Business and Life Calculus

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LSSU Math 112

Subsection 1

Exponential Functions

Exponential Functions

- An **exponential function** is one of the form

$$f(x) = a^x,$$

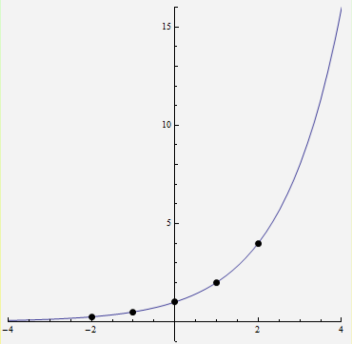
  where $0 < a \neq 1$ and x is a real number;

- Example: Consider $f(x) = 2^x$, $g(x) = (\frac{1}{4})^{1-x}$ and $h(x) = -3^x$; Compute the following values:
  - $f(\frac{3}{2}) = 2^{3/2} = \sqrt{2^3} = \sqrt{2^2}\sqrt{2} = 2\sqrt{2}$;
  - $f(-3) = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$;
  - $g(3) = (\frac{1}{4})^{1-3} = (\frac{1}{4})^{-2} = 4^2 = 16$;
  - $h(2) = -3^2 = -9$;
- Two important exponentials for applications:
  - $f(x) = 10^x$, called **common base** exponential;
  - $f(x) = e^x$, called **natural base** exponential.

Graphs of Exponentials (Exponential Growth)

- When the base a is such that $a > 1$, then $f(x) = a^x$ has an increasing graph (going up as we move from left to right);
- We sketch the graph of $f(x) = 2^x$;

x	$y = 2^x$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4

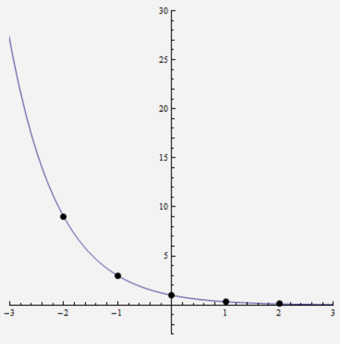


- Note that the x-axis is a horizontal asymptote as $x \to -\infty$.

Graphs of Exponentials (Exponential Decay)

- When the base a is such that $0 < a < 1$, then $f(x) = a^x$ has a decreasing graph (going down as we move from left to right);
- We sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$;

x	$y = \left(\frac{1}{3}\right)^x$
-2	9
-1	3
0	1
1	1/3
2	1/9



- Note that the x-axis is a horizontal asymptote as $x \to +\infty$.

Exponential Equations

One-to-One Property of Exponentials

$$a^m = a^n \quad \text{implies} \quad m = n.$$

- Example: Solve each equation:
  - $3^{2x-1} = 243$
    $3^{2x-1} = 243 \Rightarrow 3^{2x-1} = 3^5 \Rightarrow 2x - 1 = 5 \Rightarrow 2x = 6 \Rightarrow x = 3;$
  - $8^{|x|} = 2$
    $8^{|x|} = 2 \Rightarrow (2^3)^{|x|} = 2^1 \Rightarrow 2^{3|x|} = 2^1 \Rightarrow 3|x| = 1 \Rightarrow |x| = \frac{1}{3}$
    $\Rightarrow x = -\frac{1}{3}$ or $x = \frac{1}{3};$
  - $\frac{1}{125} = 25^{x+7}$
    $\frac{1}{125} = 25^{x+7} \Rightarrow \frac{1}{5^3} = (5^2)^{x+7} \Rightarrow 5^{-3} = 5^{2(x+7)} \Rightarrow -3 = 2x + 14$
    $\Rightarrow 2x = -17 \Rightarrow x = -\frac{17}{2};$
- Example: Suppose $f(x) = 2^x$ and $g(x) = (\frac{1}{3})^{1-x}$; Find x if
  - $f(x) = 128$
    $f(x) = 128 \Rightarrow 2^x = 2^7 \Rightarrow x = 7;$
  - $g(x) = 81$
    $g(x) = 81 \Rightarrow (\frac{1}{3})^{1-x} = 3^4 \Rightarrow 3^{-(1-x)} = 3^4 \Rightarrow -1 + x = 4 \Rightarrow x = 5.$

Application: Compounding of Interest I

- If P is the principal amount invested, i the interest rate per period and n the number of periods, then the amount A accrued at the end of n periods is

$$A = P(1+i)^n;$$

- Example: If \$1,200 is deposited in an account paying 2% compounded quarterly, how much will be in the account at the end of 5 years?

  We have $P = 1200$, $i = \frac{0.02}{4} = 0.005$ and $n = 5 \cdot 4 = 20$;
  Therefore

$$A = P(1+i)^n = 1200(1+0.005)^{20} = 1200 \cdot 1.005^{20} \approx 1,325.87.$$

Application: Compounding of Interest II

- If r is the annual interest rate and compounding is taking place m periods in a year for a total of t years, then in $A = P(1+i)^n$, we get $i = \frac{r}{m}$ and number of periods in the t years $n = m \cdot t$; Therefore, the previous formula gives

$$A(t) = P\left(1 + \frac{r}{m}\right)^{mt};$$

- Example: If \$4,000 is deposited in an account paying 10% compounded quarterly, how much will be in the account at the end of 2 years?

  We have $P = 4000$, $r = 0.1$, $m = 4$ and $t = 2$; Therefore

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 4000\left(1 + \frac{0.1}{4}\right)^{4\cdot 2} = 4000 \cdot 1.025^8 \approx 4,873.61.$$

Present Value

Present Value

Suppose an amount A is to be paid in t years, given an annual interest rate r compounded m times per year; Then, its present value P is given by

$$P = \frac{A}{\left(1+\frac{r}{m}\right)^{mt}};$$

- Example: Find the present value of \$5,000 to be paid in 10 years from now at 4% interest rate compounded semiannually;
  Now we have
  $$A = 5000, \quad r = 0.04, \quad m = 2, \quad t = 10;$$
  Thus,
  $$P = \frac{A}{(1+\frac{r}{m})^{mt}} = \frac{5000}{(1+\frac{0.04}{2})^{2\cdot 10}} = \frac{5000}{1.02^{20}} \approx 3,364.86.$$

Application: Continuous Compounding of Interest

- If P is the principal amount invested, r the annual interest rate compounded continuously and t the number of years, then the amount A accrued at the end of t years is

$$A = Pe^{rt};$$

- Example: If \$1,200 is deposited in an account paying annually 2% compounded continuously, how much will be in the account at the end of 5 years?

  We have $P = 1200$, $r = 0.02$ and $t = 5$; Therefore

$$A = Pe^{rt} = 1200 \cdot e^{0.02 \cdot 5} = 1200 \cdot e^{0.1} \approx 1,326.21.$$

Present Value With Continuous Compounding

Present Value With Continuous Compounding

Suppose an amount A is to be paid in t years, given an annual interest rate r compounded continuously; Then, its present value P is given by

$$P = \frac{A}{e^{rt}} = Ae^{-rt};$$

- Example: Find the present value of \$10,000 to be paid in 5 years from now at 3% interest rate compounded continuously;
  Now we have
  $$A = 10000, \quad r = 0.03, \quad t = 5;$$
  Thus,
  $$P = Ae^{-rt} = 10000e^{-0.03 \cdot 5} = 10000e^{-0.15} \approx 8,607.08.$$

Subsection 2

Logarithmic Functions

Logarithmic Functions

- The logarithm $y = \log_a x$ (read logarithm to base a of x) is the exponent to which one must raise the base a to get x;
- More formally,

$$y = \log_a x \quad \text{if and only if} \quad a^y = x.$$

- Example: Convert each logarithmic equation to an exponential one and vice-versa:
  - $5^3 = 125 \quad \Longleftrightarrow \quad \log_5 125 = 3$;
  - $(\frac{1}{4})^6 = x \quad \Longleftrightarrow \quad 6 = \log_{1/4} x$;
  - $(\frac{1}{2})^m = 8 \quad \Longleftrightarrow \quad m = \log_{1/2} 8$;
  - $7 = 3^z \quad \Longleftrightarrow \quad z = \log_3 7$.

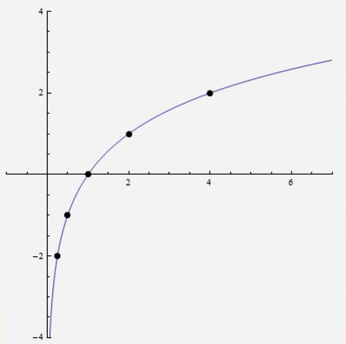
Evaluating Logarithms

- Evaluate each logarithm:
  - $\log_5 25 = 2$; (since $5^2 = 25$)
  - $\log_2 \frac{1}{8} = -3$; (since $2^{-3} = \frac{1}{8}$)
  - $\log_{1/2} 4 = -2$; (since $(\frac{1}{2})^{-2} = 4$)
  - $\log_{10} 0.001 = -3$; (since $10^{-3} = 0.001$)
  - $\log_9 3 = \frac{1}{2}$; (since $9^{1/2} = 3$)
- The logarithm to base 10 is called the **common logarithm**, denoted $\log x$ (i.e., this means $\log_{10} x$); The logarithm to base e is called the **natural logarithm**, denoted $\ln x$ (i.e., this means $\log_e x$);
- Evaluate each logarithm:
  - $\log 1000 = 3$; (since $10^3 = 1000$)
  - $\ln e = 1$; (since $e^1 = e$)
  - $\log \frac{1}{10} = -1$. (since $10^{-1} = \frac{1}{10}$)

Graphs of Logarithmic Functions to Base $a > 1$

- When the base a is such that $a > 1$, then $f(x) = \log_a x$ has an increasing graph (going up as we move from left to right);
- We sketch the graph of $f(x) = \log_2 x$;

x	$y = \log_2 x$
1/4	-2
1/2	-1
1	0
2	1
4	2

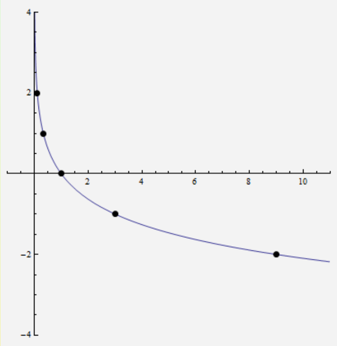


- Note that the y-axis is a vertical asymptote.

Graphs of Logarithmic Functions to Base $0 < a < 1$

- When the base a is such that $0 < a < 1$, then $f(x) = \log_a x$ has a decreasing graph (going down as we move from left to right);
- We sketch the graph of $f(x) = \log_{1/3} x$;

x	$y = \log_{1/3} x$
$1/9$	2
$1/3$	1
1	0
3	-1
9	-2



- Note that the y-axis is a vertical asymptote.

Logarithmic Equations

One-to-One Property of Logarithms

$$\log_a m = \log_a n \quad \text{implies} \quad m = n.$$

- Example: Solve each equation:
  - $\log_3 x = -2$
    $\log_3 x = -2 \Rightarrow x = 3^{-2} \Rightarrow x = \frac{1}{9}$;
  - $\log_x 8 = -3$
    $\log_x 8 = -3 \Rightarrow x^{-3} = 8 \Rightarrow (x^{-3})^{-1/3} = 8^{-1/3} \Rightarrow x = \frac{1}{8^{1/3}} \Rightarrow x = \frac{1}{\sqrt[3]{8}} \Rightarrow x = \frac{1}{2}$;
  - $\log(x^2) = \log 4$
    $\log(x^2) = \log 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm\sqrt{4} \Rightarrow x = -2$ or $x = 2$.

Application: Find Time in Continuous Compounding

- Recall: If P is the principal amount invested, r the annual interest rate compounded continuously and t the number of years, then the amount A accrued at the end of t years is
$$A = Pe^{rt};$$
- Example: How long does it take for \$200 to grow to \$600 at 3% annually compounded continuously?

  We have $P = 200$, $r = 0.03$ and $A = 600$;
  We would like to compute t;
$$\begin{aligned} A = Pe^{rt} \quad &\Rightarrow \quad 600 = 200e^{0.03t} \\ &\Rightarrow \quad e^{0.03t} = 3 \\ &\Rightarrow \quad 0.03t = \ln 3 \\ &\Rightarrow \quad t = \tfrac{100}{3}\ln 3 \approx 36.62 \text{ years.} \end{aligned}$$

The Inverse Properties

- Recall the definition of the logarithm to base a of x:

$$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$

- This property gives

$$\boxed{a^{\log_a x} = x \quad \text{and} \quad \log_a(a^y) = y;}$$

- Example: Simplify:
  - $\ln(e^7) = 7$; (recall this means $\log_e(e^7)$)
  - $2^{\log_2 13} = 13$.

The Product Rule for Logarithms

- For $M > 0$ and $N > 0$, we have

$$\boxed{\log_a (M \cdot N) = \log_a M + \log_a N.}$$

- Example: Write as a single logarithm:
  - $\log_2 7 + \log_2 5 = \log_2 (7 \cdot 5) = \log_2 35$;
  - $\ln(\sqrt{2}) + \ln(\sqrt{3}) = \ln(\sqrt{2} \cdot \sqrt{3}) = \ln(\sqrt{2 \cdot 3}) = \ln(\sqrt{6})$.

The Quotient Rule for Logarithms

- For $M > 0$ and $N > 0$, we have

$$\boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N.}$$

- Example: Write as a single logarithm:
  - $\log_2 3 - \log_2 7 = \log_2\left(\frac{3}{7}\right)$;
  - $\ln(w^8) - \ln(w^2) = \ln\left(\frac{w^8}{w^2}\right) = \ln(w^{8-2}) = \ln(w^6)$.

The Power Rule for Logarithms

- For $M > 0$, we have

$$\boxed{\log_a (M^N) = N \cdot \log_a M.}$$

- Example: Write in terms of $\log 2$:
  - $\log (2^{10}) = 10 \log 2$;
  - $\log (\sqrt{2}) = \log (2^{1/2}) = \frac{1}{2} \log 2$;
  - $\log (\frac{1}{2}) = \log (2^{-1}) = -\log 2$.

Summary of the Properties of Logarithms

Properties of Logarithms

Assuming M, N and a are positive, with $a \neq 1$, we have

1. $\log_a a = 1$;
2. $\log_a 1 = 0$;
3. $\log_a (a^y) = y$; $\quad$ (y any real)
4. $a^{\log_a x} = x$; $\quad$ (x any positive real)
5. $\log_a (M \cdot N) = \log_a M + \log_a N$;
6. $\log_a \left(\dfrac{M}{N}\right) = \log_a M - \log_a N$;
7. $\log_a \left(\dfrac{1}{N}\right) = -\log_a N$;
8. $\log_a (M^N) = N \log_a M$.

Using the Properties

- Rewrite each expression in terms of $\log 2$ and/or $\log 3$:
  - $\log 6 = \log(2 \cdot 3) = \log 2 + \log 3$;
  - $\log 16 = \log(2^4) = 4\log 2$;
  - $\log(\frac{9}{2}) = \log 9 - \log 2 = \log(3^2) - \log 2 = 2\log 3 - \log 2$;
  - $\log(\frac{1}{3}) = -\log 3$;
- Rewrite each expression as a sum or difference of multiples of logarithms:
  - $\log\left(\dfrac{xz}{y}\right) = \log(xz) - \log y = \log x + \log z - \log y$;
  - $\log_3\left(\dfrac{(x-3)^{2/3}}{\sqrt{x}}\right) = \log_3((x-3)^{2/3}) - \log_3(x^{1/2}) = \frac{2}{3}\log_3(x-3) - \frac{1}{2}\log_3 x$;
- Rewrite each expression as a single logarithm:
  - $\frac{1}{2}\log x - 2\log(x+1) = \log(x^{1/2}) - \log((x+1)^2) = \log\left(\dfrac{\sqrt{x}}{(x+1)^2}\right)$;
  - $3\log y + \frac{1}{2}\log z - \log x = \log(y^3) + \log(z^{1/2}) - \log x = \log(y^3\sqrt{z}) - \log x = \log(\frac{y^3\sqrt{z}}{x})$.

Logarithmic Equations with Only One Logarithm

- Solve $\log(x+3) = 2$;

  The technique involves rewriting as an exponential equation using the property

  $$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$

  $$\begin{aligned} & \log(x+3) = 2 \\ \Rightarrow \quad & 10^2 = x+3 \\ \Rightarrow \quad & x = 100 - 3 = 97; \end{aligned}$$

  Now check whether the solution is admissible; i.e., is $\log(97+3) = 2$? Yes! since $10^2 = 100$; So, the solution $x = 97$ is admissible.

Using the Product Rule to Solve an Equation

- Solve $\log_2(x+3) + \log_2(x-3) = 4$;

  The technique involves using the Product Rule $\log_a(M \cdot N) = \log_a M + \log_a N$ to combine the sum into a single logarithm; Then, we apply the technique for solving equations involving a single logarithm;

$$
\begin{aligned}
& \log_2(x+3) + \log_2(x-3) = 4 \\
\Rightarrow\quad & \log_2((x+3)(x-3)) = 4 \\
\Rightarrow\quad & 2^4 = (x+3)(x-3) \\
\Rightarrow\quad & 16 = x^2 - 9 \\
\Rightarrow\quad & x^2 = 25 \\
\Rightarrow\quad & x = \pm\sqrt{25} = \pm 5;
\end{aligned}
$$

  Now check whether the solutions are admissible; $x = -5$ is not admissible; Therefore, $x = 5$ is the only admissible solution!

Using the One-to-One Property to Solve an Equation

- Solve $\log x + \log(x-1) = \log(8x-12) - \log 2$;

  The technique involves using the Product/Quotient Rule for Logarithms to combine the sum/difference into a single logarithm and, then, the One-to-One Property

$$\log_a m = \log_a n \quad \text{implies} \quad m = n;$$

$$\begin{aligned}
&\log x + \log(x-1) = \log(8x-12) - \log 2 \\
\Rightarrow \quad & \log(x(x-1)) = \log\left(\tfrac{8x-12}{2}\right) \\
\Rightarrow \quad & x(x-1) = 4x - 6 \\
\Rightarrow \quad & x^2 - x = 4x - 6 \\
\Rightarrow \quad & x^2 - 5x + 6 = 0 \\
\Rightarrow \quad & (x-2)(x-3) = 0 \\
\Rightarrow \quad & x = 2 \text{ or } x = 3;
\end{aligned}$$

  Now check whether the solutions are admissible; Both $x = 2$ and $x = 3$ are admissible solutions!

A Single Exponential Equation

- Solve $2^x = 10$;

  The technique involves rewriting as a logarithmic equation using the property

  $$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$

  $$\begin{aligned} 2^x &= 10 \\ \Rightarrow \quad x &= \log_2 10; \end{aligned}$$

  Now check whether the solution is admissible; The solution $x = \log_2 10$ is admissible.

Powers of the Same Base

- Solve $2^{(x^2)} = 4^{3x-4}$;

  The technique involves rewriting the two sides as powers over the same base and, then, using the One-to-One Property

  $$a^m = a^n \quad \text{implies} \quad m = n;$$

  $$\begin{aligned} & 2^{(x^2)} = 4^{3x-4} \\ \Rightarrow \quad & 2^{(x^2)} = (2^2)^{3x-4} \\ \Rightarrow \quad & 2^{(x^2)} = 2^{2(3x-4)} \\ \Rightarrow \quad & x^2 = 6x - 8 \\ \Rightarrow \quad & x^2 - 6x + 8 = 0 \\ \Rightarrow \quad & (x-2)(x-4) = 0 \\ \Rightarrow \quad & x = 2 \text{ or } x = 4; \end{aligned}$$

  Now check whether the solutions are admissible; Both $x = 2$ and $x = 4$ are admissible solutions.

Exponential Equations Involving Different Bases

- Solve $2^{x-1} = 3^x$;

  The technique involves taking logarithms of both sides and, then, using the Power Property

$$\log_a(M^N) = N\log_a M;$$

$$\begin{aligned} & 2^{x-1} = 3^x \\ \Rightarrow \quad & \log(2^{x-1}) = \log(3^x) \\ \Rightarrow \quad & (x-1)\log 2 = x\log 3 \\ \Rightarrow \quad & x\log 2 - \log 2 = x\log 3 \\ \Rightarrow \quad & x\log 2 - x\log 3 = \log 2 \\ \Rightarrow \quad & x(\log 2 - \log 3) = \log 2 \\ \Rightarrow \quad & x = \frac{\log 2}{\log 2 - \log 3}. \end{aligned}$$

Changing the Base

Change-Of-Base Formula

If a, b are positive numbers not equal to 1 and $M > 0$,

$$\boxed{\log_a M = \frac{\log_b M}{\log_b a};}$$

- Example: Use your calculator to compute $\log_7 99$ to four decimal places;

  Calculators usually allow us to compute common logarithms (base 10) and natural logarithms (base e);
  Thus, we need to use Change-Of-Base to convert $\log_7 99$ to an expression involving either common or natural logarithms:

  $$\log_7 99 = \frac{\ln 99}{\ln 7} \approx 2.3614.$$

Strategy for Solving Equations

1. If the equation has a single logarithm or a single exponential, rewrite using
$$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$
2. Use the rules to combine logarithms as much as possible;
3. Use the One-to-One Properties
$$\log_a m = \log_a n \quad \text{implies} \quad m = n;$$
$$a^m = a^n \quad \text{implies} \quad m = n;$$
4. For exponential equations with different bases, take the common or natural logarithms of both sides of the equation.

Application: Finding Time in Finance

- If \$1,000 is deposited into an account paying 4% compounded quarterly, in how many quarters will the account have \$2,000 in it?

  Recall $A = P(1+i)^n$;
  We have $P = 1000$, $i = \frac{0.04}{4} = 0.01$ and $A = 2000$;
  We would like to determine the number of periods (quarters) n;

$$\begin{aligned} 2000 &= 1000(1+0.01)^n \\ \Rightarrow \quad 2 &= 1.01^n \\ \Rightarrow \quad n &= \log_{1.01} 2 \\ \overset{\text{Change Base}}{\Rightarrow} \quad n &= \frac{\ln 2}{\ln 1.01} \approx 69.66; \end{aligned}$$

  Thus, the account will have \$2,000 in approximately 70 quarters, i.e., in 17-and-a-half years.

Application: Tripling Time With Continuous Compounding

- A sum P is invested in account with interest rate 5% compounded continuously. How soon will the amount triple?

  Recall $A = Pe^{rt}$;
  We have $A = 3P$, and $r = 0.05$;
  We would like to determine the number of years t;

$$\begin{aligned} & A = Pe^{rt} \\ \Rightarrow \quad & 3P = Pe^{0.05t} \\ \Rightarrow \quad & 3 = e^{0.05t} \\ \Rightarrow \quad & 0.05t = \ln 3 \\ \Rightarrow \quad & t = 20 \ln 3 \approx 21.97; \end{aligned}$$

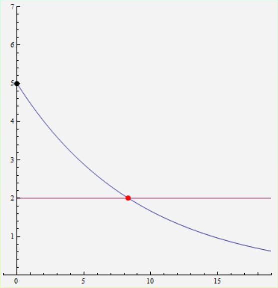
  Thus, the account will triple in approximately 22 years.

Application: Calculating Drug Dosage

- The amount of penicillin remaining in a person's bloodstream t hours after an original concentration of 5 mg/ml of blood is administered is

$$C(t) = 5e^{-0.11t};$$

Suppose that the minimum effective concentration is 2 milligrams;



After how many hours should another dose be administered to maintain an effective concentration?

We must find t for which

$$C(t) = 2 \quad \Rightarrow \quad 5e^{-0.11t} = 2 \quad \Rightarrow \quad e^{-0.11t} = \frac{2}{5}$$

$$\Rightarrow \quad -0.11t = \ln\frac{2}{5} \quad \Rightarrow \quad t = \frac{\ln 0.4}{-0.11} \approx 8.3 \text{ hours.}$$

Application: Finding the Rate of Radioactive Decay

- The number of grams of a radioactive substance that is present in an old bone after t years is given by
$$A = 8e^{rt},$$
where r is the decay rate.
  (a) How many grams were present when the bone was in a living organism at $t = 0$?
  (b) If it took 6300 years for the substance to decay from 8 grams to 4 grams, what is its decay rate?

(a) Plugging-in $t = 0$, we get $A = 8e^{r\cdot 0} = 8 \cdot e^0 = 8 \cdot 1 = 8$; Thus, when the organism was living the bone had 8 grams of the substance;

(b)
$$\begin{aligned} & 4 = 8e^{6300r} \\ \Rightarrow \quad & \tfrac{1}{2} = e^{6300r} \\ \Rightarrow \quad & 6300r = \ln\left(\tfrac{1}{2}\right) \\ \Rightarrow \quad & r = \tfrac{1}{6300}\ln\left(\tfrac{1}{2}\right) \approx -0.00011. \end{aligned}$$

Application: Learning Theory

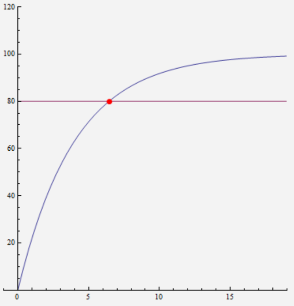
- The number of words per minute that a secretary can type after t weeks of training is given by

$$S(t) = 100(1 - e^{-0.25t});$$

In how many weeks will he be able to type 80 words per minute?

We must find t for which

$$\begin{array}{l} S(t) = 80 \quad \Rightarrow \quad 100(1 - e^{-0.25t}) = 80 \\ \Rightarrow \quad 1 - e^{-0.25t} = 0.8 \quad \Rightarrow \quad e^{-0.25t} = 0.2 \\ \Rightarrow \quad -0.25t = \ln 0.2 \quad \Rightarrow \quad t = \dfrac{\ln 0.2}{-0.25} \approx 6.4. \end{array}$$



Subsection 3

Derivatives of Exponentials and Logarithms

Derivatives of Logarithms

Derivative of $\ln x$

$$(\ln x)' = \frac{1}{x};$$

- Example: Compute $f'(x)$ if $f(x) = x^7 \ln x$;

$$f'(x) = (x^7 \ln x)' = (x^7)' \ln x + x^7 (\ln x)' = \\ 7x^6 \ln x + x^7 \cdot \tfrac{1}{x} = 7x^6 \ln x + x^6;$$

Derivative of $\ln f(x)$

$$(\ln f(x))' = \frac{f'(x)}{f(x)};$$

- Example: Compute $f'(x)$ if $f(x) = \ln (x^2 + 1)$;

$$f'(x) = [\ln (x^2 + 1)]' = \frac{(x^2 + 1)'}{x^2 + 1} = \frac{2x}{x^2 + 1}.$$

Some More Examples

- Compute the following derivatives:
  - $\left[\frac{\ln x}{x}\right]' = \frac{(\ln x)'x - \ln x \cdot (x)'}{x^2} = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$;
  - $[\ln(x^3 - 5x + 1)]' = \frac{(x^3 - 5x + 1)'}{x^3 - 5x + 1} = \frac{3x^2 - 5}{x^3 - 5x + 1}$;
  - $[\ln(x^4 - 1)^3]' = \frac{[(x^4 - 1)^3]'}{(x^4 - 1)^3} = \frac{3(x^4 - 1)^2(x^4 - 1)'}{(x^4 - 1)^3} = \frac{3(x^4 - 1)^2 \cdot 4x^3}{(x^4 - 1)^3} = \frac{12x^3}{x^4 - 1}$;
  - $[\ln(x^4 - 1)^3]' = [3\ln(x^4 - 1)]' = 3[\ln(x^4 - 1)]' = 3 \cdot \frac{(x^4 - 1)'}{x^4 - 1} = 3 \cdot \frac{4x^3}{x^4 - 1} = \frac{12x^3}{x^4 - 1}$.

Derivatives of Exponentials

Derivative of e^x

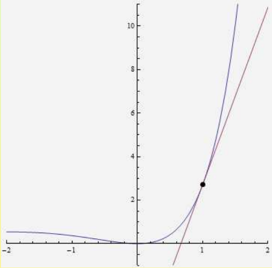
$$(e^x)' = e^x;$$

- Example: Compute $f'(x)$ if $f(x) = \frac{e^x}{x}$;

$$f'(x) = \left(\frac{e^x}{x}\right)' = \frac{(e^x)'x - e^x(x)'}{x^2} = \frac{e^x \cdot x - e^x \cdot 1}{x^2} = \frac{xe^x - e^x}{x^2};$$

- Find an equation for the tangent line to $f(x) = x^2 e^x$ at $x = 1$;

  Compute $f'(x) = (x^2 e^x)' = (x^2)' e^x + x^2 (e^x)' = 2xe^x + x^2 e^x$; Find the slope $f'(1) = 2 \cdot 1 \cdot e + 1^2 \cdot e = 2e + e = 3e$; Thus, the equation of the tangent is $y - f(1) = f'(1)(x-1) \Rightarrow y - e = 3e(x-1) \Rightarrow y = 3ex - 2e$.



The General Exponential

Derivative of $e^{f(x)}$

$$(e^{f(x)})' = e^{f(x)} \cdot f'(x);$$

- Example: Compute the derivatives:
  - $(e^{x^4+1})' = e^{x^4+1} \cdot (x^4+1)' = e^{x^4+1} \cdot 4x^3 = 4x^3 e^{x^4+1}$;
  - $(e^{x^2/2})' = e^{x^2/2} \cdot (\frac{x^2}{2})' = e^{x^2/2} \cdot \frac{2x}{2} = xe^{x^2/2}$;
  - $(e^{1+\frac{x^3}{3}})' = e^{1+\frac{x^3}{3}} \cdot (1+\frac{x^3}{3})' = e^{1+\frac{x^3}{3}} \cdot \frac{3x^2}{3} = x^2 e^{1+\frac{x^3}{3}}$.

Some Mixed Derivatives

Summary of Rules

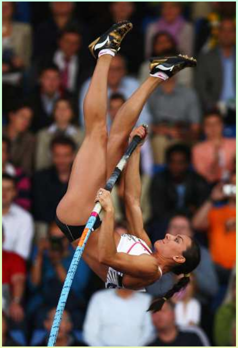
	Logarithms	Exponentials
Simple	$(\ln x)' = \frac{1}{x}$	$(e^x)' = e^x$
Chain	$(\ln f(x))' = \frac{f'(x)}{f(x)}$	$(e^{f(x)})' = e^{f(x)} f'(x)$

- Example: Compute the derivatives:
  - $[\ln(1+e^x)]' = \dfrac{(1+e^x)'}{1+e^x} = \dfrac{e^x}{1+e^x}$;
  - $(e^{1+e^x})' = e^{1+e^x}(1+e^x)' = e^{1+e^x}e^x$;
  - $[\ln(e^x+e^{-x})]' = \dfrac{(e^x+e^{-x})'}{e^x+e^{-x}} = \dfrac{e^x+e^{-x}(-x)'}{e^x+e^{-x}} = \dfrac{e^x-e^{-x}}{e^x+e^{-x}}$;
  - $(e^x\ln(x+1))' = (e^x)'\ln(x+1) + e^x(\ln(x+1))' =$
    $e^x\ln(x+1) + e^x\dfrac{(x+1)'}{x+1} = e^x\ln(x+1) + \dfrac{e^x}{x+1}$;
  - $[(2x+\ln x)^3]' = 3(2x+\ln x)^2(2x+\ln x)' = 3(2x+\ln x)^2(2+\frac{1}{x})$.

Application: Pole Vault Jump

- After t weeks of practice a pole vaulter can vault $H(t) = 15 - 11e^{-0.1t}$ feet. What is the rate of change of the athlete's jump at the beginning of her training? How about 12 weeks into her training?

  We compute the derivative, which represents the rate of change of the height with respect to time:



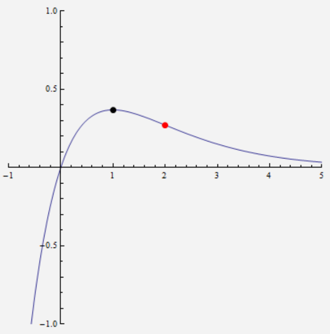
$\frac{dH}{dt} = (15 - 11e^{-0.1t})' = -11e^{-0.1t}(-0.1t)' = -11e^{-0.1t} \cdot (-0.1) = 1.1e^{-0.1t}$; Therefore, at the beginning of training $t = 0$, the rate of change is 1.1 feet/week; After 12 weeks the rate of change is $\left.\frac{dH}{dt}\right|_{t=12} = 1.1e^{-0.1 \cdot 12} = 1.1e^{-1.2} (\approx 0.33)$ feet/week.

Using Derivatives to Graph Exponentials

- Graph the function $f(x) = xe^{-x}$;
  Compute the first derivative:
  $f'(x) = (xe^{-x})' = (x)'e^{-x} + x(e^{-x})' = e^{-x} - xe^{-x} = e^{-x}(1-x)$; Find its critical numbers: $f'(x) = 0 \Rightarrow e^{-x}(1-x) = 0 \Rightarrow x = 1$;
  Find second derivative: $f''(x) = (e^{-x} - xe^{-x})' = (e^{-x})' - (xe^{-x})' = -e^{-x} - e^{-x} + xe^{-x} = xe^{-x} - 2e^{-x} = e^{-x}(x-2)$; Its zero is $x = 2$;
  Create combined sign table for first and second derivatives:

	$x < 1$	$[1,2]$	$2 < x$
f'	$+$	$-$	$-$
f''	$-$	$-$	$+$
f	$\curvearrowright$	$\searrow$	$\searrow$

  Thus f has relative max $(1, \frac{1}{e})$, and an inflection $(2, \frac{2}{e^2})$.



Using Derivatives to Graph Logarithms

- Graph the function $f(x) = x \ln x$;
  Compute the first derivative:
  $f'(x) = (x \ln x)' = (x)' \ln x + x(\ln x)' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$; Find its critical numbers: $f'(x) = 0 \Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$;
  Find second derivative: $f''(x) = (\ln x + 1)' = \frac{1}{x}$; It is undefined at $x = 0$;
  Create combined sign table for first and second derivatives:

	$0 < x < \frac{1}{e}$	$\frac{1}{e} < x$
f'	$-$	$+$
f''	$+$	$+$
f	$\searrow$	$\nearrow$

Thus f has relative min $(\frac{1}{e}, -\frac{1}{e})$.