

Calculus I

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LSSU Math 151

1 Limits

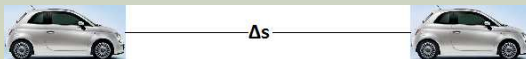
- Limits, Rates of Change and Tangent Lines
- A Graphical Approach to Limits
- Basic Limit Laws
- Limits and Continuity
- Algebraic Evaluation of Limits
- Trigonometric Limits
- Limits at Infinity
- Intermediate Value Theorem

Subsection 1

Limits, Rates of Change and Tangent Lines

Average Velocity

- An object moving on a straight line is at position $s(t)$ at time t ;
- Then in the time interval $[t_0, t_1]$ it has moved from position $s(t_0)$ to position $s(t_1)$ having a **displacement** (or **net change in position**) $\Delta s = s(t_1) - s(t_0)$;



- Its average velocity in $[t_0, t_1]$ is given by

$$v_{\text{avg}}[t_0, t_1] = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.$$

Example: If an object is at position $s(t) = 5t^2$ miles from the origin at time t in hours, what is $v_{\text{avg}}[1, 5]$?

$$v_{\text{avg}}[1, 5] = \frac{s(5) - s(1)}{5 - 1} = \frac{5 \cdot 5^2 - 5 \cdot 1^2}{4} = 30\text{mph}.$$

Instantaneous Velocity

- An object moving on a straight line is at position $s(t)$ at time t ;
- To estimate the instantaneous velocity of the object at t_0 , we consider a very short time interval $[t_0, t_1]$ and compute $v_{\text{avg}}[t_0, t_1]$;
- If $[t_0, t_1]$ is very short, then the change in velocity might be negligible and so a good approximation of the instantaneous velocity at t_0 ;
- Thus $v(t_0) \underbrace{\cong}_{\Delta t \text{ small}} \frac{\Delta s}{\Delta t}$;

Example: Estimate the instantaneous velocity $v(1)$ of the object whose position function is $s(t) = 5t^2$ miles from the origin at time t in hours.

$$v(1) \cong \frac{s(1.01) - s(1)}{1.01 - 1} = \frac{5 \cdot (1.01)^2 - 5 \cdot 1^2}{0.01} = 10.05 \text{mph.}$$

Another Example of a Rate of Change

- Suppose that the length of the side of a melting cube as function of time is given by $s(t) = \frac{1}{t+2}$ inches at t minutes since the start of the melting process. What is the average change in the volume of the ice cube from $t = 0$ to $t = 3$ minutes?



The volume $V(t)$ in cubic inches as a function of time t in minutes is given by $V(t) = s(t)^3 = \left(\frac{1}{t+2}\right)^3$.

Therefore

$$\begin{aligned} \left(\frac{\Delta V}{\Delta t}\right)_{\text{avg}} [0, 3] &= \frac{V(3) - V(0)}{3 - 0} = \frac{\left(\frac{1}{5}\right)^3 - \left(\frac{1}{2}\right)^3}{3} \\ &= \frac{\frac{1}{125} - \frac{1}{8}}{3} = \frac{\frac{8}{1000} - \frac{125}{1000}}{3} = -\frac{117}{3000} \text{ in}^3/\text{min}. \end{aligned}$$

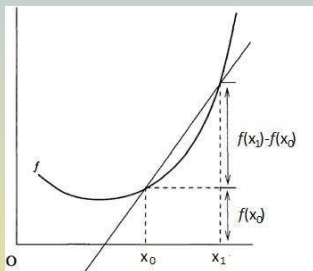
Instantaneous Rate of Change of Volume

- In the previous example, to estimate the instantaneous rate of change of the volume of the ice cube at $t = 1$, we may consider the average rate of change between $t = 1$ minute and $t = 1.01$ minute:

$$\begin{aligned}\left(\frac{\Delta V}{\Delta t}\right) \Big|_{t=1} &\cong \left(\frac{\Delta V}{\Delta t}\right)_{\text{avg}} [1, 1.01] \\ &= \frac{V(1.01) - V(1)}{1.01 - 1} \\ &= \frac{\left(\frac{1}{3.01}\right)^3 - \left(\frac{1}{3}\right)^3}{0.01} \\ &\cong -0.037 \text{in}^3/\text{min}.\end{aligned}$$

Slope of a Secant Line

- Consider the graph of $y = f(x)$ and two points on the graph $(x_0, f(x_0))$ and $(x_1, f(x_1))$;

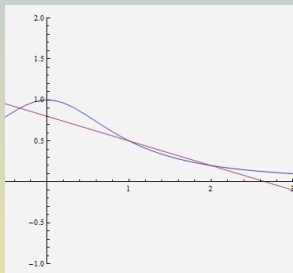


- The line passing through these two points is called the **secant line** to $y = f(x)$ through x_0 and x_1 ;
- Its slope is equal to

$$m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

An Example

- **Example:** Find an equation for the secant line to $f(x) = \frac{1}{1+x^2}$ through $x_0 = 1$ and $x_1 = 2$;



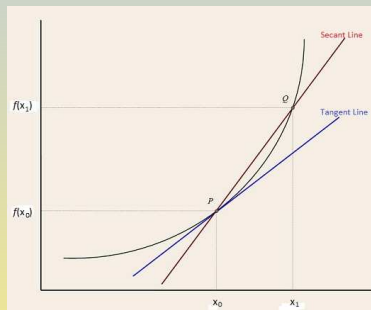
We have

$$m_f[1, 2] = \frac{f(2) - f(1)}{2 - 1} = \frac{\frac{1}{5} - \frac{1}{2}}{2 - 1} = -\frac{3}{10}.$$

Therefore $y - \frac{1}{2} = -\frac{3}{10}(x - 1)$ is the point-slope form of the equation of the secant line.

Slope of a Tangent Line

- To approximate the slope $m_f(x_0)$ of the tangent line to the graph of $y = f(x)$ at x_0 we use a process similar to that approximating the instantaneous rate of change by using the average rate of change for points x_0, x_1 very close to each other;

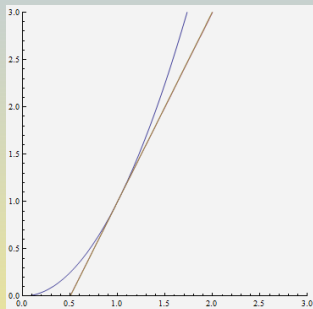


Therefore, we have

$$m_f(x_0) \underbrace{\cong}_{\Delta x \text{ small}} m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Approximating the Slope of a Tangent Line

- Let us approximate the slope to $y = x^2$ at $x = 1$ using the process outlined in the previous slide;



We have

$$m_f(1) \cong m_f[1, 1.01] = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{(1.01)^2 - 1^2}{0.01} = 2.01.$$

Subsection 2

A Graphical Approach to Limits

Definition of Limit

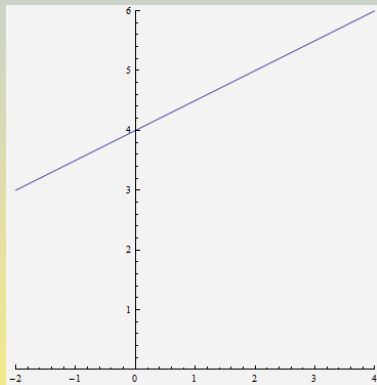
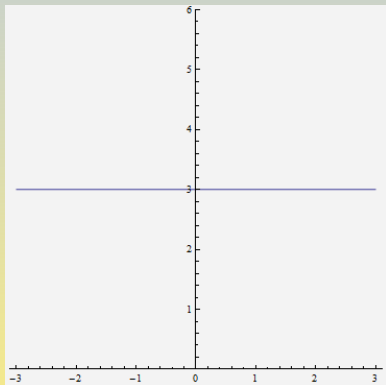
- Suppose that $f(x)$ is defined in an open interval containing a number c , but not necessarily c itself;
- The **limit of $f(x)$ as x approaches c is equal to L** if $f(x)$ has value arbitrarily close to L when x assumes values sufficiently close (**but not equal**) to c .
- In this case, we write

$$\lim_{x \rightarrow c} f(x) = L.$$

- An alternative terminology is that $f(x)$ **approaches** or **converges to L as x approaches c** .

Two Easy Examples

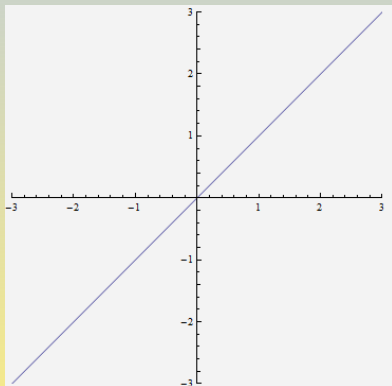
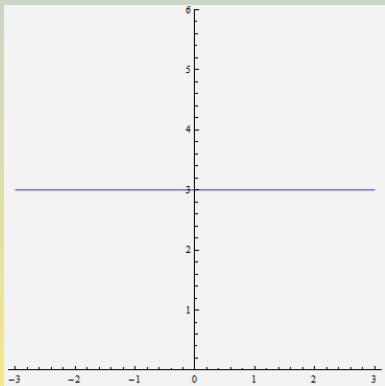
- Draw the graph of $f(x) = 3$ and find graphically the limit $\lim_{x \rightarrow c} f(x)$.
- Draw the graph of $g(x) = \frac{1}{2}x + 4$ and find graphically $\lim_{x \rightarrow 2} f(x)$.



We have $\lim_{x \rightarrow c} 3 = 3$ and $\lim_{x \rightarrow 2} (\frac{1}{2}x + 4) = 5$.

Two Easy Rules

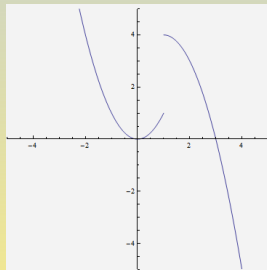
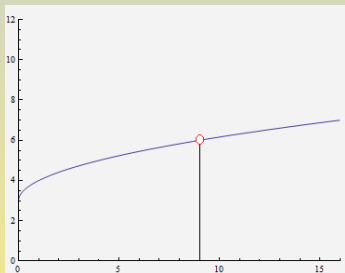
- Draw the graph of $f(x) = k$ (a constant) and find graphically the limit $\lim_{x \rightarrow c} k$.
- Draw the graph of $g(x) = x$ and find graphically $\lim_{x \rightarrow c} x$.



We have $\lim_{x \rightarrow c} k = k$ and $\lim_{x \rightarrow c} x = c$.

Two More Complicated Examples

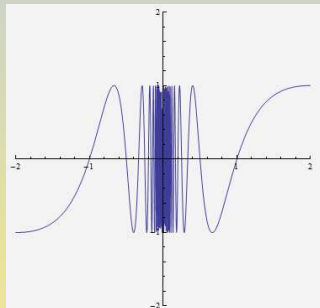
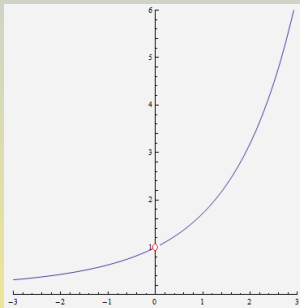
- Draw the graph of $f(x) = \frac{x-9}{\sqrt{x}-3}$ and find graphically the limit $\lim_{x \rightarrow 9} f(x)$.
- Draw the graph of $g(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ -x^2 + 2x + 3, & \text{if } x > 1 \end{cases}$ and find graphically $\lim_{x \rightarrow 1} f(x)$.



We have $\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} = 6$ and $\lim_{x \rightarrow 1} g(x)$ does not exist since $g(x)$ does not approach a single number when x approaches 1.

Two Additional Examples

- Draw the graph of $f(x) = \frac{e^x - 1}{x}$ and find graphically the limit $\lim_{x \rightarrow 0} f(x)$.
- Draw the graph of $g(x) = \sin \frac{\pi}{x}$ and find graphically $\lim_{x \rightarrow 0} g(x)$.



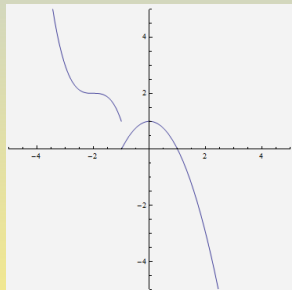
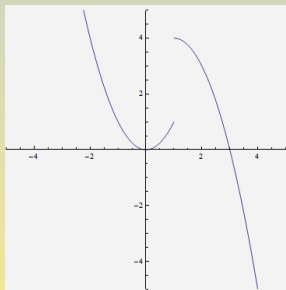
We have $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ and $\lim_{x \rightarrow 0} \sin \frac{\pi}{x}$ does not exist since the values of $g(x) = \sin \frac{\pi}{x}$ oscillate between -1 and 1 as x approaches 0 .

Definition of Side-Limits

- Suppose that $f(x)$ is defined in an open interval containing a number c , but not necessarily c itself;
- The **right-hand limit of $f(x)$ as x approaches c (from the right) is equal to L** if $f(x)$ has value arbitrarily close to L when x approaches sufficiently close (**but is not equal**) to c from the right hand side.
In this case, we write $\lim_{x \rightarrow c^+} f(x) = L$.
- The **left-hand limit of $f(x)$ as x approaches c (from the left) is equal to L** if $f(x)$ has value arbitrarily close to L when x approaches sufficiently close (**but is not equal**) to c from the left hand side.
In this case, we write $\lim_{x \rightarrow c^-} f(x) = L$.
- The limits we saw before are “two sided limits”; It is the case that $\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{x \rightarrow c^+} f(x) = L$ and $\lim_{x \rightarrow c^-} f(x) = L$, i.e., a function has limit L as x approaches c if and only if the left and right hand side limits as x approaches c exist and are equal.

Two Examples

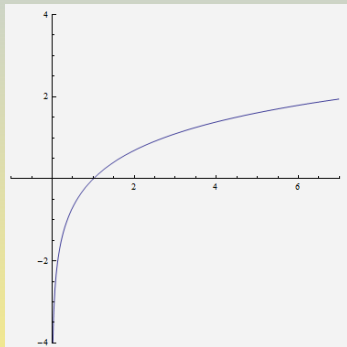
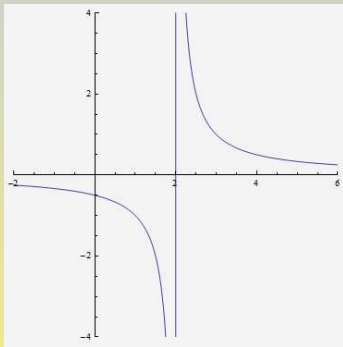
- Draw the graph of $f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ -x^2 + 2x + 3, & \text{if } x > 1 \end{cases}$ and find graphically $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$.
- Draw the graph of $g(x) = \begin{cases} -(x+2)^3 + 2, & \text{if } x < -1 \\ -x^2 + 1, & \text{if } x > -1 \end{cases}$ and find graphically $\lim_{x \rightarrow -1^-} g(x)$ and $\lim_{x \rightarrow -1^+} g(x)$.



$\lim_{x \rightarrow 1^-} f(x) = 1$, $\lim_{x \rightarrow 1^+} f(x) = 4$, so $\lim_{x \rightarrow 1} f(x)$ DNE, and
 $\lim_{x \rightarrow -1^-} g(x) = 1$, $\lim_{x \rightarrow -1^+} g(x) = 0$, so $\lim_{x \rightarrow -1} g(x)$ DNE.

Examples of Limits Involving Infinity

- Draw the graph of $f(x) = \frac{1}{x-2}$ and find graphically $\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$.
- Draw the graph of $g(x) = \ln x$ and find graphically $\lim_{x \rightarrow 0^+} g(x)$ and $\lim_{x \rightarrow +\infty} g(x)$.



$\lim_{x \rightarrow 2^-} f(x) = -\infty$, $\lim_{x \rightarrow 2^+} f(x) = +\infty$, and
 $\lim_{x \rightarrow 0^+} g(x) = -\infty$, $\lim_{x \rightarrow +\infty} g(x) = +\infty$.

Subsection 3

Basic Limit Laws

Theorem (Basic Limit Laws)

Suppose that $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist. Then

- **Sum Law:** $\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$;
- **Constant Factor Law:** $\lim_{x \rightarrow c} kf(x) = k \lim_{x \rightarrow c} f(x)$;
- **Product Law:** $\lim_{x \rightarrow c} f(x)g(x) = (\lim_{x \rightarrow c} f(x))(\lim_{x \rightarrow c} g(x))$;
- **Quotient Law:** If $\lim_{x \rightarrow c} g(x) \neq 0$, then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$;
- **Power and Root Law:** For p, q integers, with $q \neq 0$,
 $\lim_{x \rightarrow c} [f(x)]^{p/q} = (\lim_{x \rightarrow c} f(x))^{p/q}$, under the assumption that
 $\lim_{x \rightarrow c} f(x) \geq 0$ if q is even and $\lim_{x \rightarrow c} f(x) \neq 0$ if $\frac{p}{q} < 0$.

In particular, for n a positive integer,

- $\lim_{x \rightarrow c} [f(x)]^n = (\lim_{x \rightarrow c} f(x))^n$;
- $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow c} f(x)}$;

Examples of Calculating Limits I

- Compute $\lim_{x \rightarrow 2} x^3$;

We apply the power rule:

$$\lim_{x \rightarrow 2} (x^3) = (\lim_{x \rightarrow 2} x)^3 = 2^3 = 8.$$

- Compute $\lim_{x \rightarrow -1} (-2x^3 + 7x - 5)$;

We apply the sum rule, the constant factor and the power rules:

$$\begin{aligned} \lim_{x \rightarrow -1} (-2x^3 + 7x - 5) &= \lim_{x \rightarrow -1} (-2x^3) + \lim_{x \rightarrow -1} (7x) - \lim_{x \rightarrow -1} 5 \\ &= -2 \lim_{x \rightarrow -1} (x^3) + 7 \lim_{x \rightarrow -1} x - \lim_{x \rightarrow -1} 5 \\ &= -2 \cdot (-1)^3 + 7(-1) - 5 \\ &= -10. \end{aligned}$$

Examples of Calculating Limits II

- Compute $\lim_{x \rightarrow 2} \frac{x+30}{2x^4}$;

We apply the quotient rule:

$$\lim_{x \rightarrow 2} \frac{x+30}{2x^4} = \frac{\lim_{x \rightarrow 2} (x+30)}{\lim_{x \rightarrow 2} (2x^4)} = \frac{2+30}{2 \cdot 2^4} = 1.$$

- Compute $\lim_{x \rightarrow 3} (x^{-1/4}(x+5)^{1/3})$;

We apply the product and the power rules:

$$\begin{aligned} \lim_{x \rightarrow 3} (x^{-1/4}(x+5)^{1/3}) &= (\lim_{x \rightarrow 3} x^{-1/4})(\lim_{x \rightarrow 3} \sqrt[3]{x+5}) \\ &= ((\lim_{x \rightarrow 3} x)^{-1/4})(\sqrt[3]{\lim_{x \rightarrow 3} x + 5}) \\ &= 3^{-1/4} \sqrt[3]{8} \\ &= \frac{2}{\sqrt[4]{3}}. \end{aligned}$$

Treacherous Applications of the Laws

- We must take the hypotheses of the Basic Limit Laws into account when applying the rules;
- For instance, if $f(x) = x$ and $g(x) = x^{-1}$, then

$$\lim_{x \rightarrow 0} f(x)g(x) = \lim_{x \rightarrow 0} xx^{-1} = \lim_{x \rightarrow 0} 1 = 1,$$

but, if we tried to apply the product rule, we would be stuck:

$$\lim_{x \rightarrow 0} f(x)g(x) = (\lim_{x \rightarrow 0} x)(\lim_{x \rightarrow 0} x^{-1}),$$

The last limit on the right does not exist since $\lim_{x \rightarrow 0^+} x^{-1} = +\infty$ and

$$\lim_{x \rightarrow 0^-} x^{-1} = -\infty.$$

Subsection 4

Limits and Continuity

Continuity at a Point

- A function $f(x)$ defined on an open interval containing $x = c$ is **continuous at** $x = c$ if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

- If either the limit does not exist, or exists but is not equal to $f(c)$, then f has a **discontinuity** or is **discontinuous** at $x = c$.
- Not that the limit above exists if and only if $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$;
- Therefore, the condition for continuity is equivalent to

$$\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x).$$

Example: Let $f(x) = k$ a constant. Recall that $\lim_{x \rightarrow c} k = k$. Also $f(c) = k$. Therefore, $f(x) = k$ is continuous at all $x = c$.

Some Additional Examples

- Consider $f(x) = x^n$, where n is a natural number. Then $\lim_{x \rightarrow c} x^n = (\lim_{x \rightarrow c} x)^n = c^n$. Also $f(c) = c^n$. Therefore, $f(x) = x^n$ is continuous at all $x = c$.
- Consider $f(x) = x^5 + 7x - 12$. Applying some of the Limit Laws, we get

$$\begin{aligned}\lim_{x \rightarrow c} (x^5 + 7x - 12) &= (\lim_{x \rightarrow c} x)^5 + 7(\lim_{x \rightarrow c} x) - \lim_{x \rightarrow c} 12 \\ &= c^5 + 7c - 12 \\ &= f(c).\end{aligned}$$

Therefore $f(x)$ is continuous at $x = c$.

- Consider also $f(x) = \frac{x^2+5}{x+3}$. Applying some of the Limit Laws, we get

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2+5}{x+3} &= \frac{\lim_{x \rightarrow 2} (x^2+5)}{\lim_{x \rightarrow 2} (x+3)} = \frac{(\lim_{x \rightarrow 2} x)^2 + \lim_{x \rightarrow 2} 5}{\lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 3} \\ &= \frac{2^2+5}{2+3} = f(2).\end{aligned}$$

Thus $f(x)$ is continuous at $x = 2$.

Types of Discontinuities

- Recall $f(x)$ is continuous at $x = c$ if

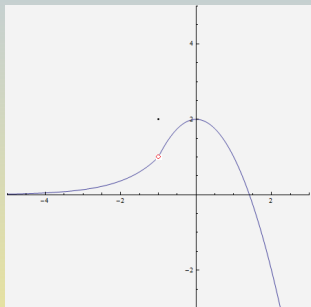
$$\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x).$$

- If $\lim_{x \rightarrow c} f(x)$ exists but is not equal to $f(c)$, then $f(x)$ has a **removable discontinuity** at $x = c$;
- If $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$ (in this case, of course, $\lim_{x \rightarrow c} f(x)$ does not exist), then f has a **jump discontinuity** at $x = c$;
- If either $\lim_{x \rightarrow c^-} f(x)$ or $\lim_{x \rightarrow c^+} f(x)$ is infinite, then f has an **infinite discontinuity** at $x = c$.

Removable Discontinuity

- Consider the piece-wise defined function

$$f(x) = \begin{cases} e^{x+1}, & \text{if } x < -1 \\ 2, & \text{if } x = -1 \\ -x^2 + 2, & \text{if } x > -1 \end{cases}$$

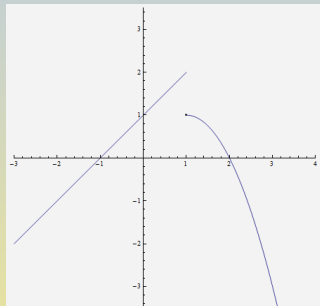


We have $\lim_{x \rightarrow -1^-} f(x) = 1$ and $\lim_{x \rightarrow -1^+} f(x) = 1$, whence $\lim_{x \rightarrow -1} f(x) = 1$. But $f(-1) = 2$. So $\lim_{x \rightarrow -1} f(x)$ exists, but it does not equal $f(-1)$. This shows that $f(x)$ has a removable discontinuity at $x = -1$.

Jump Discontinuity

- Consider the piece-wise defined function

$$f(x) = \begin{cases} x + 1, & \text{if } x < 1 \\ -x^2 + 2x, & \text{if } x \geq 1 \end{cases}$$

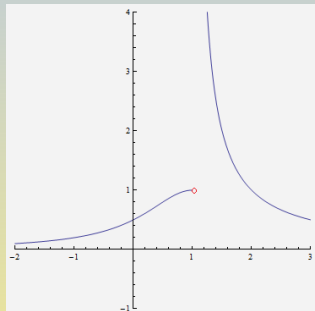


We have $\lim_{x \rightarrow 1^-} f(x) = 2$ and $\lim_{x \rightarrow 1^+} f(x) = 1$, whence $\lim_{x \rightarrow 1} f(x) = \text{DNE}$. So the side limits of $f(x)$ as x approaches 1 exist, but they are not equal. This shows that $f(x)$ has a jump discontinuity at $x = 1$.

Infinite Discontinuity

- Consider the piece-wise defined function

$$f(x) = \begin{cases} \frac{1}{x^2-2x+2}, & \text{if } x < 1 \\ \frac{1}{x-1}, & \text{if } x > 1 \end{cases}$$



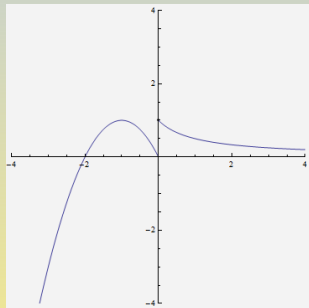
We have $\lim_{x \rightarrow 1^-} f(x) = 1$ and $\lim_{x \rightarrow 1^+} f(x) = +\infty$. Thus, at least one of the side limits as x approaches 1 is $\pm\infty$. This shows that $f(x)$ has an infinite discontinuity at $x = 1$.

One-Sided Continuity

- A function $f(x)$ is called
 - **left-continuous** at $x = c$ if $\lim_{x \rightarrow c^-} f(x) = f(c)$;
 - **right-continuous** at $x = c$ if $\lim_{x \rightarrow c^+} f(x) = f(c)$;

Example: Consider the function

$$f(x) = \begin{cases} -x^2 - 2x, & \text{if } x < 0 \\ \frac{1}{x+1}, & \text{if } x \geq 0 \end{cases}$$

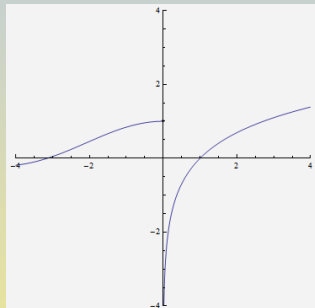


We have $\lim_{x \rightarrow 0^-} f(x) = 0$ and $\lim_{x \rightarrow 0^+} f(x) = 1$. Moreover, $f(0) = 1$. Therefore $f(x)$ is right-continuous at $x = 0$, but not left continuous at $x = 0$.

One More Example

- Consider the piece-wise defined function

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \\ \ln x, & \text{if } x > 0 \end{cases}$$



We have $\lim_{x \rightarrow 0^-} f(x) = 1$ and $\lim_{x \rightarrow 0^+} f(x) = -\infty$. Moreover, $f(0) = 1$. Therefore, $f(x)$ is left-continuous at $x = 0$, but not right-continuous at $x = 0$.

Basic Continuity Laws

Theorem (Basic Laws of Continuity)

If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following functions are also continuous at $x = c$:

$$(i) \quad f(x) \pm g(x)$$

$$(iii) \quad f(x)g(x)$$

$$(ii) \quad kf(x)$$

$$(iv) \quad \frac{f(x)}{g(x)}, \text{ if } g(c) \neq 0.$$

- For instance, knowing that $f(x) = x$ and $g(x) = k$ are continuous functions at all real numbers, the previous rules allow us to conclude that
 - any polynomial function $P(x)$ is continuous at all real numbers;
 - any rational function $\frac{P(x)}{Q(x)}$ is continuous at all values in its domain.

Example: $f(x) = 3x^4 - 2x^3 + 8x$ is continuous at all real numbers.
 $g(x) = \frac{x+3}{x^2-1}$ is continuous at all numbers $x \neq \pm 1$.

Continuity of Roots, Trig, Exp and Log Functions

Theorem (Continuity of Various Functions)

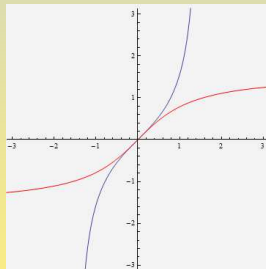
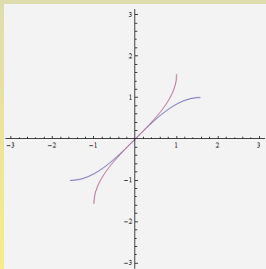
- $f(x) = \sqrt[n]{x}$ is continuous on its domain;
- $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at all real numbers;
- $f(x) = b^x$ is continuous at all real numbers ($0 < b \neq 1$);
- $f(x) = \log_b x$ is continuous at all $x > 0$ ($0 < b \neq 1$);
- Based on this theorem and the theorem on quotients, we may conclude, for example, that $\tan x = \frac{\sin x}{\cos x}$ is continuous at all points in its domain, i.e., at all $x \neq (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.
- We can also conclude that $\csc x = \frac{1}{\sin x}$ is continuous at all points in its domain, i.e., at all $x \neq k\pi$, $k \in \mathbb{Z}$.

Continuity of Inverse Functions

Theorem (Continuity of Inverse Functions)

If $f(x)$ is continuous on an interval I with range R , then if $f^{-1}(x)$ exists, then $f^{-1}(x)$ is continuous with domain R .

- For instance $f(x) = \sin x$ is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with range $[-1, 1]$ and has an inverse; So, $f^{-1}(x) = \sin^{-1} x$ is continuous on $[-1, 1]$.
- Similarly $g(x) = \tan x$ is continuous on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with range \mathbb{R} and has an inverse; Therefore $g^{-1}(x) = \tan^{-1} x$ is continuous on \mathbb{R} .

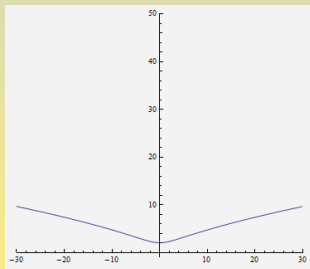
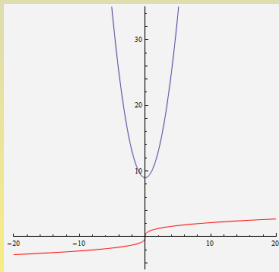


Continuity of Composite Functions

Theorem (Continuity of Composite Functions)

If $g(x)$ is continuous at the point $x = c$ and $f(x)$ is continuous at the point $x = g(c)$, then the function $F(x) = f(g(x))$ is continuous at $x = c$.

- For instance, the function $g(x) = x^2 + 9$ is continuous at all real numbers, since it is a polynomial function; Moreover, the function $f(x) = \sqrt[3]{x}$ is continuous at all real numbers as a root function; Therefore, the function $F(x) = f(g(x)) = \sqrt[3]{x^2 + 9}$ is also a continuous function, as the composite of two continuous functions.



Substitution Method: Using Continuity to Evaluate Limits

- Recall that $f(x)$ is continuous at $x = c$ if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

- Suppose that you know that $f(x)$ is continuous at $x = c$ and want to compute $\lim_{x \rightarrow c} f(x)$.

Then, because of the definition of continuity, to find $\lim_{x \rightarrow c} f(x)$, you may compute, instead, $f(c)$.

- This is called the **substitution property** (or **method**) for evaluating limits of continuous functions.

Examples of Using the Substitution Method

Example: Let us evaluate the limit $\lim_{x \rightarrow \frac{\pi}{3}} \sin x$.

Since $f(x)$ is continuous (by the basic theorem on trig functions) at all $x \in \mathbb{R}$, we may use the substitution property:

$$\lim_{x \rightarrow \frac{\pi}{3}} \sin x = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example: Let us evaluate the limit $\lim_{x \rightarrow -1} \frac{3^x}{\sqrt{x+5}}$.

Since $f(x)$ is continuous (as a ratio of an exponential over a root function, both of which are continuous in their domain), we may use the substitution property:

$$\lim_{x \rightarrow -1} \frac{3^x}{\sqrt{x+5}} = \frac{3^{-1}}{\sqrt{-1+5}} = \frac{1}{6}.$$

Subsection 5

Algebraic Evaluation of Limits

Indeterminate Forms

- The following are **Indeterminate Forms**:

- $\frac{0}{0}$

- Example: $\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$

- $\frac{\infty}{\infty}$

- Example: $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{\sec x}$

- $\infty \cdot 0$

- Example: $\lim_{x \rightarrow 2} \left(\frac{1}{2x - 4} \cdot (x - 2)^2 \right)$

- $\infty - \infty$

- Example: $\lim_{x \rightarrow 1} \left(\frac{1}{x - 1} - \frac{2}{x^2 - 1} \right)$

The Indeterminate Form $\frac{0}{0}$: Factor and Cancel

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$;

We have

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 + x - 12} &= \lim_{x \rightarrow 3} \frac{(x - 1)(x - 3)}{(x + 4)(x - 3)} \\ &= \lim_{x \rightarrow 3} \frac{x - 1}{x + 4} \\ &= \frac{3 - 1}{3 + 4} \\ &= \frac{2}{7}.\end{aligned}$$

The Indeterminate Form $\frac{0}{0}$: Another Example

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x \rightarrow 7} \frac{x - 7}{x^2 - 49}$;

We have

$$\begin{aligned}\lim_{x \rightarrow 7} \frac{x - 7}{x^2 - 49} &= \lim_{x \rightarrow 7} \frac{x - 7}{(x + 7)(x - 7)} \\ &= \lim_{x \rightarrow 7} \frac{1}{x + 7} \\ &= \frac{1}{7 + 7} \\ &= \frac{1}{14}.\end{aligned}$$

The Indeterminate Form $\frac{\infty}{\infty}$

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{\sec x}$;

We have

$$\begin{aligned}\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{\sec x} &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \\ &= \lim_{x \rightarrow \frac{\pi}{2}} \sin x \\ &= \sin \frac{\pi}{2} \\ &= 1.\end{aligned}$$

The Indeterminate Form $\frac{0}{0}$: Multiply by Conjugate

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$;

We have

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \rightarrow 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} \\&= \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} \\&= \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} \\&= \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.\end{aligned}$$

The Indeterminate Form $\frac{0}{0}$: Multiply by Conjugate

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x \rightarrow 7} \frac{x - 7}{\sqrt{x + 9} - 4}$;

$$\begin{aligned}\lim_{x \rightarrow 7} \frac{x - 7}{\sqrt{x + 9} - 4} &= \lim_{x \rightarrow 7} \frac{(x - 7)(\sqrt{x + 9} + 4)}{(\sqrt{x + 9} - 4)(\sqrt{x + 9} + 4)} \\&= \lim_{x \rightarrow 7} \frac{(x - 7)(\sqrt{x + 9} + 4)}{x + 9 - 16} \\&= \lim_{x \rightarrow 7} \frac{(x - 7)(\sqrt{x + 9} + 4)}{x - 7} = \lim_{x \rightarrow 7} (\sqrt{x + 9} + 4) \\&= \sqrt{7 + 9} + 4 = 8.\end{aligned}$$

The Indeterminate Form $\infty - \infty$

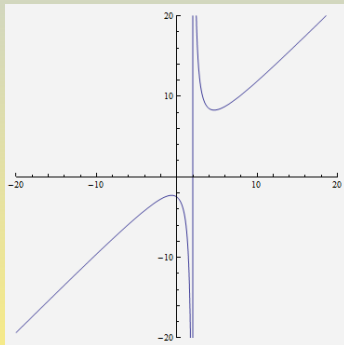
- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$;

$$\begin{aligned} \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) &= \lim_{x \rightarrow 2} \left(\frac{x+2}{(x-2)(x+2)} - \frac{4}{(x-2)(x+2)} \right) \\ &= \lim_{x \rightarrow 2} \frac{x+2-4}{(x+2)(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x-2}{(x+2)(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}. \end{aligned}$$

Forms $\frac{c}{0}$, with $c \neq 0$ are Infinite but not Indeterminate

- $\lim_{x \rightarrow 2} \frac{x^2 - x + 5}{x - 2}$ is of the form $\frac{7}{0}$;
- These forms are not indeterminate, but rather they suggest that the side-limits as $x \rightarrow 2$ are infinite;
- If $x \rightarrow 2^-$, then $x < 2$, whence $x - 2 < 0$. Thus,
$$\lim_{x \rightarrow 2^-} \frac{x^2 - x + 5}{x - 2} (= (\frac{7}{0^-})) = -\infty;$$
- If $x \rightarrow 2^+$, then $x > 2$, whence $x - 2 > 0$. Thus,
$$\lim_{x \rightarrow 2^+} \frac{x^2 - x + 5}{x - 2} (= (\frac{7}{0^+})) = \infty;$$



Subsection 6

Trigonometric Limits

The Squeeze Theorem

The Squeeze Theorem

Assume that for $x \neq c$ in some open interval containing c ,

$$\ell(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \rightarrow c} \ell(x) = \lim_{x \rightarrow c} u(x) = L.$$

Then $\lim_{x \rightarrow c} f(x)$ exists and $\lim_{x \rightarrow c} f(x) = L$.

Example: We show $\lim_{x \rightarrow 0} (x \sin \frac{1}{x}) = 0$

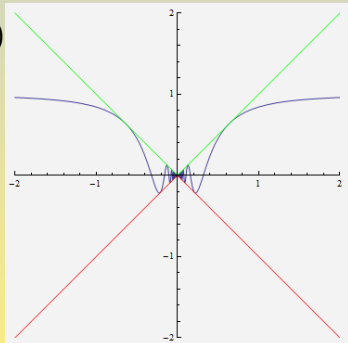
Note that $-|x| \leq x \sin \frac{1}{x} \leq |x|$;

Note, also that

$$\lim_{x \rightarrow 0} (-|x|) = \lim_{x \rightarrow 0} |x| = 0;$$

Therefore, by Squeeze,

$$\lim_{x \rightarrow 0} (x \sin \frac{1}{x}) = 0.$$

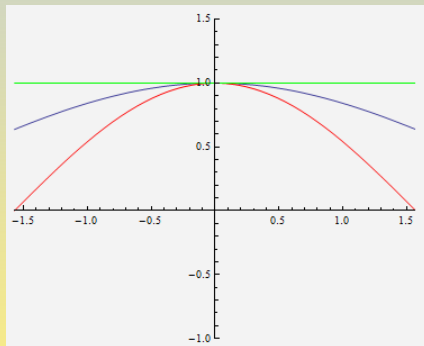


An Important Squeeze Identity

Theorem

For all $\theta \neq 0$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1.$$



Important Trigonometric Limits

Important Trigonometric Limits

We have

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0.$$

- Note that the first limit above follows by the Squeeze Theorem using the Squeeze Identity of the previous slide;
- For the second one, we have

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{\theta(1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} \right) \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{1 + \cos \theta} \\ &= 1 \cdot \frac{0}{1+1} = 0. \end{aligned}$$

Evaluation of Limits by a Change of Variable

- Compute the limit $\lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta}$;

We have

$$\begin{aligned}
 \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{4 \sin 4\theta}{4\theta} \\
 &= 4 \lim_{\theta \rightarrow 0} \frac{\sin 4\theta}{4\theta} \\
 &\stackrel{x=4\theta}{=} 4 \lim_{x \rightarrow 0} \frac{\sin x}{x} \\
 &= 4 \cdot 1 = 4.
 \end{aligned}$$

- Compute the limit $\lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{\sin 3\theta}$;

We have

$$\begin{aligned}
 \lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{\sin 3\theta} &= \lim_{\theta \rightarrow 0} \frac{7\theta \frac{\sin 7\theta}{7\theta}}{3\theta \frac{\sin 3\theta}{3\theta}} = \lim_{\theta \rightarrow 0} \frac{7 \frac{\sin 7\theta}{7\theta}}{3 \frac{\sin 3\theta}{3\theta}} \\
 &= \frac{7 \lim_{\theta \rightarrow 0} \frac{\sin 7\theta}{7\theta}}{3 \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta}} \stackrel{x=7\theta}{=} \frac{7 \lim_{x \rightarrow 0} \frac{\sin x}{x}}{3 \lim_{y=3\theta} \frac{\sin y}{y}} \\
 &= \frac{7}{3} \frac{1}{1} = \frac{7}{3}.
 \end{aligned}$$

Subsection 7

Limits at Infinity

Limits at Infinity

Limit of $f(x)$ as $x \rightarrow \pm\infty$

- We write $\lim_{x \rightarrow \infty} f(x) = L$ if $f(x)$ gets closer and closer to L as $x \rightarrow \infty$, i.e., as x increases without bound;
- We write $\lim_{x \rightarrow -\infty} f(x) = L$ if $f(x)$ gets closer and closer to L as $x \rightarrow -\infty$, i.e., as x decreases without bound;

In either case, the line $y = L$ is called a **horizontal asymptote** of $y = f(x)$.

- Horizontal asymptotes describe the asymptotic behavior of $f(x)$, i.e., the behavior of the graph as we move way out to the left or to the right.

Example of Limits at Infinity

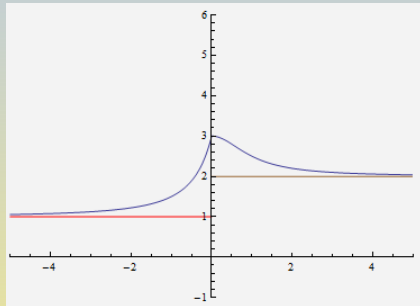
- Consider the function $f(x)$ whose graph is given on the right:

We have

$$\lim_{x \rightarrow -\infty} f(x) = 1$$

and

$$\lim_{x \rightarrow \infty} f(x) = 2.$$



Thus, both $y = 1$ and $y = 2$ are horizontal asymptotes of $y = f(x)$.

Powers of x

Theorem

Assume $n > 0$. Then we have

$$\lim_{x \rightarrow \infty} x^n = \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} x^{-n} = \lim_{x \rightarrow \infty} \frac{1}{x^n} = 0.$$

For $n > 0$ an integer,

$$\lim_{x \rightarrow -\infty} x^n = \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \lim_{x \rightarrow -\infty} x^{-n} = \lim_{x \rightarrow -\infty} \frac{1}{x^n} = 0.$$

Example: $\lim_{x \rightarrow \infty} (3 - 4x^{-3} + 5x^{-5}) =$
 $\lim_{x \rightarrow \infty} 3 - 4 \lim_{x \rightarrow \infty} x^{-3} + 5 \lim_{x \rightarrow \infty} x^{-5} = 3 - 4 \cdot 0 + 5 \cdot 0 = 3.$

Example

- Calculate $\lim_{x \rightarrow \pm\infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5}$.

We follow the method of dividing numerator and denominator by the highest power x^5 :

$$\begin{aligned}
 \lim_{x \rightarrow \pm\infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} &= \lim_{x \rightarrow \pm\infty} \frac{\frac{20x^2 - 3x}{x^5}}{\frac{3x^5 - 4x^2 + 5}{x^5}} \\
 &= \lim_{x \rightarrow \pm\infty} \frac{\frac{20x^2}{x^5} - \frac{3x}{x^5}}{\frac{3x^5}{x^5} - \frac{4x^2}{x^5} + \frac{5}{x^5}} \\
 &= \lim_{x \rightarrow \pm\infty} \frac{\frac{20}{x^3} - \frac{3}{x^4}}{3 - \frac{4}{x^3} + \frac{5}{x^5}} \\
 &= \frac{\lim_{x \rightarrow \pm\infty} \frac{20}{x^3} - \lim_{x \rightarrow \pm\infty} \frac{3}{x^4}}{\lim_{x \rightarrow \pm\infty} 3 - \lim_{x \rightarrow \pm\infty} \frac{4}{x^3} + \lim_{x \rightarrow \pm\infty} \frac{5}{x^5}} \\
 &= \frac{0-0}{3-0+0} = 0.
 \end{aligned}$$

Limits at Infinity of Rational Functions

Theorem

If $a_n, b_m \neq 0$, then it is the case that

$$\lim_{x \rightarrow \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x \rightarrow \pm\infty} x^{n-m}.$$

Example:

- $\lim_{x \rightarrow \infty} \frac{3x^4 - 7x + 9}{7x^4 - 4} = \frac{3}{7} \lim_{x \rightarrow \infty} x^0 = \frac{3}{7};$
- $\lim_{x \rightarrow \infty} \frac{3x^3 - 7x + 9}{7x^4 - 4} = \frac{3}{7} \lim_{x \rightarrow \infty} x^{-1} = \frac{3}{7} \lim_{x \rightarrow \infty} \frac{1}{x} = 0;$
- $\lim_{x \rightarrow -\infty} \frac{3x^8 - 7x + 9}{7x^3 - 4} = \frac{3}{7} \lim_{x \rightarrow -\infty} x^5 = -\infty;$
- $\lim_{x \rightarrow -\infty} \frac{3x^7 - 7x + 9}{7x^3 - 4} = \frac{3}{7} \lim_{x \rightarrow -\infty} x^4 = \infty;$

Two More Examples

- Compute the limit $\lim_{x \rightarrow \infty} \frac{3x^{7/2} + 7x^{-1/2}}{x^2 - x^{1/2}}$;

We have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x^{7/2} + 7x^{-1/2}}{x^2 - x^{1/2}} &= \lim_{x \rightarrow \infty} \frac{(x^{-2})(3x^{7/2} + 7x^{-1/2})}{(x^{-2})(x^2 - x^{1/2})} \\ &= \lim_{x \rightarrow \infty} \frac{3x^{3/2} + 7x^{-5/2}}{1 - x^{-3/2}} \\ &= \frac{\lim_{x \rightarrow \infty} 3x^{3/2} + \lim_{x \rightarrow \infty} 7x^{-5/2}}{\lim_{x \rightarrow \infty} 1 - \lim_{x \rightarrow \infty} x^{-3/2}} \\ &= \frac{\infty}{1} = \infty. \end{aligned}$$

- Compute the limit $\lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^3 + 1}}$;

We have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2}{\sqrt{x^3 + 1}} &= \lim_{x \rightarrow \infty} \frac{x^{-3/2} x^2}{x^{-3/2} \sqrt{x^3 + 1}} = \lim_{x \rightarrow \infty} \frac{x^{1/2}}{\sqrt{x^{-3}(x^3 + 1)}} \\ &= \lim_{x \rightarrow \infty} \frac{x^{1/2}}{\sqrt{1 + x^{-3}}} = \frac{\infty}{1} = \infty. \end{aligned}$$

One More Example

- Calculate the limits at infinity of $f(x) = \frac{12x + 25}{\sqrt{16x^2 + 100x + 500}}$;

We have

$$\begin{aligned}
 \lim_{x \rightarrow -\infty} \frac{12x+25}{\sqrt{16x^2+100x+500}} &= \lim_{x \rightarrow -\infty} \frac{12x(1+\frac{25}{12x})}{\sqrt{16x^2(1+\frac{100}{16x}+\frac{500}{16x^2})}} \\
 &= \lim_{x \rightarrow -\infty} \frac{12x(1+\frac{25}{12x})}{-4x\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} \\
 &= -3 \lim_{x \rightarrow -\infty} \frac{1+\frac{25}{12x}}{\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} \\
 &= -3;
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{12x+25}{\sqrt{16x^2+100x+500}} &= \lim_{x \rightarrow \infty} \frac{12x(1+\frac{25}{12x})}{\sqrt{16x^2(1+\frac{100}{16x}+\frac{500}{16x^2})}} \\
 &= \lim_{x \rightarrow \infty} \frac{12x(1+\frac{25}{12x})}{4x\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} \\
 &= 3 \lim_{x \rightarrow \infty} \frac{1+\frac{25}{12x}}{\sqrt{1+\frac{100}{16x}+\frac{500}{16x^2}}} = 3.
 \end{aligned}$$

Subsection 8

Intermediate Value Theorem

The Intermediate Value Theorem

Intermediate Value Theorem

If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$, such that $f(c) = M$.

Example: Show that $\sin x = \frac{1}{8}$ has at least one solution.

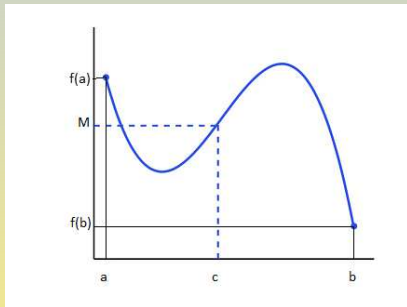
Consider $f(x) = \sin x$ in the closed interval $[0, \frac{\pi}{2}]$.

We have

$$f(0) = 0 < \frac{1}{8} < 1 = f\left(\frac{\pi}{2}\right).$$

Thus, by the Intermediate Value Theorem,

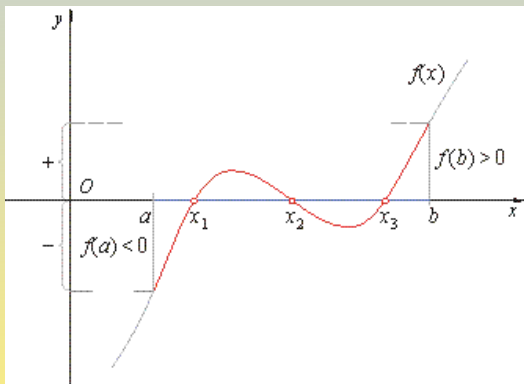
there exists $c \in (0, \frac{\pi}{2})$, such that $f(c) = \frac{1}{8}$, i.e., $\sin c = \frac{1}{8}$. This c is a solution of the equation $\sin x = \frac{1}{8}$.



Existence of Zeros

Existence of Zeros

If $f(x)$ is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then $f(x)$ has a zero in (a, b) .



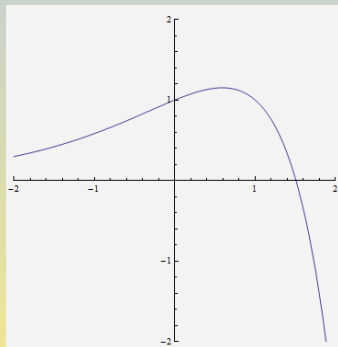
Applying the Existence of Zeros Theorem

- Show that the equation $2^x + 3^x = 4^x$ has at least one zero.

Consider $f(x) = 2^x + 3^x - 4^x$ in the closed interval $[1, 2]$.

We have $f(1) = 1 > 0$, whereas $f(2) = -3 < 0$.

Thus, by the Existence of Zeros Theorem, there exists $c \in (1, 2)$, such that $f(c) = 0$, i.e., $2^c + 3^c - 4^c = 0$. But, then, c satisfies $2^c + 3^c = 4^c$, i.e., it is a zero of $2^x + 3^x = 4^x$.

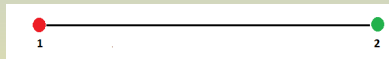


The Bisection Method

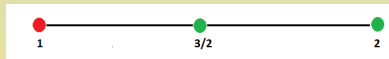
- Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$;

Consider the function $f(x) = x^7 + 3x - 10$ in $[1, 2]$.

Since $f(1) = -6 < 0$ and $f(2) = 112 > 0$, by the Existence of Zeros Theorem, it has a root in $(1, 2)$.



Since $f(1) = -6 < 0$ and $f(\frac{3}{2}) = 11.586 > 0$, it has a root in the interval $(1, \frac{3}{2})$.



Finally, since $f(\frac{5}{4}) = -1.482 < 0$ and $f(\frac{3}{2}) = 11.586 > 0$, the root is in the interval $(\frac{5}{4}, \frac{3}{2})$, which has length $\frac{1}{4}$.

