Calculus I

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 151

1. Review of Precalculus
   - Real Numbers, Functions and Graphs
   - Linear and Quadratic Functions
   - Polynomial, Rational and Algebraic Functions
   - Trigonometric Functions
   - Inverse Functions
   - Exponential and Logarithmic Functions

Subsection 1

Real Numbers, Functions and Graphs

The Absolute Value

- The **absolute value** $|a|$ of a real number a is its distance from 0 on the real line;
- Algebraically, the absolute value $|a|$ is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

- We have $|ab| = |a||b|$;
- It is not always the case that $|a+b| = |a| + |b|$; We only have $|a+b| \leq |a| + |b|$;

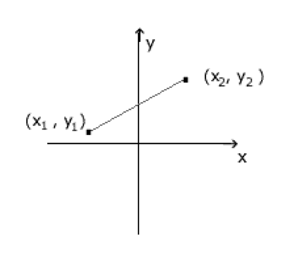
  Example: $|7 + (-5)| \neq |7| + |-5|$;

Inequalities Involving Absolute Values

- Given a real number $r > 0$, the inequality
  - $|x| < r$ has solution interval $-r < x < r$, also written $x \in (-r, r)$;
  - $|x| > r$ has solution set $x < -r$ or $x > r$; also written $x \in (-\infty, -r) \cup (r, +\infty)$;
- Given a real number c and a positive real number $r > 0$, the inequality
  - $|x - c| < r$ has solution interval $c - r < x < c + r$, also written $x \in (c - r, c + r)$;
  - $|x - c| > r$ has solution set $x < c - r$ or $x > c + r$, also written $x \in (-\infty, c - r) \cup (c + r, +\infty)$;

  Example: Solve the inequality $|7x - 2| \leq 23$;
  We have $-23 \leq 7x - 2 \leq 23$, whence $-21 \leq 7x \leq 25$ and, therefore, $-3 \leq x \leq \frac{25}{7}$. Thus $x \in [-3, \frac{25}{7}]$.

The Distance Formula



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

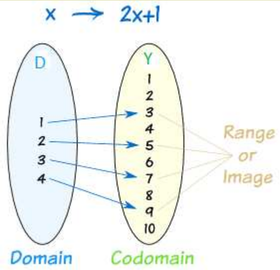
- Example: Calculate the distance between $P_1(-2, 3)$ and $P_2(5, -4)$. We have

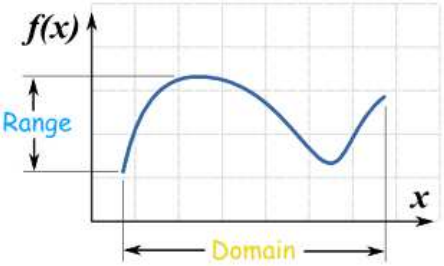
$$\begin{array}{rcl} d & = & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ & = & \sqrt{(5 - (-2))^2 + (-4 - 3)^2} \\ & = & \sqrt{7^2 + (-7)^2} \\ & = & \sqrt{2 \cdot 7^2} = 7\sqrt{2}. \end{array}$$

Definition of a Function: Domain, Range

- A **function** f from a set D to a set Y is a rule that assigns to *each* element x in D a *unique* element $y = f(x)$ in Y. Such a function is denoted by $f : D \to Y$;
- The set D is called the **domain** of f; the set of all allowable inputs;
- For every x in D, the **value** of f at x is denoted $f(x)$;
- The **range** R of f is the subset of Y consisting of all values of f;

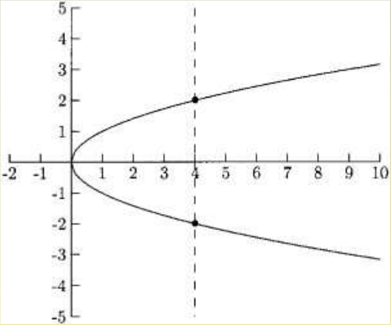
$$R = \{y \in Y : f(x) = y \text{ for some } x \in D\}.$$





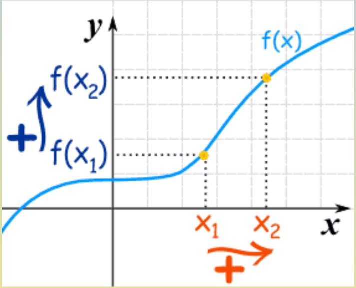
The Vertical Line Test

- The definition imposes the condition that to every x in D, there should be assigned a unique value $f(x)$ in Y;
- This implies that a curve on the plane is the graph of a function f if and only if it passes the **Vertical Line Test**, i.e., if every vertical line intersects the curve in at most one point;
- For example, the following curve is not the graph of any function:



Increasing Functions

- A function $y = f(x)$ is **increasing** on an interval $[a, b]$ if its graph goes up as we move from left to right;

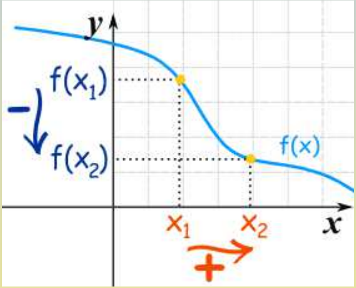


More formally, $f \nearrow [a, b]$ iff, for all x_1, x_2 in (a, b),

$$x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2).$$

Decreasing Functions

- A function $y = f(x)$ is **decreasing** on an interval $[a, b]$ if its graph goes down as we move from left to right;

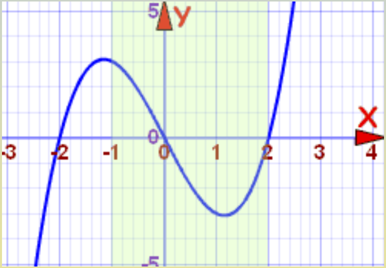


More formally, $f \searrow [a, b]$ iff, for all x_1, x_2 in (a, b),

$$x_1 < x_2 \quad \text{implies} \quad f(x_1) > f(x_2).$$

Monotonicity

- A function may be increasing over some intervals of its domain and decreasing over some other intervals;
- For example, the function f whose graph is shown here

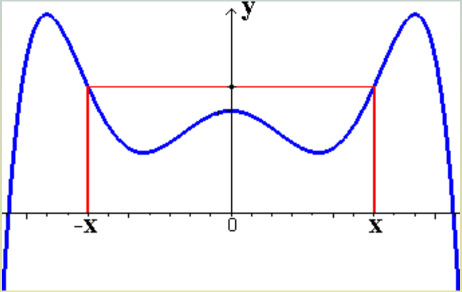


is increasing on $(-\infty, -1.2)$ and on $(1.2, +\infty)$ and decreasing on $(-1.2, 1.2)$; We summarize this behavior by writing

$$f \nearrow (-\infty, -1.2) \cup (1.2, +\infty) \quad \text{and} \quad f \searrow (-1.2, 1.2).$$

Even Functions

- A function $y = f(x)$ is **even** if its graph is symmetric with respect to the y-axis;

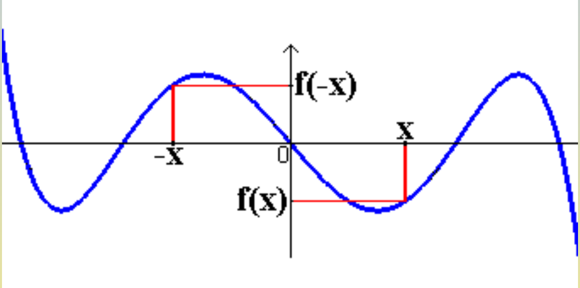


- More formally, f is even iff, for all $x \in D$, we have $-x \in D$ and

$$f(-x) = f(x);$$

Odd Functions

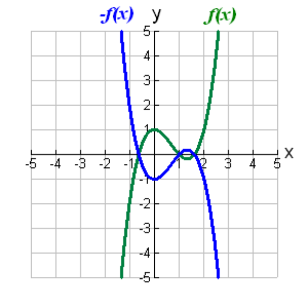
- A function $y = f(x)$ is **odd** if its graph is symmetric with respect to the origin, i.e., the point $(0,0)$;

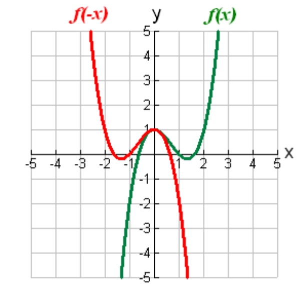


- More formally, f is odd iff, for all $x \in D$, we have $-x \in D$ and
$$f(-x) = -f(x);$$

Reflections

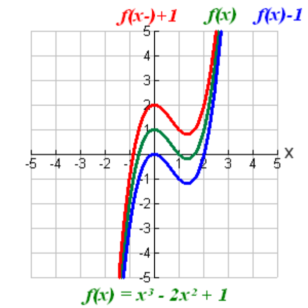
- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ with respect to the x-axis;
- The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ with respect to the y-axis;

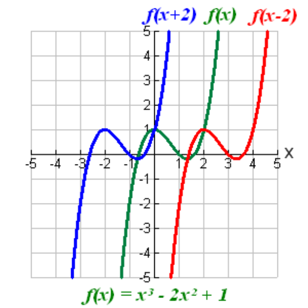




Translations (Shiftings)

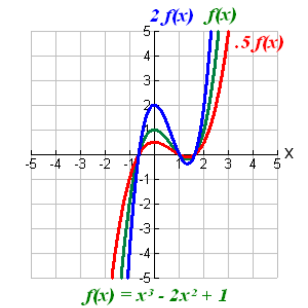
- The graph of $y = f(x) + c$ is a shift of the graph of $y = f(x)$ by $|c|$ units vertically, upward if $c > 0$ and downward if $c < 0$;
- The graph of $y = f(x + c)$ is a shift of the graph of $y = f(x)$ by $|c|$ units horizontally, to the right if $c < 0$ and to the left if $c > 0$;

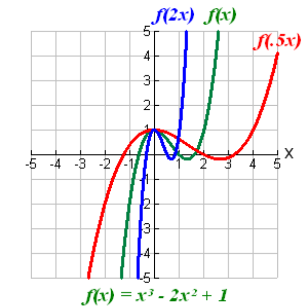




Stretchings (Scalings)

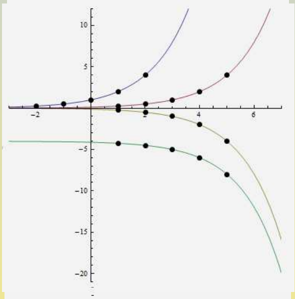
- The graph of $y = kf(x)$ is a vertical stretching of the graph of $y = f(x)$ by a scale factor of k units; It is an expansion if $k > 1$ and a compression (contraction) if $0 < k < 1$;
- The graph of $y = f(kx)$ is a horizontal stretching of the graph of $y = f(x)$ by a scale factor of k units; It is a compression (contraction) if $k > 1$ and an expansion if $0 < k < 1$;





Using Transformations to Graph

- This method of graphing starts with a known graph and through a series of reflections, shifts and, possibly, expansions/contractions leads to the graph of a function not known in advance;
- We illustrate with an example;



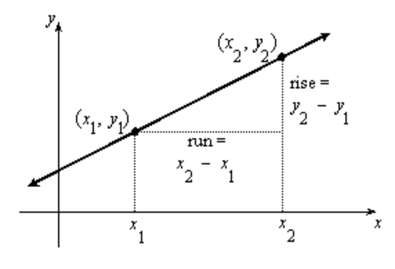
The graph of $y = 2^x$ is known to us; Suppose that, based on it, we would like to obtain the graph of $f(x) = -2^{x-3} - 4$, which is not known to us; From $y = 2^x$, by shifting 3 to the right, we get $y = 2^{x-3}$;
Then, by reflecting with respect to the x-axis, we get $y = -2^{x-3}$;
Finally, by shifting down 4, we get $y = -2^{x-3} - 4$;

Subsection 2

Linear and Quadratic Functions

The Slope of a Straight Line

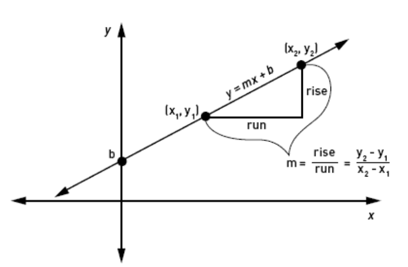
- The slope of a straight line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2.$$



Example: What is the slope of the line passing through $(-2, 10)$ and $(3, -5)$?
$$m = \frac{-5 - 10}{3 - (-2)} = \frac{-15}{5} = -3.$$

Equation of a Straight Line: Slope-Intercept Form

- If a straight line has slope m and y-intercept (point where it intersects the y-axis) $(0, b)$, then its equation is $\boxed{y = mx + b.}$

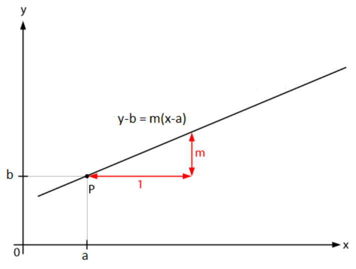


Example: Find the equation of the line passing through the points $(0, 7)$ and $(18, 1)$.

The slope is given by $m = \dfrac{1-7}{18-0} = -\dfrac{1}{3}$. The point $(0, 7)$ is the y-intercept, i.e., $b = 7$. Therefore, by the slope-intercept form, the equation of the straight line is $y = -\frac{1}{3}x + 7$.

Equation of a Straight Line: Point-Slope Form

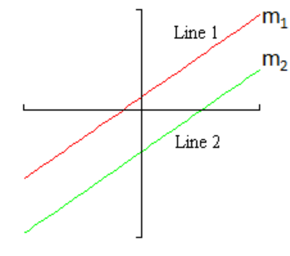
- If a straight line has slope m and passes through the point (a, b), then its equation is $\boxed{y - b = m(x - a).}$



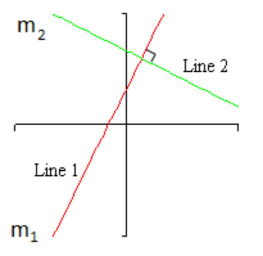
Example: Find the equation of the line passing through the points $(3, 18)$ and $(7, 2)$.
The slope is given by $m = \dfrac{2-18}{7-3} = -\dfrac{16}{4} = -4$. The point $(a, b) = (7, 2)$ is on the line. Therefore, by the point-slope form, the equation of the straight line is $y - 2 = -4(x - 7)$.

Parallel and Perpendicular Lines

- Let L_1, L_2 be lines, with slopes m_1 and m_2, respectively;
- If L_1 and L_2 are parallel (denoted $L_1 \parallel L_2$), then $m_1 = m_2$;
- If L_1 and L_2 are perpendicular (denoted $L_1 \perp L_2$), then $m_1 m_2 = -1$ (or, equivalently, $m_2 = -\frac{1}{m_1}$);



Parallel Lines



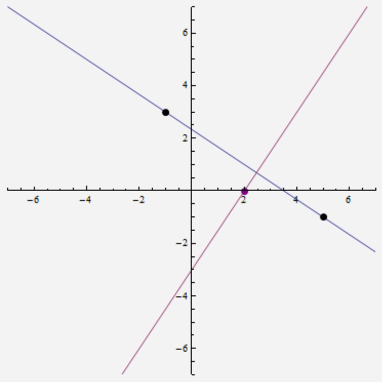
Perpendicular Lines

Example

- Find an equation for the line ℓ passing through $(2,0)$ that is perpendicular to the line ℓ' through $(5,-1)$ and $(-1,3)$;

We first compute the slope of the second line:

$$m' = \frac{3-(-1)}{-1-5} = -\frac{2}{3};$$

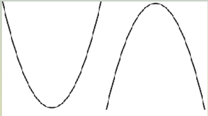


Since $\ell \perp \ell'$, we get $m = \frac{3}{2}$; Now using the point-slope form with $m = \frac{3}{2}$ and $(x_1, y_1) = (2,0)$ we get

$$y = \frac{3}{2}(x-2) = \frac{3}{2}x - 3;$$

Quadratic Functions

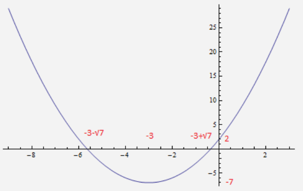
- A **quadratic function** is one of the form
$$f(x) = ax^2 + bx + c, \quad a \neq 0.$$
- The graphs of quadratic functions are parabolas opening up or down.



- To graph $y = ax^2 + bx + c$, we do the following:
  - Find the location of the vertex $V = (-\frac{b}{2a}, f(-\frac{b}{2a}))$;
  - Specify the opening direction: up if $a > 0$, and down if $a < 0$;
  - Take into account the y-intercept $f(0) = c$;
  - Discover the x-intercepts; These are the roots of the equation $ax^2 + bx + c = 0$; We find them either by factoring, if possible, or by using the quadratic formula:
$$D = b^2 - 4ac; \quad x = \frac{-b \pm \sqrt{D}}{2a}, \text{ if } D \geq 0.$$

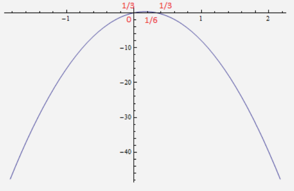
Graphing a Quadratic Function: Example I

- Graph the quadratic function $f(x) = x^2 + 6x + 2$;
- We follow the steps outlined in previous slide:
  - The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{6}{2\cdot 1} = -3$ and y-coordinate $f(-3) = (-3)^2 + 6(-3) + 2 = -7$; Thus, $V = (-3, -7)$;
  - The parabola opens up, since $a = 1 > 0$;
  - The y-intercept is $(0, 2)$;
  - The x-intercepts are the solutions of $x^2 + 6x + 2 = 0$; Since $D = b^2 - 4ac = 6^2 - 4 \cdot 1 \cdot 2 = 36 - 8 = 28 \geq 0$, the roots are given by $x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-6 \pm \sqrt{28}}{2} = -3 \pm \sqrt{7}$.
- Putting all elements together, we obtain the graph



Graphing a Quadratic Function: Example II

- Graph the quadratic function $f(x) = -12x^2 + 4x$;
- We again follow the steps outlined before:
  - The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{4}{2(-12)} = \frac{1}{6}$ and y-coordinate $f(\frac{1}{6}) = -12(\frac{1}{6})^2 + 4\frac{1}{6} = \frac{1}{3}$; Thus, $V = (\frac{1}{6}, \frac{1}{3})$;
  - The parabola opens down, since $a = -12 < 0$;
  - The y-intercept is $(0, 0)$;
  - The x-intercepts are the solutions of $-12x^2 + 4x = 0$; We have $-4x(3x - 1) = 0$, whence $x = 0$ or $x = \frac{1}{3}$.
- Putting all elements together, we obtain the graph



Subsection 3

Polynomial, Rational and Algebraic Functions

Polynomial Functions

- A **monomial function** is one of the form $f(x) = ax^n$, where a is any real number and n is a nonnegative integer;
  - For example $f(x) = 3x^7$ and $g(x) = \frac{1}{7}x^{22}$ are monomial functions;
- A **polynomial function** is a sum of monomials;, Such a sum can be expressed in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0;$$

The numbers $a_0, \ldots, a_n$ are the **coefficients**; The **degree** of P is the highest power with nonzero coefficient; If P has degree n, then a_n is called the **leading coefficient**;
All polynomial functions have domain $\mathbb{R}$;
  - For example $f(x) = 7x^5 - 9x^3 + 4x$ is a polynomial function of degree 5 and its leading coefficient is 7 (the coefficient of x^5);

Rational Functions

- A **rational function** is one of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions;
  - For example $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{7x^6 + x^3 - 3x - 1}{x^2 - 1}$ are rational functions;
- When finding the domain of a rational function we must make sure to exclude those real numbers that are roots of the denominator;
  - Thus, if $f(x) = \frac{1}{x^2}$, $\mathrm{Dom}(f) = \mathbb{R} - \{0\}$;
  - Moreover, if $g(x) = \frac{7x^6 + x^3 - 3x - 1}{x^2 - 1}$, $\mathrm{Dom}(g) = \mathbb{R} - \{-1, 1\}$;

Algebraic Functions

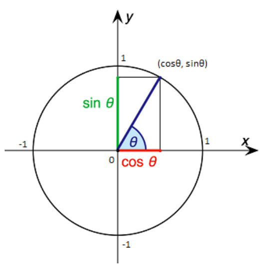
- An **algebraic function** is one generated by taking sums, products and quotients of roots of polynomial and rational functions;
  - For example $f(x) = \sqrt{-x^4 + 3x^2 + 1}$ and $g(x) = \dfrac{x + x^{-5/3}}{5x^3 - \sqrt{x}}$ are algebraic functions;
- When finding the domain of an alegbraic function we must make sure to exclude those real numbers that are roots of the denominator or make any quantity appearing under an even-order root negative;
  - Thus, if $f(x) = \sqrt{3 - 7x}$, we must ensure that $3 - 7x \geq 0$, which yields $\mathrm{Dom}(f) = (-\infty, \frac{3}{7}]$;
  - Moreover, if $g(x) = \sqrt{\dfrac{3x-1}{x+2}}$, we must ensure that $x + 2 \neq 0$ and $\frac{3x-1}{x+2} \geq 0$, which yields (using the sign table method!) $\mathrm{Dom}(g) = (-\infty, -2) \cup [\frac{1}{3}, +\infty)$;

Subsection 4

Trigonometric Functions

Trigonometric Functions

- Recall the basic definition of the trigonometric numbers of an angle x measured in radians:



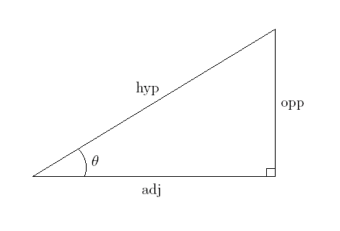
- If the angle θ is placed in standard position, then the point of intersection (x, y) of its terminal side with the unit circle is such that

$$x = \cos\theta \quad \text{and} \quad y = \sin\theta.$$

- $$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.$$

Right Triangle Trigonometry

- Let θ be an acute angle in a right triangle



- Its trigonometric numbers are given by

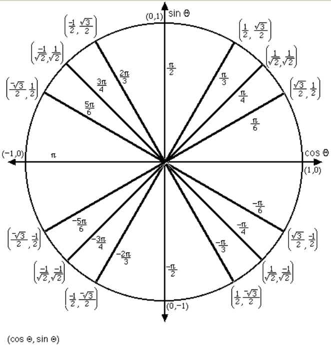
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \tan\theta = \frac{\text{opp}}{\text{adj}} \quad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \cot\theta = \frac{\text{adj}}{\text{opp}} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

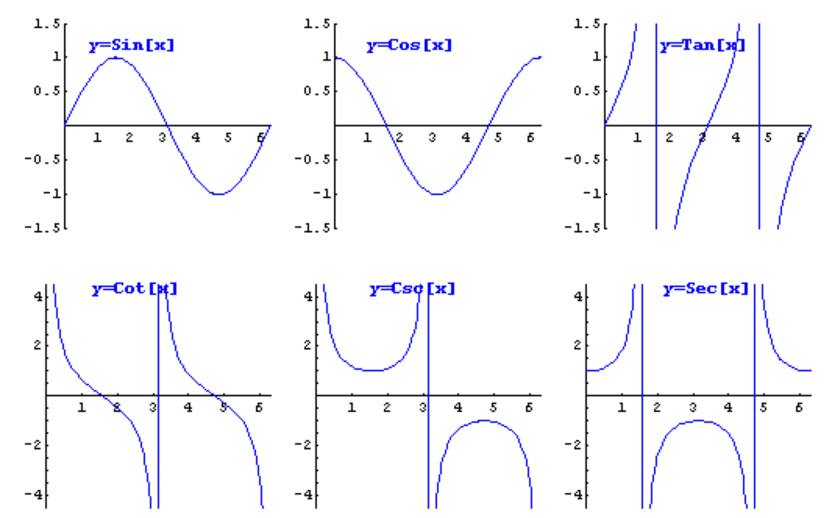
Trigonometric Numbers We Should Remember

- The following numbers we need to remember:

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot\theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\mathrm{cosec}\,\theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1



A Quick Look at Trigonometric Graphs



A Couple of Problems Involving Trig Numbers

- Find the values of $\sin\frac{4\pi}{3}, \cos\frac{4\pi}{3}$ and $\cot\frac{4\pi}{3}$;
  The angle $\frac{4\pi}{3}$ has reference angle (in the first quadrant) $\frac{\pi}{3}$ and is located in the 3rd quadrant. Therefore, $\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$ and $\cot\frac{4\pi}{3} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$.
- Find the angles θ, such that $\sec\theta = 2$;
  We need $\sec\theta = \frac{1}{\cos\theta} = 2$; Therefore, $\cos\theta = \frac{1}{2}$; All angles θ in the 1st and 4th quadrants, with reference angles $\frac{\pi}{3}$ have cosine equal to $\frac{1}{2}$. These are collectively given by $\theta = 2k\pi \pm \frac{\pi}{3}, k \in \mathbb{Z}$.

Solving Trigonometric Equations

- Solve $\sin 4x + \sin 2x = 0$ for $x \in [0, 2\pi)$;
  Recall the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$; Applying this identity, we get $2\sin 2x\cos 2x + \sin 2x = 0$; Factor out $\sin 2x$ to get $\sin 2x(2\cos 2x + 1) = 0$; By the zero-factor property, we must have $\sin 2x = 0$ or $\cos 2x = -\frac{1}{2}$; Thus, since $0 \leq x < 2\pi$, we are seeking all solutions $0 \leq 2x < 4\pi$, such that $\sin 2x = 0$ or $\cos 2x = -\frac{1}{2}$; We have:

$$\begin{array}{lllll} \sin 2x = 0: & 2x = 0 & 2x = \pi & 2x = 2\pi & 2x = 3\pi \\ \cos 2x = -\frac{1}{2}: & 2x = \frac{2\pi}{3} & 2x = \frac{4\pi}{3} & 2x = \frac{8\pi}{3} & 2x = \frac{10\pi}{3} \end{array}$$

  Therefore, dividing all these by 2, we obtain eight solutions for x:

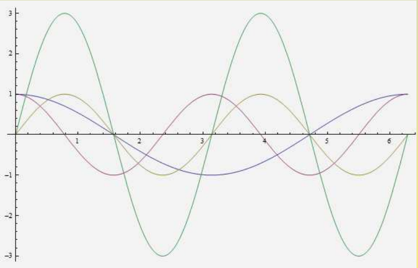
$$x = 0, \quad x = \frac{\pi}{2}, \quad x = \pi, \quad x = \frac{3\pi}{2},$$

$$x = \frac{\pi}{3}, \quad x = \frac{2\pi}{3}, \quad x = \frac{4\pi}{3}, \quad x = \frac{5\pi}{3}.$$

Using Transformations to Graph Trig Functions

- Use transformations to obtain a graph of $k(x) = 3\cos(2(x - \frac{\pi}{4}))$ starting from the graph of $f(x) = \cos x$;

Function	Move Compared to Previous Graph
$f(x) = \cos x$	This is a known graph
$f(x) = \cos 2x$	Horizontal Compression by a Scale of 2
$f(x) = \cos(2(x - \frac{\pi}{4}))$	Shift Right by $\frac{\pi}{4}$
$f(x) = 3\cos(2(x - \frac{\pi}{4}))$	Vertical Stretch by a Scale of 3



A Summary of Basic Trigonometric Identities

- There is a wide variety of trigonometric identities that you have seen in earlier classes. They are sometimes split into categories. Here we review only the ones that we will use the most among each category.
  - The Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1, \quad \tan^2 x + 1 = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x.$$

  - The Sum/Difference Formulas:

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

  - The Double-Angle Formulas:

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x,$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Using Basic Trigonometric Identities

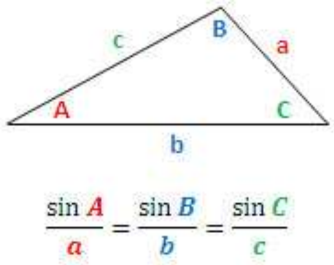
- If $\cos\theta = \frac{2}{5}$ and $\pi < \theta < 2\pi$, calculate the value of $\tan\theta$;
  Using the first Pythagorean identity $\sin^2 x + \cos^2 x = 1$ and taking into account the fact that $\pi < \theta < 2\pi$, we obtain that

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\frac{4}{25}} = -\frac{\sqrt{21}}{5}.$$

  Therefore

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{21}}{5}}{\frac{2}{5}} = -\frac{\sqrt{21}}{2}.$$

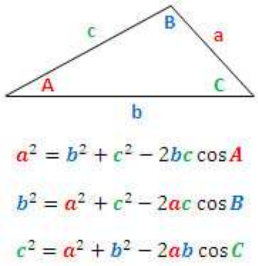
The Law of Sines



Example: If $\widehat{A} = \frac{\pi}{6}$, $\widehat{B} = \frac{2\pi}{3}$ and $b = 12$, then we can compute

$$a = \sin\widehat{A} \cdot \frac{b}{\sin\widehat{B}} = \sin\frac{\pi}{6} \cdot \frac{12}{\sin\frac{2\pi}{3}} = = \frac{1}{2} \cdot \frac{12}{\frac{\sqrt{3}}{2}} = 4\sqrt{3}.$$

The Law of Cosines



Example: If $\widehat{C} = \frac{\pi}{4}$, $a = 5$ and $b = 10$, then we can compute

$$c^2 = a^2 + b^2 - 2ab\cos\widehat{C} = 25 + 100 - 2 \cdot 5 \cdot 10\cos\frac{\pi}{4} =$$
$$= 25 + 100 - 100\frac{\sqrt{2}}{2} = 125 - 50\sqrt{2}.$$

Therefore $c = \sqrt{125 - 50\sqrt{2}}$.

Subsection 5

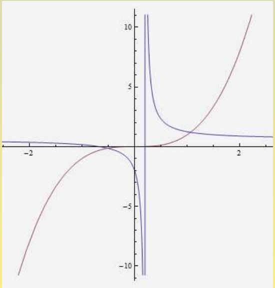
Inverse Functions

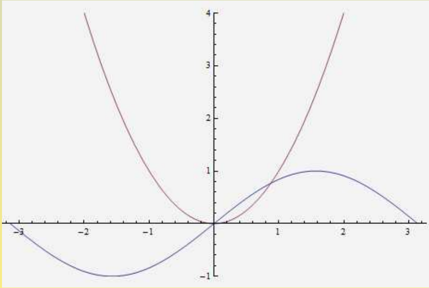
One-to-One Functions

- A function $f(x)$ is **one-to-one** on a domain D if, for all $a, b \in D$, if $a \neq b$, then $f(a) \neq f(b)$.
- A function $f(x)$ is one-to-one if and only if it passes the **Horizontal Line Test**, i.e., every horizontal line intersects its graph in at most one point;

  Example: $f(x) = \frac{3x+2}{5x-1}$ and $g(x) = x^3$ are one-to-one;

  Example: $h(x) = x^2$ and $k(x) = \sin x$ are not one-to-one;



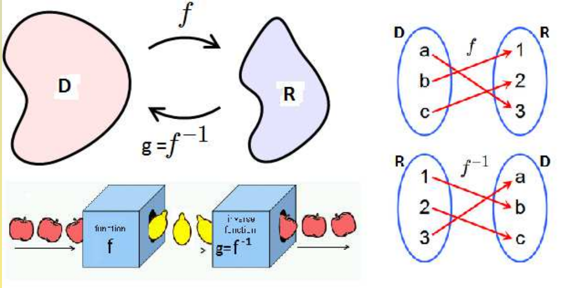


Inverse Functions

- If $f(x)$ has domain D and range R, $f(x)$ is called **invertible** if there exists a function $g(x)$ with domain R, such that

$$g(f(x)) = x, \text{ for all } x \in D, \text{ and } f(g(x)) = x, \text{ for all } x \in R.$$

In that case $g(x)$ is called the **inverse** of f and denoted by $g = f^{-1}(x)$;



Relation Between One-to-One and Inverse Functions

Theorem on the Existence of Inverses

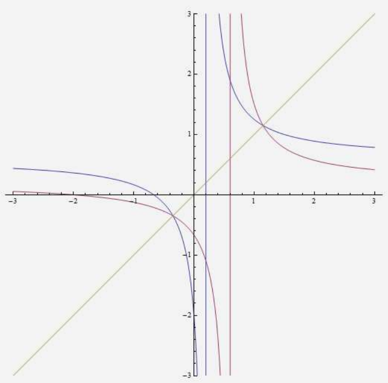
The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D. Furthermore the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

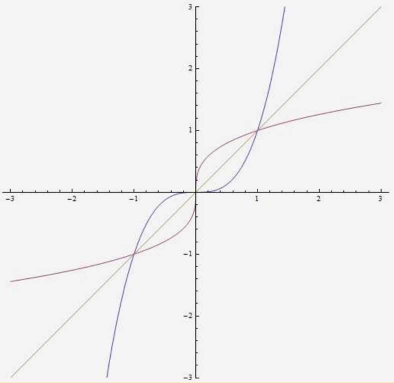
Example: We saw that $f(x) = \frac{3x+2}{5x-1}$ is a one-to-one function on $D = \mathbb{R} - \{\frac{1}{5}\}$. Therefore, according to the theorem above, it has an inverse $f^{-1}(x)$. To find that inverse, interchange x with y in $y = f(x)$, thus obtaining $x = f(y)$, and, then, solve for y; We get $f^{-1}(x) = \frac{x+2}{5x-3}$;

Example: We also saw that $f(x) = x^3$ is one-to-one on $D = \mathbb{R}$; What is $f^{-1}(x)$? Following the same technique, we get $f^{-1}(x) = \sqrt[3]{x}$.

Illustration of Inverse Functions

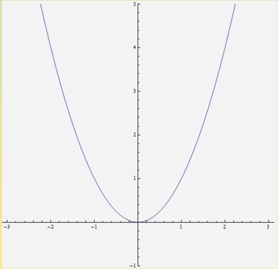
- The following graph illustrates $f(x) = \frac{3x+2}{5x-1}$ and $f^{-1}(x) = \frac{x+2}{5x-3}$; Notice symmetry with respect to $y = x$;
- The second graph illustrates $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$; Notice again the symmetry with respect to $y = x$;

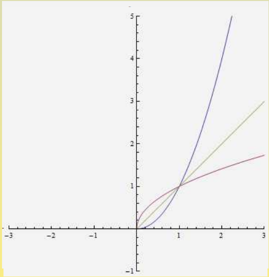




Inverses of Restrictions of Non-One-to-One Functions

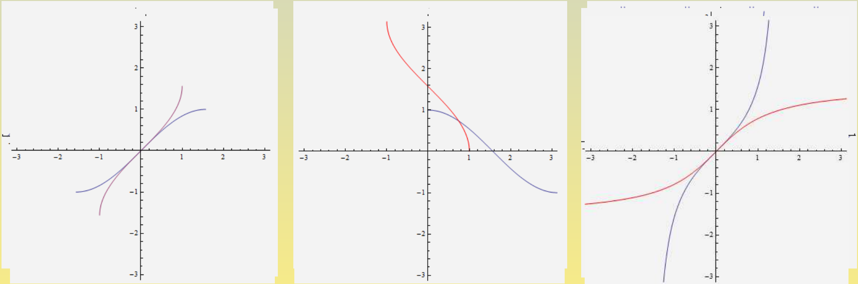
- The functions $f(x) = x^2$ and $g(x) = \sin x$ are not one-to-one;
- Thus, according to the theorem, they do not have inverses;
- However, not all hope is lost! By restricting their domain, we can force them to become one-to-one and, then, we can consider the inverses of the new functions having smaller domains;
- For example, $f(x) = x^2$ with domain $\mathbb{R}$ is not one-to-one; But $f(x) = x^2$, with domain $[0, +\infty)$ is one-to-one; This new $f(x)$ has an inverse; Which function is its inverse? $f^{-1}(x) = \sqrt{x}$;





Inverse Trigonometric Functions

- All three main trig functions are not one-to-one;
- We restrict their domains to force them to be one-to-one;
- Note that the new domains are $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for $\sin x$, $[0, \pi]$ for $\cos x$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ for $\tan x$;
- Then we may define the inverses $\sin^{-1} x, \cos^{-1} x$ and $\tan^{-1} x$ of the functions with the smaller domains!



Examples of Inverse Trigonometric Calculations

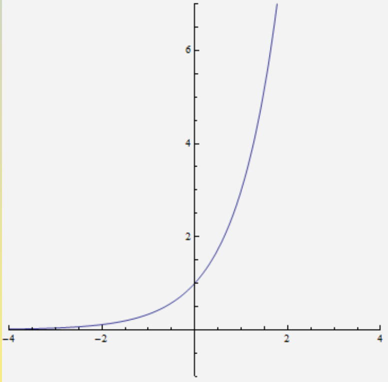
- Let us compute $\sin^{-1}(\sin(-\frac{5\pi}{6}))$;
  We have
  $$\sin^{-1}(\sin(-\frac{5\pi}{6})) = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}.$$
- Let us also compute $\tan^{-1}(\tan\frac{3\pi}{4})$;
  We have
  $$\tan^{-1}(\tan\frac{3\pi}{4}) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$
- Finally, let us do the more abstract $\tan(\cos^{-1}x)$;
  In this case, let $\phi = \cos^{-1}x$. Then, we know that $x = \cos\phi$ and $0 \leq \phi \leq \pi$; Therefore, since for angles in Quadrants I and II we have $\sin x \geq 0$, we obtain $\sin\phi = +\sqrt{1-\cos^2\phi} = \sqrt{1-x^2}$. This yields
  $$\tan(\cos^{-1}x) = \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sqrt{1-x^2}}{x}.$$

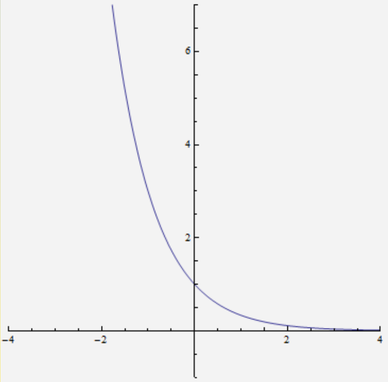
Subsection 6

Exponential and Logarithmic Functions

Exponential Functions

- An **exponential function** is one of the form $f(x) = b^x$, where $0 < b \neq 1$ is a real number, called the **base**;
- If $b > 1$, $f(x) = b^x$ is an increasing function, whereas, if $0 < b < 1$, then $f(x) = b^x$ is a decreasing function;





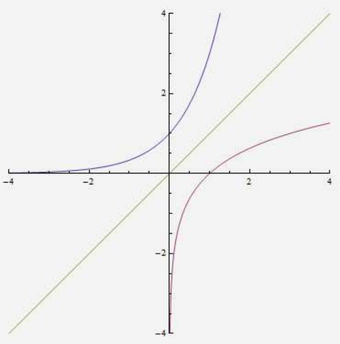
Laws of Exponents

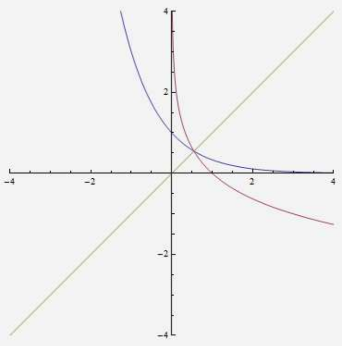
- Recall the following basic rules for algebraically handling exponential expressions:
  - $b^0 = 1$;
  - $b^x b^y = b^{x+y}$;
  - $\frac{b^x}{b^y} = b^{x-y}$;
  - $(b^x)^y = b^{xy}$;
  - $b^{-x} = \frac{1}{b^x}$;
  - $b^{1/n} = \sqrt[n]{b}$;
- Let us simplify some expressions:

$$2^5 \cdot 2^3 = 2^8, \quad \frac{4^7}{4^2} = 4^5, \quad 3^{-4} = \frac{1}{81}, \quad 5^{1/2} = \sqrt{5},$$
$$16^{-1/2} = \frac{1}{4}, \quad 27^{2/3} = 9, \quad 4^{16} \cdot 4^{-18} = \frac{1}{16}, \quad \frac{9^3}{3^7} = \frac{1}{3}.$$

Logarithms

- Recall that the **logarithm** $\log_b x$ to base b of x is the exponent y, such that $b^y = x$:
$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$
- This definition suggests that the two functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverse functions;
- So for $b > 1$ we have the graphs and for $0 < b < 1$ the graphs





Laws of Logarithms

- Recall the following basic rules for algebraically handling logarithmic expressions:
  - $\log_b 1 = 0$;
  - $\log_b b = 1$;
  - $\log_b (xy) = \log_b x + \log_b y$;
  - $\log_b (\frac{x}{y}) = \log_b x - \log_b y$;
  - $\log_b (\frac{1}{x}) = -\log_b x$;
  - $\log_b (x^n) = n \log_b x$;
- Let us simplify some expressions:

$$\log_5 (2 \cdot 3) = \log_5 2 + \log_5 3, \quad \log_2 (\frac{3}{7}) = \log_2 3 - \log_2 7,$$

$$\log_2 (\frac{1}{7}) = -\log_2 7, \quad \log_{10} (7^5) = 5 \log_{10} 7,$$

$$\log_6 9 + \log_6 4 = 2, \quad \ln (\frac{1}{\sqrt{e}}) = -\frac{1}{2},$$

$$\log_{\frac{1}{10}} 1000 = -3, \quad 10 \log_b (b^3) - 4 \log_b (\sqrt{b}) = 28.$$

Simple Exponential and Logarithmic Equations

Example: Solve $2^{5x+1} = 2^7$;
Since the exponentials on either side have the same base, we must have $5x + 1 = 7$; Therefore, $5x = 6$ and $x = \frac{6}{5}$;

Example: Solve $b^3 = 2^9$;
We have $b = (2^9)^{1/3} = 2^{9\cdot(1/3)} = 2^3 = 8$;

Example: Solve $5^{3x-7} = (\frac{1}{5})^{2x}$;
We have $5^{3x-7} = 5^{-2x}$; Therefore $3x - 7 = -2x$; This gives $5x = 7$, whence $x = \frac{7}{5}$;

Example: Solve $e^{7t-3} = 11$;
We have $7t - 3 = \ln 11$, whence $7t = 3 + \ln 11$ and, therefore, $t = \frac{3+\ln 11}{7}$.