Business and Life Calculus

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LSSU Math 112

1. Integration Techniques and Differential Equations
   - Integration By Parts
   - Improper Integrals
   - Differential Equations

Subsection 1

Integration By Parts

Integration By Parts

- Recall the Product Rule for Derivatives:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x);$$

Integrate both sides with respect to x:

$$\int (f(x)g(x))' dx = \int [f'(x)g(x) + f(x)g'(x)] dx;$$

Since integration is the reverse operation of differentiation, we have

$$\int (f(x)g(x))' dx = f(x)g(x);$$

Moreover, because of the sum rule for integrals:

$$\int [f'(x)g(x) + f(x)g'(x)] dx = \int f'(x)g(x) dx + \int f(x)g'(x) dx;$$

Putting all these together:

$$f(x)g(x) = \int f'(x)g(x) dx + \int f(x)g'(x) dx;$$

Finally, subtract to get the **Integration By Parts Formula**:

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx.$$

Alternative Form

- We came up with the formula

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

- Use two new variables u and v as follows: Set

$$\begin{array}{ll} u = g(x) & du = g'(x)dx \\ v = f(x) & dv = f'(x)dx \end{array}$$

- Now substitute into the formula above to get the uv-form of the **By Parts Rule**:

$$\int u dv = uv - \int v du.$$

Example I (fg-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}\int xe^x dx &= \int x(e^x)' dx \\ &= xe^x - \int (x)' e^x dx \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C.\end{aligned}$$

Example I (uv-form)

$$\int udv = uv - \int vdu;$$

We want to compute $\int xe^x dx$;

Set $u = x$ and $dv = e^x dx$; Then $\frac{du}{dx} = 1 \Rightarrow du = dx$; Moreover $\frac{dv}{dx} = e^x \Rightarrow v = e^x$;

$$\begin{aligned} \int xe^x dx &= \int udv \\ &= uv - \int vdu \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C. \end{aligned}$$

Example II (*fg*-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int x^7 \ln x dx &= \int (\frac{1}{8}x^8)' \ln x dx \\
&= \frac{1}{8}x^8 \ln x - \int \frac{1}{8}x^8 (\ln x)' dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^8 \cdot \frac{1}{x} dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7 dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8} \cdot \frac{1}{8}x^8 + C \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C.
\end{aligned}$$

Example II (uv-form)

$$\int u dv = uv - \int v du;$$

We want to compute $\int x^7 \ln x dx$;

Set $u = \ln x$ and $dv = x^7 dx$; Then $\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$; Moreover $\frac{dv}{dx} = x^7 \Rightarrow v = \frac{1}{8}x^8$;

$$\begin{aligned} \int x^7 \ln x dx &= \int u dv = uv - \int v du \\ &= \frac{1}{8}x^8 \ln x - \int \frac{1}{8}x^8 \cdot \frac{1}{x} dx = \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7 dx \\ &= \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C. \end{aligned}$$

Example III

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int (x-2)(x+4)^8 dx &= \int (x-2)[\frac{1}{9}(x+4)^9]' dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \int (x-2)' \frac{1}{9}(x+4)^9 dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{9}\int (x+4)^9 dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{9} \cdot \frac{1}{10}(x+4)^{10} + C \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{90}(x+4)^{10} + C.
\end{aligned}$$

Example IV: Integral of $\ln x$

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int \ln x dx &= \int (x)' \ln x dx \\
&= x \ln x - \int x(\ln x)' dx \\
&= x \ln x - \int x \cdot \frac{1}{x} dx \\
&= x \ln x - \int dx \\
&= x \ln x - x + C;
\end{aligned}$$

Present Value of Continuous Stream of Income

Present Value of Continuous Income Stream

If $C(t)$ is a continuous stream of income in dollars per year, where t is number of years from now for a period of T years, its present value assuming continuous interest rate r is given by

$$P = \int_0^T C(t)e^{-rt}dt;$$

- Example: Assume a business generates income at the rate of $2t$ million dollars per year, where t is number of years from now; What is present value of this continuous stream for the next 10 years at a continuous interest rate of 5% ?

$$C(t) = 2t, \quad T = 10, \quad r = 0.05;$$

Example (Cont'd)

$$P = \int_0^T C(t)e^{-rt}dt = \int_0^{10} 2te^{-0.05t}dt =$$
$$\int_0^{10} 2t(-20e^{-0.05t})'dt =$$
$$[2t(-20e^{-0.05t})]\Big|_0^{10} - \int_0^{10}(2t)'(-20e^{-0.05t})dt =$$
$$-40\,(te^{-0.05t})\Big|_0^{10} - \int_0^{10} -40e^{-0.05t}dt =$$
$$-40\,(te^{-0.05t})\Big|_0^{10} + 40\int_0^{10} e^{-0.05t}dt =$$
$$-40\,(te^{-0.05t})\Big|_0^{10} + 40\,(-20e^{-0.05t})\Big|_0^{10} =$$
$$-40\,(te^{-0.05t})\Big|_0^{10} - 800\,(e^{-0.05t})\Big|_0^{10} =$$
$$-40 \cdot 10e^{-0.5} - 800(e^{-0.5} - 1) =$$
$$800 - 1200e^{-0.5} \approx 72.16 \text{ million.}$$

Subsection 2

Improper Integrals

Limits as $x \to \pm\infty$

Limits as $x \to \infty$

- $\lim_{x\to\infty} \frac{1}{x^n} = 0$, if $n > 0$;
- $\lim_{x\to\infty} e^{-ax} = 0$, if $a > 0$;

Limits as $x \to -\infty$

- $\lim_{x\to-\infty} \frac{1}{x^n} = 0$, if $n > 0$;
- $\lim_{x\to-\infty} e^{ax} = 0$, if $a > 0$;

Limits Diverging

- $\lim_{x\to\infty} x^n = +\infty$, if $n > 0$;
- $\lim_{x\to\infty} e^{ax} = +\infty$, if $a > 0$;
- $\lim_{x\to\infty} \ln x = +\infty$.

Examples of Limits

- Evaluate the following limits:
  - $\lim_{b\to\infty} \frac{1}{b^2} = 0$;
  - $\lim_{b\to\infty} (3 - \frac{1}{b}) = 3 - 0 = 3$;
  - $\lim_{b\to\infty} (e^{-2b} - 5) = 0 - 5 = -5$;
  - $\lim_{b\to\infty} (b^3) = +\infty$;
  - $\lim_{b\to\infty} (\sqrt{b} - 1) = +\infty$;
  - $\lim_{b\to\infty} (1 - \frac{1}{b}) = 1 - 0 = 1$;
  - $\lim_{b\to\infty} (\sqrt[3]{b} + 3) = +\infty$.

Improper Integrals

- Evaluate the improper integral $\int_1^\infty \frac{1}{x^2} dx$;

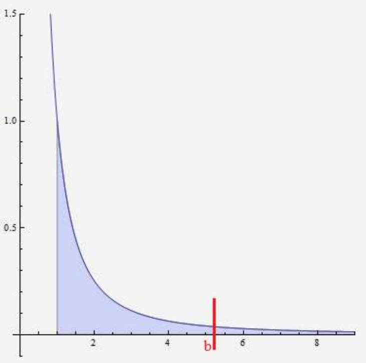
  The given integral represents the area of the unbounded shaded region in the graph below:

  To compute it, we introduce an artificial finite upper bound b; Then we compute the ordinary integral

  $$\int_1^b \frac{1}{x^2} dx;$$

  Finally, we take the limit

  $$\lim_{b\to\infty} \int_1^b \frac{1}{x^2} dx.$$



Evaluating Improper Integrals

We continue with the details of computing $\int_1^\infty \frac{1}{x^2}dx$:

$$\int_1^b \frac{1}{x^2}dx = \int_1^b x^{-2}dx = -\frac{1}{x}\Big|_1^b = 1 - \frac{1}{b};$$

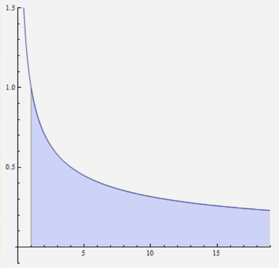
Therefore, we get

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}(1 - \frac{1}{b}) = 1.$$

Another Example of Evaluating Improper Integrals

- Evaluate the improper integral $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$;

$$\int_1^b \frac{1}{\sqrt{x}} dx =$$
$$\int_1^b x^{-1/2} dx =$$
$$2\sqrt{x}\Big|_1^b =$$
$$2\sqrt{b} - 2;$$



$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty} (2\sqrt{b} - 2) = +\infty.$$

Application: Permanent Endowment

- The size of the fund necessary to generate \$2,000 annually forever at 5% interest rate compounded continuously is $\int_0^\infty 2000e^{-0.05t}dt$; Evaluate the integral to find the size of this permanent endowment;

$$\int_0^b 2000e^{-0.05t}dt = 2000\int_0^b e^{-0.05t}dt = \\ 2000\left(-20e^{-0.05t}\right)\Big|_0^b = -40,000(e^{-0.05b}-1);$$

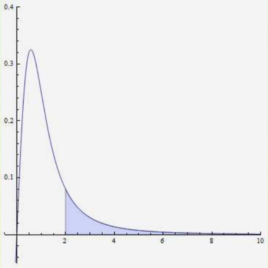
Therefore, we get

$$\int_1^\infty 2000e^{-0.05t}dt = \lim_{b\to\infty}\int_1^b 2000e^{-0.05t}dt = \\ \lim_{b\to\infty}(-40,000(e^{-0.05b}-1)) = 40,000.$$

Using Substitutions

- Evaluate the improper integral $\int_2^\infty \frac{x}{(x^2+1)^2} dx$;
  Set $u = x^2 + 1$; Then $du = 2xdx \Rightarrow \frac{1}{2}du = xdx$ and $x = 2 \Rightarrow u = 5$ and $x = b \Rightarrow u = b^2 + 1$;

$$\int_2^b \frac{x}{(x^2+1)^2} dx = \int_5^{b^2+1} \frac{1}{2} \frac{1}{u^2} du = -\frac{1}{2} \frac{1}{u}\Big|_5^{b^2+1} = -\frac{1}{2}\left(\frac{1}{b^2+1} - \frac{1}{5}\right);$$



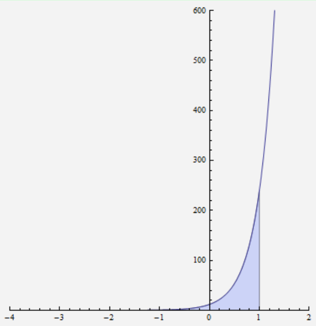
$$\int_2^\infty \frac{x}{(x^2+1)^2} dx = \lim_{b\to\infty} \int_2^b \frac{x}{(x^2+1)^2} dx = \lim_{b\to\infty} \left(\frac{1}{10} - \frac{1}{2(b^2+1)}\right) = \frac{1}{10}.$$

Integrating to $-\infty$

- Evaluate the improper integral $\int_{-\infty}^{1} 12e^{3x}\,dx$;

Recall that $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$;

$$\int_b^1 12e^{3x}\,dx = 12\int_b^1 e^{3x}\,dx =$$
$$12\,\frac{1}{3}e^{3x}\Big|_b^1 = 4(e^3 - e^{3b});$$



$$\int_{-\infty}^{1} 12e^{3x}\,dx = \lim_{b\to-\infty}\int_b^1 12e^{3x}\,dx = \lim_{b\to-\infty}(4(e^3 - e^{3b})) = 4e^3.$$

Subsection 3

Differential Equations

Differential Equations

- A **differential equation** is an equation involving derivatives;
- A simple example of a differential equation is the equation $y' = 2x$;
- To solve this equation, we integrate both sides with respect to x:

$$\int y' dx = \int 2x dx \quad \Rightarrow \quad y = x^2 + C;$$

- $y = x^2 + C$ is called the **general solution** of the differential equation $y' = 2x$ because varying C gives *all* solutions of the equation;
- For particular values of C, we obtain the **particular solutions** of the equation;
- For example, $y = x^2 + 2, y = x^2$ and $y = x^2 - 5$ are all particular solutions of $y' = 2x$ (for $C = 2$, $C = 0$ and $C = -5$, respectively).

Verifying Solutions I

- Example: Verify that $y = e^{2x} + e^{-x} - 1$ is a solution of the differential equation

$$y'' - y' - 2y = 2;$$

Compute

$$\begin{aligned} y' &= (e^{2x} + e^{-x} - 1)' = 2e^{2x} - e^{-x}; \\ y'' &= (2e^{2x} - e^{-x})' = 4e^{2x} + e^{-x}; \end{aligned}$$

Hence we have

$$\begin{aligned} y'' - y' - 2y &= (4e^{2x} + e^{-x}) - (2e^{2x} - e^{-x}) - 2(e^{2x} + e^{-x} - 1) \\ &= 4e^{2x} + e^{-x} - 2e^{2x} + e^{-x} - 2e^{2x} - 2e^{-x} + 2 \\ &= 2; \end{aligned}$$

This verifies the differential equation.

Verifying Solutions II

- Example: Verify that $y = e^{-x} + e^{3x}$ is a solution of the differential equation

$$y'' - 2y' - 3y = 0;$$

Compute

$$\begin{aligned} y' &= (e^{-x} + e^{3x})' = -e^{-x} + 3e^{3x}; \\ y'' &= (-e^{-x} + 3e^{3x})' = e^{-x} + 9e^{3x}; \end{aligned}$$

Hence we have

$$\begin{aligned} y'' - 2y' - 3y &= (e^{-x} + 9e^{3x}) - 2(-e^{-x} + 3e^{3x}) - 3(e^{-x} + e^{3x}) \\ &= e^{-x} + 9e^{3x} + 2e^{-x} - 6e^{3x} - 3e^{-x} - 3e^{3x} \\ &= 0; \end{aligned}$$

This verifies the differential equation.

Separable Differential Equations

- A differential equation involving *only* the first-derivative is called a **first-order differential equation**;

Separable Differential Equations

A first-order differential equation is **separable** if it can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)},$$

for some functions $f(x)$ and $g(y)$, with $g(y) \neq 0$; To solve it

1. Eliminate denominators by multiplying both sides by $g(y)dx$:

   $$g(y)dy = f(x)dx;$$

   This step is also called **separating variables**;

2. Integrate:

   $$\int g(y)dy = \int f(x)dx.$$

Separation of Variables

- Find the general solution of the differential equation

$$\frac{dy}{dx} = 2xy^2;$$

$$\begin{aligned} \frac{dy}{dx} = 2xy^2 & \\ \Rightarrow \quad & \frac{1}{y^2}dy = 2xdx \\ \Rightarrow \quad & \int \frac{1}{y^2}dy = \int 2xdx \\ \Rightarrow \quad & -\frac{1}{y} = x^2 + C \\ \Rightarrow \quad & y = -\frac{1}{x^2 + C}. \end{aligned}$$

Finding a Particular Solution I

- Solve the differential equation $y' = \frac{6x}{y^2}$ with the initial condition $y(1) = 2$;

  We first find the general solution:

$$\frac{dy}{dx} = \frac{6x}{y^2} \quad \Rightarrow \quad y^2 dy = 6x dx$$
$$\Rightarrow \quad \int y^2 dy = \int 6x dx \quad \Rightarrow \quad \frac{1}{3} y^3 = 3x^2 + c$$
$$\Rightarrow \quad y^3 = 9x^2 + 3c \quad \Rightarrow \quad y = \sqrt[3]{9x^2 + C};$$

  Since $y(1) = 2$, we get

$$2 = \sqrt[3]{9 \cdot 1^2 + C} \quad \Rightarrow \quad 8 = 9 + C \quad \Rightarrow \quad C = -1;$$

  Therefore, the particular solution is $y = \sqrt[3]{9x^2 - 1}$.

Finding a Particular Solution II

- Solve the differential equation $y' = \frac{6x^2}{y^4}$ with the initial condition $y(0) = 2$;

  We first find the general solution:

$$\begin{aligned} \frac{dy}{dx} = \frac{6x^2}{y^4} \quad &\Rightarrow \quad y^4 dy = 6x^2 dx \\ &\Rightarrow \quad \int y^4 dy = \int 6x^2 dx \quad \Rightarrow \quad \frac{1}{5}y^5 = 2x^3 + c \\ &\Rightarrow \quad y^5 = 10x^3 + 5c \quad \Rightarrow \quad y = \sqrt[5]{10x^3 + C}; \end{aligned}$$

  Since $y(0) = 2$, we get

$$2 = \sqrt[5]{10 \cdot 0^3 + C} \quad \Rightarrow \quad 32 = 0 + C \quad \Rightarrow \quad C = 32;$$

  Therefore, the particular solution is $y = \sqrt[5]{10x^3 + 32}$.

Finding a Particular Solution III

- Solve the differential equation $y' = xy$ with the initial condition $y(0) = 2$ (assume $y > 0$);

  We first find the general solution:

$$\frac{dy}{dx} = xy \quad \Rightarrow \quad \frac{1}{y}dy = xdx \quad \Rightarrow \quad \int \frac{1}{y}dy = \int xdx$$
$$\Rightarrow \quad \ln y = \frac{1}{2}x^2 + c \quad \Rightarrow \quad y = e^{\frac{1}{2}x^2 + c}$$
$$\Rightarrow \quad y = e^c e^{x^2/2} \quad \Rightarrow \quad y = Ce^{x^2/2};$$

  Since $y(0) = 2$, we get

$$2 = Ce^{\frac{1}{2}\cdot 0^2} \quad \Rightarrow \quad 2 = C \cdot 1 \quad \Rightarrow \quad C = 2;$$

  Therefore, the particular solution is $y = 2e^{x^2/2}$.

Finding a Particular Solution IV

- Solve the differential equation $y' = x^2 y$ with the initial condition $y(0) = 5$ (assume $y > 0$);

  We first find the general solution:

$$\frac{dy}{dx} = x^2 y \quad \Rightarrow \quad \frac{1}{y} dy = x^2 dx \quad \Rightarrow \quad \int \frac{1}{y} dy = \int x^2 dx$$
$$\Rightarrow \quad \ln y = \frac{1}{3} x^3 + c \quad \Rightarrow \quad y = e^{\frac{1}{3}x^3 + c}$$
$$\Rightarrow \quad y = e^c e^{x^3/3} \quad \Rightarrow \quad y = Ce^{x^3/3};$$

  Since $y(0) = 5$, we get

$$5 = Ce^{\frac{1}{3} \cdot 0^3} \quad \Rightarrow \quad 5 = C \cdot 1 \quad \Rightarrow \quad C = 5;$$

  Therefore, the particular solution is $y = 5e^{x^3/3}$.

Finding a General Solution

- Find the general solution of the differential equation

$$yy' - x = 0;$$

$$\begin{aligned} y\frac{dy}{dx} - x = 0 & \\ \Rightarrow \quad & y\frac{dy}{dx} = x \\ \Rightarrow \quad & ydy = xdx \\ \Rightarrow \quad & \int ydy = \int xdx \\ \Rightarrow \quad & \frac{1}{2}y^2 = \frac{1}{2}x^2 + c \\ \Rightarrow \quad & y^2 = x^2 + 2c \\ \Rightarrow \quad & y^2 = x^2 + C. \end{aligned}$$

Using Substitution

- Solve the differential equation $y' = xy - x$ with the initial condition $y(0) = 4$ (assume $y > 1$);

  We first find the general solution:

$$\frac{dy}{dx} = xy - x \quad \Rightarrow \quad \frac{dy}{dx} = x(y-1) \quad \Rightarrow \quad \frac{1}{y-1}dy = xdx$$
$$\Rightarrow \quad \int \frac{1}{y-1}dy = \int xdx \quad \Rightarrow \quad \ln(y-1) = \frac{1}{2}x^2 + c$$
$$\Rightarrow \quad y - 1 = e^{\frac{1}{2}x^2 + c} \quad \Rightarrow \quad y = e^c e^{x^2/2} + 1$$
$$\Rightarrow \quad y = Ce^{x^2/2} + 1;$$

  Since $y(0) = 4$, we get
  $4 = Ce^{\frac{1}{2} \cdot 0^2} + 1 \quad \Rightarrow \quad 3 = C \cdot 1 \quad \Rightarrow \quad C = 3;$
  Therefore, the particular solution is $y = 3e^{x^2/2} + 1$.

Application: Predicting Wealth

- Suppose we have saved \$ 5,000 and expect to have an additional \$3,000 during each year; If we deposit these savings into an account paying 5% interest compounded continuously, What will the balance be after t years?
  Let $y(t)$ be the amount in the account at time t; Then, the rate of change of y in thousands of dollars per year is

$$\begin{aligned} \frac{dy}{dt} = 3 + 0.05y \quad &\Rightarrow \quad \frac{1}{0.05y+3} dy = dt \\ \Rightarrow \quad \int \frac{1}{0.05y+3} dy = \int dt \quad &\Rightarrow \quad \frac{1}{0.05} \ln(0.05y+3) = t + c \\ \Rightarrow \quad \ln(0.05y+3) = 0.05t + c' \quad &\Rightarrow \quad 0.05y + 3 = e^{0.05t+c'} \\ \Rightarrow \quad 0.05y = e^{c'} e^{0.05t} - 3 \quad &\Rightarrow \quad y = 20c'' e^{0.05t} - 60 \\ \Rightarrow \quad y = Ce^{0.05t} - 60; & \end{aligned}$$

  Since $y(0) = 5$, we get $5 = Ce^0 - 60 \Rightarrow C = 65$; Therefore $y(t) = 65e^{0.05t} - 60$.

Application: Appreciation of a Building

- Suppose that $y(t)$ is the value of a commercial building in millions of dollars after t years;
  - Write a differential equation saying that the rate of growth of the value equals two times the one-half power of its present value;
  $$\frac{dy}{dt} = 2 \cdot y^{1/2};$$
  - Write an initial condition that says that at time 0 the value is \$9 million;
  $$y(0) = 9;$$
  - Solve the differential equation and initial condition:
  $$\frac{dy}{dt} = 2y^{1/2} \quad \Rightarrow \quad \frac{1}{y^{1/2}} dy = 2dt \quad \Rightarrow \quad \int y^{-1/2} dy = \int 2dt$$
  $$\Rightarrow \quad 2\sqrt{y} = 2t + c \quad \Rightarrow \quad \sqrt{y} = t + C \quad \Rightarrow \quad y = (t + C)^2;$$
  Since $y(0) = 9$, we get $9 = (0 + C)^2 \Rightarrow C = 3$; So $y(t) = (t + 3)^2$.