Business and Life Calculus

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LSSU Math 112

1. Integration and Its Applications
   - Antiderivatives and Indefinite Integrals
   - Integration Using Exponential and Logarithmic Functions
   - Definite Integrals and Areas
   - Average Value and Area Between Curves
   - Integration By Substitution

Subsection 1

Antiderivatives and Indefinite Integrals

Antiderivatives and Indefinite Integrals

- The reverse process of differentiation (i.e., of taking derivatives) is called **antidifferentiation**;
- Recall

  the derivative of x^2 is $2x$; So reversing
  an **antiderivative** of $2x$ is x^2;

- Notice that there are other functions that would work as antiderivatives of $2x$: e.g., $x^2 + 1$ or $x^2 - 49$ also work;
- In fact, if $F(x)$ is an antiderivative of $f(x)$ (i.e., $f(x) = F'(x)$), the **most general antiderivative** of $f(x)$ is $F(x) + C$, C any constant;
- The most general antiderivative of $f(x)$ is called the **indefinite integral of** $f(x)$ and is denoted by $\int f(x)dx$;
- Thus, if $F'(x) = f(x)$, then

$$\int f(x)dx = F(x) + C.$$

The Power Rule for Integrals

- Recall that

Derivatives	Integrals
$(x)' = 1$	$\int 1dx = x + C$
$(x^2)' = 2x$	$\int xdx = \frac{1}{2}x^2 + C$
$(x^3)' = 3x^2$	$\int x^2dx = \frac{1}{3}x^3 + C$
$(x^4)' = 4x^3$	$\int x^3dx = \frac{1}{4}x^4 + C$

- In general

Power Rule for Derivatives	Power Rule for Integrals
$(x^n)' = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + C.$

More Examples on the Power Rule for Integrals

- Example: Compute the integrals:
  - $\int \sqrt{x}dx = \int x^{1/2}dx = \frac{1}{\frac{3}{2}}x^{3/2} + C = \frac{2}{3}x^{3/2} + C;$
  - $\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{1}{-1}x^{-1} + C = -\frac{1}{x} + C;$
  - $\int x^7 dx = \frac{1}{8}x^8 + C;$
  - $\int \frac{1}{x^{11}}dx = \int x^{-11}dx = \frac{1}{-10}x^{-10} + C = -\frac{1}{10x^{10}} + C;$
  - $\int \frac{1}{\sqrt[3]{x^2}}dx = \int \frac{1}{x^{2/3}}dx = \int x^{-2/3}dx = \frac{1}{\frac{1}{3}}x^{1/3} + C = 3\sqrt[3]{x} + C.$

The Sum/Difference Rule for Integrals

Integral of a Sum is Sum of Integrals

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx;$$

- Example: Compute the integrals:
  - $\int (x^8 - x^5)dx = \int x^8 dx - \int x^5 dx = \frac{1}{9}x^9 - \frac{1}{6}x^6 + C;$
  - $\int (\sqrt{x} + \frac{1}{\sqrt{x}})dx = \int \sqrt{x}dx + \int \frac{1}{\sqrt{x}}dx = \int x^{1/2}dx + \int x^{-1/2}dx =$ $\frac{1}{\frac{3}{2}}x^{3/2} + \frac{1}{\frac{1}{2}}x^{1/2} + C = \frac{2}{3}\sqrt{x^3} + 2\sqrt{x} + C.$

Constant Factor Rule For Integrals

Constant Factor Rule for Integrals

$$\int k \cdot f(x)dx = k \int f(x)dx;$$

- Example: Compute the integrals:
  - $\int 15x^4 dx = 15 \int x^4 dx = 15 \cdot \frac{1}{5}x^5 + C = 3x^5 + C;$
  - $\int 13dx = 13 \int dx = 13x + C;$
  - $\int (6x^2 - \frac{3}{x^2} + 5)dx \overset{\text{sum}}{=} \int 6x^2 dx - \int \frac{3}{x^2} dx + \int 5dx \overset{\text{constant}}{=} 6 \int x^2 dx - 3 \int x^{-2} dx + 5 \int dx = 6 \cdot \frac{1}{3}x^3 - 3 \cdot \frac{1}{-1}x^{-1} + 5x + C = 2x^3 + \frac{3}{x} + 5x + C;$
  - $\int (\sqrt[3]{x} - \frac{4}{x^3})dx = \int \sqrt[3]{x} dx - \int \frac{4}{x^3} dx = \int x^{1/3} dx - 4 \int x^{-3} dx = \frac{1}{\frac{4}{3}}x^{4/3} - 4 \cdot \frac{1}{-2}x^{-2} + C = \frac{3}{4}\sqrt[3]{x^4} + \frac{2}{x^2} + C.$

Simplifying to Compute

- Compute the integrals:
  - $\int x^2(x+6)^2 dx = \int x^2(x^2+12x+36)dx = \int (x^4+12x^3+36x^2)dx = \int x^4 dx + \int 12x^3 dx + \int 36x^2 dx = \frac{1}{5}x^5 + 3x^4 + 12x^3 + C;$
  - $\int \frac{6x^2 - x}{x} dx = \int (\frac{6x^2}{x} - \frac{x}{x})dx = \int (6x-1)dx = 3x^2 - x + C;$
  - $\int (1-7x)\sqrt[3]{x}dx = \int (1-7x)x^{1/3}dx = \int (x^{1/3} - 7x^{4/3})dx = \frac{1}{\frac{4}{3}}x^{4/3} - 7 \cdot \frac{1}{\frac{7}{3}}x^{7/3} + C = \frac{3}{4}\sqrt[3]{x^4} - 3\sqrt[3]{x^7} + C;$
  - $\int \frac{4x^4 + 4x^2 - x}{x} dx = \int (\frac{4x^4}{x} + \frac{4x^2}{x} - \frac{x}{x})dx = \int (4x^3+4x-1)dx = x^4 + 2x^2 - x + C.$

Application: Cost From Marginal Cost

- Recall

$$C(x) \xrightarrow{\text{Derivative}} \mathrm{MC}(x) = C'(x)$$
$$C(x) = \int \mathrm{MC}(x)dx \xleftarrow{\text{Integral}} \mathrm{MC}(x)$$

- Application: If a company's marginal cost function is $\mathrm{MC}(x) = 6\sqrt{x}$ and its fixed costs are \$1,000, what is its cost function $C(x)$?

$$C(x) = \int \mathrm{MC}(x)dx = \int 6x^{1/2}dx = 6\int x^{1/2}dx = 6 \cdot \frac{1}{\frac{3}{2}}x^{3/2} + C = 4\sqrt{x^3} + C;$$

The company has fixed costs \$1000; This means that $C(0) = 4\sqrt{0^3} + C = 1000$; So we get $C = 1000$; Therefore $C(x) = 4\sqrt{x^3} + 1000$.

Application: Quantity From Rate of Change

- Recall

$$Q(t) \xrightarrow{\text{Derivative}} \frac{dQ}{dt}$$

$$Q(t) = \int \frac{dQ}{dt} dt \xleftarrow{\text{Integral}} \frac{dQ}{dt}$$

- Application: Suppose GDP of a country is \$78 billion and growing at the rate of $4.4t^{-1/3}$ billion dollars per year after t years; What will the GDP be after t years?

$$P(t) = \int P'(t)dt = \int 4.4t^{-1/3}dt = 4.4\int t^{-1/3}dt = 4.4 \cdot \frac{1}{\frac{2}{3}}t^{2/3} + C = 6.6\sqrt[3]{t^2} + C;$$

The country has current GDP \$78 billion; This means that $P(0) = 6.6\sqrt[3]{0^2} + C = 78$; So we get $C = 78$; Therefore $P(t) = 6.6\sqrt[3]{t^2} + 78.$

Subsection 2

Integration Using Exponential and Logarithmic Functions

The Integral $\int e^{ax} dx$

Integrating an Exponential Function

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C;$$

- Example: Compute the integrals:
  - $\int e^{\frac{1}{2}x} dx = \frac{1}{\frac{1}{2}} e^{\frac{1}{2}x} + C = 2e^{\frac{1}{2}x} + C;$
  - $\int 6e^{-3x} dx = 6 \int e^{-3x} dx = 6 \cdot \frac{1}{-3} e^{-3x} + C = -2e^{-3x} + C;$
  - $\int e^{x} dx = \frac{1}{1} e^{x} + C = e^{x} + C;$
  - $\int 21e^{7x} dx = 21 \int e^{7x} dx = 21 \cdot \frac{1}{7} e^{7x} + C = 3e^{7x} + C.$

Application: Number of Flu Cases From Rate of Spreading

- Suppose that an influenza epidemic spreads at the rate of $12e^{0.2t}$ new cases per day, where t is number of days since epidemic began; Suppose, also, that at the beginning 4 cases existed;
  - Find a formula for the total number of cases after t days;
    To get the number from the rate of change we integrate:

    $$N(t) = \int 12e^{0.2t}dt = 12\int e^{0.2t}dt = 12 \cdot \frac{1}{0.2}e^{0.2t} + C = 60e^{0.2t} + C;$$

    Since at $t = 0,\ N(0) = 4,$ we get

    $$60e^{0.2 \cdot 0} + C = 4 \Rightarrow 60 + C = 4 \Rightarrow C = -56;$$

    Therefore $N(t) = 60e^{0.2t} - 56;$
  - How many cases will there be during the first 30 days?

    $$N(30) = 60e^{0.2 \cdot 30} - 56 = 60e^{6} - 56 \approx 24,150 \text{ cases.}$$

Evaluating C

- Suppose that an initial condition $f(x_0) = y_0$ is provided;
- To evaluate the constant C in the given problem:
  1. Evaluate the integral at the given number x_0 and set the result equal to the given initial value y_0;
  2. Solve the resulting equation for C;
  3. Write the answer with C replaced by the value found;
- Suppose that the initial condition $N(0) = 4$ for $N(t) = 60e^{0.2t} + C$ is provided;
  1. $N(0) = 4 \Rightarrow 60e^{0.2 \cdot 0} + C = 4 \Rightarrow 60 + C = 4$;
  2. $C = -56$;
  3. $N(t) = 60e^{0.2 \cdot 0} - 56$.

The Integral $\int \frac{1}{x} dx$

The integral of $\frac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C;$$

- Example: Compute the integrals:
  - $\int \frac{5}{2x} dx = \frac{5}{2} \int \frac{1}{x} dx = \frac{5}{2} \ln|x| + C;$
  - $\int (x^{-1} + x^{-2}) dx = \int x^{-1} dx + \int x^{-2} dx = \ln|x| + \frac{1}{-1} x^{-1} + C = \ln|x| - \frac{1}{x} + C;$
  - $\int \frac{xe^x - 1}{x} dx = \int (\frac{xe^x}{x} - \frac{1}{x}) dx = \int (e^x - \frac{1}{x}) dx = \int e^x dx - \int \frac{1}{x} dx = e^x - \ln|x| + C.$

An Additional Example

- $$\begin{aligned}\int \frac{5x^3 - 7x + 11}{x^2} dx &= \int \left(\frac{5x^3}{x^2} - \frac{7x}{x^2} + \frac{11}{x^2}\right) dx \\ &= \int \left(5x - \frac{7}{x} + 11x^{-2}\right) dx \\ &= 5\int x dx - 7\int \frac{1}{x} dx + 11\int x^{-2} dx \\ &= 5 \cdot \frac{1}{2}x^2 - 7\ln|x| + 11 \cdot \frac{1}{-1}x^{-1} + C \\ &= \frac{5}{2}x^2 - 7\ln|x| - \frac{11}{x} + C.\end{aligned}$$

Application: Total Sales From Rate of Sales

- Suppose that during month t of a computer sale, a computer will sell at a rate of approximately $\frac{25}{t}$ per month, where $t = 1$ corresponds to the beginning of the sale, at which time no computers have yet been sold;
  - Find a formula for the total number of computers that will be sold up to the month t;
    $$N(t) = \int \frac{25}{t} dt = 25 \int \frac{1}{t} dt = 25 \ln t + C;$$
    Since $N(1) = 0$, we get
    $$25 \ln 1 + C = 0 \Rightarrow 25 \cdot 0 + C = 0 \Rightarrow C = 0;$$
    Hence $N(t) = 25 \ln t$;
  - Will the store's inventory of 64 computers be sold by month $t = 12$?
    $$N(12) = 25 \ln 12 \approx 62;$$
    All but 2 of the 64 computers will be sold by $t = 12$.

Application: Consumption of Raw Materials

- The rate of consumption of tin is predicted to be $0.26e^{0.01t}$ million metric tons per year, where t is counted in years since 2008;
  - Find a formula for the total tin consumption within t years of 2008;

$$T(t) = \int 0.26e^{0.01t}dt = 0.26\int e^{0.01t}dt =$$
$$0.26 \cdot \frac{1}{0.01}e^{0.01t} + C = 26e^{0.01t} + C;$$

    Since $T(0) = 0$, we get $26e^{0.01 \cdot 0} + C = 0 \Rightarrow 26 + C = 0 \Rightarrow C = -26$;
    Hence $T(t) = 26e^{0.01 \cdot t} - 26$;
  - Estimate when the known world reserves of 6.1 million metric tons will be exhausted;
    We must estimate t, so that $T(t) = 6.1$;

$$26e^{0.01t} - 26 = 6.1 \Rightarrow 26e^{0.01t} = 32.1 \Rightarrow e^{0.01t} = \frac{32.1}{26}$$
$$\Rightarrow 0.01t = \ln\frac{32.1}{26} \Rightarrow t = 100\ln\frac{32.1}{26} \approx 21.1;$$

    Thus the tin reserves will be exhausted in about 21 years after 2008, or around the year 2029.

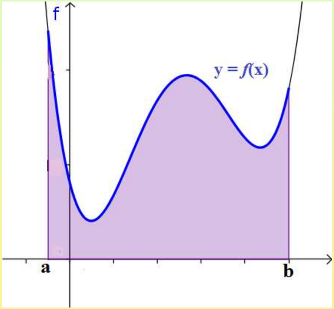
Subsection 3

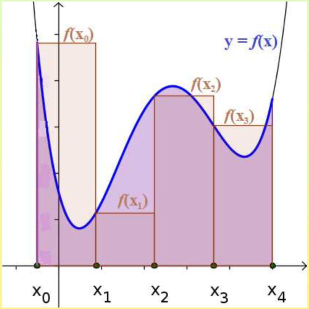
Definite Integrals and Areas

Area Under a Curve

- Let $y = f(x)$ be a continuous nonnegative function on $[a, b]$;
- To compute the area under $y = f(x)$ from $x = a$ to $x = b$;
- A technique to approximate the area is to consider it as approximately equal to the sum of the area of rectangles as shown here:
- If all bases are of equal length Δx, the area is equal to

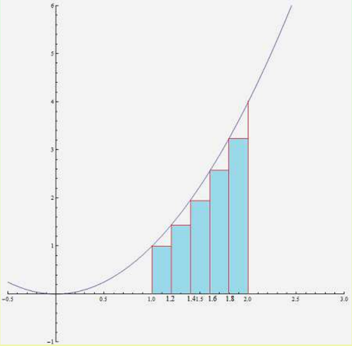
$$\begin{aligned} A &\approx \Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) \\ &= \Delta x[f(x_0) + f(x_1) + f(x_2) + f(x_3)]. \end{aligned}$$





Approximating Area By Rectangles

Approximate the area under $f(x) = x^2$ from 1 to 2 by five rectangles with equal bases and heights equal to the height of the curve at the left end of the rectangles;



The length of the base is

$$\Delta x = \frac{2-1}{5} = 0.2;$$

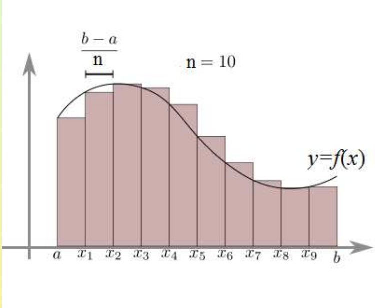
Thus the approximating sum is

$$\begin{aligned} A &\approx \Delta x[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)] \\ &= 0.2[f(1) + f(\tfrac{6}{5}) + f(\tfrac{7}{5}) + f(\tfrac{8}{5}) + f(\tfrac{9}{5})] \\ &= 0.2[1^2 + (\tfrac{6}{5})^2 + (\tfrac{7}{5})^2 + (\tfrac{8}{5})^2 + (\tfrac{9}{5})^2] = 2.04. \end{aligned}$$

Approximating Area Under $y = f(x)$ by Rectangles

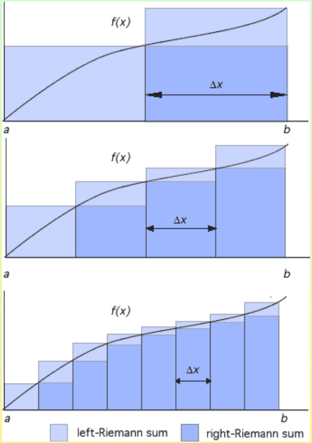
Area Under $y = f(x)$ from a to b Approximated by n Left Rectangles

1. Calculate the rectangle width $\Delta x = \frac{b-a}{n}$;
2. Find the x-values $x_1, x_2, \ldots, x_{n-1}$ by adding Δx at each step starting from $x_0 = a$;
3. Calculate the sum $A \approx \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$.



Definite Integral of f from a to b

- The expression $\Delta x[f(x_0)+f(x_1)+\cdots+f(x_{n-1})]$ is called the n**-th Riemann sum** of $f(x)$ on $[a,b]$;
- Take a closer look at what happens when the number n increases
  - The length of each interval decreases;
  - The "error areas" also decrease;
  - Thus, when $n\to\infty$ the Riemann sum becomes equal to the actual area A of the region under $y=f(x)$ from $x=a$ to $x=b$;
  - That quantity is called the **definite integral of** f **from** a **to** b, denoted $\int_a^b f(x)dx = \lim_{n\to\infty}[\Delta x(f(x_0)+\cdots+f(x_{n-1}))]$.



Fundamental Theorem of Calculus

Definition of the symbol $F(x)|_a^b$

$$F(x)|_a^b = \underbrace{F(b)}_{\text{Evaluate at Upper}} - \underbrace{F(a)}_{\text{Evaluate at Lower}} ;$$

- Example:

$$\sqrt{x}\Big|_4^{25} = \sqrt{25} - \sqrt{4} = 5 - 2 = 3;$$

Fundamental Theorem of Integral Calculus

For a continuous f on $[a, b]$,

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b,$$

where F is an antiderivative of f.

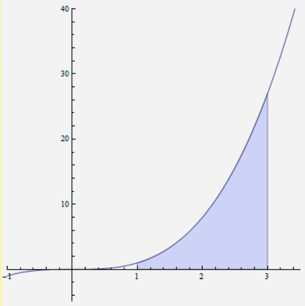
Computing Definite Integrals I

- Example: Calculate $\int_1^2 x^2 dx$;

$$\int_1^2 x^2 dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}\cdot 2^3 - \frac{1}{3}\cdot 1^3 = \frac{7}{3};$$

- Example: Find the exact area under $y = x^3$ from 1 to 3;
  Recall

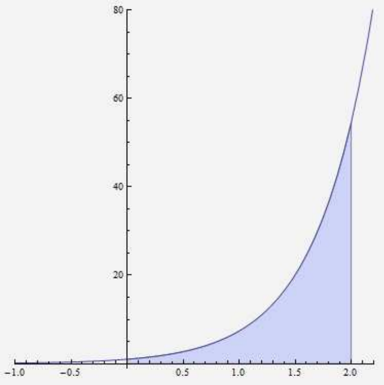
$$\begin{aligned} A &= \int_1^3 x^3 dx \\ &= \frac{1}{4}x^4\Big|_1^3 \\ &= \frac{1}{4}\cdot 3^4 - \frac{1}{4}\cdot 1^4 \\ &= 20. \end{aligned}$$



Computing Areas I

- Find the exact area under $y = e^{2x}$ from 0 to 2;

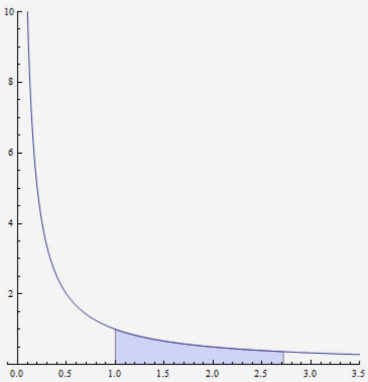
$$\begin{aligned} A &= \int_0^2 e^{2x} dx \\ &= \frac{1}{2} e^{2x} \Big|_0^2 \\ &= \frac{1}{2} \cdot e^4 - \frac{1}{2} \cdot e^0 \\ &= \frac{e^4 - 1}{2}. \end{aligned}$$



Computing Areas II

- Find the exact area under $y = \frac{1}{x}$ from 1 to e;

$$\begin{aligned} A &= \int_1^e \frac{1}{x} dx \\ &= \ln x|_1^e \\ &= \ln e - \ln 1 \\ &= 1 - 0 = 1. \end{aligned}$$



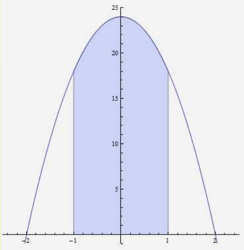
Properties of Definite Integrals

Properties of Definite Integrals

1. $\int_a^b c \cdot f(x)dx = c\int_a^b f(x)dx$;
2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;

- Example: Find the area under $f(x) = 24 - 6x^2$ from -1 to 1;

$$\begin{aligned} A &= \int_{-1}^1 (24 - 6x^2)dx \\ &= 24\int_{-1}^1 dx - 6\int_{-1}^1 x^2dx \\ &= 24\ x\Big|_{-1}^1 - 6\ \frac{1}{3}x^3\Big|_{-1}^1 \end{aligned}$$



$$= 24(1-(-1)) - 6(\frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot (-1)^3) = 48 - 4 = 44.$$

Computing Total Cost for a Number of Successive Units

Cost of a Succession of Units

$$\text{Total Cost of Units } a \text{ to } b = \int_a^b \text{MC}(x)dx;$$

- Example: If the marginal cost function is $\text{MC}(x) = \frac{75}{\sqrt{x}}$, where x is the number of units, what is the total cost of producing units 100 to 400?

$$C(100, 400) = \int_{100}^{400} \text{MC}(x)dx = \int_{100}^{400} \frac{75}{\sqrt{x}}dx =$$
$$75\int_{100}^{400} x^{-1/2}dx = 75 \cdot 2x^{1/2}\Big|_{100}^{400} = 75(2\sqrt{400} - 2\sqrt{100}) =$$
$$75(40 - 20) = 1500.$$

Computing Total Accumulation Given the Rate of Change

Total Accumulation at a Given Rate

$$\text{Total Accumulation at rate } f \text{ from } a \text{ to } b = \int_a^b f(x)dx;$$

- Example: A technician can test computer chips at the rate of $-3t^2 + 18t + 15$ chips per hour ($0 \leq t \leq 6$), where t is number of hours after 9:00am. How many chips can be tested between 10:00am and 1:00pm?

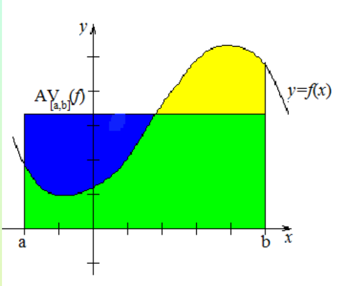
$$C(1,4) = \int_1^4 (-3t^2 + 18t + 15)dt = -3\int_1^4 t^2 dt + 18\int_1^4 t dt$$
$$+15\int_1^4 dt = -3 \cdot \frac{1}{3}t^3\Big|_1^4 + 18 \cdot \frac{1}{2}t^2\Big|_1^4 + 15 \cdot t|_1^4 =$$
$$-(4^3 - 1^3) + 9(4^2 - 1^2) + 15(4-1) = -63 + 135 + 45 = 117.$$

Subsection 4

Average Value and Area Between Curves

Average Value of a Function

- Consider a function $f(x)$ continuous on $[a, b]$;
- The **average value** $\text{AV}_{[a,b]}(f)$ of f on $[a, b]$ is the height of a rectangle with base $[a, b]$ that has the same area as the area under the curve from a to b;



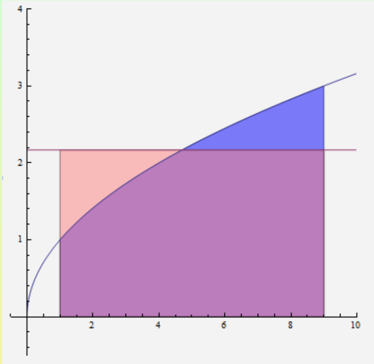
- Since the area under the curve is $\int_a^b f(x)dx$ and the area of the rectangle is $(b-a)\text{AV}_{[a,b]}(f)$, and these are equal: $\int_a^b f(x)dx = (b-a)\text{AV}_{[a,b]}(f)$, we get

$$\text{AV}_{[a,b]}(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Computing Average

- Find the average value of $f(x) = \sqrt{x}$ from $x = 1$ to $x = 9$;

$$\begin{aligned} \mathrm{AV}_{[1,9]}(f) &= \frac{1}{9-1}\int_1^9 \sqrt{x}dx \\ &= \frac{1}{8}\cdot\frac{2}{3}x^{3/2}\Big|_1^9 \\ &= \frac{1}{12}\sqrt{x}^3\Big|_1^9 \\ &= \frac{1}{12}(27-1) \\ &= \frac{13}{6}. \end{aligned}$$

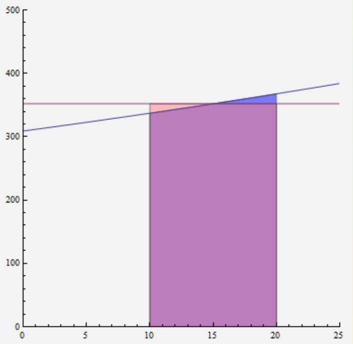


Computing Average: An Application

- The U.S. population t years after 2010 is predicted to be $P(t) = 309e^{0.0087t}$ million people; What is the average population between the years 2020 and 2030?

$\mathrm{AV}_{[10,20]}(P)$

$$
\begin{aligned}
&= \frac{1}{20-10}\int_{10}^{20} 309e^{0.0087t}\,dt \\
&= \frac{1}{10}\cdot 309\int_{10}^{20} e^{0.0087t}\,dt \\
&= 30.9\cdot \frac{1}{0.0087}e^{0.0087t}\Big|_{10}^{20} \\
&= \frac{30.9}{0.0087}(e^{0.0087\cdot 20} - e^{0.0087\cdot 10}) \\
&\approx 352 \text{ millions.}
\end{aligned}
$$

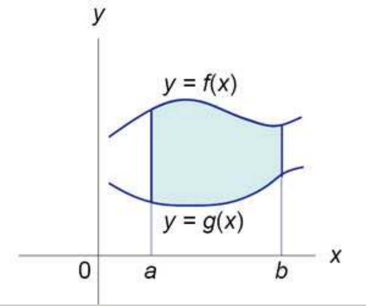


Area Between Curves

Area Between Two Curves

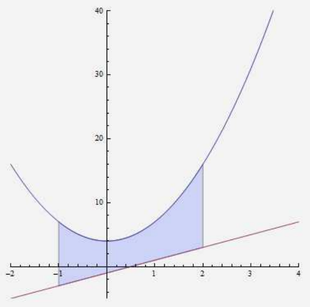
The area between two continuous curves $f(x) \geq g(x)$ on over an interval $[a, b]$ is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$



Computing Area Between Two Curves I

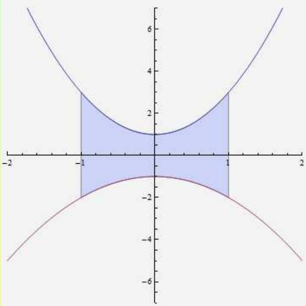
- Example: Find the area between $y = 3x^2 + 4$ and $y = 2x - 1$ from $x = -1$ to $x = 2$;



$$A = \int_{-1}^{2} [(3x^2 + 4) - (2x - 1)]dx = \int_{-1}^{2} (3x^2 - 2x + 5)dx = (x^3 - x^2 + 5x)\Big|_{-1}^{2} = 2^3 - 2^2 + 5 \cdot 2 - ((-1)^3 - (-1)^2 + 5(-1)) = 14 - (-7) = 21.$$

Computing Area Between Two Curves II

- Example: Find the area between $y = 2x^2 + 1$ and $y = -x^2 - 1$ from $x = -1$ to $x = 1$;

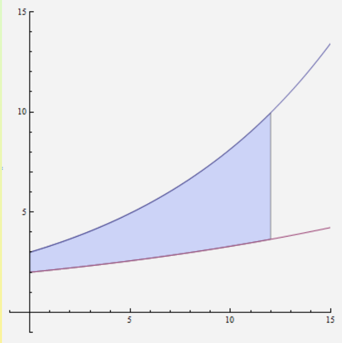


$$A = \int_{-1}^{1} [(2x^2 + 1) - (-x^2 - 1)]dx = \int_{-1}^{1} (3x^2 + 2)dx = (x^3 + 2x)\Big|_{-1}^{1} = 1^3 + 2 - ((-1)^3 + 2(-1)) = 3 + 3 = 6.$$

Computing Area Between Two Curves: Application

- TV sets are expected to sell at the rate of $2e^{0.05t}$ thousands per month, where t is number of months since they become available; With additional advertising, they could sell at the rate of $3e^{0.1t}$ thousands per month; How many additional sales would result from the extra advertisement during the first year?

$$\int_0^{12}(3e^{0.1t}-2e^{0.05t})dt =$$
$$3\int_0^{12}e^{0.1t}dt - 2\int_0^{12}e^{0.05t}dt =$$
$$3\,\frac{1}{0.1}e^{0.1t}\Big|_0^{12} - 2\,\frac{1}{0.05}e^{0.05t}\Big|_0^{12} =$$
$$30(e^{1.2}-1)-40(e^{0.6}-1)=$$
$$30e^{1.2}-40e^{0.6}+10$$
$$\approx 36.7 \text{ thoudands.}$$

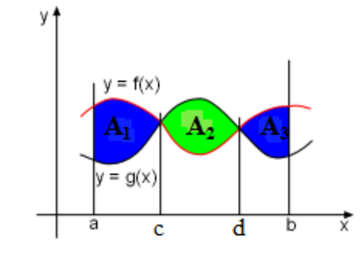


Area Between Curves that Cross

Area Between Two Crossing Curves

Suppose that two continuous curves $y = f(x)$ and $y = g(x)$ cross at $x = c$ and $x = d$ as shown over an interval $[a, b]$; Then the total area between the curves from a to b is given by

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= \int_a^c [f(x) - g(x)]dx + \int_c^d [g(x) - f(x)]dx + \int_d^b [f(x) - g(x)]dx. \end{aligned}$$

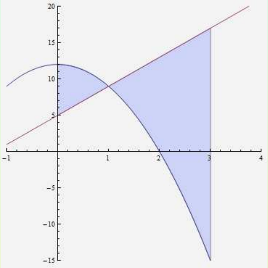


Computing Area Between Two Crossing Curves

- Example: Find the area between $y = 12 - 3x^2$ and $y = 4x + 5$ from $x = 0$ to $x = 3$;

  Find point of intersection in $[0, 3]$:

$$\begin{aligned} &12 - 3x^2 = 4x + 5 \\ &\Rightarrow 3x^2 + 4x - 7 = 0 \\ &\Rightarrow (3x + 7)(x - 1) = 0 \\ &\Rightarrow x = -\frac{7}{3} \text{ or } x = 1; \end{aligned}$$



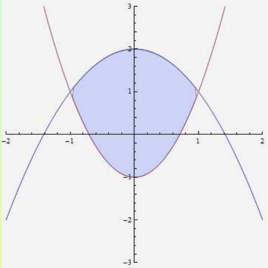
$$\begin{aligned} A &= \int_0^1 [(12 - 3x^2) - (4x + 5)]dx + \int_1^3 [(4x + 5) - (12 - 3x^2)]dx \\ &= \int_0^1 (-3x^2 - 4x + 7)dx + \int_1^3 (3x^2 + 4x - 7)dx \\ &= (-x^3 - 2x^2 + 7x)\Big|_0^1 + (x^3 + 2x^2 - 7x)\Big|_1^3 \\ &= (-1 - 2 + 7 - 0) + (27 + 18 - 21 - (1 + 2 - 7)) = 32. \end{aligned}$$

Area Bounded by Curves

- Example: Find the area between $y = 2x^2 - 1$ and $y = 2 - x^2$;

  Find the points of intersection:

$$\begin{aligned} &2x^2 - 1 = 2 - x^2 \\ &\Rightarrow 3x^2 - 3 = 0 \\ &\Rightarrow 3(x^2 - 1) = 0 \\ &\Rightarrow 3(x+1)(x-1) = 0 \\ &\Rightarrow x = -1 \text{ or } x = 1; \end{aligned}$$



$$A = \int_{-1}^{1} [(2 - x^2) - (2x^2 - 1)]dx = \int_{-1}^{1} (3 - 3x^2)dx$$
$$= (3x - x^3)\Big|_{-1}^{1} = 3 - 1 - (3(-1) - (-1)^3) = 4.$$

Technique Summary

- To find the area between two curves:
  1. If the x-values are not given, set the functions equal to each other and solve to find the points of intersection;
  2. Use a test point in each interval between points of intersection to determine which curve is the "upper" curve and which is the "lower" curve in that interval;
  3. Integrate "upper minus lower" on each interval.

Subsection 5

Integration By Substitution

Differentials

Differentials

For $f(x)$ a differentiable function, we define the **differential** df by

$$df = f'(x)dx;$$

- Note that this definition is consistent with the notation $f'(x) = \frac{df}{dx}$ used for the derivative $f'(x)$ of $f(x)$ with respect to x;
- Example:

Function $f(x)$	Differential df
$f(x) = x^2$	$df = 2xdx$
$f(x) = \ln x$	$df = \frac{1}{x}dx$
$f(x) = e^{x^2}$	$df = 2xe^{x^2}dx$
$f(x) = x^4 - 5x + 2$	$df = (4x^3 - 5)dx$

Substitution Method

Important Substitution Formulas

1. $\int u^n du = \frac{1}{n+1} u^{n+1} + C,$ if $n \neq -1$;
2. $\int e^u du = e^u + C$;
3. $\int \frac{1}{u} du = \ln |u| + C$;

- Example: Integrate $\int (x^2+1)^3 2x dx$;

  Substitute $u = x^2 + 1$; Compute the derivative $\frac{du}{dx} = (x^2+1)' = 2x$; Multiply both sides by dx: $du = 2xdx$; Go back to the integral and perform a careful substitution:

$$\int (x^2+1)^3 2xdx = \int u^3 du = \frac{1}{4} u^4 + C = \frac{1}{4}(x^2+1)^4 + C.$$

Example

- Integrate $\int (2x^5-7)^{11}10x^4dx$;
  Substitute $u=2x^5-7$; Compute the derivative $\frac{du}{dx}=(2x^5-7)'=10x^4$; Multiply both sides by dx:

$$du=10x^4dx;$$

  Go back to the integral and perform a careful substitution:

$$\begin{aligned}\int (2x^5-7)^{11}10x^4dx &= \int u^{11}du \\ &= \frac{1}{12}u^{12}+C \\ &= \frac{1}{12}(2x^5-7)^{12}+C.\end{aligned}$$

Multiplying by Constants

- Integrate $\int (x^3+1)^9 x^2 dx$;

  Substitute $u = x^3+1$; Compute the derivative $\frac{du}{dx} = (x^3+1)' = 3x^2$; Multiply both sides by dx: $du = 3x^2 dx$; Therefore, we get

$$\frac{1}{3} du = x^2 dx;$$

Go back to the integral and perform a careful substitution:

$$\begin{aligned}\int (x^3+1)^9 x^2 dx &= \int u^9 \frac{1}{3} du = \frac{1}{3} \int u^9 du \\ &= \frac{1}{3} \cdot \frac{1}{10} u^{10} + C = \frac{1}{30} (x^3+1)^{10} + C.\end{aligned}$$

An Exponential Integral

- Integrate $\int e^{x^5-2}x^4dx$;

  Substitute $u = x^5 - 2$; Compute the derivative $\frac{du}{dx} = (x^5-2)' = 5x^4$;
  Multiply both sides by dx: $du = 5x^4dx$; Therefore, we get

$$\frac{1}{5}du = x^4dx;$$

Go back to the integral and perform a careful substitution:

$$\begin{aligned}\int e^{x^5-2}x^4dx &= \int e^u \frac{1}{5}du = \frac{1}{5}\int e^u du \\ &= \frac{1}{5} \cdot e^u + C = \frac{1}{5}e^{x^5-2} + C.\end{aligned}$$

Cost From Marginal Cost

- A company's marginal cost is $\text{MC}(x) = \dfrac{x^3}{x^4+1}$ and its fixed costs are \$ 1,000; Find the company's cost function;

  Recall that $C(x) = \int \text{MC}(x)dx = \int \dfrac{x^3}{x^4+1}dx$;

  Substitute $u = x^4 + 1$; Compute the derivative $\dfrac{du}{dx} = (x^4+1)' = 4x^3$; Multiply both sides by dx: $du = 4x^3 dx$;

  Therefore, we get $\dfrac{1}{4}du = x^3 dx$; Go back to the integral and perform a careful substitution:

$$\int \frac{x^3}{x^4+1}dx = \int \frac{1}{u}\frac{1}{4}du = \frac{1}{4}\int \frac{1}{u}du = \frac{1}{4}\cdot \ln u + C = \frac{1}{4}\ln(x^4+1) + C;$$

Now, note $C(0) = 1000 \Rightarrow \dfrac{1}{4}\ln 1 + C = 1000 \Rightarrow C = 1000$;

Therefore $C(x) = \dfrac{1}{4}\ln(x^4+1) + 1000.$

Another Example

- Integrate $\int \sqrt{x^3 - 3x}(x^2 - 1)dx$;
  Substitute $u = x^3 - 3x$; Compute the derivative
  $\frac{du}{dx} = (x^3 - 3x)' = 3x^2 - 3 = 3(x^2 - 1)$; Multiply both sides by dx:
  $du = 3(x^2 - 1)dx$; Therefore, we get
  $$\frac{1}{3}du = (x^2 - 1)dx;$$
  Go back to the integral and perform a careful substitution:
  $$\begin{aligned} \int \sqrt{x^3 - 3x}(x^2 - 1)dx &= \int \sqrt{u}\frac{1}{3}du = \frac{1}{3}\int u^{1/2}du \\ &= \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{9}\sqrt{(x^3 - 3x)^3} + C. \end{aligned}$$

Yet Another Example

- Integrate $\int e^{\sqrt{x}} x^{-1/2} dx$;

  Substitute $u = \sqrt{x}$; Compute the derivative $\frac{du}{dx} = (x^{1/2})' = \frac{1}{2} x^{-1/2}$; Multiply both sides by dx: $du = \frac{1}{2} x^{-1/2} dx$; Therefore, we get

$$2du = x^{-1/2} dx;$$

  Go back to the integral and perform a careful substitution:

$$\begin{aligned} \int e^{\sqrt{x}} x^{-1/2} dx &= \int e^u 2du = 2 \int e^u du \\ &= 2e^u + C = 2e^{\sqrt{x}} + C. \end{aligned}$$

Definite Integration By Substitution

- Integrate $\int_4^5 \frac{1}{3-x} dx$;

  Substitute $u = 3 - x$; Compute the derivative $\frac{du}{dx} = (3-x)' = -1$; Multiply both sides by dx: $du = -dx$ or $-du = dx$; Also, for $x = 4$, we get $u = 3 - 4 = -1$ and for $x = 5$, we get $u = 3 - 5 = -2$; Go back to the integral and perform a careful substitution:

$$\begin{aligned} \int_4^5 \frac{1}{3-x} dx &= \int_{-1}^{-2} \frac{1}{u}(-du) \\ &= -\int_{-1}^{-2} \frac{1}{u} du \\ &= -(\ln |u|)\Big|_{-1}^{-2} \\ &= -(\ln 2 - \ln 1) = -\ln 2. \end{aligned}$$

Application: Total Pollution from the Rate

- A lake is being polluted at the rate of $r(t) = 400te^{t^2}$ tons of pollutants per year, where t is number of years since measurements began; Find the total amount of pollutants discharged into the lake during the first 2 years;

  We must compute $P(t) = \int_0^2 r(t)dt = \int_0^2 400te^{t^2}dt$;

  Substitute $u = t^2$; Compute the derivative $\frac{du}{dt} = (t^2)' = 2t$; Multiply both sides by dt: $du = 2tdt$ or $\frac{1}{2}du = tdt$; Also, for $t = 0$, we get $u = 0^2 = 0$ and for $t = 2$, we get $u = 2^2 = 4$; Go back to the integral and perform a careful substitution:

$$\begin{aligned}\int_0^2 400te^{t^2}dt &= \int_0^4 400e^u \tfrac{1}{2}du = 200\int_0^4 e^u du = 200\,(e^u)\big|_0^4 \\ &= 200(e^4 - e^0) = 200(e^4 - 1) \text{ tons.}\end{aligned}$$

Application: Average Water Depth

- After x months the water level in a reservoir is $L(x) = 40x(x^2+9)^{-1/2}$ feet; Find the average depth during the first 4 months;

  We need $\text{AV}_{[0,4]}(L) = \dfrac{1}{4-0}\displaystyle\int_0^4 L(x)dx = \frac{1}{4}\int_0^4 40x(x^2+9)^{-1/2}dx$;

  Substitute $u = x^2+9$; Compute the derivative $\dfrac{du}{dx} = (x^2+9)' = 2x$;

  Multiply both sides by dx: $du = 2xdx$ or $\frac{1}{2}du = xdx$; Also, for $x = 0$, we get $u = 0^2+9 = 9$ and for $x = 4$, we get $u = 4^2+9 = 25$; So

$$\begin{aligned}\text{AV}_{[0,4]}(L) &= \frac{1}{4}\int_0^4 40x(x^2+9)^{-1/2}dx = 10\int_9^{25} u^{-1/2}\tfrac{1}{2}du \\ &= 5\int_9^{25} u^{-1/2}du = 5\left.(2\sqrt{u})\right|_9^{25} \\ &= 10(\sqrt{25}-\sqrt{9}) = 20 \text{ feet.}\end{aligned}$$