Business and Life Calculus

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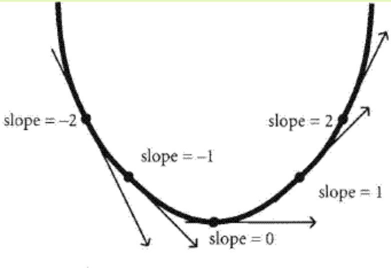
LSSU Math 112

Subsection 1

Graphing Using First Derivative

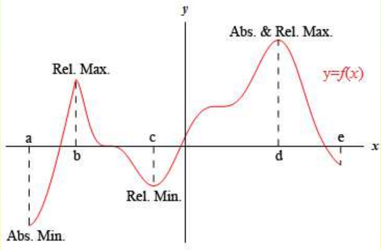
Increasing and Decreasing Functions

- Recall that a function $f(x)$ is
  - **increasing** on $[a, b]$ if its graph rises as we move left to right in $[a, b]$;
  - **decreasing** on $[a, b]$ if its graph falls as we move left to right in $[a, b]$;
- Note that
  - if f is increasing ($\nearrow$) on $[a, b]$, the slopes of the tangents are positive, so $f'(x) > 0$, for x in $[a, b]$;
  - if f is decreasing ($\searrow$) on $[a, b]$, the slopes of the tangents are negative, so $f'(x) < 0$, for x in $[a, b]$;



Relative Maxima and Minima

- A function $f(x)$ has a
  - **relative maximum** $y = f(d)$ **at** $x = d$ if $f(d)$ is at least as high as all its neighboring points on the curve;
  - **relative minimum** $y = f(c)$ **at** $x = c$ if $f(c)$ is at least as low as all its neighboring points on the curve;
- Note that:
  - if $y = f(x)$ has a tangent at a relative extremum, with $x = c$, then its slope is equal to 0; Thus, in this case, $f'(c) = 0$;
  - But f may also have a relative extremum at a point $x = b$, where the tangent, and, hence, its slope $f'(b)$ is not defined!



Critical Numbers

Critical Numbers

A **critical number** of a function f is an x-value in the domain of f at which either $f'(x) = 0$ or $f'(x)$ is undefined;

- Thus, according to our previous analysis, if f has a relative minimum or a relative maximum at x, then x must be a critical number of f;
- It is not true that all critical numbers give rise to relative minimum or relative maximum points!

Graphing Using the First Derivative

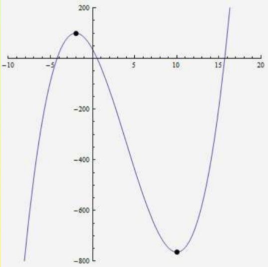
- Suppose we would like to roughly sketch the graph of $y = f(x)$ using the first-derivative $f'(x)$ as an aid. Then we perform the following steps:
  1. We find the critical numbers, i.e., the points where $f'(x) = 0$ or $f'(x)$ is undefined;
  2. Using those points, we construct the sign table for the first derivative $f'(x)$;
  3. If in an interval $[a, b]$,
     - $f'(x) > 0$, then $f \nearrow [a, b]$;
     - $f'(x) < 0$, then $f \searrow [a, b]$;
  4. We use this information to draw conclusions about the relative maxima and the relative minima;
  5. Using these extreme points and the increasing/decreasing properties of $f(x)$, we roughly sketch the graph $y = f(x)$;
- We will see some examples in following slides.

Graphing a Function I

- Use the first derivative to sketch the graph of the function $f(x) = x^3 - 12x^2 - 60x + 36$;
  1. Compute first derivative and find the critical numbers: $f'(x) = 3x^2 - 24x - 60 = 3(x^2 - 8x - 20) = 3(x+2)(x-10)$; So $f'(x) = 0 \Rightarrow 3(x+2)(x-10) = 0 \Rightarrow x = -2$ or $x = 10$;
  2. Create the sign table for $f'(x)$:

	$x < -2$	$-2 < x < 10$	$10 < x$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

  3. From the table, we see that f has a relative maximum at $(-2, 100)$ and a relative minimum at $(10, -764)$;
  4. We plot these two points and use them to roughly sketch the graph.

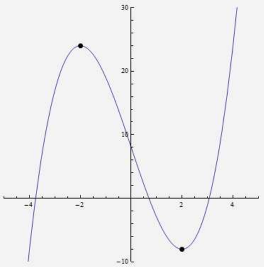


Graphing a Function II

- Use the first derivative to sketch the graph of the function $f(x) = x^3 - 12x + 8$;
  1. Compute first derivative and find the critical numbers: $f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$; So $f'(x) = 0 \Rightarrow 3(x+2)(x-2) = 0 \Rightarrow x = -2$ or $x = 2$;
  2. Create the sign table for $f'(x)$:

	$x < -2$	$-2 < x < 2$	$2 < x$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

  3. From the table, we see that f has a relative maximum at $(-2, 24)$ and a relative minimum at $(2, -8)$;
  4. We plot these two points and use them to roughly sketch the graph.



First Derivative Test for Relative Extrema

First Derivative Test

If f has a critical number at c, then at $x = c$, it has a

- a *relative maximum* if $f' > 0$ just before c and $f' < 0$ just after c;
- a *relative minimum* if $f' < 0$ just before c and $f' > 0$ just after c;

- In summary, we have the following four cases as far as a critical number c is concerned:

	$x < c$	$x > c$
f'	$+$	$-$
f	$\nearrow$	$\searrow$

Relative Maximum

	$x < c$	$x > c$
f'	$-$	$+$
f	$\searrow$	$\nearrow$

Relative Minimum

	$x < c$	$x > c$
f'	$+$	$+$
f	$\nearrow$	$\nearrow$

Neither

	$x < c$	$x > c$
f'	$-$	$-$
f	$\searrow$	$\searrow$

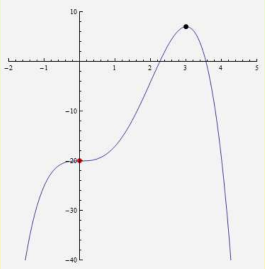
Neither

Graphing a Function III

- Use the first derivative to sketch the graph of the function $f(x) = -x^4 + 4x^3 - 20$;
  1. Compute first derivative and find the critical numbers:
     $f'(x) = -4x^3 + 12x^2 = -4x^2(x-3)$; So
     $f'(x) = 0 \Rightarrow -4x^2(x-3) = 0 \Rightarrow x = 0$ or $x = 3$;
  2. Create the sign table for $f'(x)$:

	$x < 0$	$0 < x < 3$	$3 < x$
$f'(x)$	$+$	$+$	$-$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$

  3. From the table, we see that f has a relative maximum at $(3, 7)$ and no relative minimum;
  4. We plot the relative maximum and use it to roughly sketch the graph.



Graphing Rational Functions

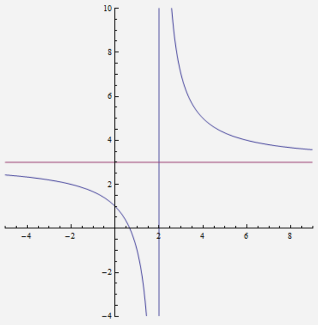
- Suppose we would like to roughly sketch the graph of a *rational function* $y = f(x)$ using the first-derivative $f'(x)$ as an aid. Then we perform the following steps:
  1. First, find the domain;
  2. Then, find the horizontal and the vertical asymptotes;
  3. We, then, find the critical numbers, i.e., the points where $f'(x) = 0$ or $f'(x)$ is undefined;
  4. Using those points, we construct the sign table for the first derivative $f'(x)$ and draw conclusions about monotonicity;
  5. We use this information and the first derivative test to find the relative maxima and the relative minima;
  6. Using the asymptotes, the extreme points and the increasing / decreasing properties of $f(x)$, we roughly sketch the graph $y = f(x)$;
- We will see some examples in following slides; But note that the domain and the asymptotes are the main additions here!

Graphing a Rational Function I

- Use the first derivative to sketch the graph of $f(x) = \frac{3x-2}{x-2}$;
  1. The domain is $\mathrm{Dom}(f) = \mathbb{R} - \{2\}$;
  2. Vertical asymptote $x = 2$, where f is undefined;
  3. Horizontal asymptote $y = \lim_{x\to\infty} \frac{3x-2}{x-2} = \frac{3}{1} = 3$;
  4. Compute first derivative and find the critical numbers: $f'(x) = \frac{(3x-2)'(x-2)-(3x-2)(x-2)'}{(x-2)^2} = \frac{3(x-2)-(3x-2)}{(x-2)^2} = \frac{-4}{(x-2)^2}$; So only number to consider is $x = 2$;
  5. Create the sign table for $f'(x)$:

	$x < 2$	$2 < x$
$f'(x)$	$-$	$-$
$f(x)$	$\searrow$	$\searrow$

  6. From the table, we see that f does not have extrema;
  7. Plot the asymptotes $x = 2$ and $y = 3$ and use them to graph.

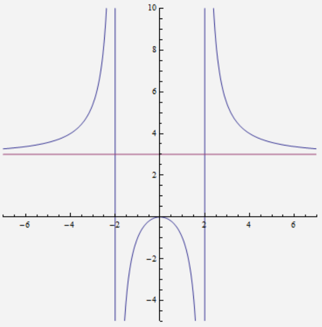


Graphing a Rational Function II

- Use the first derivative to sketch the graph of $f(x) = \frac{3x^2}{x^2-4}$;
  1. The domain is $\mathrm{Dom}(f) = \mathbb{R} - \{-2, 2\}$;
  2. Vertical asymptotes $x = -2$ and $x = 2$, where f is undefined;
  3. Horizontal asymptote $y = \lim_{x\to\infty} \frac{3x^2}{x^2-4} = \frac{3}{1} = 3$;
  4. Compute first derivative and find the critical numbers: $f'(x) = \frac{(3x^2)'(x^2-4)-3x^2(x^2-4)'}{(x^2-4)^2} = \frac{6x(x^2-4)-3x^2\cdot 2x}{(x^2-4)^2} = \frac{-24x}{(x^2-4)^2}$; So we consider the numbers $x = 0$ and $x = \pm 2$;
  5. Create the sign table for $f'(x)$:

	$x < -2$	$(-2, 0)$	$(0, 2)$	$2 < x$
$f'(x)$	$+$	$+$	$-$	$-$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$

  4. From the table, we see that f does has a local maximum $(0, 0)$;
  5. Plot $(0, 0)$, $x = -2, x = 2$ and $y = 3$ and use them to graph.

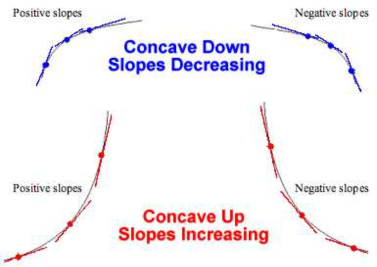


Subsection 2

Graphing Using First and Second Derivatives

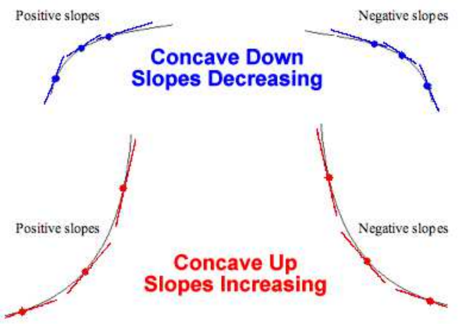
Concavity

- We say that the graph of $y = f(x)$ on $[a, b]$ is
  - **concave down** if the slopes of the tangent lines are decreasing;
  - **concave up** if the slopes of the tangent lines are increasing;
- Recall $f \nearrow [a, b]$ if $f' > 0$ in $[a, b]$ and $f \searrow [a, b]$ if $f' < 0$ in $[a, b]$;
- Note the following about concavity:
  - Concave down means slopes decreasing, which means $f' \searrow [a, b]$, which means $f'' < 0$ in $[a, b]$;
  - Concave up means slopes increasing, which means $f' \nearrow [a, b]$, which means $f'' > 0$ in $[a, b]$.



Inflection Points

- An **inflection point** is a point on the graph where the convavity changes, i.e., we go either from concave up to concave down or vice-versa;



- Note that at an inflection point x, f'' changes sign, so we must have that $f''(x) = 0$ or $f''(x)$ is undefined.

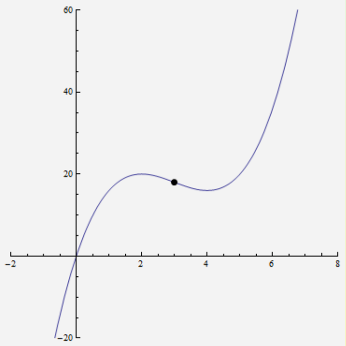
Second Derivative for Concavity and Inflection

- Find the intervals of concavity and the inflection points of $f(x) = x^3 - 9x^2 + 24x$;
  Compute first derivative: $f'(x) = (x^3 - 9x^2 + 24x)' = 3x^2 - 18x + 24$;
  Compute second derivative:
  $f''(x) = (3x^2 - 18x + 24)' = 6x - 18 = 6(x - 3)$; Solve
  $f''(x) = 0 \Rightarrow 6(x - 3) = 0 \Rightarrow x = 3$;

  Form the sign table of the second derivative:

	$x < 3$	$3 < x$
f''	$-$	$+$
f	$\frown$	$\smile$

  Thus, f has an inflection point $(3, 18)$.



Combining First and Second Derivatives

- If we combine the signs of the first and the second derivatives in specific intervals we get the following four cases:

	$[a, b]$
f'	$+$
f''	$+$
f	

Increasing
Concave Up

	$[a, b]$
f'	$+$
f''	$-$
f	

Increasing
Concave Down

	$[a, b]$
f'	$-$
f''	$+$
f	

Decreasing
Concave Up

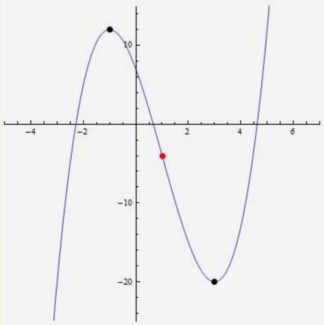
	$[a, b]$
f'	$-$
f''	$-$
f	

Decreasing
Concave Down

Graphing Using Both First and Second Derivatives

- Graph the function $f(x) = x^3 - 3x^2 - 9x + 7$
  Compute the first derivative:
  $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$; Find its critical numbers: $f'(x) = 0 \Rightarrow 3(x+1)(x-3) = 0 \Rightarrow x = -1$ or $x = 3$;
  Find second derivative: $f''(x) = 6x - 6 = 6(x-1)$; Its zero is $x = 1$;
  Create combined sign table for first and second derivatives:

	$x < -1$	$[-1, 1]$	$[1, 3]$	$3 < x$
f'	$+$	$-$	$-$	$+$
f''	$-$	$-$	$+$	$+$
f	$\curvearrowright$	$\curvearrowright$	$\hookrightarrow$	⤴



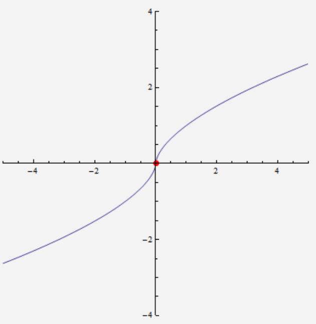
Thus f has relative max $(-1, 12)$, relative min $(3, -20)$ and inflection $(1, -4)$.

Graphing II

- Graph the function $f(x) = x^{3/5} (= \sqrt[5]{x^3})$
  Compute the first derivative: $f'(x) = (x^{3/5})' = \frac{3}{5}x^{-2/5} = \frac{3}{5\sqrt[5]{x^2}}$; f' is undefined at $x = 0$; So $x = 0$ is critical point;
  Find second derivative: $f''(x) = (\frac{3}{5}x^{-2/5})' = \frac{3}{5} \cdot (-\frac{2}{5})x^{-7/5} = -\frac{6}{25\sqrt[5]{x^7}}$;
  Also undefined at $x = 0$;
  Create combined sign table for first and second derivatives:

	$x < 0$	$0 < x$
f'	$+$	$+$
f''	$+$	$-$
f	$\nearrow$	$\nearrow$

Thus f has no relative extrema and inflection $(0, 0)$.

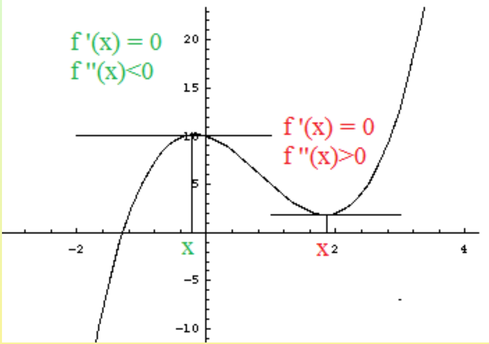


The Second-Derivative Test

Second-Derivative Test for Relative Extrema

If $x = c$ is a critical number of f at which f'' is defined, then

- if $f''(c) > 0$, then f has a relative minimum at $x = c$;
- if $f''(c) < 0$, then f has a relative maximum at $x = c$.

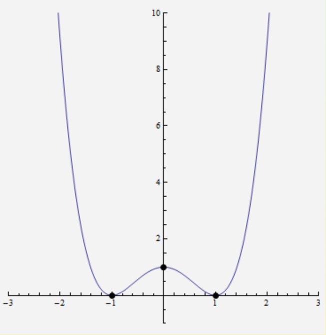


Applying the Second Derivative Test

- Example: Use second derivative test to find all relative extrema of $f(x) = x^4 - 2x^2 + 1$;
  Compute first derivative:
  $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$; Find all critical numbers $f'(x) = 0 \Rightarrow 4x(x+1)(x-1) = 0 \Rightarrow x = 0$ or $x = -1$ or $x = 1$;
  Now compute the second derivative: $f''(x) = 12x^2 - 4$;

  Evaluate the second derivative at each of the critical points:

x	$f''(x)$	Point
-1	$8 > 0$	Min
0	$-4 < 0$	Max
1	$8 > 0$	Min

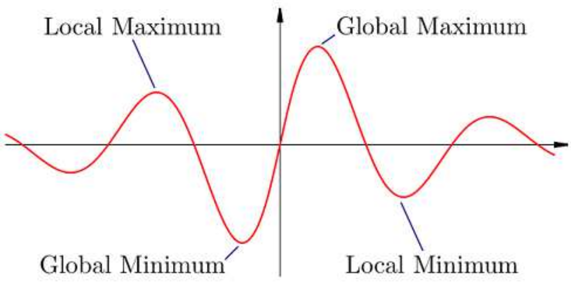


Subsection 3

Optimization

Absolute Extrema in a Closed Interval

- **Optimizing** a function means finding its maximum or its minimum value;
- The **absolute max/min** value of a function is the largest/smallest value of the function on its domain;
- An **absolute extremum** is either an absolute max or an absolute min.



Optimizing a Continuous Function on a Closed Interval

- A continuous function f on a closed interval $[a, b]$ has both an absolute max and an absolute min value;
- To compute these values
  1. Find all critical numbers of f in $[a, b]$;
  2. Evaluate f at the critical numbers and at a and b;
  3. The largest and smallest values found in previous step are the absolute extrema of f on $[a, b]$.

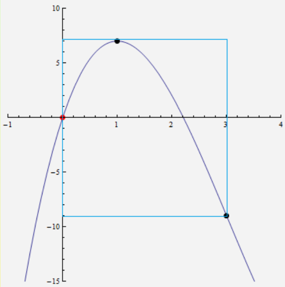
Example

- Find absolute extrema of $f(x) = x^3 - 9x^2 + 15x$ on $[0, 3]$;
  Compute $f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5) = 3(x-1)(x-5)$;
  Set $f'(x) = 0 \Rightarrow 3(x-1)(x-5) = 0 \Rightarrow x = 1$ or $x = 5$; The only critical number in $[0, 3]$ is $x = 1$;

  Compute

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 7, \\ f(3) &= -9; \end{aligned}$$

  Thus, absolute max is $f(1) = 7$ and absolute min $f(3) = -9$.

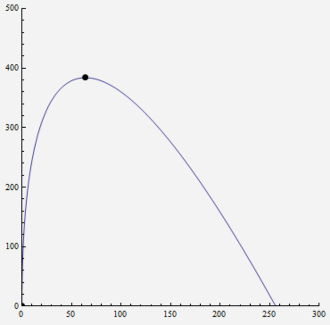


Application: Timber Forest

- The value of a timber forest after t years is $V(t) = 96\sqrt{t} - 6t$ thousand dollars $(t > 0)$. When is the value maximized?

  Find $V'(t) = (96\sqrt{t} - 6t)' = 96(t^{1/2})' - 6(t)' = 96 \cdot \frac{1}{2}t^{-1/2} - 6 = \frac{48}{\sqrt{t}} - 6 = \frac{48}{\sqrt{t}} - \frac{6\sqrt{t}}{\sqrt{t}} = \frac{48-6\sqrt{t}}{\sqrt{t}} = \frac{6(8-\sqrt{t})}{\sqrt{t}}$;
  Thus, $V(t) = 0 \Rightarrow 8 - \sqrt{t} = 0 \Rightarrow \sqrt{t} = 8 \Rightarrow t = 64$;

  Note that the critical numbers of $V(t)$ are $t = 0$ and $t = 64$; However, $V(0) = 0$ will not give max; The max is $V(64) = 96\sqrt{64} - 6 \cdot 64 = 384$ thousands of dollars.



Application: Maximum Profit

- Suppose it costs \$ 8,000 to produce a car and fixed costs are \$ 20,000 per week; Suppose, also, the price function is $p(x) = 22,000 - 70x$, where p is the price at which exactly x cars are sold;
  - What is the revenue, the cost and the profit function?

$$R(x) = xp(x) = x(22000 - 70x) = -70x^2 + 22000x;$$
$$C(x) = 8000x + 20000;$$
$$P(x) = R(x) - C(x) = -70x^2 + 22000x - (8000x + 20000) =$$
$$-70x^2 + 14000x - 20000;$$

  - How many cars should be produced each week to maximize profit? Compute $P'(x) = -140x + 14000$; Set $P'(x) = 0 \Rightarrow -140x + 14000 = 0 \Rightarrow x = 100$;
  - For what price should they be sold?

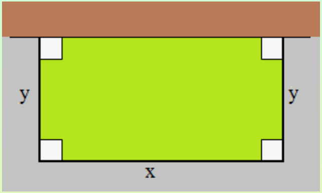
$$p(100) = 22000 - 70 \cdot 100 = 15000;$$

  - What is the maximum profit? $P(100) = -70 \cdot 100^2 + 14000 \cdot 100 - 20000 =$ $-700000 + 1400000 - 20000 = \$680,000.$

Application: Maximum Area

- A farmer has 1000 feet of fence and wants to build a rectangular enclosure along a straight wall.

  If the side along the wall needs no fencing, find the dimensions that make the enclosure as large as possible and the maximum area;

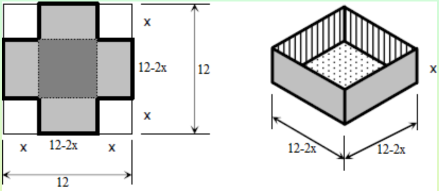
  

  Suppose x is the length and y the width of the rectangular area; Then, since the length of the fence is 1000 feet, we must have $x + 2y = 1000 \Rightarrow x = 1000 - 2y$; Moreover, the area enclosed is $A = xy = (1000 - 2y)y = -2y^2 + 1000y$; Compute $A'(y) = -4y + 1000$; Set $A'(y) = 0 \Rightarrow -4y + 1000 = 0 \Rightarrow y = 250$; Thus, the dimensions that maximize the area are 500 feet $\times$ 250 feet and the maximum area is $A(250) = 500 \cdot 250 = 125000$ feet^2.

Application: Maximum Volume

- An open top box is to be made from a square sheet of metal 12 inches on each side by cutting a square from each corner and folding up the sides;

  Find the volume of the largest box that can be made;



  Suppose x is the length of the side of the corner square; Then, the volume must be
  $V(x) = (12-2x)(12-2x)x = (144-48x+4x^2)x = 4x^3-48x^2+144x$;
  Compute
  $V'(x) = 12x^2-96x+144 = 12(x^2-8x+12) = 12(x-2)(x-6)$;
  Set $V'(x) = 0 \Rightarrow 12(x-2)(x-6) = 0 \Rightarrow x = 2$ or $x = 6$; However, $x = 6$ cannot be, so $x = 2$; Thus, the dimensions that maximize the volume are $8 \times 8 \times 2$ inches and max volume is $V(2) = 8 \cdot 8 \cdot 2 = 128$ in^3.

Subsection 4

Further Applications of Optimization

Price and Quantity Functions

- A store can sell 20 bikes per week at \$ 400 each; The manager estimates that for each \$ 10 reduction in price she can sell two more bikes per week; The bikes cost the store \$ 200 each; Let x stand for the number of \$ 10 reductions;
- Find an expression for the price p as a function of x;
$$p(x) = 400 - 10x;$$
- Find an expression for the quantity q sold as a function of x;
$$q(x) = 20 + 2x;$$
- Find the revenue, cost and profit as functions of x;
$$\begin{aligned} R(x) &= q(x)p(x) = (20 + 2x)(400 - 10x) = -20x^2 + 600x + 8000; \\ C(x) &= 200q(x) = 200(20 + 2x) = 400x + 4000; \\ P(x) &= R(x) - C(x) = -20x^2 + 600x + 8000 - (400x + 4000) = \\ &\quad -20x^2 + 200x + 4000; \end{aligned}$$

Maximizing Profit

- We found $p(x) = 400 - 10x$ and $q(x) = 20 + 2x$; We also computed $P(x) = -20x^2 + 200x + 4000$;
- What is the price and the quantity that maximize profit? What is maximum profit?
  Compute $P'(x) = -40x + 200$;
  Set $P'(x) = 0 \Rightarrow -40x + 200 = 0 \Rightarrow x = 5$;
  Therefore, the price that maximizes profit is $p(5) = 350$; Moreover, the quantity that maximizes profit is $q(5) = 30$ bikes per week; Finally the max profit is

$$P(5) = -20 \cdot 5^2 + 200 \cdot 5 + 4000 = -500 + 1000 + 4000 = \$4,500.$$

Maximizing Harvest Size

- An orange grower finds that, if he plants 80 orange trees per acre, each tree will yield 60 bushels of oranges; For each additional tree planted per acre, the yield of each tree will decrease by 2 bushels; How many trees should he plant per acre to maximize harvest?

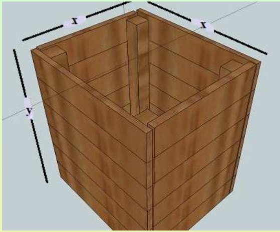
  Let x be the number of additional trees per acre; Then, there are $T(x) = 80 + x$ trees per acre; Each tree would yield $Y(x) = 60 - 2x$ bushels of oranges; Thus, the total harvest per acre is

  $$H(x) = T(x) \cdot Y(x) = (80 + x)(60 - 2x) = -2x^2 - 100x + 4800 \text{ bushels;}$$

  To maximize, compute $H'(x) = -4x - 100$; Set $H'(x) = 0 \Rightarrow -4x - 100 = 0 \Rightarrow x = -25$;
  Thus, the number of trees that should be planted per acre is $T(-25) = 80 - 25 = 55$.

Minimizing Packaging Materials

- A moving company wants to design an open-top box with a square base whose volume is exactly 32 feet^3; Find the dimensions of the box requiring the least amount of materials;
  Suppose that the box has dimensions of base $x \times x$ feet and height y feet;



  Then, since the volume is 32 feet^3, we must have $x^2 y = 32 \Rightarrow y = \frac{32}{x^2}$; Moreover, the amount of materials, given by the surface area, is $A = \overbrace{x^2}^{\text{bottom}} + \overbrace{4xy}^{\text{sides}} = x^2 + 4x \cdot \frac{32}{x^2} = x^2 + \frac{128}{x}$;
  To minimize compute $A'(x) = (x^2 + 128x^{-1})' = 2x - 128x^{-2} = 2x - \frac{128}{x^2} = \frac{2x^3}{x^2} - \frac{128}{x^2} = \frac{2(x^3-64)}{x^2}$; Set $A'(x) = 0 \Rightarrow x^3 - 64 = 0 \Rightarrow x^3 = 64 \Rightarrow x = \sqrt[3]{64} = 4$; Thus, the dimensions that minimize the amount of materials are $4 \times 4 \times 2$ feet.

Maximizing Tax Revenue

- Suppose that the relationship between the tax rate t on an item and the total sales S of the item in millions of dollars is $S(t) = 9 - 20\sqrt{t}$; What is the tax rate that maximizes the government revenue?

  We have $R(t) = tS(t) = t(9 - 20\sqrt{t}) = 9t - 20t^{3/2}$;
  To maximize, compute
  $R'(t) = (9t - 20t^{3/2})' = 9 - 20 \cdot \frac{3}{2}t^{1/2} = 9 - 30\sqrt{t}$;
  Set $R'(t) = 0 \Rightarrow 9 - 30\sqrt{t} = 0 \Rightarrow \sqrt{t} = 0.3 \Rightarrow t = 0.09$;
  The second derivative is $R''(t) = (9 - 30t^{1/2})' = -15t^{-1/2} = -\frac{15}{\sqrt{t}}$;
  Since $R''(0.09) < 0$, at $t = 0.09$ $R(t)$ has indeed a maximum (and not a minimum);
  Thus, the rate that maximizes revenue is in fact 9%.

Subsection 5

Implicit Differentiation and Related Rates

Implicit Definition of y in Terms of x

- An expression giving $y = f(x)$, i.e., that is solved for y, is said to **define** y **explicitly in terms of** x; E.g., $y = \sqrt{x}$ or $y = x^2 - 5x + 7$ define y explicitly in terms of x;
- On the other hand, an expression of the form $f(x, y) = 0$, that is not explicitly solved for y, is said to **define** y **implicitly in terms of** x; E.g., $x^2 + y^2 = 25$ or $xy^3 + x^3y - 1 = 0$ define y implicitly in terms of x;
- Note that, even in cases where it is possible to solve for y, as for example in $x^2 + y^2 = 25$, we might want to avoid doing this;

  In this specific case, we would have

$$y = \pm\sqrt{25 - x^2},$$

  which would force us to deal with two, instead of with just one, formulas.

Implicit Differentiation

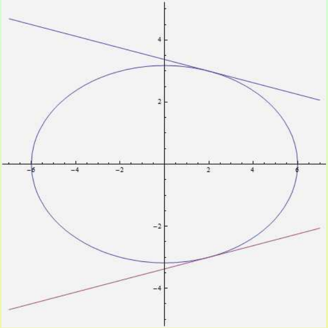
- To compute the derivative $y' = \frac{dy}{dx}$ of y when y is given implicitly in terms of x, we
  - take derivatives of both sides with respect to x;
  - use the general power rule $[f(x)^n]' = n \cdot f(x)^{n-1} \cdot f'(x)$ very carefully; i.e., when we take the derivative of a power y^n, with respect to x, we must use the *general power rule* $(y^n)' = ny^{n-1}y'$.
- Suppose we want to compute $y' = \frac{dy}{dx}$ if $x^2 + y^2 = 25$;

$$
\begin{aligned}
&(x^2 + y^2)' = (25)' \\
&\stackrel{\text{sum rule}}{\Rightarrow} (x^2)' + (y^2)' = 0 \\
&\stackrel{\text{power rule}}{\Rightarrow} 2x + 2yy' = 0 \\
&\Rightarrow 2yy' = -2x \\
&\Rightarrow y' = -\frac{x}{y}.
\end{aligned}
$$

An Additional Example

- Find the slope of the tangent lines to the ellipse $\frac{x^2}{36} + \frac{8y^2}{81} = 1$ at $(2,3)$ and at $(2,-3)$;

  We differentiate implicitly:



$$\left(\frac{x^2}{36} + \frac{8y^2}{81}\right)' = (1)'$$
$$\Rightarrow \left(\frac{x^2}{36}\right)' + \left(\frac{8y^2}{81}\right)' = 0$$
$$\Rightarrow \frac{x}{18} + \frac{16yy'}{81} = 0 \Rightarrow \frac{16yy'}{81} = -\frac{x}{18}$$
$$\Rightarrow y' = -\frac{x}{18} \cdot \frac{81}{16y} \Rightarrow y' = -\frac{9x}{32y};$$

  Thus, we get $y'(2,3) = -\frac{9\cdot 2}{32\cdot 3} = -\frac{3}{16}$ and $y'(2,-3) = -\frac{9\cdot 2}{32\cdot(-3)} = \frac{3}{16}$.

The General Method

Finding $\frac{dy}{dx}$ by Implicit Differentiation

1. Differentiate both sides with respect to x; When differentiating a y, include $\frac{dy}{dx}$ (Chain Rule);
2. Collect all terms involving $\frac{dy}{dx}$ on one side and all others on the other;
3. Factor out the $\frac{dy}{dx}$ and solve for it by dividing.

- Example: If $y^4 + x^4 - 2x^2y^2 = 9$, find $\frac{dy}{dx}$;

  $(y^4 + x^4 - 2x^2y^2)' = (9)' \Rightarrow (y^4)' + (x^4)' - (2x^2y^2)' = 0 \Rightarrow$ $4y^3y' + 4x^3 - 2((x^2)'y^2 + x^2(y^2)') = 0 \Rightarrow 4y^3y' + 4x^3 - 2(2xy^2 + x^2 \cdot 2yy') =$ $0 \Rightarrow 4y^3y' + 4x^3 - 4xy^2 - 4x^2yy' = 0 \Rightarrow 4y^3y' - 4x^2yy' = 4xy^2 - 4x^3 \Rightarrow$ $(4y^3 - 4x^2y)y' = 4xy^2 - 4x^3 \Rightarrow y' = \dfrac{4(xy^2 - x^3)}{4(y^3 - x^2y)} = \dfrac{xy^2 - x^3}{y^3 - x^2y}$.

Finding and Interpreting the Implicit Derivative

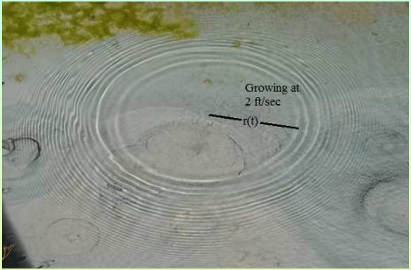
- The demand equation gives the quantity x of some commodity to be consumed as a function of the price p at which it is offered;
- If $x = \sqrt{1900 - p^3}$, use implicit differentiation to find $\frac{dp}{dx}$; Evaluate this at $p = 10$ and interpret the answer;

$$\begin{aligned}
&\frac{dx}{dx} = \frac{d}{dx}\sqrt{1900 - p^3} \\
&\Rightarrow 1 = \frac{1}{2}(1900 - p^3)^{-1/2}\frac{d}{dx}(1900 - p^3) \\
&\Rightarrow 1 = \frac{1}{2\sqrt{1900-p^3}}\left(-3p^2\frac{dp}{dx}\right) \\
&\Rightarrow 1 = -\frac{3p^2}{2\sqrt{1900-p^3}} \cdot \frac{dp}{dx} \\
&\Rightarrow \frac{dp}{dx} = -\frac{2\sqrt{1900-p^3}}{3p^2};
\end{aligned}$$

Therefore, $\left.\frac{dp}{dx}\right|_{p=10} = -\frac{2\sqrt{1900-1000}}{3\cdot 100} = -\frac{2\sqrt{900}}{300} = -\frac{60}{300} = -0.2$; This is the approximate price decrease per 1 unit increase in quantity; Put differently, each 20¢ decrease in price will result in approximately one additional unit of the commodity being sold.

Related Rates

A pebble thrown into a pond causes circular ripples to radiate outward; If the radius is growing by 2 feet/second, how fast is the area of the circle growing at the moment when the radius is exactly 10 feet?



Recall formula for the area $A = \pi r^2$; To find the rate at which area is changing with respect to time, i.e., $\frac{dA}{dt}$, we differentiate both sides with respect to time t:

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) \Rightarrow \frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt} \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt};$$

Therefore, for $r = 10$ and $\frac{dr}{dt} = 2$, we get $\frac{dA}{dt} = 2\pi \cdot 10 \cdot 2 = 40\pi$ feet^2/second.

General Method

To Solve a Related Rates Problem

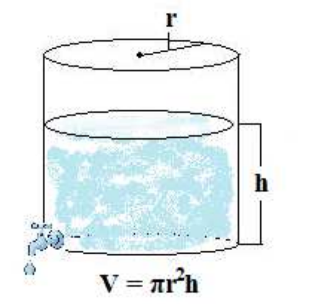
1. Determine which quantities are changing with time;
2. Find an equation that relates these quantities;
3. Differentiate both sides of the equation implicitly with respect to t;
4. Substitute into the resulting equation any given values for the variables and for the derivatives (interpreted as rates of change);
5. Solve for the remaining derivative and interpret the answer as a rate of change.

In the example above:

1. The Area A and the radius r were changing with time;
2. The equation that related those was $A = \pi r^2$;
3. We took derivatives with respect to t and found $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$;
4. We substituted $r = 10$ and $\frac{dr}{dt}$ to get $\frac{dA}{dt}$;
5. This was interpreted as the rate of change of the area.

Application: Emptying a Cylindric Tank

A tap at the bottom of a cylindric tank of radius $r = 5$ inches is turned on; If the tap causes the water to drain at a rate of 5π inches^3/second, how fast is the level of the water falling in the tank?



Recall formula for the volume $V = \pi r^2 h$; To find the rate at which the level h of the water is changing with respect to time, i.e., $\frac{dh}{dt}$, we differentiate both sides with respect to time t:

$$\frac{dV}{dt} = \frac{d}{dt}(\pi r^2 h) \Rightarrow \frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt};$$

Therefore, for $r = 5$ and $\frac{dV}{dt} = -5\pi$, we get $\frac{dh}{dt} = \frac{1}{\pi \cdot 5^2} \cdot (-5\pi) = -\frac{1}{5}$ in/sec.

Application: Profit Growth

A boat yard's total profit from selling x boat motors is $P(x) = -x^2 + 1000x - 2000$. If the motors are selling at the rate of 20 per week, how fast is the profit changing when 400 motors have been sold?

The changing quantities are x and P and they are related by the given equation;

We differentiate both sides with respect to t:

$$\frac{dP}{dt} = \frac{d}{dt}(-x^2 + 1000x - 2000) \Rightarrow \frac{dP}{dt} = -2x\frac{dx}{dt} + 1000\frac{dx}{dt};$$

Therefore, for $x = 400$ and $\frac{dx}{dt} = 20$, we get

$$\frac{dP}{dt} = -2 \cdot 400 \cdot 20 + 1000 \cdot 20 = \$4000 \text{ per week.}$$

Application: Predicting Pollution

Sulfur oxide emissions in a city will be $S = 2 + 20x + 0.1x^2$ tons, where x is the population in thousands; If the population t years from now is expected to be $x = 800 + 20\sqrt{t}$ thousand people, how rapidly will the pollution be increasing 4 years from now?

The changing quantities are x and S and they are related by the equation $S = 2 + 20x + 0.1x^2$; We differentiate both sides with respect to t:

$$\frac{dS}{dt} = \frac{d}{dt}(2 + 20x + 0.1x^2) \Rightarrow \frac{dS}{dt} = 20\frac{dx}{dt} + 0.2x\frac{dx}{dt};$$

Note that for $t = 4$, $x = 800 + 20\sqrt{4} = 840$ and also that $\frac{dx}{dt} = \frac{d}{dt}(800 + 20\sqrt{t}) = 20 \cdot \frac{1}{2}t^{-1/2} = \frac{10}{\sqrt{t}}$, whence $\left.\frac{dx}{dt}\right|_{t=4} = 5$;
Therefore, we get $\frac{dS}{dt} = 20 \cdot 5 + 0.2 \cdot 840 \cdot 5 = 940$ tons/year.