Business and Life Calculus

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LSSU Math 112

Subsection 1

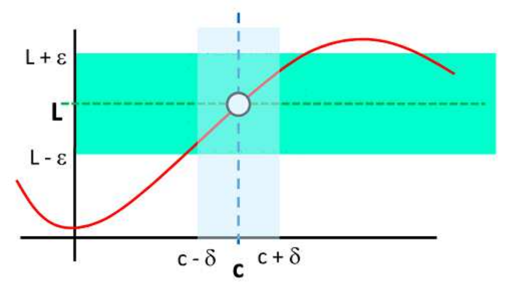
Limits and Continuity

Limits

- The statement

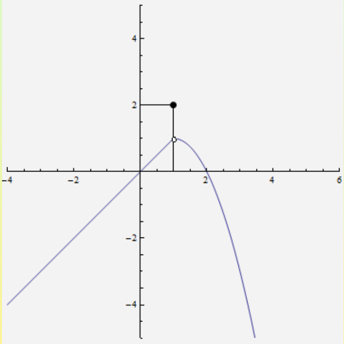
$$\lim_{x \to c} f(x) = L,$$

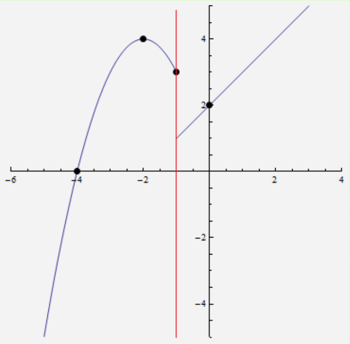
read the limit of $f(x)$ as x approaches c is L, means that the value of $y = f(x)$ approaches arbitrarily close to L as x approaches sufficiently close from either side (*but is not equal to*) c.



Limits Using Graphs

- Consider the function $f(x)$ whose graph is shown below: Find
  - $f(1) = 2$;
  - $\lim_{x \to 1} f(x) = 1$;
- Consider the function $g(x)$ whose graph is given below: Find
  - $g(-1) = 3$;
  - $\lim_{x \to -1} g(x) =$ Does Not Exist.

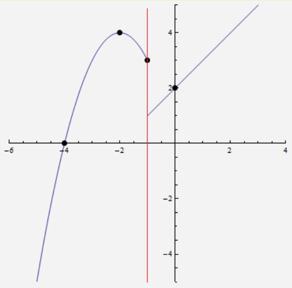




One-Sided Limits

- The statement $\lim_{x \to c^-} f(x) = L$, read the limit of $f(x)$ as x approaches c from the left is L, means that the value of $y = f(x)$ approaches arbitrarily close to L as x approaches sufficiently close from the left (*but is not equal to*) c;
- The statement $\lim_{x \to c^+} f(x) = L$, read the limit of $f(x)$ as x approaches c from the right is L, means that the value of $y = f(x)$ approaches arbitrarily close to L as x approaches sufficiently close from the right (*but is not equal to*) c;
- Revisiting the function g:

  We have $\lim_{x \to -1^-} g(x) = 3$; and $\lim_{x \to -1^+} g(x) = 1$.

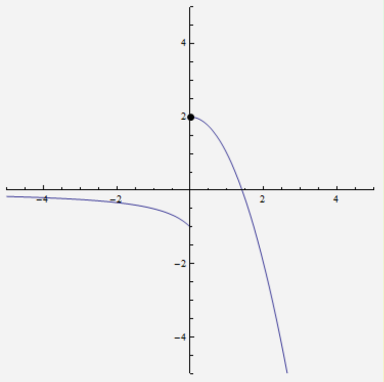


Limits Using Graphs I

- Consider the function f whose graph is given below:

We have

$$\begin{aligned} f(0) &= 2; \\ \lim_{x\to 0^-} f(x) &= -1; \\ \lim_{x\to 0^+} f(x) &= 2; \\ \lim_{x\to 0} f(x) &= \text{DNE}. \end{aligned}$$

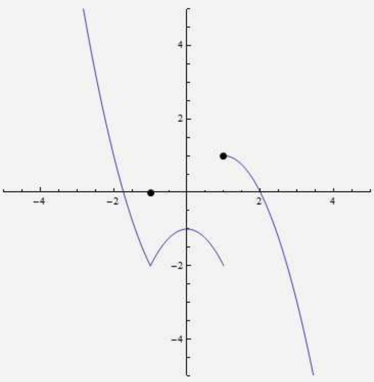


Limits Using Graphs II

- Consider the function f whose graph is given below:

We have

$$
\begin{aligned}
f(-1) &= 0; \\
\lim_{x\to -1^-} f(x) &= -2; \\
\lim_{x\to -1^+} f(x) &= -2; \\
\lim_{x\to -1} f(x) &= -2; \\
f(1) &= 1; \\
\lim_{x\to 1^-} f(x) &= -2; \\
\lim_{x\to 1^+} f(x) &= 1; \\
\lim_{x\to 1} f(x) &= \text{DNE}.
\end{aligned}
$$

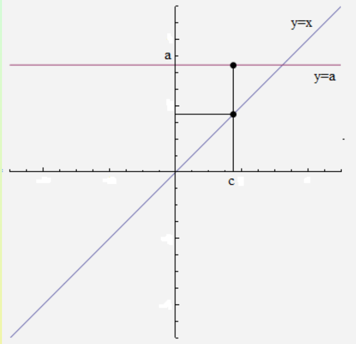


Two Important Simple Limits

- Consider the function $f(x) = a$, a constant;
- Consider, also $g(x) = x$;

What is

$$\begin{aligned} \lim_{x \to c} f(x) &= a \\ \lim_{x \to c} g(x) &= c \end{aligned}$$



- So, for all real numbers a, c, we have the rules

$$\lim_{x \to c} a = a \quad \text{and} \quad \lim_{x \to c} x = c.$$

Rules for Limits

- We would like to be able to find limits without having to graph;
- For this reason we develop some algebraic rules for computing limits;

Rules for Limits

1. $\lim_{x\to c} a = a$; and $\lim_{x\to c} x = c$;
2. $\lim_{x\to c} [f(x)^n] = (\lim_{x\to c} f(x))^n$;
3. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, if $\lim_{x\to c} f(x) \geq 0$, when n is even;
4. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then
   a. $\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;
   b. $\lim_{x\to c} [f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$;
   c. $\lim_{x\to c} [f(x) \cdot g(x)] = [\lim_{x\to c} f(x)] \cdot [\lim_{x\to c} g(x)]$;
   d. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, if $\lim_{x\to c} g(x) \neq 0$.

Using the Rules to Find Limits

- Use the rules to compute the following limits:
  - $\lim_{x\to -3}(2x^2-3x+1)$
    $\stackrel{\text{sum}}{=} \lim_{x\to -3}(2x^2) - \lim_{x\to -3}(3x) + \lim_{x\to -3}1$
    $\stackrel{\text{product}}{=} (\lim_{x\to -3}2)\cdot(\lim_{x\to -3}(x^2)) - (\lim_{x\to -3}3)\cdot(\lim_{x\to -3}x) + \lim_{x\to -3}1$
    $\stackrel{\text{power}}{=} (\lim_{x\to -3}2)\cdot((\lim_{x\to -3}x)^2) - (\lim_{x\to -3}3)\cdot(\lim_{x\to -3}x) + \lim_{x\to -3}1$
    $\stackrel{\text{basic}}{=} 2\cdot(-3)^2 - 3\cdot(-3) + 1 = 28;$
  - $\lim_{x\to 11}\sqrt{x-2} \stackrel{\text{root}}{=} \sqrt{\lim_{x\to 11}(x-2)} \stackrel{\text{differ}}{=} \sqrt{\lim_{x\to 11}x - \lim_{x\to 11}2} \stackrel{\text{basic}}{=} \sqrt{11-2} = 3;$
  - $\lim_{x\to 6}\dfrac{x^2}{x+3} \stackrel{\text{quotient}}{=} \dfrac{\lim_{x\to 6}(x^2)}{\lim_{x\to 6}(x+3)} \stackrel{\text{sum/prod}}{=} \dfrac{(\lim_{x\to 6}x)^2}{\lim_{x\to 6}x + \lim_{x\to 6}3} \stackrel{\text{basic}}{=} \dfrac{6^2}{6+3} = \dfrac{36}{9} = 4.$

Summary of the Rules: The Substitution Principle

- For functions composed of additions, subtractions, multiplications, divisions, powers and roots, limits may be evaluated by direct substitution, provided that the resulting expression is defined:

$$\lim_{x \to c} f(x) = f(c);$$

- For instance, as we saw in previous slide:
  - $\lim_{x \to -3} (2x^2 - 3x + 1) = 2(-3)^2 - 3(-3) + 1 = 28;$
  - $\lim_{x \to 11} \sqrt{x-2} = \sqrt{11-2} = 3;$
  - $\lim_{x \to 6} \frac{x^2}{x+3} = \frac{6^2}{6+3} = 4;$
- The problem arises when, in attempting to apply the rules the resulting expression is not defined; In that case, we may not conclude that the limit does not exist; Since the rules are not applicable, we simply have to employ some other technique to find it!

Examples Where Rules do not Apply

- Let $f(x) = \dfrac{x^2+6x-7}{x-1}$; Since $\lim\limits_{x\to 1}(x-1) = 0$, in computing $\lim\limits_{x\to 1}\dfrac{x^2+6x-7}{x-1}$, we cannot apply the quotient rule, i.e., we cannot write $\lim\limits_{x\to 1}\dfrac{x^2+6x-7}{x-1} = \dfrac{\lim\limits_{x\to 1}(x^2+6x-7)}{\lim\limits_{x\to 1}(x-1)}$;
  The expression on the right does not even make sense;
- Let $f(x) = \dfrac{x-11}{\sqrt{x-2}-3}$; Since $\lim\limits_{x\to 11}(\sqrt{x-2}-3) = 0$, in computing $\lim\limits_{x\to 11}\dfrac{x-11}{\sqrt{x-2}-3}$, we cannot apply the quotient rule, i.e., we cannot write $\lim\limits_{x\to 11}\dfrac{x-11}{\sqrt{x-2}-3} = \dfrac{\lim\limits_{x\to 11}(x-11)}{\lim\limits_{x\to 11}(\sqrt{x-2}-3)}$;
  The expression on the right does not make sense either;
- So what are we supposed to do in these cases?

Finding Limits by Factoring and Simplifying

- Compute the following limits:
  - $\lim_{x\to 1}\frac{x^2-1}{x-1}(\overset{\text{quotient}}{=}\frac{0}{0})\overset{\text{factor}}{=}\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}\overset{\text{simplify}}{=}\lim_{x\to 1}(x+1)\overset{\text{substitute}}{=}1+1=2$;
  - $\lim_{x\to 5}\frac{2x^2-10x}{x-5}(\overset{\text{quotient}}{=}\frac{0}{0})\overset{\text{factor}}{=}\lim_{x\to 5}\frac{2x(x-5)}{x-5}\overset{\text{simplify}}{=}\lim_{x\to 5}(2x)\overset{\text{substitute}}{=}2\cdot 5=10$;
  - $\lim_{x\to 1}\frac{x^2+6x-7}{x-1}(\overset{\text{quotient}}{=}\frac{0}{0})\overset{\text{factor}}{=}\lim_{x\to 1}\frac{(x-1)(x+7)}{x-1}\overset{\text{simplify}}{=}\lim_{x\to 1}(x+7)\overset{\text{substitute}}{=}1+7=8$;
  - $\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}(\overset{\text{quotient}}{=}\frac{0}{0})\overset{\text{subtract}}{=}\lim_{x\to 1}\frac{\frac{1}{x}-\frac{x}{x}}{x-1}=\lim_{x\to 1}\frac{\frac{1-x}{x}}{x-1}\overset{\text{divide}}{=}\lim_{x\to 1}\frac{-(x-1)}{x(x-1)}\overset{\text{simplify}}{=}\lim_{x\to 1}\frac{-1}{x}\overset{\text{substitute}}{=}\frac{-1}{1}=-1.$

Finding Limits by Multiplying by the Conjugate I

- Compute the following limit:
  - $\lim_{x\to 9}\frac{2\sqrt{x+7}-8}{x-9}\left(\stackrel{\text{quotient}}{=}\frac{0}{0}\right)\stackrel{\text{conjugate}}{=}$
    $\lim_{x\to 9}\frac{(2\sqrt{x+7}-8)(2\sqrt{x+7}+8)}{(x-9)(2\sqrt{x+7}+8)}\stackrel{\text{multiply top}}{=}$
    $\lim_{x\to 9}\frac{4(x+7)-64}{(x-9)(2\sqrt{x+7}+8)}\stackrel{\text{simplify}}{=}\lim_{x\to 9}\frac{4x+28-64}{(x-9)(2\sqrt{x+7}+8)}=$
    $\lim_{x\to 9}\frac{4x-36}{(x-9)(2\sqrt{x+7}+8)}\stackrel{\text{factor}}{=}\lim_{x\to 9}\frac{4(x-9)}{(x-9)(2\sqrt{x+7}+8)}\stackrel{\text{simplify}}{=}$
    $\lim_{x\to 9}\frac{4}{2\sqrt{x+7}+8}\stackrel{\text{substitute}}{=}\frac{4}{2\sqrt{9+7}+8}=\frac{1}{4}.$

Finding Limits by Multiplying by the Conjugate II

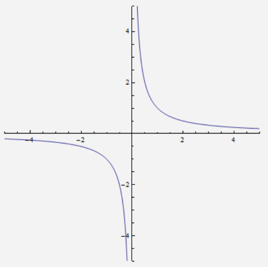
- Compute the following limit:
  - $$\lim_{x\to 11}\frac{x-11}{\sqrt{x-2}-3}\left(\stackrel{\text{quotient}}{=}\ \frac{0}{0}\right)\stackrel{\text{conjugate}}{=}$$
    $$\lim_{x\to 11}\frac{(x-11)(\sqrt{x-2}+3)}{(\sqrt{x-2}-3)(\sqrt{x-2}+3)}\stackrel{\text{multiply bottom}}{=}$$
    $$\lim_{x\to 11}\frac{(x-11)(\sqrt{x-2}+3)}{x-2-9}=\lim_{x\to 11}\frac{(x-11)(\sqrt{x-2}+3)}{x-11}\stackrel{\text{simplify}}{=}$$
    $$\lim_{x\to 11}(\sqrt{x-2}+3)\stackrel{\text{substitute}}{=}\sqrt{11-2}+3=6.$$

Limits Where $y \to \pm\infty$

- $\lim_{x \to c^-} f(x) = \infty$ means that the values of $f(x)$ grow arbitrarily large as x approaches c from the left;
- $\lim_{x \to c^+} f(x) = \infty$ means that the values of $f(x)$ grow arbitrarily large as x approaches c from the right;
- $\lim_{x \to c} f(x) = \infty$ means that both $\lim_{x \to c^-} f(x) = \infty$ and $\lim_{x \to c^+} f(x) = \infty$ are true;
- Consider the function $f(x) = \frac{1}{x}$ whose graph is given below:

  We have

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= -\infty; \\ \lim_{x \to 0^+} f(x) &= \infty; \\ \lim_{x \to 0} f(x) &= \text{DNE}. \end{aligned}$$

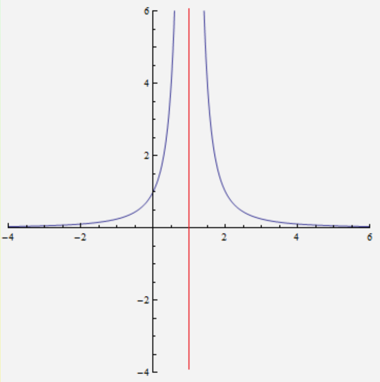


Another Example

- Consider the function $f(x) = \frac{1}{(x-1)^2}$ whose graph is given below:

  We have

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \infty; \\ \lim_{x \to 1^+} f(x) &= \infty; \\ \lim_{x \to 1} f(x) &= \infty. \end{aligned}$$

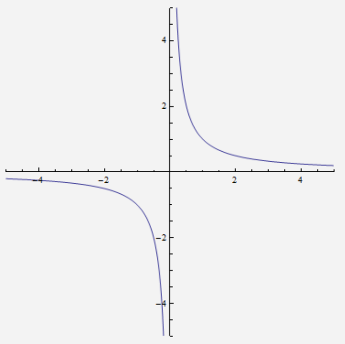


Limits Where $x \to \pm\infty$

- $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ get arbitrarily close to L as x grows arbitrarily small;
- $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ get arbitrarily close to L as x grows arbitrarily large;
- Consider, again $f(x) = \frac{1}{x}$ whose graph is given below:

  We have

$$\begin{aligned} \lim_{x\to-\infty} f(x) &= 0; \\ \lim_{x\to\infty} f(x) &= 0. \end{aligned}$$



Graphical Examples of Limits Involving Infinity

- Consider the function $f(x)$ whose graph is given below:

We have

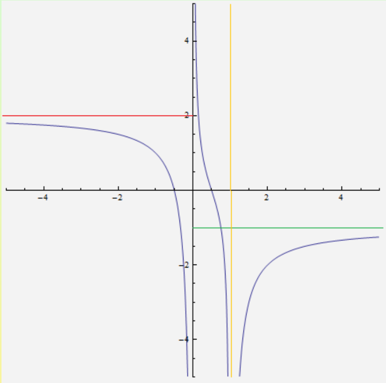
$$\lim_{x\to-\infty} f(x) = 2; \qquad \lim_{x\to\infty} f(x) = -1;$$

$$\lim_{x\to 0^-} f(x) = -\infty; \qquad \lim_{x\to 0^+} f(x) = \infty;$$

$$\lim_{x\to 0} f(x) = \text{DNE};$$

$$\lim_{x\to 1^-} f(x) = -\infty; \qquad \lim_{x\to 1^+} f(x) = -\infty;$$

$$\lim_{x\to 1} f(x) = -\infty.$$



Algebraic Computation of Limits Involving Infinity

- Suppose that you would like to compute $\lim_{x \to c} f(x)$; Try plugging in c for x to see if the resulting expression is defined; If yes, you may use the substitution property; In addition, be aware of the following cases (a is a fixed real):
  - The form $\dfrac{a}{\pm\infty}$: Always approaches 0:
    - For example, $\lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{5}{x+2} = 0$;
  - The form $\dfrac{a}{0}$ with $a \neq 0$: Always approaches $\pm\infty$:
    - For example, $\lim_{x \to 0^+} \frac{1}{x} = +\infty$, $\lim_{x \to 5^-} \frac{3}{x-5} = -\infty$;
  - The forms $\dfrac{0}{0}$ and $\dfrac{\pm\infty}{\pm\infty}$: These are undetermined; But this does not necessarily mean that limits do not exist; It simply means that we have to work harder to reveal what really happens;
    - For example, $\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x = \infty$, $\lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to 0^-} \frac{x}{x^2} = \lim_{x \to 0^-} \frac{1}{x} = -\infty$.

More Involved Examples

- Compute the limits of the following rational functions:
  - $\lim_{x\to\infty} \dfrac{3x^3 - x + 1}{x^3 + 2x} \overset{\text{divide by } x^3}{=} \lim_{x\to\infty} \dfrac{\frac{3x^3-x+1}{x^3}}{\frac{x^3+2x}{x^3}} \overset{\text{break}}{=} \lim_{x\to\infty} \dfrac{\frac{3x^3}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}}{\frac{x^3}{x^3} + \frac{2x}{x^3}} \overset{\text{simplify}}{=}$
    $\lim_{x\to\infty} \dfrac{3 - \frac{1}{x^2} + \frac{1}{x^3}}{1 + \frac{2}{x^2}} \overset{\text{take limits}}{=} \dfrac{3 - 0 + 0}{1 + 0} = 3;$
  - $\lim_{x\to\infty} \dfrac{5x + 3}{x^2 - 1} \overset{\text{divide by } x^2}{=} \lim_{x\to\infty} \dfrac{\frac{5x+3}{x^2}}{\frac{x^2-1}{x^2}} \overset{\text{break}}{=} \lim_{x\to\infty} \dfrac{\frac{5x}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} \overset{\text{simplify}}{=}$
    $\lim_{x\to\infty} \dfrac{\frac{5}{x} + \frac{3}{x^2}}{1 - \frac{1}{x^2}} \overset{\text{take limits}}{=} \dfrac{0 + 0}{1 - 0} = 0;$
  - $\lim_{x\to\infty} \dfrac{x^3 + 5x}{2x^2 + 7} \overset{\text{divide by } x^2}{=} \lim_{x\to\infty} \dfrac{\frac{x^3+5x}{x^2}}{\frac{2x^2+7}{x^2}} \overset{\text{break}}{=} \lim_{x\to\infty} \dfrac{\frac{x^3}{x^2} + \frac{5x}{x^2}}{\frac{2x^2}{x^2} + \frac{7}{x^2}} \overset{\text{simplify}}{=}$
    $\lim_{x\to\infty} \dfrac{x + \frac{5}{x}}{2 + \frac{7}{x^2}} \overset{\text{take limits}}{=} \dfrac{\infty + 0}{2 + 0} = \infty.$

Limits of Rational Functions as $x \to \pm\infty$

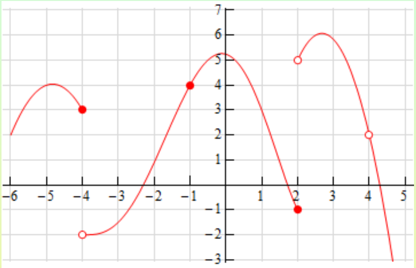
- The examples in the previous slide suggest the following general rule for finding
$$\lim_{x\to\infty} \frac{P(x)}{Q(x)},$$
for $P(x)$ and $Q(x)$ polynomial functions:
  1. If the degree of the numerator is greater than the degree of the denominator, then the limit is $\pm\infty$;
     - $\lim_{x\to\infty} \frac{x^3+5x}{2x^2+7} = \infty$;
  2. If the degree of the numerator and the degree of the denominator are equal, then the limit is equal to the ratio of the leading coefficients;
     - $\lim_{x\to\infty} \frac{3x^3-x+1}{x^3+2x} = \frac{3}{1} = 3$;
  3. If the degree of the numerator is less than the degree of the denominator, then the limit is 0;
     - $\lim_{x\to\infty} \frac{5x+3}{x^2-1} = 0$.

Subsection 2

Continuity

Continuity at a Point Geometrically

- A function is **continuous at** $x = c$ if its graph passes through $(c, f(c))$ without a "hole" or a "jump";
- Let us take a look at $y = f(x)$



The function $f(x)$ depicted is discontinuous at the points -4, 2 and 4, but it is continuous at -1;

- At points of continuity, we can draw graph moving from left to right passing through the point without having to lift pencil from paper;
- This is not true at the points of discontinuity; There the "hole" or the "jump" forces us to reposition the pencil by lifting it from the paper as we move from left to right through the point.

Continuity at a Point Algebraically

- A function is **continuous at** $x = c$ if
  - $f(c)$ is defined;

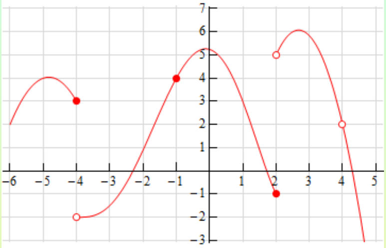
    This means a graph has a point at $x = c$.
  - $\lim_{x \to c} f(x)$ exists;

    This means that $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$; So, as x approaches c from left, $f(x)$ stays at same level as when x approaches c from the right. Thus, f does not "jump" passing through c.
  - $\lim_{x \to c} f(x) = f(c)$.

    The point of the graph at c has to be at the same level as that approached by $f(x)$ when x approaches c; So, no "hole" occurs over c.

Example Revisited

- Let us look again at $y = f(x)$:



$$\begin{array}{lll} f(-4) = 3; & f(2) = -1; & f(4) = \text{DNE}; \\ \lim_{x \to -4^-} f(x) = 3; & \lim_{x \to 2^-} f(x) = -1; & \lim_{x \to 4^-} f(x) = 2; \\ \lim_{x \to -4^+} f(x) = -2; & \lim_{x \to 2^+} f(x) = 5; & \lim_{x \to 4^+} f(x) = 2; \\ \lim_{x \to -4} f(x) = \text{DNE}; & \lim_{x \to 2} f(x) = \text{DNE}; & \lim_{x \to 4} f(x) = 2. \end{array}$$

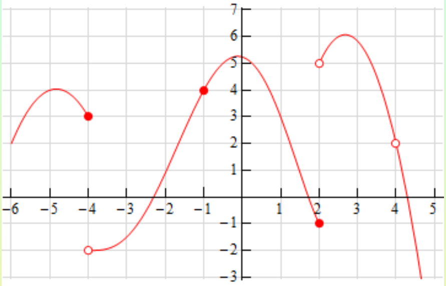
Left and Right Continuity at a Point

- If $\lim_{x\to c^-} f(x) = f(c)$, then we say that f is **left continuous at** c;
- If $\lim_{x\to c^+} f(x) = f(c)$, then we say that f is **right continuous at** c;
- Since for continuity at c we require that $\lim_{x\to c} f(x) = f(c)$ (which means $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$), it is clear that:

  f is continuous at c if and only if it is both left and right continuous at c.

Example Revisited

- Let us look again at $y = f(x)$:



  - At $x = -4$, $\lim_{x \to -4^-} f(x) = 3 = f(-4)$. So f is left continuous at $x = -4$;
  - At $x = 2$, $\lim_{x \to 2^-} f(x) = -1 = f(2)$. So f is left continuous at $x = 2$;
  - At $x = 4$, $f(4)$ does not exist. So f can be neither left nor right continuous at $x = 4$.

Algebraic Example

- Consider the piece-wise defined function
$$f(x) = \begin{cases} x+2, & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 2 \\ -x^2+7, & \text{if } x \ge 2 \end{cases}$$
- We first investigate whether f is continuous, left/right continuous or discontinuous at $x = -1$.

$$\begin{aligned} & f(-1) = (-1)^2 = 1; \\ & \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x+2) = -1+2 = 1; \\ & \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 = (-1)^2 = 1; \\ & \lim_{x \to -1} f(x) = 1; \end{aligned}$$

Therefore $\lim_{x \to -1} f(x) = 1 = f(-1)$ and f is continuous at $x = -1$;

Algebraic Example (Cont'd)

- Consider, again, the piece-wise defined function
$$f(x) = \begin{cases} x+2, & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 2 \\ -x^2+7, & \text{if } x \geq 2 \end{cases}$$
- We next investigate whether f is continuous, left/right continuous or discontinuous at $x = 2$.

$$\begin{aligned} & f(2) = -2^2+7 = 3; \\ & \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 2^2 = 4; \\ & \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (-x^2+7) = -2^2+7 = 3; \\ & \lim_{x \to 2} f(x) = \text{DNE}; \end{aligned}$$

Therefore, $\lim_{x \to 2^+} f(x) = 3 = f(2)$ and f is right continuous at $x = 2$;
But since the $\lim_{x \to 2} f(x)$ does not exist f is not continuous at $x = 2$;

Combinations of Continuous Functions

- If two functions f and g are continuous at a point c, then the following are also continuous at c:
  1. $f \pm g$;
  2. $a \cdot f$, for any constant a;
  3. $f \cdot g$;
  4. $\frac{f}{g}$, if $g(c) \neq 0$;
  5. $f(g(x))$, if f is continuous at $g(c)$;
- Consequences of these rules are the following facts:
  - Every polynomial function is continuous at all real numbers;
  - Every rational function is continuous at all points where it is defined.

Functions That Are Continuous

- We saw that:
  - Every polynomial function is continuous everywhere;
  - Every rational function is continuous at all points where it is defined;
- From the graphs, it is easy to see that, in addition:
  - Every exponential function $f(x) = a^x$, $0 < a \neq 1$, is continuous everywhere.
- Example: Determine where each function is continuous or discontinuous:
  - $f(x) = x^5 + 5x^3 - 11x$ is continuous everywhere since it is a polynomial;
  - $f(x) = \frac{1}{x^2 - 36}$ is continuous at all points except ± 6, since it is a rational function with domain $\mathrm{Dom}(f) = \mathbb{R} - \{-6, 6\}$;
  - $f(x) = e^{x-5}$ is continuous everywhere because it is the composite of $x - 5$ and e^x, which are continuous - a polynomial and an exponential - everywhere.

Subsection 3

Rates of Change, Slopes and Derivatives

Average and Instantaneous Rate of Change

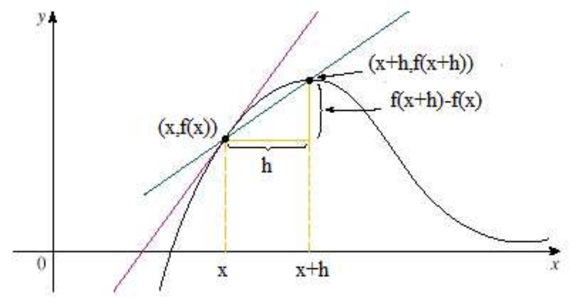
- The **average rate of change** of a function f between x and $x+h$ is
$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h};$$
- To find the instantaneous rate of change at x, we take $h \to 0$, so that the interval $(x, x+h)$ becomes so small that f has very little chance to change.
- The **instantaneous rate of change** of f at x is
$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Average and Instantaneous Rate of Change: Example

- Find the average rate of change of the temperature $T(t) = t^2$ (in degrees) between $t = 1$ and $t = 3$ hours;
  We have
  $$\frac{T(3) - T(1)}{3 - 1} = \frac{3^2 - 1^2}{3 - 1} = \frac{8}{2} = 4;$$
  Thus, the average rate of change was 4 degrees per hour;
- Example: Find the instantaneous rate of change of the temperature $T(t) = t^2$ (in degrees) at $t = 1$;
  We have
  $$\begin{aligned} \lim_{h \to 0} \frac{T(1+h) - T(1)}{h} &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} \\ &= \lim_{h \to 0} (2 + h) = 2; \end{aligned}$$
  Thus, the instantaneous rate of change was 2 degrees/hour at $t = 1$.

Slope of Secant and Tangent Lines

- Consider the secant line to the graph of $y = f(x)$ through $(x, f(x))$ and $(x+h, f(x+h))$;



- It has slope $\dfrac{f(x+h)-f(x)}{h}$;
- The tangent line to $y = f(x)$ at x has slope $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

Slope of a Tangent Line

- Find the slope of the tangent line to $f(x) = \frac{1}{x+1}$ at $x = 3$;

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\frac{1}{(3+h)+1} - \frac{1}{3+1}}{h} =$$

$$\lim_{h \to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h} = \lim_{h \to 0} \frac{\frac{4}{4(4+h)} - \frac{4+h}{4(4+h)}}{h} =$$

$$\lim_{h \to 0} \frac{\frac{4-(4+h)}{4(4+h)}}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{4(4+h)} =$$

$$\lim_{h \to 0} \left(-\frac{1}{4(4+h)}\right) = -\frac{1}{16}.$$

Equation of a Tangent Line I

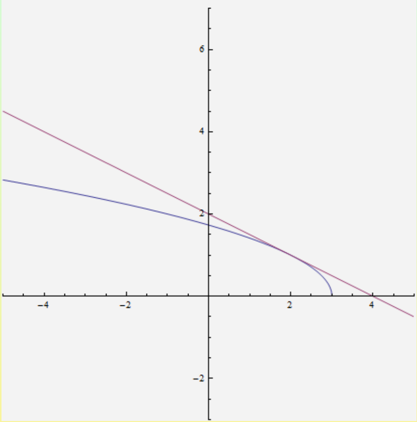
- Find an equation of the tangent line to $f(x) = \sqrt{3-x}$ at $x = 2$;
  First, find the slope of the tangent line

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\sqrt{3-(2+h)}-\sqrt{3-2}}{h} =$$
$$\lim_{h\to 0} \frac{\sqrt{1-h}-1}{h} = \lim_{h\to 0} \frac{(\sqrt{1-h}-1)(\sqrt{1-h}+1)}{h(\sqrt{1-h}+1)} =$$
$$\lim_{h\to 0} \frac{(\sqrt{1-h})^2-1^2}{h(\sqrt{1-h}+1)} = \lim_{h\to 0} \frac{1-h-1}{h(\sqrt{1-h}+1)} =$$
$$\lim_{h\to 0} \frac{-h}{h(\sqrt{1-h}+1)} = \lim_{h\to 0} \frac{-1}{\sqrt{1-h}+1} =$$
$$\frac{-1}{\sqrt{1-0}+1} = -\frac{1}{2};$$

Thus, the equation of the tangent line is $y - f(2) = -\frac{1}{2}(x-2)$ or $y - 1 = -\frac{1}{2}(x-2)$.

Equation of a Tangent Line I (Illustration)

- The equation of the tangent line to $f(x) = \sqrt{3-x}$ at $x = 2$ is $y - 1 = -\frac{1}{2}(x-2)$.



Equation of a Tangent Line II

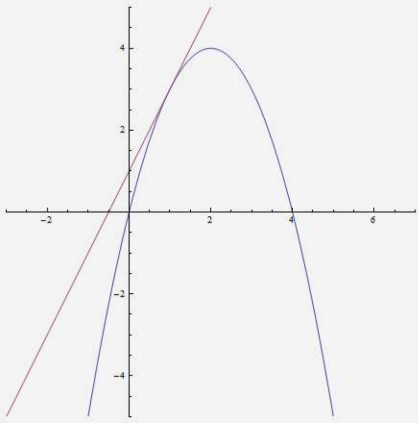
- Find an equation of the tangent line to $f(x) = -x^2 + 4x$ at $x = 1$; First, find the slope of the tangent line

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} =$$
$$\lim_{h\to 0} \frac{-(1+h)^2 + 4(1+h) - (-1^2 + 4 \cdot 1)}{h} =$$
$$\lim_{h\to 0} \frac{-(1^2 + 2h + h^2) + 4 + 4h - 3}{h} =$$
$$\lim_{h\to 0} \frac{-1 - 2h - h^2 + 4 + 4h - 3}{h} =$$
$$\lim_{h\to 0} \frac{2h - h^2}{h} = \lim_{h\to 0} \frac{h(2-h)}{h} = \lim_{h\to 0} (2-h) = 2;$$

Thus, the equation of the tangent line is $y - f(1) = 2(x-1)$ or $y - 3 = 2(x-1)$.

Equation of a Tangent Line II (Illustration)

- The equation of the tangent line to $f(x) = -x^2 + 4x$ at $x = 1$ is $y - 3 = 2(x - 1)$.



The Derivative

Definition of the Derivative

The **derivative of** f **at** x is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limits exists; The derivative $f'(x)$ gives the instantaneous rate of change of f at x and, also, the slope of the tangent line to $y = f(x)$ at x;

- Example: Compute $f'(x)$ if $f(x) = x^2 - 3x$;

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h} =$$
$$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} =$$
$$\lim_{h \to 0} \frac{h(2x + h - 3)}{h} = \lim_{h \to 0} (2x + h - 3) = 2x + 0 - 3 = 2x - 3.$$

Another Example of a Tangent Line

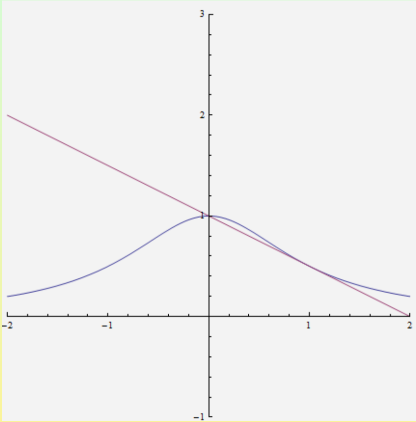
- Find an equation of the tangent line to $f(x) = \dfrac{1}{1+x^2}$ at $x = 1$;
  First, find the slope of the tangent line $f'(1)$:

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\frac{1}{1+(1+h)^2}-\frac{1}{1+1^2}}{h} =$$
$$\lim_{h\to 0}\frac{\frac{1}{1+1+2h+h^2}-\frac{1}{2}}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2+2h+h^2}-\frac{1}{2}\right) =$$
$$\lim_{h\to 0}\frac{1}{h}\cdot\frac{2-(2+2h+h^2)}{2(2+2h+h^2)} = \lim_{h\to 0}\frac{1}{h}\cdot\frac{-h(2+h)}{2(2+2h+h^2)} =$$
$$\lim_{h\to 0}\frac{-(2+h)}{2(2+2h+h^2)} = \frac{-(2+0)}{2(2+2\cdot 0+0^2)} = -\frac{1}{2};$$

  Thus, the equation of the tangent line is $y - f(1) = -\frac{1}{2}(x-1)$ or $y - \frac{1}{2} = -\frac{1}{2}(x-1)$.

Tangent Line (Illustration)

- The equation of the tangent line to $f(x) = \dfrac{1}{1+x^2}$ at $x = 1$ is $y - \frac{1}{2} = -\frac{1}{2}(x-1)$.



Alternative Notation for the Derivative

- We should all be aware that the following alternative notation is sometimes used for the derivative $f'(x)$ of f at a point x:

$$f'(x) = \frac{df}{dx} = y' = \frac{dy}{dx};$$

- Also, when the value of $f'(x)$ at a specific point $x = c$ is considered, we write

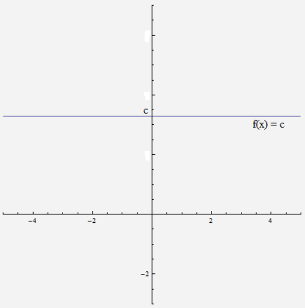
$$f'(c) = \left.\frac{df}{dx}\right|_{x=c} = y'(c) = \left.\frac{dy}{dx}\right|_{x=c};$$

Subsection 4

Some Differentiation Formulas

Derivative of a Constant

- Consider a constant function $f(x) = c$;



The tangent to $y = c$ at any point is horizontal; Thus, its slope is zero:

$$(c') = 0;$$

- Algebraically,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

Power Rule

- For any exponent n,

$$\boxed{(x^n)' = n \cdot x^{n-1};}$$

- Example: Find the following derivatives:
  - $(x^7)' = 7x^{7-1} = 7x^6$;
  - $(x^{94})' = 94x^{94-1} = 94x^{93}$;
  - $(\frac{1}{x^5})' = (x^{-5})' = (-5)x^{-5-1} = -5x^{-6} = -\frac{5}{x^6}$;
  - $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$;
  - $(\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3\sqrt[3]{x^2}}$;
  - $(x)' = (x^1)' = 1x^{1-1} = 1x^0 = 1$.

Constant Factor

- Constant Factor or Constant Multiple Rule:

$$\boxed{(c \cdot f(x))' = c \cdot f'(x);}$$

- Example: Find the following derivatives:
  - $(5x^7)' = 5(x^7)' = 5 \cdot 7x^6 = 35x^6$;
  - $(\frac{7}{x^3})' = (7x^{-3})' = 7(x^{-3})' = 7 \cdot (-3)x^{-4} = -\frac{21}{x^4}$;
  - $(\frac{8}{\sqrt{x}})' = (8x^{-1/2})' = 8(x^{-1/2})' = 8 \cdot (-\frac{1}{2})x^{-3/2} = -\frac{4}{x^{3/2}} = -\frac{4}{\sqrt{x^3}}$;
  - $(17x)' = 17(x)' = 17 \cdot 1 = 17$.

Sum/Difference Rule

- Sum/Difference Rule:

$$\boxed{(f(x) \pm g(x))' = f'(x) \pm g'(x);}$$

- Example: Find the following derivatives:
  - $(x^3 - x^7)' = (x^3)' - (x^7)' = 3x^2 - 7x^6;$
  - $(7x^{-5} - 3x^{1/3} + 17)' = (7x^{-5})' - (3x^{1/3})' + (17)' =$
    $7(x^{-5})' - 3(x^{1/3})' + 0 = 7(-5)x^{-6} - 3 \cdot \frac{1}{3}x^{-2/3} = -35x^{-6} - x^{-2/3}.$

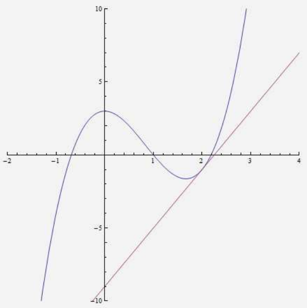
Example I

- Find an equation for the tangent line to the graph of $f(x) = 2x^3 - 5x^2 + 3$ at $x = 2$;
  First compute the slope $f'(2)$ of the tangent line:

$$f'(x) = (2x^3 - 5x^2 + 3)' = (2x^3)' - (5x^2)' + (3)' = 2(x^3)' - 5(x^2)' + 0 = 2 \cdot 3x^2 - 5 \cdot 2x = 6x^2 - 10x;$$

Thus, $f'(2) = 6 \cdot 2^2 - 10 \cdot 2 = 4$; Therefore the tangent line has equation

$$y - f(2) = f'(2)(x - 2)$$
$$\Rightarrow y - (-1) = 4(x - 2)$$
$$\Rightarrow y = 4x - 9.$$



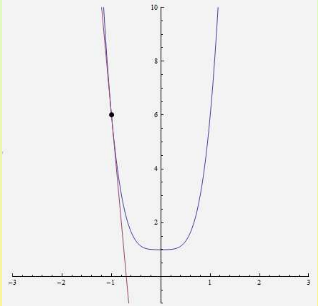
Example II

- Find an equation for the tangent line to the graph of $f(x) = 5x^4 + 1$ at $x = -1$;
  First compute the slope $f'(-1)$ of the tangent line:

$$f'(x) = (5x^4 + 1)' = (5x^4)' + (1)' = 5(x^4)' + 0 = 5 \cdot 4x^3 = 20x^3;$$

Thus, $f'(-1) = 20 \cdot (-1)^3 = -20$; Therefore the tangent line has equation

$$y - f(-1) = f'(-1)(x - (-1)) \Rightarrow y - 6 = -20(x + 1) \Rightarrow y = -20x - 14.$$



Business: Marginal Analysis

- Let x denote the number of items produced and sold by a company;
- Suppose that $C(x), R(x)$ and $P(x) = R(x) - C(x)$ are the cost, revenue and profit function, respectively;
- The **marginal cost** at x is the cost for producing one more unit:

$$\text{Marginal Cost}(x) = C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1}$$
$$\approx \lim_{h \to 0} \frac{C(x+h) - C(x)}{h} = C'(x);$$

- Because of this, in calculus we define the **marginal cost function** $\text{MC}(x)$ by

$$\boxed{\text{MC}(x) = C'(x);}$$

- Similarly, for **marginal revenue** and for **marginal profit**:

$$\text{MR}(x) = R'(x) \quad \text{and} \quad \text{MP}(x) = P'(x).$$

Application: Marginal Cost

- The cost function in dollars for producing x items is given by

$$C(x) = 8\sqrt[4]{x^3} + 300;$$

  - Find the marginal cost function $\mathrm{MC}(x)$;

$$\mathrm{MC}(x) = C'(x) = (8\sqrt[4]{x^3} + 300)' = (8x^{3/4})' = 8(x^{3/4})' = 8 \cdot \frac{3}{4}x^{-1/4} = \frac{6}{\sqrt[4]{x}};$$

  - Find the marginal cost when 81 items are produced; Interpret the answer;

$$\mathrm{MC}(81) = \frac{6}{\sqrt[4]{81}} = \frac{6}{3} = 2;$$

  This is the approximate additional cost for producing the 82nd item.

Application: Learning Rate

- A psychology researcher found that the number of names a person can memorize in t minutes is approximately
$$N(t) = 6\sqrt[3]{t^2};$$
Find the instantaneous rate of change of this function after 8 minutes and interpret your answer;

$$N'(t) = (6\sqrt[3]{t^2})' = (6t^{2/3})' = 6(t^{2/3})' = 6 \cdot \frac{2}{3} t^{-1/3} = \frac{4}{\sqrt[3]{t}};$$

Therefore,
$$N'(8) = \frac{4}{\sqrt[3]{8}} = \frac{4}{2} = 2;$$
Thus, a person can memorize approximately 2 additional names/minute after 8 minutes.

Subsection 5

The Product and Quotient Rules

Product Rule

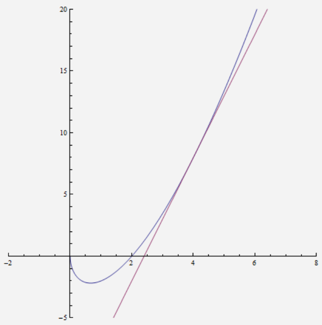
- The Product Rule for Derivatives:

$$\boxed{(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x);}$$

- Example: Use the product rule to calculate the derivatives:
  - $(x^4 \cdot x^7)' = (x^4)'x^7 + x^4(x^7)' = 4x^3x^7 + x^4 \cdot 7x^6 = 4x^{10} + 7x^{10} = 11x^{10}$;
  - $[(x^2 - x + 2)(x^3 + 5)]' = (x^2 - x + 2)'(x^3 + 5) + (x^2 - x + 2)(x^3 + 5)' = (2x - 1)(x^3 + 5) + (x^2 - x + 2)(3x^2) = 2x^4 - x^3 + 10x - 5 + 3x^4 - 3x^3 + 6x^2 = 5x^4 - 4x^3 + 6x^2 + 10x - 5$;
  - $[x^3(x^2 - x)]' = (x^3)'(x^2 - x) + x^3(x^2 - x)' = 3x^2(x^2 - x) + x^3(2x - 1) = 3x^4 - 3x^3 + 2x^4 - x^3 = 5x^4 - 4x^3$.

Using Product Rule

- Find an equation for the tangent line to the graph of $f(x) = \sqrt{x}(2x-4)$ at $x = 4$;
  First compute the slope $f'(4)$ of the tangent line:
  $$f'(x) = [\sqrt{x}(2x-4)]' = [x^{1/2}(2x-4)]' =$$
  $$(x^{1/2})'(2x-4) + x^{1/2}(2x-4)' = \frac{1}{2}x^{-1/2}(2x-4) + 2x^{1/2} =$$
  $$\frac{2x-4}{2\sqrt{x}} + 2\sqrt{x} = \frac{x-2}{\sqrt{x}} + 2\sqrt{x};$$
  Thus, $f'(4) = \dfrac{4-2}{\sqrt{4}} + 2\sqrt{4} = 1 + 4 = 5;$
  Therefore the tangent line has equation
  $$y - f(4) = f'(4)(x-4)$$
  $$\Rightarrow y - 8 = 5(x-4)$$
  $$\Rightarrow y = 5x - 12.$$



Quotient Rule

- The Quotient Rule for Derivatives:

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2};}$$

- Example: Use the quotient rule to calculate the derivatives:
  - $\left(\frac{x^{13}}{x^5}\right)' = \frac{(x^{13})'x^5 - x^{13}(x^5)'}{(x^5)^2} = \frac{13x^{12}x^5 - x^{13}\cdot 5x^4}{x^{10}} = \frac{13x^{17} - 5x^{17}}{x^{10}} = \frac{8x^{17}}{x^{10}} = 8x^7;$
  - $\left(\frac{x^2}{x+1}\right)' = \frac{(x^2)'(x+1) - x^2(x+1)'}{(x+1)^2} = \frac{2x(x+1) - x^2 1}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}.$

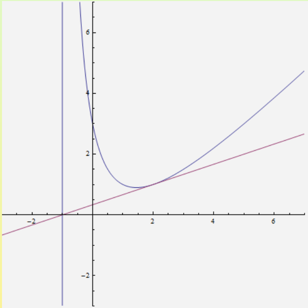
Using Quotient Rule

- Find an equation for the tangent line to the graph of $f(x) = \dfrac{x^2 - 2x + 3}{x + 1}$ at $x = 2$;
  First compute the slope $f'(2)$ of the tangent line:

  $$f'(x) = \left(\frac{x^2-2x+3}{x+1}\right)' = \frac{(x^2-2x+3)'(x+1)-(x^2-2x+3)(x+1)'}{(x+1)^2} = \frac{(2x-2)(x+1)-(x^2-2x+3)\cdot 1}{(x+1)^2} = \frac{2x^2+2x-2x-2-x^2+2x-3}{(x+1)^2} = \frac{x^2+2x-5}{(x+1)^2};$$

  Thus, $f'(2) = \frac{2^2+2\cdot 2-5}{(2+1)^2} = \frac{3}{9} = \frac{1}{3}$; Therefore the tangent line has equation

  $$y - f(2) = f'(2)(x-2)$$
  $$\Rightarrow y - 1 = \frac{1}{3}(x-2)$$
  $$\Rightarrow y = \frac{1}{3}x + \frac{1}{3}.$$



Application: Cost of Cleaner Water

- Suppose that the cost of purifying a gallon of water to a purity of x percent is $C(x) = \frac{2}{100-x}$, for $80 < x < 100$, in dollars; What is the rate of change of the purification costs when purity is 90% and 98%?

$$\begin{aligned} C'(x) &= \left(\frac{2}{100-x}\right)' \\ &= \frac{(2)'(100-x) - 2(100-x)'}{(100-x)^2} \\ &= \frac{0 \cdot (100-x) - 2(-1)}{(100-x)^2} \\ &= \frac{2}{(100-x)^2}; \end{aligned}$$

$$C'(90) = \frac{2}{10^2} = 0.02 \text{ \$/gallon and } C'(98) = \frac{2}{2^2} = 0.50 \text{ \$/gallon.}$$

Marginal Average Cost/Revenue/Profit

- If $C(x)$ is the cost for producing x items, then the **average cost per item** is $\mathrm{AC}(x) = \frac{C(x)}{x}$;
- The **marginal average cost** is defined by $\mathrm{MAC} = \mathrm{AC}'(x) = \left(\frac{C(x)}{x}\right)'$;
- Similarly, is $R(x)$ and $P(x)$ are the revenue and profit from selling x items, the **average revenue** and **average profit** per item are $\mathrm{AR}(x) = \frac{R(x)}{x}$ and $\mathrm{AP}(x) = \frac{P(x)}{x}$;
- And the **marginal average revenue** and **marginal average profit** are given by

$$\mathrm{MAR}(x) = \mathrm{AR}'(x) \quad \text{and} \quad \mathrm{MAP}(x) = \mathrm{AP}'(x);$$

- The meaning of marginal average cost is the approximate additional average cost per item for producing one more item;
- Similar interpretations apply for marginal average revenue and marginal average profit.

Application: Marginal Average Cost

- On-demand printing a typical 200 page book would cost \$ 18 per copy, with fixed costs of \$ 1500. Therefore, the cost function is $C(x) = 18x + 1500$;
  - Find the average cost function;
  $$\text{AC}(x) = \frac{C(x)}{x} = \frac{18x + 1500}{x} = \frac{18x}{x} + \frac{1500}{x} = 18 + 1500x^{-1};$$
  - Find the marginal average cost function;
  $$\text{MAC}(x) = (18 + 1500x^{-1})' = 1500(x^{-1})' = 1500 \cdot (-1)x^{-2} = -\frac{1500}{x^2};$$
  - What is the marginal average cost at $x = 100$? Interpret the answer;
  $$\text{MAC}(100) = -\frac{1500}{(100)^2} = -0.15;$$
  When 100 books are produced, the average cost per book is decreasing by about 15 cents per additional book produced.

Application: Time Saved by Speeding

- Chris drives 25 miles to his office every day. If he drives at a constant speed of v miles per hour, then his driving time is $T(v) = \frac{25}{v}$ hours; Compute $T'(55)$ and interpret the answer;

$$T'(v) = \left(\frac{25}{v}\right)' = (25v^{-1})' = 25(v^{-1})' = 25 \cdot (-1)v^{-2} = -\frac{25}{v^2};$$

So $T'(55) = -\frac{25}{55^2} = -0.00826;$

Thus, Chris would save approximately 0.00826 hours (around half a minute) per extra mile/hour of speed when driving at 55 mph.

Summary of Differentiation Rules

Rules For Taking Derivatives

- $(c)' = 0$;
- $(x^n)' = n \cdot x^{n-1}$;
- $(c \cdot f)' = c \cdot f'$;
- $(f \pm g)' = f' \pm g'$;
- $(f \cdot g)' = f' \cdot g + f \cdot g'$;
- $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}$.

Subsection 6

Higher-Order Derivatives

Higher-Order Derivatives

- Given a function $f(x)$, the derivative $(f')'$ of its first derivative f' is called its **second derivative** and denoted $f''(x)$;
- The derivative $(f'')'$ of its second derivative is called its **third derivative** and denoted f''';
- From the **fourth derivative** up, instead of piling $'$ up in the notation, we use $f^{(4)}(x), f^{(5)}(x), f^{(6)}(x)$, etc.
- Thus, since the $(n+1)$-st derivative of f is the first derivative of the n-th derivative, we have the definition

$$f^{(n+1)}(x) = (f^{(n)}(x))';$$

- In the alternative notation for derivatives, the first, second, third, fourth etc, derivatives are written

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}, \ldots$$

Calculating Higher-Order Derivatives I

- Find all derivatives of $f(x) = x^3 - 9x^2 + 5x - 17$;

$$\begin{aligned}
f'(x) &= (x^3 - 9x^2 + 5x - 17)' = 3x^2 - 18x + 5; \\
f''(x) &= (3x^2 - 18x + 5)' = 6x - 18; \\
f'''(x) &= (6x - 18)' = 6; \\
f^{(4)}(x) &= (6)' = 0; \\
f^{(5)}(x) &= (0)' = 0; \\
&\vdots
\end{aligned}$$

So we have $f^{(n)}(x) = 0$, for all $n \geq 4$.

Calculating Higher-Order Derivatives II

- Find all derivatives of $f(x) = \frac{1}{x}$;

$$\begin{aligned}
f'(x) &= (\tfrac{1}{x})' = (x^{-1})' = -x^{-2} = -\tfrac{1}{x^2}; \\
f''(x) &= (-x^{-2})' = -(-2)x^{-3} = \tfrac{2}{x^3}; \\
f'''(x) &= (2x^{-3})' = -2 \cdot 3x^{-4} = -\tfrac{2\cdot 3}{x^4}; \\
f^{(4)} &= (-2 \cdot 3x^{-4})' = 2 \cdot 3 \cdot 4x^{-5} = \tfrac{2\cdot 3\cdot 4}{x^5}; \\
f^{(5)}(x) &= (2 \cdot 3 \cdot 4x^{-5})' = -2 \cdot 3 \cdot 4 \cdot 5x^{-6} = -\tfrac{2\cdot 3\cdot 4\cdot 5}{x^6}; \\
&\vdots
\end{aligned}$$

Thus

$$f^{(n)}(x) = (-1)^n \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{x^{n+1}} = (-1)^n \frac{n!}{x^{n+1}}.$$

Computing a Second Derivative

- Compute $f''(x)$ if $f(x) = \dfrac{x^2+1}{x}$;

$$f'(x) = \left(\frac{x^2+1}{x}\right)' = \frac{(x^2+1)'x - (x^2+1)(x)'}{x^2} = \frac{2x \cdot x - x^2 - 1}{x^2} = \frac{x^2-1}{x^2};$$

$$f''(x) = \left(\frac{x^2-1}{x^2}\right)' = \frac{(x^2-1)'x^2 - (x^2-1)(x^2)'}{(x^2)^2} = \frac{2x \cdot x^2 - 2x(x^2-1)}{x^4} = \frac{2x^3 - 2x^3 + 2x}{x^4} = \frac{2x}{x^4} = \frac{2}{x^3}.$$

Evaluating a Second Derivative

- Evaluate $f''(\frac{1}{8})$ if $f(x) = \dfrac{1}{\sqrt[3]{x}}$;

$$f'(x) = \left(\frac{1}{\sqrt[3]{x}}\right)' = (x^{-1/3})' = -\tfrac{1}{3}x^{-4/3} = -\frac{1}{3\sqrt[3]{x^4}};$$

$$f''(x) = (-\tfrac{1}{3}x^{-4/3})' = -\tfrac{1}{3}\cdot(-\tfrac{4}{3}x^{-7/3}) = \tfrac{4}{9}x^{-7/3} = \frac{4}{9\sqrt[3]{x^7}};$$

Therefore, $f''(\frac{1}{8}) = \dfrac{4}{9(\sqrt[3]{\frac{1}{8}})^7} = \dfrac{4}{9(\frac{1}{2})^7} = \dfrac{4}{\frac{9}{128}} = \dfrac{4\cdot 128}{9}$.

Application: Velocity and Acceleration

- Suppose that a moving object covers distance $s(t)$ at time t;
- Then its velocity $v(t)$ at time t is the derivative of its distance

$$v(t) = s'(t);$$

- Moreover, its acceleration $a(t)$ is the derivative of its velocity

$$a(t) = v'(t) = s''(t).$$

Velocity and Acceleration: Example

- Suppose that a delivery truck covers distance $s(t) = 24t^2 - 4t^3$ miles in t hours, for $0 \leq t \leq 6$;
  - Find the velocity of the truck at $t = 2$ hours;
  $$v(t) = s'(t) = (24t^2 - 4t^3)' = 48t - 12t^2;$$
  So $v(2) = 48 \cdot 2 - 12 \cdot 2^2 = 48$ mph;
  - Find the acceleration of the truck at $t = 1$ hour;
  $$a(t) = v'(t) = (48t - 12t^2)' = 48 - 24t;$$
  Therefore, $a(1) = 48 - 24 \cdot 1 = 24$ miles/hours^2.

Application: Growth Speeding Up or Slowing Down

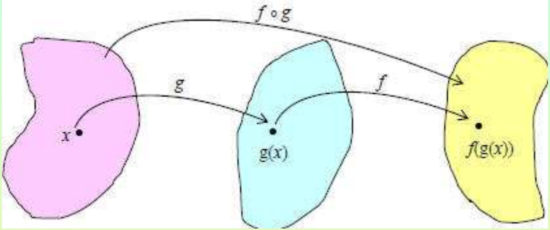
- Suppose that the world population t years from the year 2000 was predicted to be
$$P(t) = 6250 + 160t^{3/4} \text{ millions;}$$
Find $P'(16), P''(16)$ and interpret the answers;
$$P'(t) = (6250 + 160t^{3/4})' = 160 \cdot \frac{3}{4} t^{-1/4} = \frac{120}{\sqrt[4]{t}};$$
$$P''(t) = (120t^{-1/4})' = 120 \cdot (-\frac{1}{4} t^{-5/4}) = -\frac{30}{\sqrt[4]{t^5}};$$
Thus, $P'(16) = \frac{120}{\sqrt[4]{16}} = \frac{120}{2} = 60$ millions/year and
$P''(16) = -\frac{30}{(\sqrt[4]{16})^5} = -\frac{30}{32} = -0.94$ millions/year^2;
The first number shows that in 2016 the population will be increasing at the rate of 60 million people per year;
The second number shows that the growth will be slowing down at 0.94 million/year^2.

Subsection 7

Chain and Generalized Power Rules

Composite Functions

- Recall the definition of **composition**: $(f \circ g)(x) = f(g(x))$;



- Example: Find formulas for the composites $(f \circ g)(x)$ and $(g \circ f)(x)$, if $f(x) = x^7$ and $g(x) = x^2 + 2x - 3$;

$$(f \circ g)(x) = f(g(x)) = f(x^2 + 2x - 3) = (x^2 + 2x - 3)^7;$$

$$(g \circ f)(x) = g(f(x)) = g(x^7) = (x^7)^2 + 2(x^7) - 3 = x^{14} + 2x^7 - 3.$$

Decomposing Functions

- Example: Find two functions $f(x)$ and $g(x)$, such that $(x^3-7)^5$ is the composition $f(g(x))$;
  One way of doing this is to think of the series of transformations that produce output $(x^3-7)^5$ from input x;

$$x \quad \overset{3}{\mapsto} \quad x^3 \quad \overset{-7}{\mapsto} \quad x^3-7 \quad \overset{5}{\mapsto} \quad (x^3-7)^5;$$

  These transformations suggest two ways of decomposing $(x^3-7)^5$:

  1. Apply the first two steps together and, then, the last step: $g(x)=x^3-7$ and $f(x)=x^5$;
  2. Apply the first step alone and, then, the last two steps together: $g(x)=x^3$ and $f(x)=(x-7)^5$.

The Chain Rule

- To compute the derivative of the composite $f(g(x))$, we apply the Chain Rule:

$$\boxed{(f(g(x)))' = f'(g(x)) \cdot g'(x);}$$

- Example: Use the Chain Rule to find the derivatives:
  - $[(x^2-5x+1)^{10}]'$
    Let us decompose $(x^2-5x+1)^{10}$ as $f(g(x))$; Set $f(x)=x^{10}$ and $g(x)=x^2-5x+1$; Then $f'(x)=10x^9$ and $g'(x)=2x-5$; Now apply the Chain Rule:

$$[(x^2-5x+1)^{10}]' = [f(g(x))]' = f'(g(x)) \cdot g'(x) = 10(x^2-5x+1)^9(2x-5);$$

  - $[(5x-2x^3)^{16}]'$
    Let us decompose $(5x-2x^3)^{16}$ as $f(g(x))$; Set $f(x)=x^{16}$ and $g(x)=5x-2x^3$; Then $f'(x)=16x^{15}$ and $g'(x)=5-6x^2$; Now apply the Chain Rule:

$$[(5x-2x^3)^{16}]' = [f(g(x))]' = f'(g(x)) \cdot g'(x) = 16(5x-2x^3)^{15}(5-6x^2).$$

General Power Rule

- Example: Use the Chain Rule to find the derivative:
  - $[(x^3+7x)^5]'$
    Let us decompose $(x^3+7x)^5$ as $f(g(x))$; Set $f(x)=x^5$ and $g(x)=x^3+7x$; Then $f'(x)=5x^4$ and $g'(x)=3x^2+7$; Now apply the Chain Rule:

$$[(x^3+7x)^5]' = [f(g(x))]' = f'(g(x))\cdot g'(x) = 5(x^3+7x)^4(3x^2+7);$$

- Note how the derivative is taken:

$$[(x^3+7x)^5]' = \underbrace{5}_{\text{power down}}\ \underbrace{(x^3+7x)^4}_{\text{reduce power by 1}}\ \underbrace{(3x^2+7)}_{\text{derivative of the inside}};$$

- This pattern suggests the General Power Rule:

$$\boxed{[g(x)^n]' = n\cdot g(x)^{n-1}\cdot g'(x).}$$

Applying the General Power Rule

- Example: Use the Chain Rule to find the derivative of $\sqrt{x^4+3x^2}$;

$$(\sqrt{x^4+3x^2})' = [(x^4+3x^2)^{1/2}]' = \tfrac{1}{2}(x^4+3x^2)^{-1/2}(x^4+3x^2)' = \frac{4x^3+6x}{2\sqrt{x^4+3x^2}} = \frac{2x^3+3x}{\sqrt{x^4+3x^2}};$$

- Example: Use the Chain Rule to find the derivative of $\left(\frac{1}{x^2+1}\right)^8$;

$$\left[\left(\frac{1}{x^2+1}\right)^8\right]' = [(x^2+1)^{-8}]' = -8(x^2+1)^{-9}(x^2+1)' = -8\cdot\frac{1}{(x^2+1)^9}\cdot 2x = \frac{-16x}{(x^2+1)^9}.$$

Chain Rule: Alternative Notation

- In the alternative notation for derivatives, if $y = f(u)$ and $u = g(x)$, then $y = f(u) = f(g(x))$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx};$$

- This says exactly the same thing as

$$[f(g(x))]' = f'(g(x)) \cdot g'(x);$$

It is simply written in the alternative notation for derivatives.

Application: Environmental Disaster

- An oil tanker hits a reef and t days later the radius of the oil slick is $r(t) = \sqrt{4t+1}$ miles; How fast is the radius of the slick expanding after 2 days?

$$\begin{aligned} r'(t) &= (\sqrt{4t+1})' \\ &= [(4t+1)^{1/2}]' \\ &= \tfrac{1}{2}(4t+1)^{-1/2}(4t+1)' \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{4t+1}} \cdot 4 \\ &= \frac{2}{\sqrt{4t+1}}; \end{aligned}$$

Thus,

$$r'(2) = \frac{2}{\sqrt{9}} = \frac{2}{3} \text{ miles/day.}$$

Two More Complicated Examples

$$[(5x-2)^4(9x+2)^7]' \overset{\text{Product}}{=} [(5x-2)^4]'(9x+2)^7 + (5x-2)^4[(9x+2)^7]'$$
$$\overset{\text{Power}}{=} 4(5x-2)^3(5x-2)'(9x+2)^7 + (5x-2)^4 \cdot 7(9x+2)^6(9x+2)' =$$
$$4(5x-2)^3 \cdot 5 \cdot (9x+2)^7 + (5x-2)^4 \cdot 7(9x+2)^6 \cdot 9 =$$
$$20(5x-2)^3(9x+2)^7 + 63(5x-2)^4(9x+2)^6;$$

$$\left[\left(\frac{x}{x+1}\right)^4\right]' \overset{\text{Power}}{=} 4\left(\frac{x}{x+1}\right)^3\left(\frac{x}{x+1}\right)'$$
$$\overset{\text{Quotient}}{=} 4\left(\frac{x}{x+1}\right)^3 \cdot \frac{(x)'(x+1) - x(x+1)'}{(x+1)^2} =$$
$$4\left(\frac{x}{x+1}\right)^3 \cdot \frac{x+1-x}{(x+1)^2} = 4\frac{x^3}{(x+1)^3} \cdot \frac{1}{(x+1)^2} = \frac{4x^3}{(x+1)^5}.$$

One Last Example

$$
\begin{aligned}
&[[x^5+(x^2-1)^3]^7]' \\
&\overset{\text{Power}}{=} 7[x^5+(x^2-1)^3]^6[x^5+(x^2-1)^3]' \\
&\overset{\text{Sum}}{=} 7[x^5+(x^2-1)^3]^6[(x^5)'+[(x^2-1)^3]'] \\
&\overset{\text{Power}}{=} 7[x^5+(x^2-1)^3]^6[5x^4+3(x^2-1)^2(x^2-1)'] \\
&\overset{\text{Power}}{=} 7[x^5+(x^2-1)^3]^6[5x^4+3(x^2-1)^2\cdot 2x] \\
&= 7[x^5+(x^2-1)^3]^6[5x^4+6x(x^2-1)^2].
\end{aligned}
$$