Business and Life Calculus

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 112

Subsection 1

Functions of Several Variables

Functions of Two Variables

Functions of Two Variables

A function f of two variables is a rule that assigns to each ordered pair (x, y) in the domain of f a unique number $f(x, y)$;

- As with functions of a single variable, if the function is specified by a formula, the domain is taken to be the largest set of ordered pairs for which the formula is defined;
- Example: Suppose $f(x, y) = \dfrac{\sqrt{x}}{y^2}$; Find the domain and the value $f(9, -1)$;

  We must have $x \geq 0$ and $y \neq 0$; Therefore, the domain is the set

$$\text{Dom}(f) = \{(x, y) : x \geq 0, y \neq 0\};$$

Finally, $f(9, -1) = \dfrac{\sqrt{9}}{(-1)^2} = 3$;

Another Example

- Example: Suppose $f(x,y) = e^{xy} - \ln x$; Find the domain and the value $f(1,2)$;

  We must have $x > 0$; Therefore, the domain is the set

$$\mathrm{Dom}(f) = \{(x,y) : x > 0\};$$

  Finally, $f(1,2) = e^{1\cdot 2} - \ln 1 = e^2$;

An Applied Example

- It costs \$100 to a bike company to make a three-speed bike and \$150 to make a ten-speed bike; The company's fixed costs are \$2,500; Find the company's cost function and use it to compute the cost of producing 15 three-speed and 20 ten-speed bikes;

  Suppose that x is the number of 3-speed and y the number of 10-speed bikes that the company produces; Then

$$C(x,y) = \underbrace{100x}_{\text{3-speed cost}} + \underbrace{150y}_{\text{10-speed cost}} + \underbrace{2500}_{\text{fixed costs}};$$

Thus, the cost for producing 15 3-speed and 20 10-speed bikes is

$$C(15,20) = 100 \cdot 15 + 150 \cdot 20 + 2500 = \$7,000;$$

Functions of Three or More Variables

- In analogy with functions of two variables one may define functions of three or more variables:

$$V(l, w, h) = l \cdot w \cdot h; \quad \text{(Volume of a rectangular solid)}$$
$$A(P, r, t) = Pe^{rt}; \quad \text{(Future Value in Continuous Compounding)}$$
$$f(x, y, z, w) = \frac{x + y + z + w}{4}; \quad \text{(Average Value)}$$

- Example: Let $f(x, y, z) = \frac{\sqrt{x}}{y} + \ln \frac{1}{z}$; Find the domain and the value $f(4, -1, 1)$;

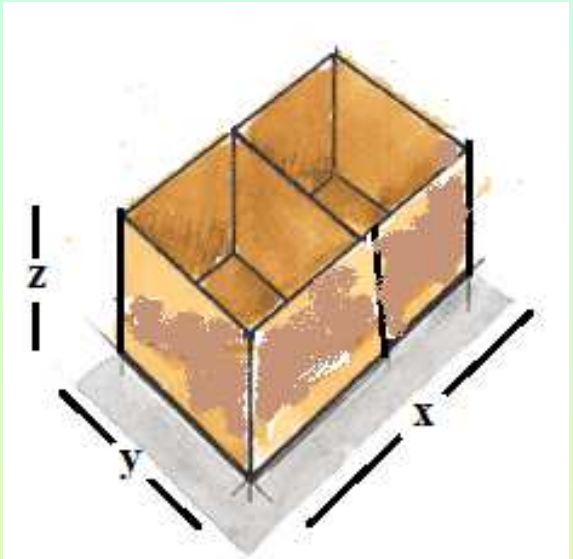
We must have $x \geq 0$, $y \neq 0$ and $z > 0$; Therefore, the domain is the set

$$\text{Dom}(f) = \{(x, y, z) : x \geq 0, y \neq 0, z > 0\};$$

Finally, $f(4, -1, 1) = \frac{\sqrt{4}}{-1} + \ln \frac{1}{1} = -2;$

Volume and Area of a Divided Box

- An open top box is to have a center divider, as shown in the diagram; Find formulas for the volume V of the box and the total amount of material M needed to construct the box;



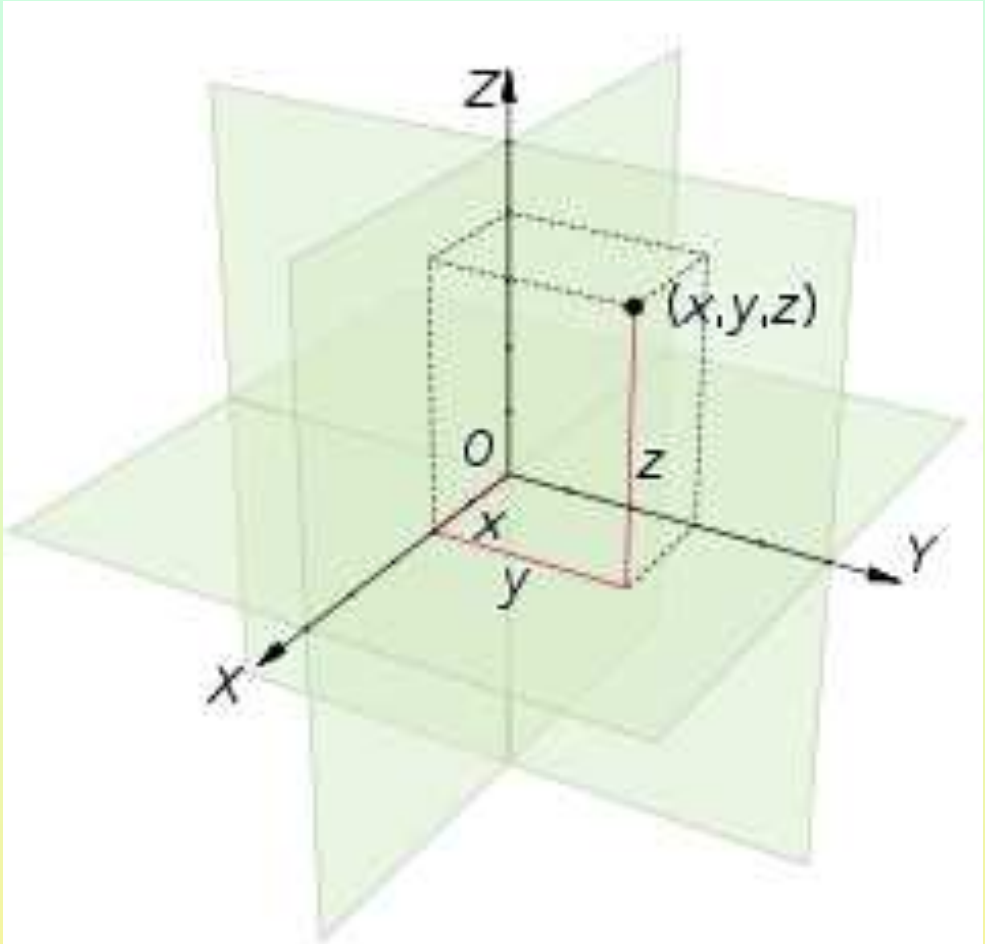
Suppose that x, y and z are the dimensions as shown in the diagram; Then, the volume is

$$V(x, y, z) = xyz;$$

The amount of material, calculated as the surface area, is given by

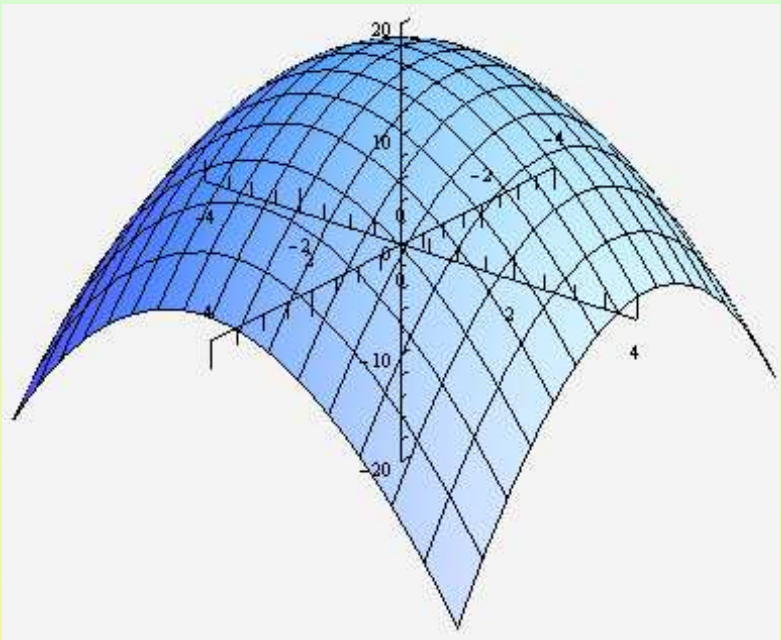
$$M(x, y, z) = \underbrace{xy}_{\text{bottom}} + \underbrace{2xz}_{\text{back and front}} + \underbrace{3yz}_{\text{sides and divider}};$$

Three-Dimensional Coordinate System



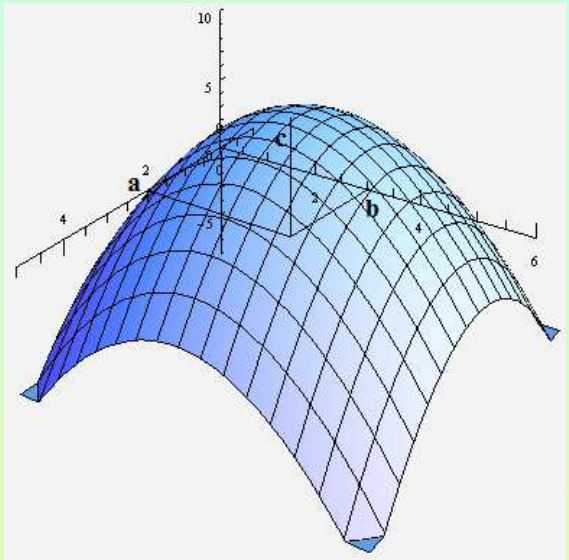
Graphs of Functions of Two Variables

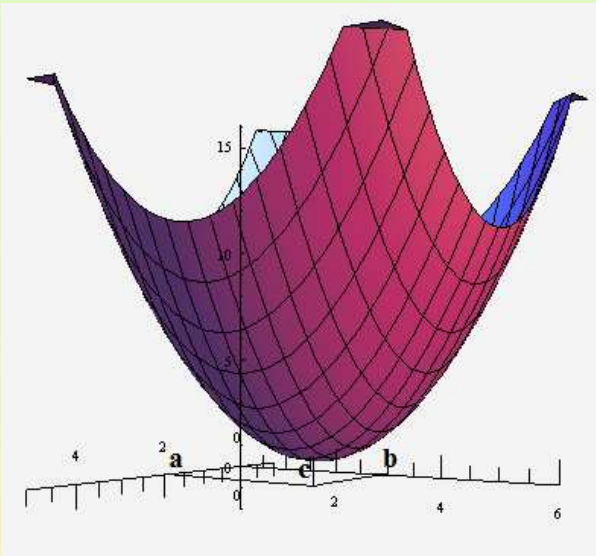
- Suppose we want to sketch the graph of $f(x, y) = 18 - x^2 - y^2$;
- We may first look at the cross-sections:
  - For $x = c$, $z = (18 - c^2) - y^2$ is a parabola opening down;
  - For $y = c$, $z = (18 - c^2) - x^2$ is also a parabola opening down;
  - For $z = c$, $x^2 + y^2 = 18 - c$ is a circle centered at the origin;



Relative Extreme Points

- A point (a, b, c) on a surface $z = f(x, y)$ is a **relative maximum point** if $f(a, b) \geq f(x, y)$, for all (x, y) in some region surrounding (a, b);

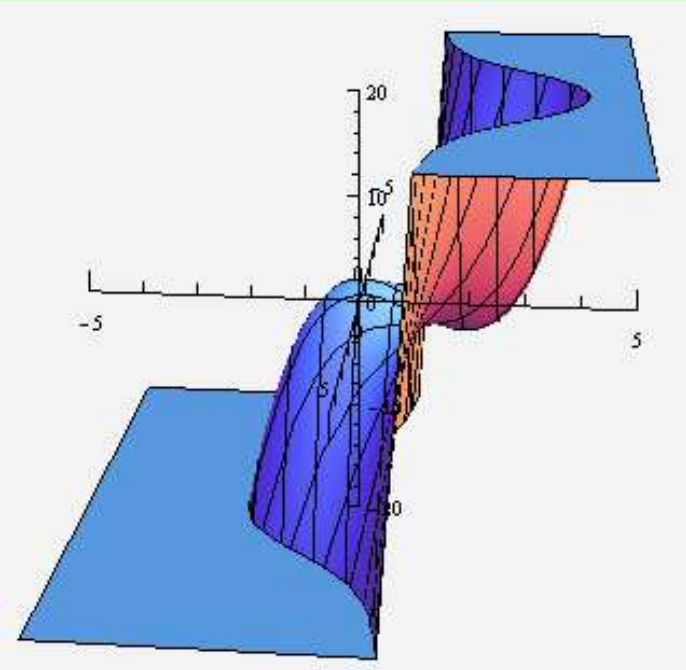


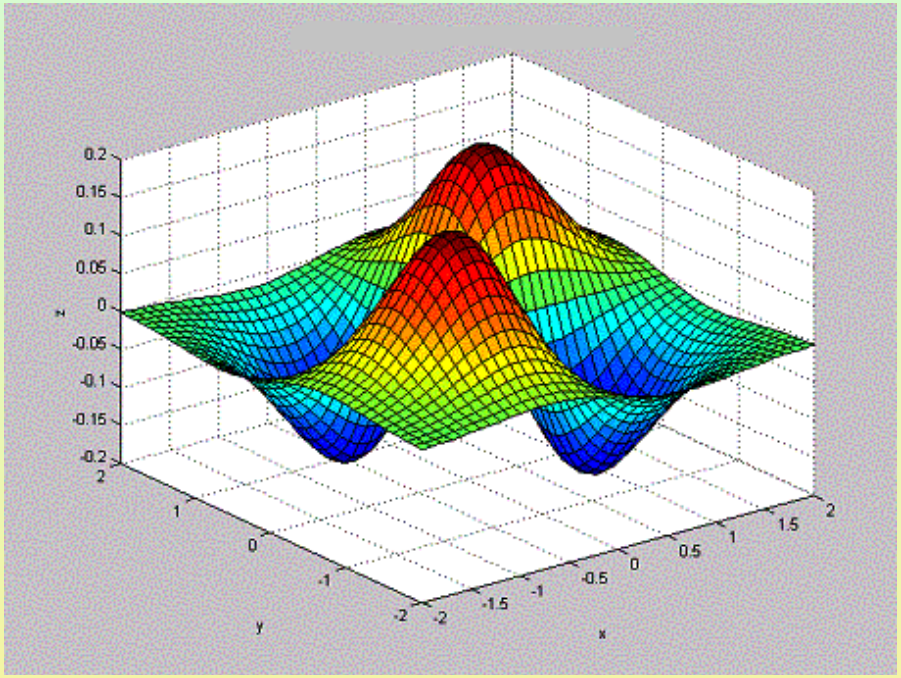


- A point (a, b, c) on a surface $z = f(x, y)$ is a **relative minimum point** if $f(a, b) \leq f(x, y)$, for all (x, y) in some region surrounding (a, b);

Relative Extrema

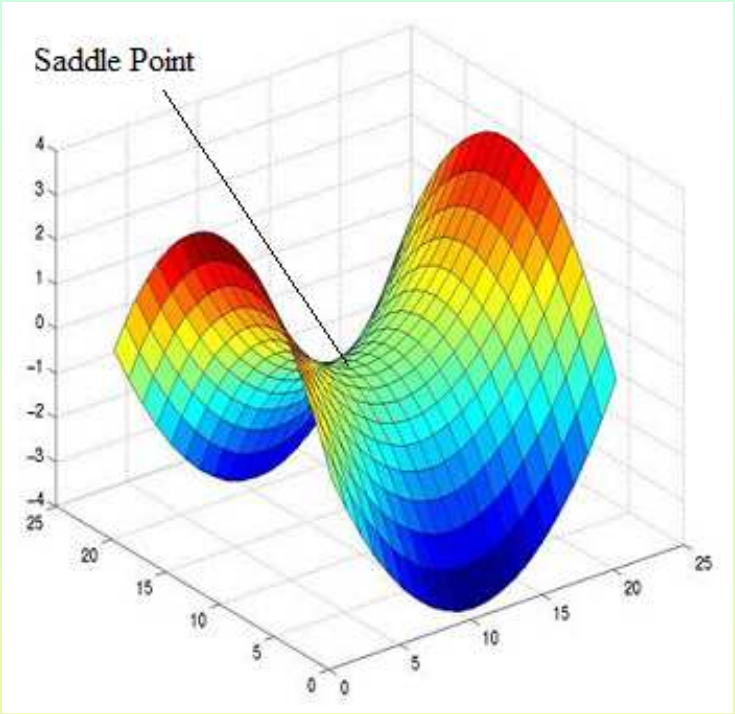
- Needless to say a function may have both relative maxima and relative minima at various points of its domain:





Saddle Points

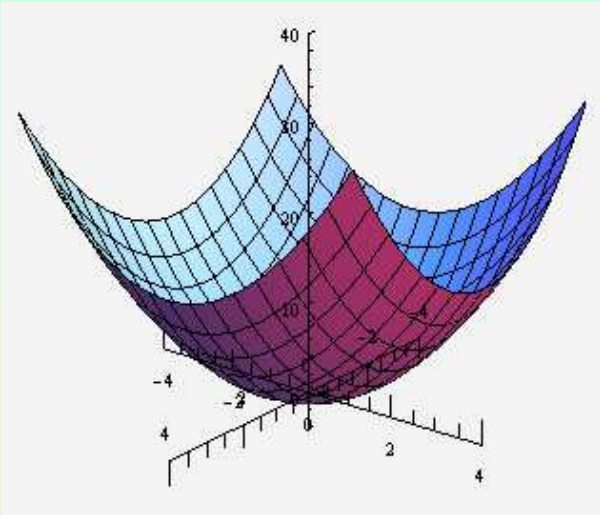
- A point like the one shown on the right is called a **saddle point**;
- It is the highest point along one curve on the surface and the lowest along another curve;
- Saddle points are neither maxima nor minima;



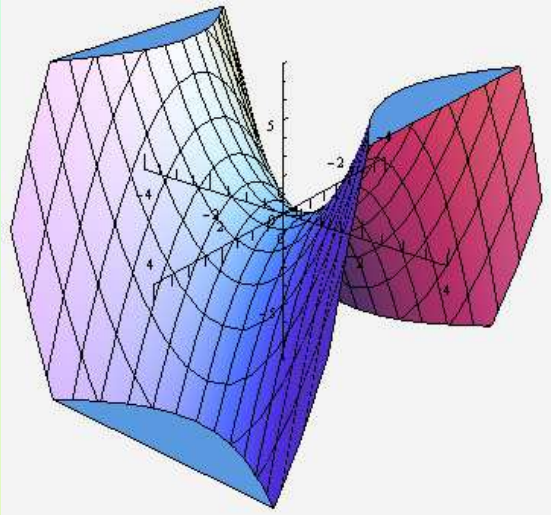
- Intuitively speaking, we think of
  - relative maxima as "hilltops";
  - relative minima as "valley bottoms";
  - saddle points as "mountain passes" between two peaks;

Gallery of Various Cases

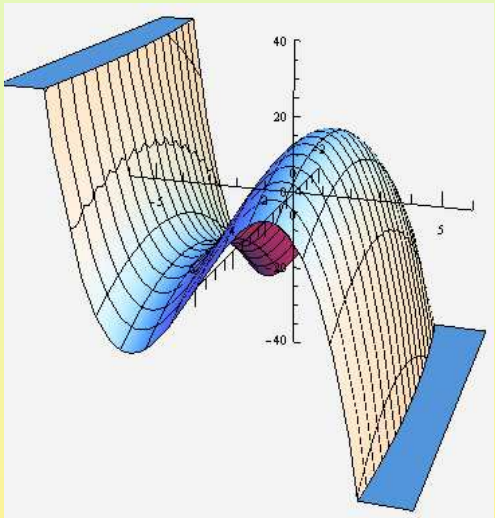
$f(x,y) = x^2 + y^2$;



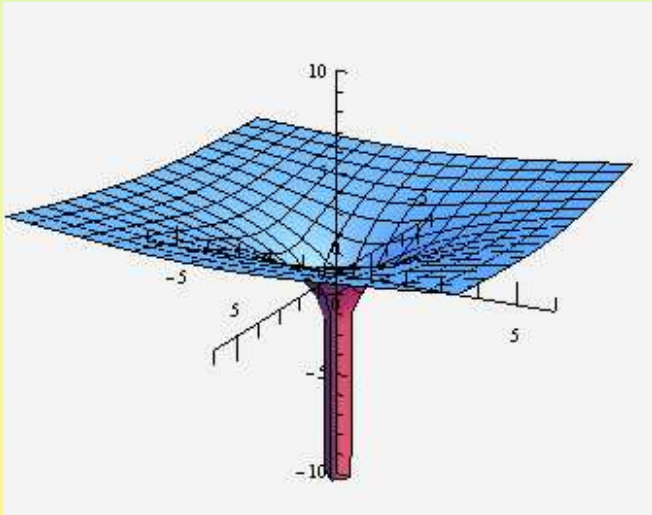
$f(x,y) = y^2 - x^2$;



$f(x,y) = 12y + 6x - x^2 - y^3$;



$f(x,y) = \ln(x^2 + y^2)$;



Subsection 2

Partial Derivatives

Partial Derivatives

- A function $f(x, y)$ has two partial derivatives, one with respect to x and the other with respect to y;

Partial Derivatives

- The **partial derivative of f with respect to** x is defined by
$$\frac{\partial}{\partial x} f(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h};$$
- The **partial derivative of f with respect to** y is defined by
$$\frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h};$$

- To compute
  - $\frac{\partial}{\partial x} f(x, y)$ we take the derivative of f with respect to x assuming that y is constant;
  - $\frac{\partial}{\partial y} f(x, y)$ we take the derivative of f with respect to y assuming that x is constant;

Computing Partial Derivatives

- Compute the following partial derivatives:
  - $\frac{\partial}{\partial x}x^3y^4 = y^4\frac{\partial}{\partial x}x^3 = 3x^2y^4;$
  - $\frac{\partial}{\partial y}x^3y^4 = x^3\frac{\partial}{\partial y}y^4 = 4x^3y^3;$
  - $\frac{\partial}{\partial x}x^4y^2 = y^2\frac{\partial}{\partial x}x^4 = 4x^3y^2;$
  - $\frac{\partial}{\partial y}x^4y^2 = x^4\frac{\partial}{\partial y}y^2 = 2x^4y;$
  - $\frac{\partial}{\partial x}(2x^4 - 3x^3y^3 - y^2 + 4x + 1) =$
    $\frac{\partial}{\partial x}2x^4 - \frac{\partial}{\partial x}3x^3y^3 - \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial x}4x + \frac{\partial}{\partial x}1 =$
    $8x^3 - 9x^2y^3 - 0 + 4 + 0 = 8x^3 - 9x^2y^3 + 4;$

Subscript Notation

- The following is an alternative notation for partial derivatives using subscripts:

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) \quad \text{and} \quad f_y(x,y) = \frac{\partial}{\partial y} f(x,y);$$

- Example: Compute the partial derivatives:
  - $f_x(x,y)$, if $f(x,y) = 5x^4 - 2x^2y^3 - 4y^2$;

$$f_x(x,y) = \frac{\partial}{\partial x} 5x^4 - \frac{\partial}{\partial x} 2x^2y^3 - \frac{\partial}{\partial x} 4y^2 = 20x^3 - 4xy^3;$$

  - Both partials of $f(x,y) = e^x \ln y$;

$$f_x(x,y) = e^x \ln y; \qquad f_y(x,y) = \frac{e^x}{y};$$

  - $f_y(x,y)$ if $f(x,y) = (xy^2+1)^4$;

$$f_y(x,y) = \frac{\partial}{\partial y}[(xy^2+1)^4] = 4(xy^2+1)^3 \cdot \frac{\partial}{\partial y}(xy^2+1) =$$
$$4(xy^2+1)^3 \cdot 2xy = 8xy(xy^2+1)^3;$$

More Partial Derivatives

- Compute the partial derivatives:
  - $\frac{\partial g}{\partial x}$ if $g(x,y)=\frac{xy}{x^2+y^2}$;

    $\frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right)=\frac{\frac{\partial(xy)}{\partial x}(x^2+y^2)-xy\frac{\partial(x^2+y^2)}{\partial x}}{(x^2+y^2)^2}=$

    $\frac{y(x^2+y^2)-xy\cdot 2x}{(x^2+y^2)^2}=\frac{x^2y+y^3-2x^2y}{(x^2+y^2)^2}=\frac{y^3-x^2y}{(x^2+y^2)^2}$;
  - $f_x(x,y)$ if $f(x,y)=\ln(x^2+y^2)$;

    $f_x(x,y)=\frac{\partial}{\partial x}(\ln(x^2+y^2))=\frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2)=\frac{2x}{x^2+y^2}$;
  - $f_y(1,3)$ if $f(x,y)=e^{x^2+y^2}$;

    $f_y(x,y)=\frac{\partial}{\partial y}(e^{x^2+y^2})=e^{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2)=2ye^{x^2+y^2}$; Thus,

    $f_y(1,3)=2\cdot 3e^{1^2+3^2}=6e^{10}$;

Partial Derivatives in Three or More Variables

- Compute the partial derivatives:
  - $\frac{\partial}{\partial x}(x^3y^4z^5) = y^4z^5\frac{\partial}{\partial x}x^3 = 3x^2y^4z^5$;
  - $\frac{\partial}{\partial y}(x^3y^4z^5) = x^3z^5\frac{\partial}{\partial y}y^4 = 4x^3y^3z^5$;
  - $\frac{\partial}{\partial z}e^{x^2+y^2+z^2} = e^{x^2+y^2+z^2}\frac{\partial}{\partial z}(x^2+y^2+z^2) = 2ze^{x^2+y^2+z^2}$;

Partial Derivatives as Rates of Change and Marginals

Partials as Rates of Change

- The partial $f_x(x, y)$ represents the instantaneous rate of change of f with respect to x when y is held constant;
- The partial $f_y(x, y)$ represents the instantaneous rate of change of f with respect to y when x is held constant;

Partials as Marginals

Suppose $C(x, y)$ is the cost function for producing x units of product A and y units of product B; Then

- $C_x(x, y)$ is the marginal cost function for product A, when production of B is held constant;
- $C_y(x, y)$ is the marginal cost function for product B, when production of A is held constant;

An Application

- A company's profit from producing x radios and y televisions per day is $P(x,y) = 4x^{3/2} + 6y^{3/2} + xy$;
  - Find the marginal profit functions;

$$P_x(x,y) = \frac{\partial}{\partial x}4x^{3/2} + \frac{\partial}{\partial x}6y^{3/2} + \frac{\partial}{\partial x}xy = 4\cdot\tfrac{3}{2}x^{1/2} + 0 + y = 6x^{1/2} + y;$$

$$P_y(x,y) = \frac{\partial}{\partial y}4x^{3/2} + \frac{\partial}{\partial y}6y^{3/2} + \frac{\partial}{\partial y}xy = 0 + 6\cdot\tfrac{3}{2}y^{1/2} + x = 9y^{1/2} + x;$$

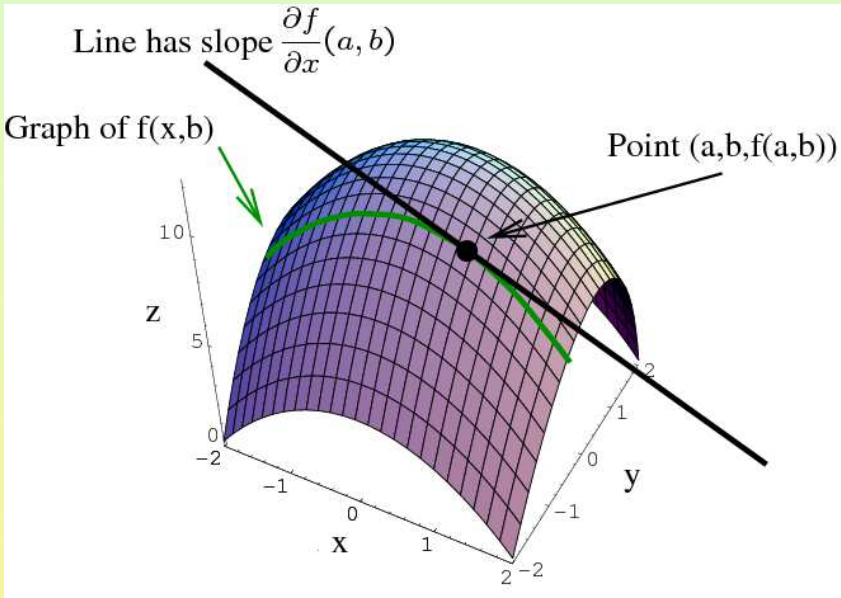
  - Find and interpret $P_y(25,36)$;

$$P_y(25,36) = 9\sqrt{36} + 25 = 79;$$

This is the approximate increase in profit per additional television produced when 25 radios and 36 televisions are produced;

Partial Derivatives Geometrically

- The equation $y = f(x, y)$ of a function in two variables represents a surface in three-dimensional space;
- Its partial derivatives represent the slopes of the tangent lines to the surface in different directions;
- For instance, $f_x(a, b)$ represents the slope of the tangent line at the point (a, b) in the x direction, when a cross-section of the surface on the plane $y = b$ (y held constant) is considered;



Higher-Order Partial Derivatives

Second-Order Partial Derivatives

Subscript	∂-Notation	Description
f_{xx}	$\frac{\partial^2}{\partial x^2} f$	Differentiate Twice w.r.t. x
f_{yy}	$\frac{\partial^2}{\partial y^2} f$	Differentiate Twice w.r.t. y
f_{xy}	$\frac{\partial^2}{\partial y \partial x} f$	Differentiate First w.r.t. x, Then w.r.t. y
f_{yx}	$\frac{\partial^2}{\partial x \partial y} f$	Differentiate First w.r.t. y, Then w.r.t. x

- Note that in both notations, we differentiate first with respect to the variable appearing closest to f;
- To calculate a second partial, we must perform a two-step calculation;

Computing Second-Order Partial Derivatives

- Find all second-order partial derivatives of $f(x,y) = x^4 + 2x^2y^2 + x^3y + y^4$;
  We must first compute the two first-order partial derivatives:

$$f_x(x,y) = \frac{\partial}{\partial x}(x^4 + 2x^2y^2 + x^3y + y^4) = 4x^3 + 4xy^2 + 3x^2y$$
$$f_y(x,y) = \frac{\partial}{\partial y}(x^4 + 2x^2y^2 + x^3y + y^4) = 4x^2y + x^3 + 4y^3;$$

  Now we proceed with the four second-order partial derivatives:

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(4x^3 + 4xy^2 + 3x^2y) = 12x^2 + 4y^2 + 6xy$$
$$f_{xy}(x,y) = \frac{\partial}{\partial y}(4x^3 + 4xy^2 + 3x^2y) = 8xy + 3x^2$$
$$f_{yx}(x,y) = \frac{\partial}{\partial x}(4x^2y + x^3 + 4y^3) = 8xy + 3x^2$$
$$f_{yy}(x,y) = \frac{\partial}{\partial y}(4x^2y + x^3 + 4y^3) = 4x^2 + 12y^2;$$

Some Remarks on Second Partial Derivarives

- Note that $f_{xy} = f_{yx}$;
- Even though this is not true for all functions, it holds for those that we will be dealing with;
- It is also true for most functions arising from applications;
- Example: Calculate the second derivatives of $f(x, y) = 2x^3 e^{-5y}$;

$$f_x(x, y) = 6x^2 e^{-5y} \quad \text{and} \quad f_y(x, y) = 2x^3(-5e^{-5y}) = -10x^3 e^{-5y};$$

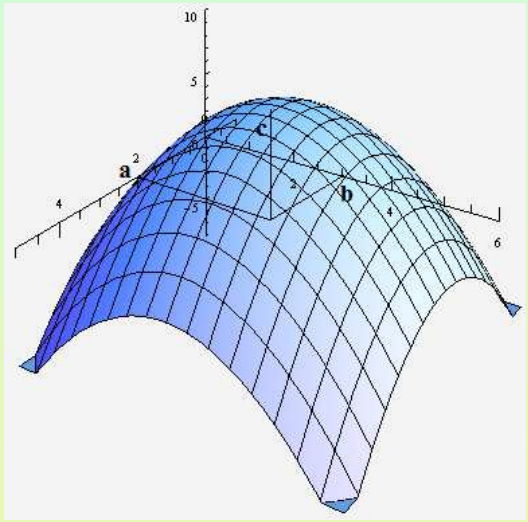
$$\begin{aligned} f_{xx}(x, y) &= 12xe^{-5y} \\ f_{xy}(x, y) &= 6x^2(-5e^{-5y}) = -30x^2 e^{-5y} \\ f_{yx}(x, y) &= -30x^2 e^{-5y} \\ f_{yy}(x, y) &= -10x^3(-5e^{-5y}) = 50x^3 e^{-5y}; \end{aligned}$$

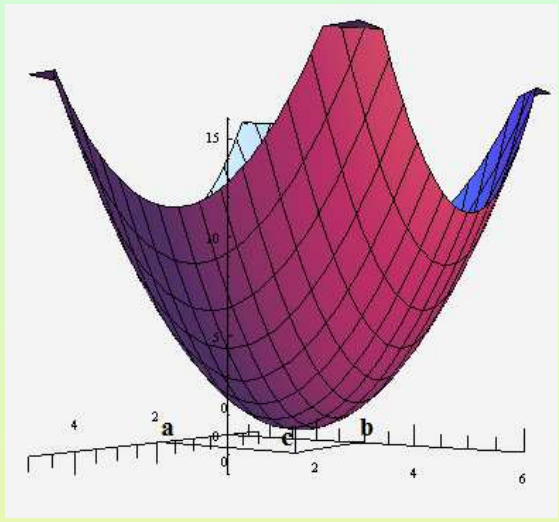
Subsection 3

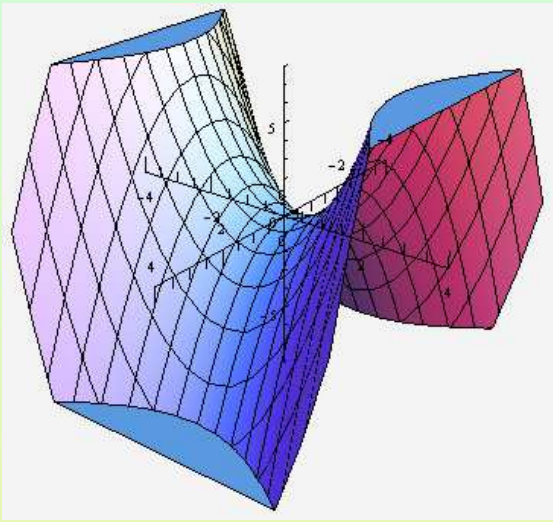
Optimizing Functions of Several Variables

Critical Points

- Recall the concepts of relative maxima, relative minima and saddle points for functions of two variables:







- Relative max and min values can occur only at **critical points**, i.e., points (a, b) where

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0;$$

Finding Critical Points

- Find all critical points of $f(x,y) = 3x^2 + y^2 + 3xy + 3x + y + 6$;
  Compute first-order partials:

$$f_x(x,y) = 6x + 3y + 3 \quad \text{and} \quad f_y(x,y) = 2y + 3x + 1;$$

Set first-order partials equal to zero and solve the resulting system for (x,y):

$$\left\{ \begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 6x+3y+3 & = & 0 \\ 2y+3x+1 & = & 0 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} 2x+y & = & -1 \\ 3x+2y & = & -1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} y & = & -2x-1 \\ 3x+2(-2x-1) & = & -1 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} y & = & -2x-1 \\ -x-2 & = & -1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} y & = & 1 \\ x & = & -1 \end{array} \right\}$$

Thus, the only critical point is $(x,y) = (-1,1)$;

Second Derivative Test: The D-Test

D-Test

Suppose (a, b) is a critical point of f and $D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$; Then, f at the point (a, b) has a:

i. relative maximum if $D > 0$ and $f_{xx}(a, b) < 0$;

ii. relative minimum if $D > 0$ and $f_{xx}(a, b) > 0$;

iii. saddle point if $D < 0$;

- Some Remarks Concerning D-Test:
  1. First, find all critical points; Then apply D-test to each critical point;
  2. $D > 0$ guarantees a relative extremum; Value of f_{xx} tells what kind it is;
  3. $D < 0$ means saddle point regardless of sign of f_{xx};
  4. If $D = 0$, the D-test is inconclusive; Function may have a maximum, minimum or saddle point at the critical point;

Finding Relative Extrema of Polynomial Functions

- Find the relative extrema of $f(x,y) = 2x^2 + y^2 + 2xy + 4x + 2y + 5$;
  First for critical points:

$$f_x(x,y) = 4x + 2y + 4 \quad \text{and} \quad f_y(x,y) = 2y + 2x + 2;$$

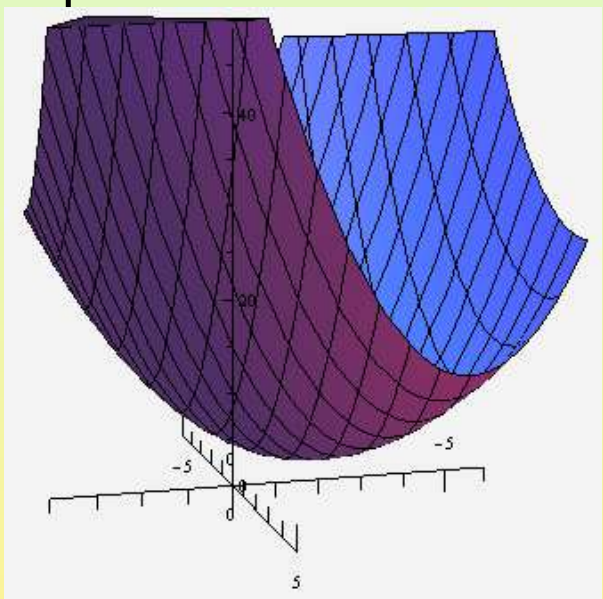
$$\left\{ \begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 4x + 2y + 4 & = & 0 \\ 2y + 2x + 2 & = & 0 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} 2x + y & = & -2 \\ x + y & = & -1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & -1 \\ y & = & 0 \end{array} \right\}$$

Compute

$$f_{xx} = 4, \quad f_{xy} = 2, \quad f_{yy} = 2;$$

Thus, $D = f_{xx}f_{yy} - f_{xy}^2 = 4 \cdot 2 - 2^2 = 4 > 0$ and $f_{xx} = 4 > 0$, which show that at $(x,y) = (-1,0)$ f has a relative minimum;



Finding Relative Extrema of Exponential Functions

- Find the relative extrema of $f(x,y) = e^{x^2-y^2}$;
  First for the critical points:

$$f_x(x,y) = 2xe^{x^2-y^2} \quad \text{and} \quad f_y(x,y) = -2ye^{x^2-y^2};$$

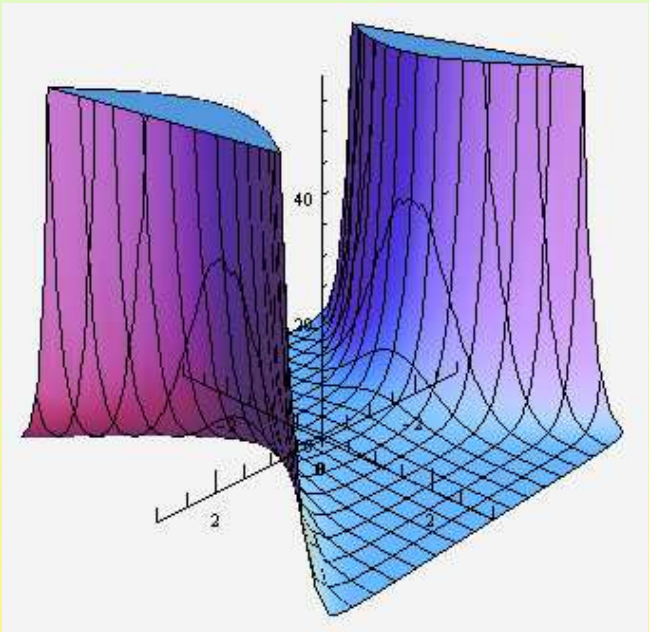
$$\left\{ \begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 2xe^{x^2-y^2} & = & 0 \\ -2ye^{x^2-y^2} & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & 0 \\ y & = & 0 \end{array} \right\}$$

Compute $f_{xx} = 2e^{x^2-y^2} + 4x^2e^{x^2-y^2}$, $f_{xy} = -4xye^{x^2-y^2}$, $f_{yy} = -2e^{x^2-y^2} + 4y^2e^{x^2-y^2}$;

Thus,

$$D = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2$$
$$= 2 \cdot (-2) - 0^0$$
$$= -4 < 0,$$

which shows that at $(x,y) = (0,0)$ f has a saddle point;



Application: Maximizing Profit

- A motor company makes compact and midsized cars. The price function for compacts is $p = 17 - 2x$ (for $0 \leq x \leq 8$) and for midsized $q = 20 - y$ (for $0 \leq y \leq 20$), both in thousands of dollars, where x, y are the number of compact and midsized cars produced per hour; Assume that the company's cost function is $C(x, y) = 15x + 16y - 2xy + 5$ thousand dollars; How many of each type of car should be produced and how should each be priced to maximize the company's profit? What will be the maximum profit? First find the profit function

$$\begin{aligned} P(x,y) &= \underbrace{R(x,y)}_{\text{Revenue}} - \underbrace{C(x,y)}_{\text{Cost}} \\ &= xp + yq - (15x + 16y - 2xy + 5) \\ &= x(17 - 2x) + y(20 - y) - (15x + 16y - 2xy + 5) \\ &= 17x - 2x^2 + 20y - y^2 - 15x - 16y + 2xy - 5 \\ &= -2x^2 - y^2 + 2xy + 2x + 4y - 5; \end{aligned}$$

Application: Maximizing Profit (Cont'd)

$$P(x,y) = -2x^2 - y^2 + 2xy + 2x + 4y - 5;$$

Compute first derivatives

$$P_x(x,y) = -4x + 2y + 2 \quad \text{and} \quad P_y(x,y) = -2y + 2x + 4;$$

Find critical points:

$$\left\{ \begin{array}{rcl} P_x(x,y) & = & 0 \\ P_y(x,y) & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} -4x + 2y + 2 & = & 0 \\ -2y + 2x + 4 & = & 0 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} -2x + y & = & -1 \\ x - y & = & -2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & 3 \\ y & = & 5 \end{array} \right\}$$

Thus $(x,y) = (3,5)$ is the critical point;

Application: Maximizing Profit (Cont'd)

$$P(x,y) = -2x^2 - y^2 + 2xy + 2x + 4y - 5;$$
$$P_x(x,y) = -4x + 2y + 2;$$
$$P_y(x,y) = -2y + 2x + 4;$$

Finally, we verify that at $(3,5)$ we indeed have a local max; We have

$$P_{xx} = -4, \quad P_{xy} = 2, \quad P_{yy} = -2;$$

Therefore, $D = P_{xx}P_{yy} - P_{xy}^2 = (-4)\cdot(-2) - 2^2 = 4 > 0$ and $P_{xx} = -4 < 0$, which show that at $(3,5)$ P has a max; The prices and the maximum profit are given by

$$p = 17 - 2x = 17 - 2\cdot 3 = 11 \text{ thousand}$$
$$q = 20 - y = 20 - 5 = 15 \text{ thousand}$$
$$P = -2x^2 - y^2 + 2xy + 2x + 4y - 5 =$$
$$-2\cdot 3^2 - 5^2 + 2\cdot 3\cdot 5 + 2\cdot 3 + 4\cdot 5 - 5 = 8 \text{ thousand};$$

Finding Relative Extrema I

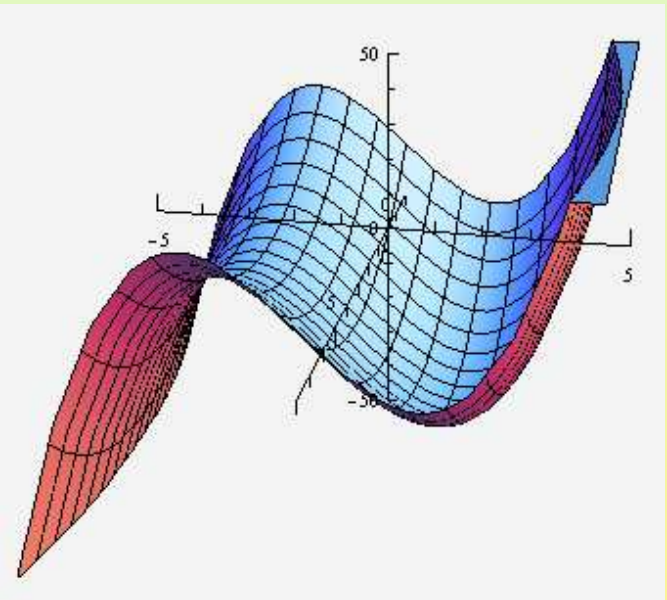
- Find the relative extrema of $f(x,y) = x^2 + y^3 - 6x - 12y$;
  First for critical points:

  $f_x(x,y) = 2x-6 = 2(x-3)$ and $f_y(x,y) = 3y^2-12 = 3(y+2)(y-2)$;

  $$\left\{ \begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 2(x-3) & = & 0 \\ 3(y+2)(y-2) & = & 0 \end{array} \right\} \Rightarrow$$

  $$\left\{ \begin{array}{l} x = 3 \\ y = -2 \text{ or } y = 2 \end{array} \right\}$$ Now compute $f_{xx} = 2, \quad f_{xy} = 0, \quad f_{yy} = 6y$;

  Thus, for $(x,y) = (3,-2)$, we get $D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 6 \cdot (-2) - 0^2 = -24 < 0$; So, this is a saddle point; For $(x,y) = (3,2)$, $D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 6 \cdot 2 - 0^2 = 24 > 0$ and $f_{xx} = 2 > 0$; so at $(x,y) = (3,2)$ f has a relative minimum;



Finding Relative Extrema II

- Find the relative extrema of $f(x,y) = 16xy - x^4 - 2y^2$;
  First for critical points:

  $f_x(x,y) = 16y - 4x^3 = 4(4y - x^3)$ and $f_y(x,y) = 16x - 4y = 4(4x - y)$;

$$\left\{ \begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 4(4y - x^3) & = & 0 \\ 4(4x - y) & = & 0 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} 4(4x) - x^3 & = & 0 \\ y & = & 4x \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x(16 - x^2) & = & 0 \\ y & = & 4x \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} x(4+x)(4-x) & = & 0 \\ y & = & 4x \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} x & = & 0 \\ y & = & 0 \end{array} \right\} \text{ or } \left\{ \begin{array}{rcl} x & = & -4 \\ y & = & -16 \end{array} \right\} \text{ or } \left\{ \begin{array}{rcl} x & = & 4 \\ y & = & 16 \end{array} \right\}$$

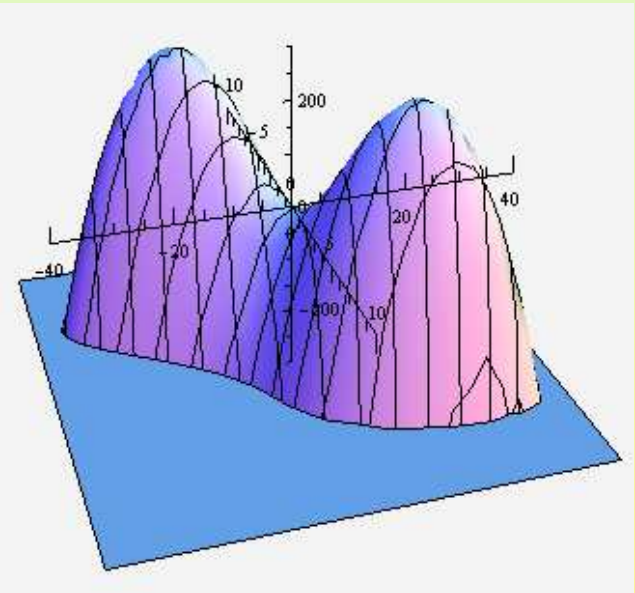
Now compute $f_{xx} = -12x^2, \quad f_{xy} = 16, \quad f_{yy} = -4;$

Finding Relative Extrema II (Cont'd)

$$f_{xx} = -12x^2, \quad f_{xy} = 16, \quad f_{yy} = -4;$$

Thus, for $(x, y) = (0, 0)$, we get $D = f_{xx}f_{yy} - f_{xy}^2 = -12 \cdot 0^2 \cdot (-4) - 16^2 = -256 < 0$; So, this is a saddle point; For $(x, y) = (-4, -16)$, $D = f_{xx}f_{yy} - f_{xy}^2 = -12 \cdot (-4)^2 \cdot (-4) - 16^2 = 512 > 0$ and $f_{xx} = -12 \cdot (-4)^2 < 0$; so at $(x, y) = (-4, -16)$ f has a relative maximum;

Finally, for $(x, y) = (4, 16)$, we get $D = f_{xx}f_{yy} - f_{xy}^2 = -12 \cdot 4^2 \cdot (-4) - 16^2 = 512 > 0$; and $f_{xx} = -12 \cdot 4^2 < 0$; So, this is a relative maximum;

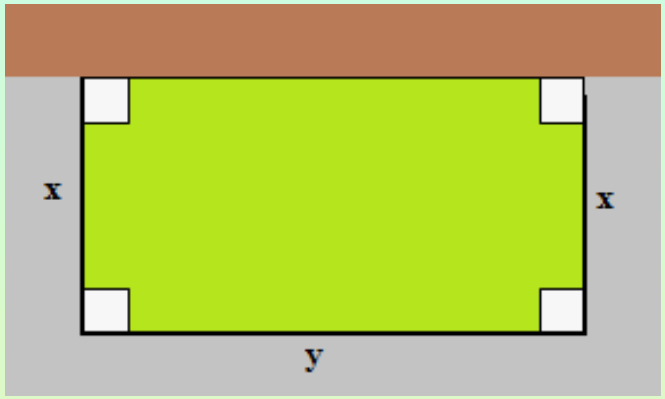


Subsection 4

Lagrange Multipliers and Constrained Optimization

Example: Maximizing Area

We want to build a rectangular enclosure along an existing stone wall; The side along the wall needs no fence; What are the dimensions of the largest enclosure that can be built using only 400 feet of fence?



Suppose that the width is x feet and the length is y feet; Since the length of the fence is $2x + y = 400$, we get the problem

$$\begin{array}{ll} \text{maximize} & A = xy \\ \text{subject to} & 2x + y - 400 = 0 \end{array}$$

Form a new function, called a **Lagrange function**,

$$\begin{aligned} F(x, y, \lambda) &= (\text{Quantity To Optimize}) + \lambda \cdot (\text{The Constraint}) \\ &= xy + \lambda(2x + y - 400) \\ &= xy + 2\lambda x + \lambda y - 400\lambda; \end{aligned}$$

Example: Maximizing Area (Cont'd)

$$F(x, y, \lambda) = xy + 2\lambda x + \lambda y - 400\lambda;$$

To optimize, compute partial derivatives and find critical points:

$$F_x = y + 2\lambda, \quad F_y = x + \lambda, \quad F_\lambda = 2x + y - 400;$$

$$\left\{ \begin{array}{rcl} F_x & = & 0 \\ F_y & = & 0 \\ F_\lambda & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} y + 2\lambda & = & 0 \\ x + \lambda & = & 0 \\ 2x + y - 400 & = & 0 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} \lambda & = & -\frac{1}{2}y \\ \lambda & = & -x \\ 2x + y - 400 & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & \frac{1}{2}y \\ 2 \cdot \frac{1}{2}y + y - 400 & = & 0 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{rcl} x & = & \frac{1}{2}y \\ 2y & = & 400 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & 100 \\ y & = & 200 \end{array} \right\}$$

The Lagrange Multiplier Method

- The function to be optimized is called the **objective function**;
- The variable λ is called the **Lagrange multiplier**;

Lagrange Multiplier Method

To optimize the function $f(x, y)$ subject to a constraint $g(x, y) = 0$:

1. Write a new function $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$;
2. Set the partial derivatives of F equal to zero: $F_x = 0, F_y = 0, F_\lambda = 0$ and solve to find the critical points;
3. The solution of the original problem (if one exists) will occur at one of these critical points;

- A possible strategy for solving the system $F_x = 0, F_y = 0, F_\lambda = 0$ could involve:
  1. Solve each of $F_x = 0, F_y = 0$ for λ;
  2. Set the two expressions for λ equal to each other;
  3. Solve the equation of Step 2 together with $F_\lambda = 0$ for x and y;

Example: Minimizing Amount of Materials

A company wants to design an aluminum can that requires the least amount of aluminum but that can hold exactly 12 fluid ounces (21.3 in^3); Find the radius r and and the height h of the can.

The objective function is the surface area of the can:

$$A = \underbrace{2\pi r^2}_{\text{top and bottom}} + \underbrace{2\pi rh}_{\text{side}};$$

The constraint has to do with the volume

$$V = 21.3 \quad \Rightarrow \quad \pi r^2 h = 21.3 \quad \Rightarrow \quad \pi r^2 h - 21.3 = 0;$$

Therefore the new function $F(r, h, \lambda)$ is

$$F(r, h, \lambda) = 2\pi r^2 + 2\pi rh + \lambda(\pi r^2 h - 21.3);$$

Example: Minimizing Amount of Materials (Cont'd)

$$F(r,h,\lambda) = 2\pi r^2 + 2\pi rh + \lambda(\pi r^2 h - 21.3);$$

Take the partial derivatives

$$F_r = 4\pi r + 2\pi h + \lambda 2\pi rh, \quad F_h = 2\pi r + \lambda \pi r^2, \quad F_\lambda = \pi r^2 h - 21.3;$$

Set these equal to zero to find critical points of F:

$$4\pi r + 2\pi h + \lambda 2\pi rh = 0, \quad 2\pi r + \lambda \pi r^2 = 0, \quad \pi r^2 h - 21.3 = 0;$$

Solve the first two for λ:

$$\lambda = -\frac{4\pi r + 2\pi h}{2\pi rh} = -\frac{2r+h}{rh}, \quad \lambda = -\frac{2\pi r}{\pi r^2} = -\frac{2}{r};$$

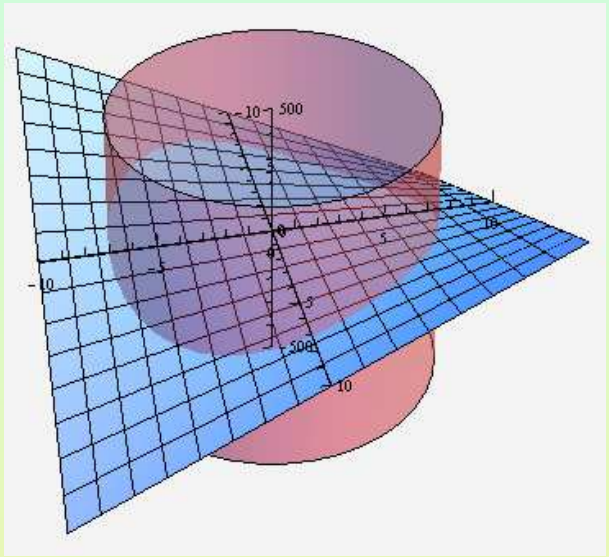
Set these equal to get

$$\frac{2r+h}{rh} = \frac{2}{r} \quad \Rightarrow \quad 2r^2 + rh = 2rh \quad \Rightarrow \quad 2r^2 = rh \quad \Rightarrow \quad 2r = h;$$

Thus, we have $\pi r^2 h = 21.3 \quad \Rightarrow \quad \pi r^2(2r) = 21.3 \quad \Rightarrow \quad 2\pi r^3 = 21.3 \quad \Rightarrow \quad r = \sqrt[3]{\frac{21.3}{2\pi}} \approx 1.5$ in; and, hence, $h \approx 3$ in.;

An Abstract Problem I

- Maximize and minimize $f(x, y) = 4xy$ subject to the constraint $x^2 + y^2 = 50$;



The objective function is $f(x, y) = 4xy$ and the constraint function is $g(x, y) = x^2 + y^2 - 50$; Thus, the new function is

$$F(x, y, \lambda) = 4xy + \lambda(x^2 + y^2 - 50);$$

Compute the three partials:

$$F_x = 4y + 2\lambda x, \quad F_y = 4x + 2\lambda y, \quad F_\lambda = x^2 + y^2 - 50;$$

Set the partials equal to zero to find the critical points:

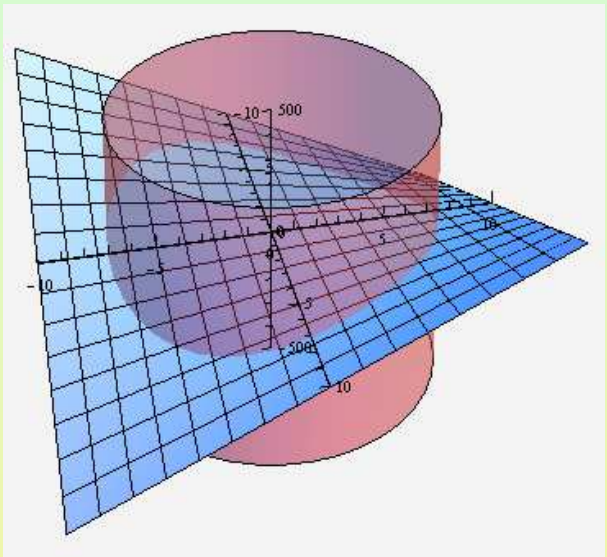
$$4y + 2\lambda x = 0, \quad 4x + 2\lambda y = 0, \quad x^2 + y^2 - 50 = 0;$$

An Abstract Problem I (Cont'd)

- We set the partials equal to zero to find the critical points:

$$4y + 2\lambda x = 0, \quad 4x + 2\lambda y = 0, \quad x^2 + y^2 - 50 = 0;$$

Therefore, $\lambda = -\frac{2y}{x}$ and $\lambda = -\frac{2x}{y}$, whence $\frac{2y}{x} = \frac{2x}{y} \quad \Rightarrow \quad x^2 = y^2 \quad \Rightarrow$ $y = \pm x$; The last equation now gives $2x^2 = 50 \quad \Rightarrow \quad x^2 = 25 \quad \Rightarrow \quad x = \pm 5$;



Thus, there are four critical points: $(-5,-5), (-5,5), (5,-5), (5,5)$; Since $f(-5,-5) = f(5,5) = 100$ and $f(-5,5) = f(5,-5) = -100$, we conclude that $f_{\max} = 100$ occurring at $(-5,-5)$ and $(5,5)$ and $f_{\min} = -100$ occurring at $(-5,5)$ and $(5,-5)$;

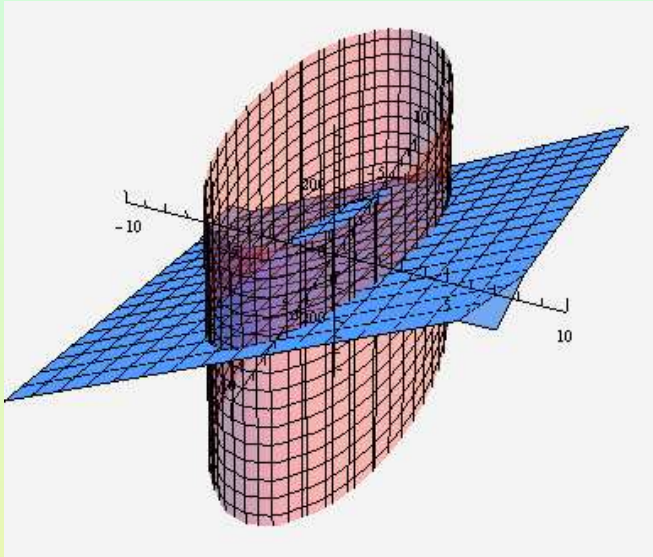
An Abstract Problem II

- Maximize and minimize $f(x,y) = 12x + 30y$ subject to the constraint $x^2 + 5y^2 = 81$;

The objective function is

$$f(x,y) = 12x + 30y$$

and the constraint function is $g(x,y) = x^2 + 5y^2 - 81$; Thus, the new function is



$F(x,y,\lambda) = 12x+30y+\lambda(x^2+5y^2-81)$;
Compute the three partials:

$$F_x = 12 + 2\lambda x, \quad F_y = 30 + 10\lambda y, \quad F_\lambda = x^2 + 5y^2 - 81;$$

Set the partials equal to zero to find the critical points:

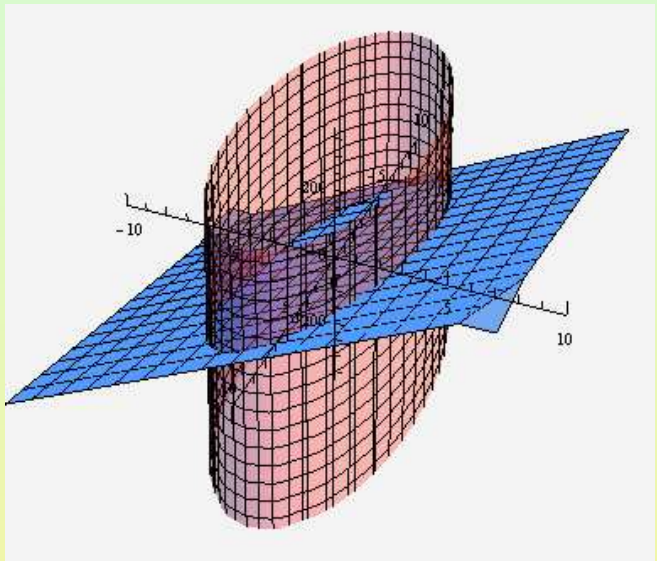
$$12 + 2\lambda x = 0, \quad 30 + 10\lambda y = 0, \quad x^2 + 5y^2 - 81 = 0;$$

An Abstract Problem II (Cont'd)

- We set the partials equal to zero to find the critical points:

$$12 + 2\lambda x = 0, \quad 30 + 10\lambda y = 0, \quad x^2 + 5y^2 - 81 = 0;$$

Therefore, $\lambda = -\frac{6}{x}$ and $\lambda = -\frac{3}{y}$, whence $\frac{6}{x} = \frac{3}{y} \Rightarrow x = 2y$; The last equation now gives $4y^2 + 5y^2 = 81 \Rightarrow 9y^2 = 81 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$;



Thus, there are two critical points: $(-6,-3), (6,3)$; Since $f(-6,-3) = -162$ and $f(6,3) = 162$, we conclude that $f_{\max} = 162$ occurring at $(6,3)$ and $f_{\min} = -162$ occurring at $(-6,-3)$;

Example: Largest Postal Service Package

The USPS will accept a package if the length plus its girth is not more than 84 inches; What are the dimensions and the volume of the largest package with a square end that can be mailed?

The objective function is the volume of the box:

$$V = xy^2;$$

The constraint has to do with the length plus girth

$$\text{Length} + \text{Girth} = 84 \quad \Rightarrow \quad x + 4y = 84 \quad \Rightarrow \quad x + 4y - 84 = 0;$$

Therefore the new function $F(x, y, \lambda)$ is

$$F(x, y, \lambda) = xy^2 + \lambda(x + 4y - 84);$$

Example: Minimizing Amount of Materials (Cont'd)

$$F(x, y, \lambda) = xy^2 + \lambda(x + 4y - 84);$$

Take the partial derivatives

$$F_x = y^2 + \lambda, \quad F_y = 2xy + 4\lambda, \quad F_\lambda = x + 4y - 84;$$

Set these equal to zero to find critical points of F:

$$y^2 + \lambda = 0, \quad 2xy + 4\lambda = 0, \quad x + 4y - 84 = 0;$$

Solve the first two for λ:

$$\lambda = -y^2, \qquad \lambda = -\frac{1}{2}xy;$$

Set these equal to get $y^2 = \frac{1}{2}xy \quad \Rightarrow \quad y = \frac{1}{2}x;$

Thus, we have $x + 2x - 84 = 0 \quad \Rightarrow \quad 3x = 84 \quad \Rightarrow \quad x = 28$ in and, hence, $y = 14$ in; Thus, the max volume is $V_{\max} = 28 \cdot 14 \cdot 14 = 5488$ in^3;