that is perpendicular to the line ℓ' passing through the points $(-2, 7)$ and $(3, 2)$.

Line ℓ' has slope

$$m' = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-7}{3-(-2)} = \frac{-5}{5} = -1;$$



Since $\ell \perp \ell'$, we must have $m = -\frac{1}{m'} = 1$; Using point-slope form, we get for ℓ: $y - 4 = 1(x-1)$ or $y = x + 3$;

Subsection 2

Exponents

Positive Integer Exponents

- For any positive integer n,

$$x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}};$$

Properties of Exponents:

- $x^m \cdot x^n = x^{m+n}$;
- $\frac{x^m}{x^n} = x^{m-n}$;
- $(x^m)^n = x^{m \cdot n}$;
- $(xy)^n = x^n \cdot y^n$;
- $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$;

Example: Simplify

- $x^2 x^3 = x^5$;
- $\frac{x^7}{x^3} = x^4$;
- $(x^3)^5 = x^{15}$;
- $(3x^2)^3 = 3^3 (x^2)^3 = 27x^6$;
- $\left(\frac{x}{2}\right)^4 = \frac{x^4}{2^4} = \frac{x^4}{16}$;

Zero and Negative Exponents

- If $x \neq 0$ and n is a positive integer,

$$x^0 = 1 \quad \text{and} \quad x^{-n} = \frac{1}{x^n};$$

- Example: Simplify:
  - $5^0 = 1$;
  - $7^{-1} = \frac{1}{7}$;
  - $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$;
  - $(-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$.
- Example: Simplify:
  - $\left(\frac{3}{5}\right)^{-2} = \left(\frac{5}{3}\right)^2 = \frac{5^2}{3^2} = \frac{25}{9}$;
  - $\left(\frac{1}{2}\right)^{-5} = 2^5 = 32$.

Roots and Fractional Exponents

- If n is a positive integer,

$$x^{1/n} = \sqrt[n]{x};$$

- Example: Evaluate
  - $9^{1/2} = \sqrt{9} = 3;$
  - $125^{1/3} = \sqrt[3]{125} = 5;$
  - $(-16)^{1/4} = \sqrt[4]{-16} =$ undefined!
  - $(-32)^{1/5} = \sqrt[5]{-32} = -2;$
  - $\left(\frac{4}{25}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5};$
  - $\left(-\frac{27}{8}\right)^{\frac{1}{3}} = \sqrt[3]{-\frac{27}{8}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{8}} = -\frac{3}{2}.$

Fractional Exponents

- If n, m are positive integers,

$$x^{m/n} = (\sqrt[n]{x})^m = \sqrt[n]{x^m};$$

- Example: Evaluate
  - $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4;$
  - $25^{3/2} = (\sqrt{25})^3 = 5^3 = 125;$
  - $\left(\frac{-27}{8}\right)^{2/3} = \left(\sqrt[3]{\frac{-27}{8}}\right)^2 = \left(\frac{-3}{2}\right)^2 = \frac{9}{4}.$

Negative Fractional Exponents

- If n, m are positive integers,

$$x^{-m/n} = \frac{1}{x^{m/n}} = \frac{1}{(\sqrt[n]{x})^m} = \frac{1}{\sqrt[n]{x^m}};$$

- Example: Evaluate
  - $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4};$
  - $\left(\frac{9}{4}\right)^{-3/2} = \left(\frac{4}{9}\right)^{3/2} = \left(\sqrt{\frac{4}{9}}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27};$
  - $25^{-3/2} = \frac{1}{25^{3/2}} = \frac{1}{(\sqrt{25})^3} = \frac{1}{125};$
  - $\left(\frac{1}{4}\right)^{-5/2} = 4^{5/2} = (\sqrt{4})^5 = 2^5 = 32.$

Subsection 3

Linear and Quadratic Functions

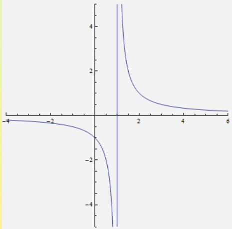
Functions

- A **function** is a rule assigning to every number x in a set, a unique number $f(x)$;
- The set of all allowable values of x is called the **domain**;
- The set of all values $f(x)$ for x in the domain is called the **range**;
- Sometimes, we write $\mathrm{Dom}(f)$ for the domain and $\mathrm{Ran}(f)$ for the range of a function f;
- When a function is defined by a formula, its domain is understood to be the *largest* set of numbers for which the formula is defined;
- The **graph** of a function f consists of all points (x, y), such that x is in the domain and $y = f(x)$;
- In this context, we call x the **independent variable** and y the **dependent variable** (since it depends on x).

An Example

- Consider the function defined by the formula $f(x) = \frac{1}{x-1}$;
  - What is $f(8)$?
  - What is the domain $\mathrm{Dom}(f)$?
  - If the graph is the one shown below, what is the range $\mathrm{Ran}(f)$?

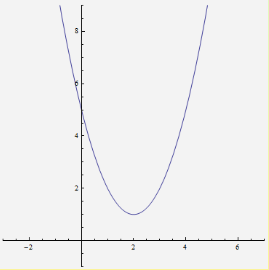
  - We set $x = 8$ and compute: $f(8) = \frac{1}{8-1} = \frac{1}{7}$;
  - The formula has a denominator; In this case, the only potential problem is dividing by zero; Set $x - 1 = 0 \Rightarrow x = 1$; Thus, we must exclude $x = 1$ from the domain; In set notation, $\mathrm{Dom}(f) = \mathbb{R} - \{1\}$ and in interval notation $\mathrm{Dom}(f) = (-\infty, 1) \cup (1, \infty)$;
  - The only value that y does not assume is zero; In set notation, we have $\mathrm{Ran}(f) = \mathbb{R} - \{0\}$ and in interval notation $\mathrm{Ran}(f) = (-\infty, 0) \cup (0, \infty)$.



Another Example

- Consider the function defined by the formula $f(x) = x^2 - 4x + 5$;
  - What is $f(-3)$?
  - What is the domain $\mathrm{Dom}(f)$?
  - If the graph is the one shown below, what is the range $\mathrm{Ran}(f)$?

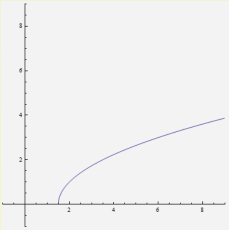
  - We set $x = -3$ and compute: $f(-3) = (-3)^2 - 4 \cdot (-3) + 5 = 26$;
  - This formula has neither denominators nor roots; In this case, no problem can potentially arise; Thus, no number needs to be excluded; In set notation, we have $\mathrm{Dom}(f) = \mathbb{R}$ and in interval notation $\mathrm{Dom}(f) = (-\infty, \infty)$;
  - y assumes only values greater than or equal to 1; Thus, in set notation, we have $\mathrm{Ran}(f) = \{y : y \geq 1\}$ and in interval notation $\mathrm{Ran}(f) = [1, \infty)$.



A Third Example

- Consider the function defined by the formula $f(x) = \sqrt{2x-3}$;
  - What is $f(\frac{19}{2})$?
  - What is the domain $\mathrm{Dom}(f)$?
  - If the graph is the one shown below, what is the range $\mathrm{Ran}(f)$?

  - We set $x = \frac{19}{2}$ and compute: $f(\frac{19}{2}) = \sqrt{2 \cdot \frac{19}{2} - 3} = \sqrt{16} = 4$;
  - This formula has an even-index root; In this case, a potential problem is having to compute the square root of a negative number; Thus, we must ensure that $2x - 3 \geq 0 \Rightarrow 2x \geq 3 \Rightarrow x \geq \frac{3}{2}$; In set notation, we have $\mathrm{Dom}(f) = \{x : x \geq \frac{3}{2}\}$ and in interval notation $\mathrm{Dom}(f) = [\frac{3}{2}, \infty)$;
  - y assumes only values greater than or equal to 0; Thus, in set notation, we have $\mathrm{Ran}(f) = \{y : y \geq 0\}$ and in interval notation $\mathrm{Ran}(f) = [0, \infty)$.



Linear Functions

- A **linear function** is a function that can be expressed in the form
$$f(x) = mx + b,$$
where m and b are constants;
- The graph of $y = f(x)$ is a straight line with slope m and y-intercept the point $(0, b)$;
- Example: Suppose that a manufacturer has fixed costs \$400 and variable costs \$10 per item produced. What is the **cost function** $C(x)$ for producing x items? What are the meanings of its slope and its y-intercept?

  We have
$$C(x) = \underbrace{10x}_{\text{variable}} + \underbrace{400}_{\text{fixed}};$$
  The slope $m = 10$ represents the variable cost and the y-intercept $b = 400$ the fixed cost.

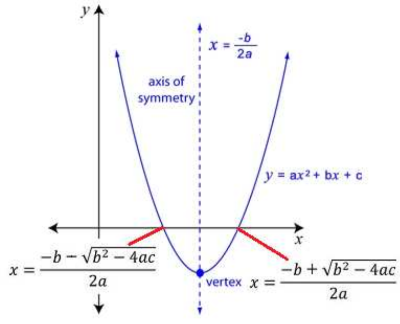
Quadratic Functions

- A **quadratic function** is a function that can be expressed in the form

$$f(x) = ax^2 + bx + c,$$

where a, b, c are constants, with $a \neq 0$;

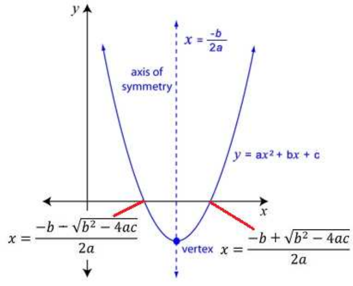
- The graph of $y = ax^2 + bx + c$ is called a **parabola** and looks like



Graphing Quadratic Functions

- The graph of the quadratic function is a **parabola** opening either up or down;

1. The **vertex** is the lowest or highest point; Its x-coordinate is $x = -\frac{b}{2a}$;
2. The parabola opens up if $a > 0$ and down if $a < 0$;
3. Its y-intercept is the point $(0, c)$;

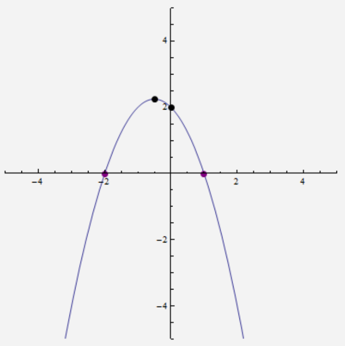


4. Finally, its x-intercepts are the points with $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; This is called the **quadratic formula**; The quantity $D = b^2 - 4ac$ is called the **discriminant**.

Examples of Quadratic Function Graphs I

- Find the vertex, the opening direction, the intercepts and sketch the graph of $f(x) = -x^2 - x + 2$;

1. The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{1}{2}$; Its y-coordinate, therefore, is $y = f(-\frac{1}{2}) = -(-\frac{1}{2})^2 - (-\frac{1}{2}) + 2 = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{9}{4}$;
2. The parabola opens down since $a = -1 < 0$;
3. Its y-intercept is $(0, 2)$;

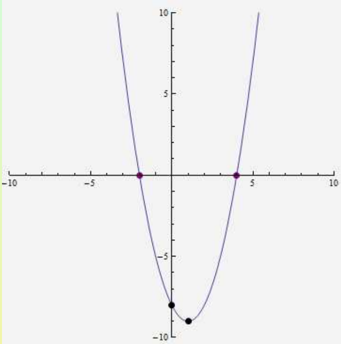


4. Finally, its x-intercepts are the solutions of $-x^2 - x + 2 = 0 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x + 2 = 0$ or $x - 1 = 0 \Rightarrow x = -2$ or $x = 1$.

Examples of Quadratic Function Graphs II

- Find the vertex, the opening direction, the intercepts and sketch the graph of $f(x) = x^2 - 2x - 8$;

1. The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-2}{2\cdot 1} = 1$; Its y-coordinate, therefore, is $y = f(1) = 1^2 - 2\cdot 1 - 8 = 1 - 2 - 8 = -9$;
2. The parabola opens up since $a = 1 > 0$;
3. Its y-intercept is $(0, -8)$;



4. Finally, its x-intercepts are the solutions of $x^2 - 2x - 8 = 0 \Rightarrow (x+2)(x-4) = 0 \Rightarrow x + 2 = 0$ or $x - 4 = 0 \Rightarrow x = -2$ or $x = 4$.

Summary of Methods for Solving $ax^2 + bx + c = 0$

- Recall: there are several methods for solving $ax^2 + bx + c = 0$:
  1. **Even-Root Property**: This, we use when $b = 0$, i.e., there is no x-term; E.g., $(x-2)^2 = 8 \Rightarrow x - 2 = \pm\sqrt{8} \Rightarrow x = 2 \pm 2\sqrt{2}$;
  2. **Factoring**: This we use whenever we are able to factor; E.g., $x^2 + 5x + 6 = 0 \Rightarrow (x+3)(x+2) = 0 \Rightarrow x + 3 = 0$ or $x + 2 = 0 \Rightarrow$ $x = -3$ or $x = -2$;
  3. **Quadratic Formula**: This solves any quadratic equation (the most powerful weapon); E.g.,
  $$x^2 + 5x + 3 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \Rightarrow x = \frac{-5 \pm \sqrt{13}}{2};$$
  4. **Completing Square**: Also solves any quadratic, but is slower than the quadratic formula; E.g., $x^2 - 6x + 7 = 0 \Rightarrow x^2 - 6x = -7 \Rightarrow$ $x^2 - 6x + 9 = -7 + 9 \Rightarrow (x-3)^2 = 2 \Rightarrow x - 3 = \pm\sqrt{2} \Rightarrow x = 3 \pm \sqrt{2}$.

Number of Solutions

- A byproduct of computing $D = b^2 - 4ac$ in the application of the quadratic formula is that we can tell right away how many solutions $ax^2 + bx + c = 0$ has:
  - If $D > 0$, it has two real solutions;
  - If $D = 0$, it has one real solution;
  - If $D < 0$, it does not have any real solutions;
- Example: Determine the number of real solutions of the given quadratic; You do not need to find the solutions (if there are any);
  - $x^2 - 3x - 5 = 0$
    $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-5) = 9 + 20 = 29 > 0$; Therefore, $x^2 - 3x - 5 = 0$ has two real solutions;
  - $x^2 = 3x - 9$ Rewrite $x^2 - 3x + 9 = 0$;
    $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot 9 = 9 - 36 = -27 < 0$; Therefore, $x^2 = 3x - 9 = 0$ has no real solutions;
  - $4x^2 - 12x + 9 = 0$
    $D = b^2 - 4ac = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$; Therefore, $4x^2 - 12x + 9 = 0$ has one real solution.

Application: Revenue, Cost (Break-Even Points)

- If the cost function is $C(x) = 120x + 4800$ and the revenue function is $R(x) = -2x^2 + 400x$, where x is the number of items produced and sold, what are the company's **break-even points** (i.e., points where its revenue equals its cost)?

  We set $C(x) = R(x)$ and solve for x:

$$\begin{aligned} & 120x + 4800 = -2x^2 + 400x \\ \Rightarrow \quad & 2x^2 - 280x + 4800 = 0 \\ \Rightarrow \quad & x^2 - 140x + 2400 = 0 \\ \Rightarrow \quad & (x - 20)(x - 120) = 0 \\ \Rightarrow \quad & x - 20 = 0 \text{ or } x - 120 = 0 \\ \Rightarrow \quad & x = 20 \text{ or } x = 120; \end{aligned}$$

  Thus the company breaks even when it produces and sells either 20 or 120 items;

Application: Revenue, Cost (Max Profit)

- If the cost function is $C(x) = 120x + 4800$ and the revenue function is $R(x) = -2x^2 + 400x$, where x is the number of items produced and sold, how many units should be produced to maximize profit and what is the max profit?

  Profit is given by
  $$\text{Profit} = \text{Revenue} - \text{Cost},$$
  in symbols $P(x) = R(x) - C(x)$; Thus,
  $$P(x) = -2x^2 + 400x - (120x + 4800) = -2x^2 + 280x - 4800;$$
  This is a parabola opening down, so the maximum occurs at
  $$x = -\frac{b}{2a} = -\frac{280}{2 \cdot (-2)} = 70;$$
  The max profit is $P(70) = -2 \cdot 70^2 + 280 \cdot 70 - 4800 = 5000.$

Subsection 4

Polynomial, Rational, Piece-wise and Composite Functions

Polynomial Functions

- A **polynomial function** is one that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real constants;
- The expressions $a_n x^n, a_{n-1} x^{n-1}, \ldots, a_1 x, a_0$ are the **terms**;
- The numbers $a_0, a_1, \ldots, a_n$ are the **coefficients**;
- The **degree** is the highest power of the variable;
- The **leading coefficient** is the one of the highest power term;
- Examples are

Polynomial	Degree	Leading Coef.
$f(x) = 2x^8 - 3x^6 + 13x^3 - 7$	8	2
$f(x) = -4x^2 + \frac{1}{7}x - 11$	2	-4
$f(x) = x + 5$	1	1
$f(x) = 2013$	0	2013

Solving Polynomial Equations

- Solve the equation $3x^4 - 6x^3 = 24x^2$;
- Solving involves
  - Making one side zero;
  - Factoring the non-zero side;
  - Using the zero-factor property;
  - Solving the simpler equations;
- We write

$$\begin{aligned} & 3x^4 - 6x^3 = 24x^2 \\ \Rightarrow \quad & 3x^4 - 6x^3 - 24x^2 = 0 \\ \Rightarrow \quad & 3x^2(x^2 - 2x - 8) = 0 \\ \Rightarrow \quad & 3x^2(x+2)(x-4) = 0 \\ \Rightarrow \quad & x = 0 \text{ or } x + 2 = 0 \text{ or } x - 4 = 0 \\ \Rightarrow \quad & x = 0 \text{ or } x = -2 \text{ or } x = 4. \end{aligned}$$

Rational Functions and Domains

- A **rational function** is a function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions, such that $Q(x) \neq 0$;
- Examples:
$$f(x) = \frac{3x+2}{x-2}, \qquad g(x) = \frac{1}{x^2+1};$$
- Example: Find the domain of the rational function
$f(x) = \frac{18}{x^2 - 2x - 24}$;

  We must have $x^2 - 2x - 24 \neq 0$; Let us solve
$$\begin{aligned} x^2 - 2x - 24 = 0 \Rightarrow \quad & (x+4)(x-6) = 0 \\ \Rightarrow \quad x+4 = 0 \text{ or } x-6 = 0 \Rightarrow \quad & x = -4 \text{ or } x = 6; \end{aligned}$$

  Thus, we must exclude $x = -4$ and $x = 6$ from the domain, i.e., we have $\mathrm{Dom}(f) = \mathbb{R} - \{-4, 6\} = (-\infty, -4) \cup (-4, 6) \cup (6, \infty)$.

Exponential Functions

- An **exponential function** is one of the form

$$f(x) = a^x,$$

  where $0 < a \neq 1$ and x is a real number;

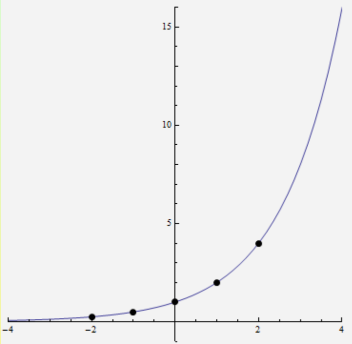
- Example: Consider $f(x) = 2^x$, $g(x) = (\frac{1}{4})^{1-x}$ and $h(x) = -3^x$; Compute the following values:
  - $f(\frac{3}{2}) = 2^{3/2} = \sqrt{2^3} = \sqrt{2^2}\sqrt{2} = 2\sqrt{2}$;
  - $f(-3) = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$;
  - $g(3) = (\frac{1}{4})^{1-3} = (\frac{1}{4})^{-2} = 4^2 = 16$;
  - $h(2) = -3^2 = -9$;

- Two important exponentials for applications are the base 10 exponential $f(x) = 10^x$ (called **common base**), and the base e exponential $f(x) = e^x$ (called **natural base**).

Graphs of Exponentials (Exponential Growth)

- When the base a is such that $a > 1$, then $f(x) = a^x$ has an increasing graph (going up as we move from left to right);
- As an example, we'll use a few points to sketch the graph of $f(x) = 2^x$;

x	$y = 2^x$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4

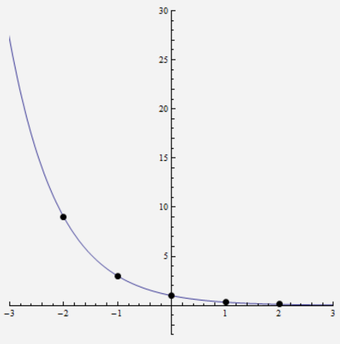


- Note that the x-axis is a horizontal asymptote as $x \to -\infty$.

Graphs of Exponentials (Exponential Decay)

- When the base a is such that $0 < a < 1$, then $f(x) = a^x$ has a decreasing graph (going down as we move from left to right);
- As an example, we'll use a few points to sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$;

x	$y = \left(\frac{1}{3}\right)^x$
-2	9
-1	3
0	1
1	1/3
2	1/9



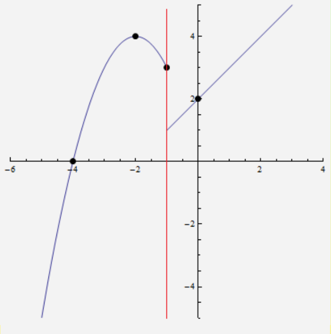
- Note that the x-axis is a horizontal asymptote as $x \to +\infty$.

Piece-wise Defined Functions

- A **piece-wise defined function** is one defined by different formulas over different parts of its domain;
- The graph of a piece-wise defined function is plotted by piecing together the graphs of the various parts;
- Example: Plot the graph of the function

$$f(x) = \begin{cases} -x^2 - 4x, & \text{if } x \leq -1 \\ x + 2, & \text{if } x > -1 \end{cases}$$

First, graph $y = -x^2 - 4x$; Then, graph $y = x + 2$; Finally, keep only the part of $y = -x^2 - 4x$ for $x \leq -1$ and the part of $y = x + 2$ for $x > -1$; This gives the graph of $y = f(x)$.

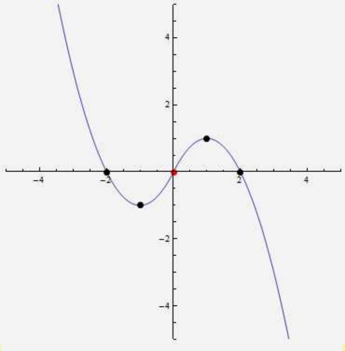


Another Example

- Plot the graph of the function

$$f(x) = \begin{cases} x^2 + 2x, & \text{if } x < 0 \\ -x^2 + 2x, & \text{if } x \geq 0 \end{cases}$$

First, graph $y = x^2 + 2x$; Then, graph $y = -x^2 + 2x$; Finally, keep only the part of $y = x^2 + 2x$ for $x < 0$ and the part of $y = -x^2 + 2x$ for $x \geq 0$; This gives the graph of $y = f(x)$.

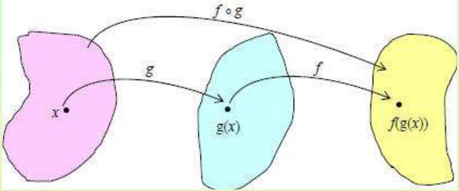


Composition of Functions

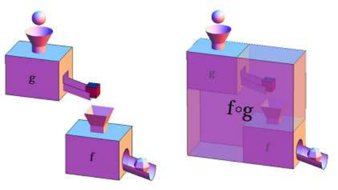
- The **composition of g and f** is the function $g \circ f$, defined by

$$(f \circ g)(x) = f(g(x));$$

In set diagram, we have



In machine diagram, we have

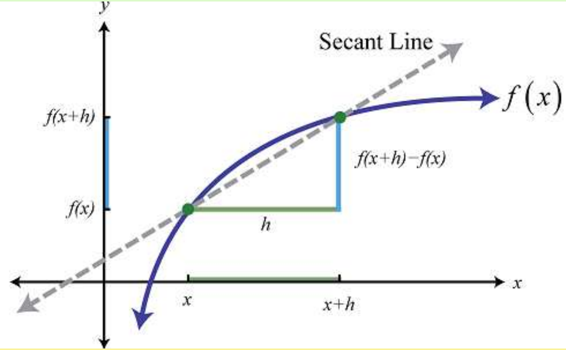


Examples of Composition

- If $f(x) = x^7$ and $g(x) = x^3 - 2x$, find
  - $f(g(x)) = f(x^3 - 2x) = (x^3 - 2x)^7$;
  - $g(f(x)) = g(x^7) = (x^7)^3 - 2(x^7) = x^{21} - 2x^7$;
  - $f(f(x)) = f(x^7) = (x^7)^7 = x^{49}$;
- If $f(x) = \frac{x+8}{x-1}$ and $g(x) = \sqrt{x}$, find
  - $f(g(x)) = f(\sqrt{x}) = \frac{\sqrt{x}+8}{\sqrt{x}-1}$;
  - $g(f(x)) = g\left(\frac{x+8}{x-1}\right) = \sqrt{\frac{x+8}{x-1}}$.

Difference Quotient

- Given a function f, the expression $\dfrac{f(x+h)-f(x)}{h}$ is called the **difference quotient of f at x**;
- Geometrically, the difference quotient is the slope of the secant line of $y = f(x)$ through the points $(x, f(x))$ and $(x+h, f(x+h))$:



Computing Difference Quotients

- Find the difference quotient of $f(x) = 3x^2 - 2x + 1$ at x and simplify:

$$\frac{f(x+h)-f(x)}{h} = \frac{(3(x+h)^2 - 2(x+h)+1) - (3x^2 - 2x + 1)}{h} =$$
$$\frac{3(x^2+2xh+h^2) - 2x - 2h + 1 - 3x^2 + 2x - 1}{h} =$$
$$\frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{2x} - 2h + \cancel{1} - \cancel{3x^2} + \cancel{2x} - \cancel{1}}{h} =$$
$$\frac{6xh + 3h^2 - 2h}{h} = \frac{h(6x+3h-2)}{h} = 6x + 3h - 2;$$

- Find the difference quotient of $f(x) = \frac{1}{x}$ at x and simplify:

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} =$$
$$\frac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}} = \frac{-h}{hx(x+h)} = \frac{-1}{x(x+h)}.$$