Calculus I

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LSSU Math 151
1 Limits

- Limits, Rates of Change and Tangent Lines
- A Graphical Approach to Limits
- Basic Limit Laws
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- Algebraic Evaluation of Limits
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Subsection 1

Limits, Rates of Change and Tangent Lines
Average Velocity

- An object moving on a straight line is at position \( s(t) \) at time \( t \);
- Then in the time interval \( [t_0, t_1] \) it has moved from position \( s(t_0) \) to position \( s(t_1) \) having a displacement (or net change in position) \( \Delta s = s(t_1) - s(t_0) \);

![](image1.png)

- Its average velocity in \( [t_0, t_1] \) is given by
  \[
  v_{\text{avg}}[t_0, t_1] = \frac{\Delta s}{\Delta t} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}.
  \]

**Example:** If an object is at position \( s(t) = 5t^2 \) miles from the origin at time \( t \) in hours, what is \( v_{\text{avg}}[1, 5] \)?

\[
v_{\text{avg}}[1, 5] = \frac{s(5) - s(1)}{5 - 1} = \frac{5 \cdot 5^2 - 5 \cdot 1^2}{4} = 30 \text{mph}.
\]
Instantaneous Velocity

- An object moving on a straight line is at position $s(t)$ at time $t$;
- To estimate the instantaneous velocity of the object at $t_0$, we consider a very short time interval $[t_0, t_1]$ and compute $v_{\text{avg}}[t_0, t_1]$;
- If $[t_0, t_1]$ is very short, then the change in velocity might be negligible and so a good approximation of the instantaneous velocity at $t_0$;
- Thus $v(t_0) \approx \frac{\Delta s}{\Delta t}$; 
  \[ \Delta t \text{ small} \]

**Example**: Estimate the instantaneous velocity $v(1)$ of the object whose position function is $s(t) = 5t^2$ miles from the origin at time $t$ in hours.

$$v(1) \approx \frac{s(1.01) - s(1)}{1.01 - 1} = \frac{5 \cdot (1.01)^2 - 5 \cdot 1^2}{0.01} = 10.05 \text{mph}.$$
Another Example of a Rate of Change

Suppose that the length of the side of a melting cube as function of time is given by \( s(t) = \frac{1}{t+2} \) inches at \( t \) minutes since the start of the melting process. What is the average change in the volume of the ice cube from \( t = 0 \) to \( t = 3 \) minutes?

The volume \( V(t) \) in cubic inches as a function of time \( t \) in minutes is given by \( V(t) = s(t)^3 = \left( \frac{1}{t+2} \right)^3 \).

Therefore

\[
\left( \frac{\Delta V}{\Delta t} \right)_{\text{avg}}^{[0,3]} = \frac{V(3) - V(0)}{3 - 0} = \frac{\left( \frac{1}{5} \right)^3 - \left( \frac{1}{2} \right)^3}{3}
\]

\[
= \frac{\frac{1}{125} - \frac{1}{8}}{3} = \frac{\frac{8}{1000} - \frac{125}{1000}}{3} = -\frac{117}{3000} \text{ in}^3/\text{min}.
\]
Instantaneous Rate of Change of Volume

In the previous example, to estimate the instantaneous rate of change of the volume of the ice cube at $t = 1$, we may consider the average rate of change between $t = 1$ minute and $t = 1.01$ minute:

$$
\left( \frac{\Delta V}{\Delta t} \right) \bigg|_{t=1} \approx \left( \frac{\Delta V}{\Delta t} \right)_{\text{avg}} [1, 1.01]
$$

$$
= \frac{V(1.01) - V(1)}{1.01 - 1}
$$

$$
= \frac{\left( \frac{1}{3.01} \right)^3 - \left( \frac{1}{3} \right)^3}{0.01}
$$

$$
\approx -0.037\text{in}^3/\text{min}.
$$
Slope of a Secant Line

- Consider the graph of \( y = f(x) \) and two points on the graph \((x_0, f(x_0))\) and \((x_1, f(x_1))\);

- The line passing through these two points is called the **secant line** to \( y = f(x) \) through \( x_0 \) and \( x_1 \);

- Its slope is equal to

\[
m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.
\]
An Example

Example: Find an equation for the secant line to \( f(x) = \frac{1}{1+x^2} \) through \( x_0 = 1 \) and \( x_1 = 2 \);

We have

\[
m_f[1, 2] = \frac{f(2) - f(1)}{2 - 1} = \frac{\frac{1}{5} - \frac{1}{2}}{2 - 1} = -\frac{3}{10}.
\]

Therefore \( y - \frac{1}{2} = -\frac{3}{10}(x - 1) \) is the point-slope form of the equation of the secant line.
Slope of a Tangent Line

To approximate the slope $m_f(x_0)$ of the tangent line to the graph of $y = f(x)$ at $x_0$ we use a process similar to that approximating the instantaneous rate of change by using the average rate of change for points $x_0, x_1$ very close to each other;

$$m_f(x_0) \approx m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Therefore, we have

$$m_f(x_0) \underset{\Delta x \text{ small}}{\approx} m_f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$
Approximating the Slope of a Tangent Line

Let us approximate the slope to \( y = x^2 \) at \( x = 1 \) using the process outlined in the previous slide;

We have

\[
m_f(1) \approx m_f[1, 1.01] = \frac{f(1.01) - f(1)}{1.01 - 1} = \frac{(1.01)^2 - 1^2}{0.01} = 2.01.
\]
Subsection 2

A Graphical Approach to Limits
Definition of Limit

- Suppose that \( f(x) \) is defined in an open interval containing a number \( c \), but not necessarily \( c \) itself;
- The **limit of \( f(x) \) as \( x \) approaches \( c \) is equal to \( L \) if \( f(x) \) has value arbitrarily close to \( L \) when \( x \) assumes values sufficiently close (but not equal) to \( c \).
- In this case, we write
  \[
  \lim_{{x \to c}} f(x) = L.
  \]
- An alternative terminology is that \( f(x) \) **approaches** or **converges to** \( L \) as \( x \) **approaches** \( c \).
Two Easy Examples

- Draw the graph of $f(x) = 3$ and find graphically the limit $\lim_{x \to c} f(x)$.
- Draw the graph of $g(x) = \frac{1}{2}x + 4$ and find graphically $\lim_{x \to 2} f(x)$.

We have $\lim_{x \to c} 3 = 3$ and $\lim_{x \to 2} \left(\frac{1}{2}x + 4\right) = 5$. 
Two Easy Rules

- Draw the graph of \( f(x) = k \) (a constant) and find graphically the limit \( \lim_{x \to c} k \).
- Draw the graph of \( g(x) = x \) and find graphically \( \lim_{x \to c} x \).

We have \( \lim_{x \to c} k = k \) and \( \lim_{x \to c} x = c \).
Two More Complicated Examples

• Draw the graph of \( f(x) = \frac{x-9}{\sqrt{x-3}} \) and find graphically the limit \( \lim_{x \to 9} f(x) \).

• Draw the graph of \( g(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ -x^2 + 2x + 3, & \text{if } x > 1 \end{cases} \) and find graphically \( \lim_{x \to 1} f(x) \).

We have \( \lim_{x \to c} \frac{x-9}{\sqrt{x-3}} = 6 \) and \( \lim_{x \to 1} g(x) \) does not exist since \( g(x) \) does not approach a single number when \( x \) approaches 1.
Two Additional Examples

- Draw the graph of $f(x) = \frac{e^x - 1}{x}$ and find graphically the limit $\lim_{x \to 0} f(x)$.
- Draw the graph of $g(x) = \sin\frac{\pi}{x}$ and find graphically $\lim_{x \to 0} g(x)$.

We have $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$ and $\lim_{x \to 0} \sin\frac{\pi}{x}$ does not exist since the values of $g(x) = \sin\frac{\pi}{x}$ oscillate between $-1$ and $1$ as $x$ approaches $0$. 
Definition of Side-Limits

- Suppose that \( f(x) \) is defined in an open interval containing a number \( c \), but not necessarily \( c \) itself;
- The **right-hand limit of \( f(x) \) as \( x \) approaches \( c \) (from the right) is equal to \( L \)** if \( f(x) \) has value arbitrarily close to \( L \) when \( x \) approaches sufficiently close (but is not equal) to \( c \) from the right hand side. In this case, we write \( \lim_{x \to c^+} f(x) = L \).
- The **left-hand limit of \( f(x) \) as \( x \) approaches \( c \) (from the left) is equal to \( L \)** if \( f(x) \) has value arbitrarily close to \( L \) when \( x \) approaches sufficiently close (but is not equal) to \( c \) from the left hand side. In this case, we write \( \lim_{x \to c^-} f(x) = L \).
- The limits we saw before are “two sided limits”; It is the case that \( \lim_{x \to c} f(x) = L \) if and only if \( \lim_{x \to c^+} f(x) = L \) and \( \lim_{x \to c^-} f(x) = L \), i.e., a function has limit \( L \) as \( x \) approaches \( c \) if and only if the left and right hand side limits as \( x \) approaches \( c \) exist and are equal.
Two Examples

- Draw the graph of \( f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ -x^2 + 2x + 3, & \text{if } x > 1 \end{cases} \) and find graphically \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^+} f(x) \).
- Draw the graph of \( g(x) = \begin{cases} -(x + 2)^3 + 2, & \text{if } x < -1 \\ -x^2 + 1, & \text{if } x > -1 \end{cases} \) and find graphically \( \lim_{x \to -1^-} g(x) \) and \( \lim_{x \to -1^+} g(x) \).

\[
\lim_{x \to 1^-} f(x) = 1, \quad \lim_{x \to 1^+} f(x) = 4, \quad \text{so } \lim_{x \to 1} f(x) \text{ DNE, and}
\]
\[
\lim_{x \to -1^-} g(x) = 1, \quad \lim_{x \to -1^+} g(x) = 0, \quad \text{so } \lim_{x \to -1} g(x) \text{ DNE.}
\]
Examples of Limits Involving Infinity

- Draw the graph of $f(x) = \frac{1}{x-2}$ and find graphically $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$.
- Draw the graph of $g(x) = \ln x$ and find graphically $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to +\infty} g(x)$.

$\lim_{x \to 2^-} f(x) = -\infty$, $\lim_{x \to 2^+} f(x) = +\infty$, and $\lim_{x \to 0^+} g(x) = -\infty$, $\lim_{x \to +\infty} g(x) = +\infty$. 
Subsection 3

Basic Limit Laws
Theorem (Basic Limit Laws)

Suppose that \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) exist. Then

- **Sum Law:** \( \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) \);

- **Constant Factor Law:** \( \lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) \);

- **Product Law:** \( \lim_{x \to c} f(x)g(x) = (\lim_{x \to c} f(x))(\lim_{x \to c} g(x)) \);

- **Quotient Law:** If \( \lim_{x \to c} g(x) \neq 0 \), then \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \);

- **Power and Root Law:** For \( p, q \) integers, with \( q \neq 0 \),
  \( \lim_{x \to c} [f(x)]^{p/q} = (\lim_{x \to c} f(x))^{p/q} \), under the assumption that
  \( \lim_{x \to c} f(x) \geq 0 \) if \( q \) is even and \( \lim_{x \to c} f(x) \neq 0 \) if \( \frac{p}{q} < 0 \).

  In particular, for \( n \) a positive integer,
  - \( \lim_{x \to c} [f(x)]^n = (\lim_{x \to c} f(x))^n \);
  - \( \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \);
Examples of Calculating Limits I

- Compute \( \lim_{x \to 2} x^3; \)
  
  We apply the power rule:

  \[
  \lim_{x \to 2} (x^3) = (\lim_{x \to 2} x)^3 = 2^3 = 8.
  \]

- Compute \( \lim_{x \to -1} (-2x^3 + 7x - 5); \)
  
  We apply the sum rule, the constant factor and the power rules:

  \[
  \begin{align*}
  \lim_{x \to -1} (-2x^3 + 7x - 5) &= \lim_{x \to -1} (-2x^3) + \lim_{x \to -1} (7x) - \lim_{x \to -1} 5 \\
  &= -2 \lim_{x \to -1} (x^3) + 7 \lim_{x \to -1} x - \lim_{x \to -1} 5 \\
  &= -2 \cdot (-1)^3 + 7(-1) - 5 \\
  &= -10.
  \end{align*}
  \]
Examples of Calculating Limits II

- Compute \( \lim_{x \to 2} \frac{x + 30}{2x^4} \);
  
  We apply the quotient rule:
  
  \[
  \lim_{x \to 2} \frac{x + 30}{2x^4} = \frac{\lim_{x \to 2} (x + 30)}{\lim_{x \to 2} (2x^4)} = \frac{2 + 30}{2 \cdot 2^4} = 1.
  \]

- Compute \( \lim_{x \to 3} (x^{-1/4}(x + 5)^{1/3}) \);
  
  We apply the product and the power rules:
  
  \[
  \lim_{x \to 3} (x^{-1/4}(x + 5)^{1/3}) = (\lim_{x \to 3} x^{-1/4})(\lim_{x \to 3} (x + 5)^{1/3})
  = ((\lim_{x \to 3} x)^{-1/4})(\lim_{x \to 3} (x + 5)^{1/3})
  = 3^{-1/4} \sqrt[3]{\lim_{x \to 3} (x + 5)}
  = 3^{-1/4} \sqrt[3]{8}
  = \frac{2}{\sqrt[3]{3}}.
  \]
Treacherous Applications of the Laws

- We must take the hypotheses of the Basic Limit Laws into account when applying the rules;
- For instance, if \( f(x) = x \) and \( g(x) = x^{-1} \), then

\[
\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} xx^{-1} = \lim_{x \to 0} 1 = 1,
\]

but, if we tried to apply the product rule, we would be stuck:

\[
\lim_{x \to 0} f(x)g(x) = (\lim_{x \to 0} x)(\lim_{x \to 0} x^{-1}),
\]

The last limit on the right does not exist since \( \lim_{x \to 0^+} x^{-1} = +\infty \) and \( \lim_{x \to 0^-} x^{-1} = -\infty \).
Subsection 4

Limits and Continuity
A function \( f(x) \) defined on an open interval containing \( x = c \) is **continuous at** \( x = c \) if

\[
\lim_{{x \to c}} f(x) = f(c).
\]

If either the limit does not exist, or exists but is not equal to \( f(c) \), then \( f \) has a **discontinuity** or is **discontinuous** at \( x = c \).

Not that the limit above exists if and only if

\[
\lim_{{x \to c^-}} f(x) = \lim_{{x \to c^+}} f(x);
\]

Therefore, the condition for continuity is equivalent to

\[
\lim_{{x \to c^-}} f(x) = f(c) = \lim_{{x \to c^+}} f(x).
\]

**Example:** Let \( f(x) = k \) a constant. Recall that \( \lim_{{x \to c^-}} k = k \). Also \( f(c) = k \). Therefore, \( f(x) = k \) is continuous at all \( x = c \).
Some Additional Examples

- Consider \( f(x) = x^n \), where \( n \) is a natural number. Then \[ \lim_{x \to c} x^n = (\lim_{x \to c} x)^n = c^n. \] Also \( f(c) = c^n \). Therefore, \( f(x) = x^n \) is continuous at all \( x = c \).
- Consider \( f(x) = x^5 + 7x - 12 \). Applying some of the Limit Laws, we get
  \[
  \lim_{x \to c} (x^5 + 7x - 12) = (\lim_{x \to c} x)^5 + 7(\lim_{x \to c} x) - \lim_{x \to c} 12
  = c^5 + 7c - 12
  = f(c).
  \]
  Therefore \( f(x) \) is continuous at \( x = c \).
- Consider also \( f(x) = \frac{x^2+5}{x+3} \). Applying some of the Limit Laws, we get
  \[
  \lim_{x \to 2} \frac{x^2+5}{x+3} = \frac{\lim_{x \to 2} (x^2+5)}{\lim_{x \to 2} (x+3)} = \frac{\lim_{x \to 2} x^2 + \lim_{x \to 2} 5}{\lim_{x \to 2} x + \lim_{x \to 2} 3}
  = \frac{2^2 + 5}{2 + 3} = f(2).
  \]
  Thus \( f(x) \) is continuous at \( x = 2 \).
Types of Discontinuities

- Recall $f(x)$ is continuous at $x = c$ if

$$
\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).
$$

- If $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$, then $f(x)$ has a **removable discontinuity** at $x = c$;

- If $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$ (in this case, of course, $\lim_{x \to c} f(x)$ does not exist), then $f$ has a **jump discontinuity** at $x = c$;

- If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ is infinite, then $f$ has an **infinite discontinuity** at $x = c$. 
Consider the piece-wise defined function

\[ f(x) = \begin{cases} 
  e^{x+1}, & \text{if } x < -1 \\
  2, & \text{if } x = -1 \\
  -x^2 + 2, & \text{if } x > -1 
\end{cases} \]

We have \( \lim_{x \to -1^-} f(x) = 1 \) and \( \lim_{x \to -1^+} f(x) = 1 \), whence \( \lim_{x \to -1} f(x) = 1 \). But \( f(-1) = 2 \). So \( \lim_{x \to -1} f(x) \) exists, but it does not equal \( f(-1) \). This shows that \( f(x) \) has a removable discontinuity at \( x = -1 \).
Consider the piece-wise defined function

\[ f(x) = \begin{cases} 
  x + 1, & \text{if } x < 1 \\
  -x^2 + 2x, & \text{if } x \geq 1 
\end{cases} \]

We have \( \lim_{x \to 1^-} f(x) = 2 \) and \( \lim_{x \to 1^+} f(x) = 1 \), whence \( \lim_{x \to 1} f(x) = \text{DNE} \). So the side limits of \( f(x) \) as \( x \) approaches 1 exist, but they are not equal. This shows that \( f(x) \) has a jump discontinuity at \( x = 1 \).
Infinite Discontinuity

Consider the piece-wise defined function

\[ f(x) = \begin{cases} 
\frac{1}{x^2 - 2x + 2}, & \text{if } x < 1 \\
\frac{1}{x - 1}, & \text{if } x > 1 
\end{cases} \]

We have \( \lim_{x \to 1^-} f(x) = 1 \) and \( \lim_{x \to 1^+} f(x) = +\infty \). Thus, at least one of the side limits as \( x \) approaches \( 1 \) is \( \pm \infty \). This shows that \( f(x) \) has an infinite discontinuity at \( x = 1 \).
One-Sided Continuity

A function $f(x)$ is called

- **left-continuous** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$;
- **right-continuous** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$;

**Example:** Consider the function

$$f(x) = \begin{cases} 
-x^2 - 2x, & \text{if } x < 0 \\
\frac{1}{x+1}, & \text{if } x \geq 0
\end{cases}$$

We have $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$. Moreover, $f(0) = 1$. Therefore $f(x)$ is right-continuous at $x = 0$, but not left continuous at $x = 0$. 
One More Example

Consider the piece-wise defined function

\[ f(x) = \begin{cases} 
  \frac{\sin x}{x}, & \text{if } x < 0 \\
  1, & \text{if } x = 0 \\
  \ln x, & \text{if } x > 0 
\end{cases} \]

We have \( \lim_{x \to 0^-} f(x) = 1 \) and \( \lim_{x \to 0^+} f(x) = -\infty \). Moreover, \( f(0) = 1 \). Therefore, \( f(x) \) is left-continuous at \( x = 0 \), but not right-continuous at \( x = 0 \).
Basic Continuity Laws

Theorem (Basic Laws of Continuity)

If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following functions are also continuous at $x = c$:

1. $f(x) \pm g(x)$
2. $k f(x)$
3. $f(x)g(x)$
4. $\frac{f(x)}{g(x)}$, if $g(c) \neq 0$.

For instance, knowing that $f(x) = x$ and $g(x) = k$ are continuous functions at all real numbers, the previous rules allow us to conclude that

- any polynomial function $P(x)$ is continuous at all real numbers;
- any rational function $\frac{P(x)}{Q(x)}$ is continuous at all values in its domain.

Example: $f(x) = 3x^4 - 2x^3 + 8x$ is continuous at all real numbers. $g(x) = \frac{x+3}{x^2-1}$ is continuous at all numbers $x \neq \pm 1$. 
Continuity of Roots, Trig, Exp and Log Functions

Theorem (Continuity of Various Functions)

- \( f(x) = \sqrt[n]{x} \) is continuous on its domain;
- \( f(x) = \sin x \) and \( g(x) = \cos x \) are continuous at all real numbers;
- \( f(x) = b^x \) is continuous at all real numbers \((0 < b \neq 1)\);
- \( f(x) = \log_b x \) is continuous at all \( x > 0 \) \((0 < b \neq 1)\);

Based on this theorem and the theorem on quotients, we may conclude, for example, that \( \tan x = \frac{\sin x}{\cos x} \) is continuous at all points in its domain, i.e., at all \( x \neq (2k + 1)\frac{\pi}{2}, k \in \mathbb{Z} \).

We can also conclude that \( \csc x = \frac{1}{\sin x} \) is continuous at all points in its domain, i.e., at all \( x \neq k\pi, k \in \mathbb{Z} \).
Limits and Continuity

Continuity of Inverse Functions

Theorem (Continuity of Inverse Functions)

If \( f(x) \) is continuous on an interval \( I \) with range \( R \), then if \( f^{-1}(x) \) exists, then \( f^{-1}(x) \) is continuous with domain \( R \).

- For instance \( f(x) = \sin x \) is continuous on \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) with range \([-1, 1]\) and has an inverse; So, \( f^{-1}(x) = \sin^{-1} x \) is continuous on \([-1, 1]\).
- Similarly \( g(x) = \tan x \) is continuous on \(( -\frac{\pi}{2}, \frac{\pi}{2}) \) with range \( \mathbb{R} \) and has an inverse; Therefore \( g^{-1}(x) = \tan^{-1} x \) is continuous on \( \mathbb{R} \).
Continuity of Composite Functions

Theorem (Continuity of Composite Functions)

If \( g(x) \) is continuous at the point \( x = c \) and \( f(x) \) is continuous at the point \( x = g(c) \), then the function \( F(x) = f(g(x)) \) is continuous at \( x = c \).

For instance, the function \( g(x) = x^2 + 9 \) is continuous at all real numbers, since it is a polynomial function; Moreover, the function \( f(x) = \sqrt[3]{x} \) is continuous at all real numbers as a root function; Therefore, the function \( F(x) = f(g(x)) = \sqrt[3]{x^2 + 9} \) is also a continuous function, as the composite of two continuous functions.
Substitution Method: Using Continuity to Evaluate Limits

- Recall that \( f(x) \) is continuous at \( x = c \) if

\[
\lim_{x \to c} f(x) = f(c).
\]

- Suppose that you know that \( f(x) \) is continuous at \( x = c \) and want to compute \( \lim_{x \to c} f(x) \).

Then, because of the definition of continuity, to find \( \lim_{x \to c} f(x) \), you may compute, instead, \( f(c) \).

- This is called the **substitution property** (or **method**) for evaluating limits of continuous functions.
Examples of Using the Substitution Method

Example: Let us evaluate the limit \( \lim_{x \to \pi/3} \sin x \).
Since \( f(x) \) is continuous (by the basic theorem on trig functions) at all \( x \in \mathbb{R} \), we may use the substitution property:

\[
\lim_{x \to \pi/3} \sin x = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.
\]

Example: Let us evaluate the limit \( \lim_{x \to -1} \frac{3^x}{\sqrt{x+5}} \).
Since \( f(x) \) is continuous (as a ratio of an exponential over a root function, both of which are continuous in their domain), we may use the substitution property:

\[
\lim_{x \to -1} \frac{3^x}{\sqrt{x+5}} = \frac{3^{-1}}{\sqrt{-1+5}} = \frac{1}{6}.
\]
Subsection 5

Algebraic Evaluation of Limits
The following are **Indeterminate Forms**:

- $0 \div 0$
  - Example: $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$

- $\infty \div \infty$
  - Example: $\lim_{x \to \pi/2} \frac{\tan x}{\sec x}$

- $\infty \cdot 0$
  - Example: $\lim_{x \to 2} \frac{1}{2x - 4} \cdot (x - 2)^2$

- $\infty - \infty$
  - Example: $\lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right)$
The Indeterminate Form $\frac{0}{0}$: Factor and Cancel

To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

**Example:** Compute $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$;

We have

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12} = \lim_{x \to 3} \frac{(x - 1)(x - 3)}{(x + 4)(x - 3)}$$

$$= \lim_{x \to 3} \frac{x - 1}{x + 4}$$

$$= \frac{3 - 1}{3 + 4}$$

$$= \frac{2}{7}.$$
The Indeterminate Form \( \frac{0}{0} \): Another Example

To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

**Example:** Compute \( \lim_{x \to 7} \frac{x - 7}{x^2 - 49} \);

We have

\[
\lim_{x \to 7} \frac{x - 7}{x^2 - 49} = \lim_{x \to 7} \frac{x - 7}{(x + 7)(x - 7)} = \lim_{x \to 7} \frac{1}{x + 7} = \frac{1}{7 + 7} = \frac{1}{14}.
\]
The Indeterminate Form $\frac{\infty}{\infty}$

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

**Example:** Compute $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x}$;

We have

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x}$$

$$= \sin \frac{\pi}{2}$$

$$= 1.$$
The Indeterminate Form $\frac{0}{0}$: Multiply by Conjugate

To lift the indeterminancy, we transform algebraically, cancel and, finally, use the substitution property;

Example: Compute $\lim_{x\to4} \frac{\sqrt{x} - 2}{x - 4}$;

We have

$$\lim_{x\to4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to4} \frac{\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)}$$

$$= \lim_{x\to4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)}$$

$$= \lim_{x\to4} \frac{1}{\sqrt{x} + 2}$$

$$= \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.$$
The Indeterminate Form \(\frac{0}{0}:\) Multiply by Conjugate

To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

**Example:** Compute \(\lim_{{x \to 7}} \frac{x - 7}{\sqrt{x + 9} - 4};\)

\[
\lim_{{x \to 7}} \frac{x - 7}{\sqrt{x + 9} - 4} = \lim_{{x \to 7}} \frac{(x - 7)(\sqrt{x + 9} + 4)}{(\sqrt{x + 9} - 4)(\sqrt{x + 9} + 4)}
\]

\[
= \lim_{{x \to 7}} \frac{(x - 7)(\sqrt{x + 9} + 4)}{x + 9 - 16}
\]

\[
= \lim_{{x \to 7}} \frac{(x - 7)(\sqrt{x + 9} + 4)}{x - 7}
\]

\[
= \sqrt{7 + 9 + 4} = 8.
\]
The Indeterminate Form $\infty - \infty$

- To lift the indeterminacy, we transform algebraically, cancel and, finally, use the substitution property;

**Example**: Compute $\lim_{x \to 2} \left( \frac{1}{x - 2} - \frac{4}{x^2 - 4} \right)$;

$$
\lim_{x \to 2} \left( \frac{1}{x - 2} - \frac{4}{x^2 - 4} \right) = \lim_{x \to 2} \left( \frac{x + 2}{(x - 2)(x + 2)} - \frac{4}{(x - 2)(x + 2)} \right) = \lim_{x \to 2} \frac{x + 2 - 4}{(x + 2)(x - 2)} = \lim_{x \to 2} \frac{x - 2}{(x + 2)(x - 2)} = \lim_{x \to 2} \frac{1}{x + 2} = \frac{1}{4}.
$$
Forms $\frac{c}{0}$, with $c \neq 0$ are Infinite but not Indeterminate

- $\lim_{{x \to 2}} \frac{x^2 - x + 5}{x - 2}$ is of the form $\frac{7}{0}$;
- These forms are not indeterminate, but rather they suggest that the side-limits as $x \to 2$ are infinite;
- If $x \to 2^-$, then $x < 2$, whence $x - 2 < 0$. Thus,
  \[ \lim_{{x \to 2^-}} \frac{x^2 - x + 5}{x - 2} = \frac{7}{0^-} = -\infty; \]
- If $x \to 2^+$, then $x > 2$, whence $x - 2 > 0$. Thus,
  \[ \lim_{{x \to 2^+}} \frac{x^2 - x + 5}{x - 2} = \frac{7}{0^+} = \infty; \]
Subsection 6

Trigonometric Limits
The Squeeze Theorem

Assume that for \( x \neq c \) in some open interval containing \( c \),

\[
\ell(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = L.
\]

Then \( \lim_{x \to c} f(x) \) exists and \( \lim_{x \to c} f(x) = L \).

**Example:** We show \( \lim_{x \to 0} (x \sin \frac{1}{x}) = 0 \).

Note that \( -|x| \leq x \sin \frac{1}{x} \leq |x| \);

Note, also that

\[
\lim_{x \to 0} (-|x|) = \lim_{x \to 0} |x| = 0;
\]

Therefore, by Squeeze,

\[
\lim_{x \to 0} (x \sin \frac{1}{x}) = 0.
\]
An Important Squeeze Identity

Theorem

For all $\theta \neq 0$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have

$$\cos \theta \leq \frac{\sin \theta}{\theta} \leq 1.$$
Important Trigonometric Limits

We have

\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0. \]

- Note that the first limit above follows by the Squeeze Theorem using the Squeeze Identity of the previous slide;
- For the second one, we have

\[
\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = \lim_{\theta \to 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{\theta(1 + \cos \theta)} = \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} = \lim_{\theta \to 0} \left( \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} \right) = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{1 + \cos \theta} = 1 \cdot \frac{0}{1+1} = 0.
\]
Evaluation of Limits by a Change of Variable

• Compute the limit \( \lim_{\theta \to 0} \frac{\sin 4\theta}{\theta} \);
  We have
  \[
  \lim_{\theta \to 0} \frac{\sin 4\theta}{\theta} = \lim_{\theta \to 0} \frac{4 \sin 4\theta}{4\theta} = 4 \lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta} = 4 \cdot 1 = 4.
  \]

• Compute the limit \( \lim_{\theta \to 0} \frac{\sin 7\theta}{\sin 3\theta} \);
  We have
  \[
  \lim_{\theta \to 0} \frac{\sin 7\theta}{\sin 3\theta} = \lim_{\theta \to 0} \frac{7 \theta \sin 7\theta}{3 \theta \sin 3\theta} = \lim_{\theta \to 0} \frac{7 \sin 7\theta}{3 \sin 3\theta} = \frac{7}{3} \lim_{\theta \to 0} \frac{\sin \frac{7\theta}{3}}{\sin \frac{3\theta}{3}} = \frac{7}{3} \lim_{\frac{7\theta}{3} \to 0} \frac{\sin \frac{7\theta}{3}}{\sin \frac{7\theta}{3}} \cdot \frac{7}{3} \lim_{\frac{3\theta}{3} \to 0} \frac{\sin \frac{3\theta}{3}}{\sin \frac{3\theta}{3}} = \frac{7}{3} \cdot 1 = \frac{7}{3}.
  \]
Subsection 7

Limits at Infinity
Limits at Infinity

#### Limit of $f(x)$ as $x \to \pm \infty$

- We write $\lim_{x \to \infty} f(x) = L$ if $f(x)$ gets closer and closer to $L$ as $x \to \infty$, i.e., as $x$ increases without bound;
- We write $\lim_{x \to -\infty} f(x) = L$ if $f(x)$ gets closer and closer to $L$ as $x \to -\infty$, i.e., as $x$ decreases without bound;

In either case, the line $y = L$ is called a **horizontal asymptote** of $y = f(x)$.

- Horizontal asymptotes describe the asymptotic behavior of $f(x)$, i.e., the behavior of the graph as we move way out to the left or to the right.
Example of Limits at Infinity

Consider the function \( f(x) \) whose graph is given on the right:

We have

\[
\lim_{x \to -\infty} f(x) = 1
\]

and

\[
\lim_{x \to \infty} f(x) = 2.
\]

Thus, both \( y = 1 \) and \( y = 2 \) are horizontal asymptotes of \( y = f(x) \).
Powers of $x$

**Theorem**

Assume $n > 0$. Then we have

\[
\lim_{x \to \infty} x^n = \infty \quad \text{and} \quad \lim_{x \to \infty} x^{-n} = \lim_{x \to \infty} \frac{1}{x^n} = 0.
\]

For $n > 0$ an integer,

\[
\lim_{x \to -\infty} x^n = \begin{cases} 
\infty, & \text{if } n \text{ is even} \\
-\infty, & \text{if } n \text{ is odd}
\end{cases}
\quad \text{and} \quad \lim_{x \to -\infty} x^{-n} = \lim_{x \to -\infty} \frac{1}{x^n} = 0.
\]

**Example:**

\[
\lim_{x \to \infty} (3 - 4x^{-3} + 5x^{-5}) = \\
\lim_{x \to \infty} 3 - 4 \lim_{x \to \infty} x^{-3} + 5 \lim_{x \to \infty} x^{-5} = 3 - 4 \cdot 0 + 5 \cdot 0 = 3.
\]
Example

Calculate \( \lim_{x \rightarrow \pm \infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} \).

We follow the method of dividing numerator and denominator by the highest power \( x^5 \):

\[
\lim_{x \rightarrow \pm \infty} \frac{20x^2 - 3x}{3x^5 - 4x^2 + 5} = \lim_{x \rightarrow \pm \infty} \frac{\frac{20x^2}{x^5} - \frac{3x}{x^5}}{\frac{3x^5}{x^5} - \frac{4x^2}{x^5} + \frac{5}{x^5}}
\]

\[
= \lim_{x \rightarrow \pm \infty} \frac{\frac{20}{x^3} - \frac{3}{x^4}}{3 - \frac{4}{x^3} + \frac{5}{x^5}}
\]

\[
= \lim_{x \rightarrow \pm \infty} \frac{20}{\frac{3}{x^3} - \frac{4}{x^3} + \frac{5}{x^5}} - \lim_{x \rightarrow \pm \infty} \frac{3}{x^4}
\]

\[
= \lim_{x \rightarrow \pm \infty} \frac{20}{3 - 0 + 0} - \lim_{x \rightarrow \pm \infty} \frac{3}{x^4} + \lim_{x \rightarrow \pm \infty} \frac{5}{x^5}
\]

\[
= \frac{0 - 0}{3 - 0 + 0} = 0.
\]
Limits at Infinity of Rational Functions

**Theorem**

If \( a_n, b_m \neq 0 \), then it is the case that

\[
\lim_{x \to \pm \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x \to \pm \infty} x^{n-m}.
\]

**Example:**

- \( \lim_{x \to \infty} \frac{3x^4 - 7x + 9}{7x^4 - 4} = \frac{3}{7} \lim_{x \to \infty} x^0 = \frac{3}{7} \);
- \( \lim_{x \to \infty} \frac{3x^3 - 7x + 9}{7x^4 - 4} = \frac{3}{7} \lim_{x \to \infty} x^{-1} = \frac{3}{7} \lim_{x \to \infty} \frac{1}{x} = 0 \);
- \( \lim_{x \to -\infty} \frac{3x^8 - 7x + 9}{7x^3 - 4} = \frac{3}{7} \lim_{x \to -\infty} x^5 = -\infty \);
- \( \lim_{x \to -\infty} \frac{3x^7 - 7x + 9}{7x^3 - 4} = \frac{3}{7} \lim_{x \to -\infty} x^4 = \infty \);
Two More Examples

- Compute the limit \( \lim_{x \to \infty} \frac{3x^{7/2} + 7x^{-1/2}}{x^2 - x^{1/2}} \);

  We have

  \[
  \lim_{x \to \infty} \frac{3x^{7/2} + 7x^{-1/2}}{x^2 - x^{1/2}} = \lim_{x \to \infty} \frac{(x^{-2})(3x^{7/2} + 7x^{-1/2})}{(x^{-2})(x^2 - x^{1/2})} = \lim_{x \to \infty} \frac{3x^{3/2} + 7x^{-5/2}}{1 - x^{-3/2}} = \lim_{x \to \infty} \frac{3x^{3/2} + \lim_{x \to \infty} 7x^{-5/2}}{\lim_{x \to \infty} 1 - \lim_{x \to \infty} x^{-3/2}} = \frac{\infty}{1} = \infty.
  \]

- Compute the limit \( \lim_{x \to \infty} \frac{x^2}{\sqrt{x^3 + 1}} \);

  We have

  \[
  \lim_{x \to \infty} \frac{x^2}{\sqrt{x^3 + 1}} = \lim_{x \to \infty} \frac{x^{-3/2}x^2}{x^{-3/2}\sqrt{x^3 + 1}} = \lim_{x \to \infty} \frac{x^{1/2}}{\sqrt{x^{-3}(x^3 + 1)}} = \lim_{x \to \infty} \frac{x^{1/2}}{\sqrt{1 + x^{-3}}} = \frac{\infty}{1} = \infty.
  \]
One More Example

Calculate the limits at infinity of \( f(x) = \frac{12x + 25}{\sqrt{16x^2 + 100x + 500}} \);

We have

\[
\lim_{x \to -\infty} \frac{12x + 25}{\sqrt{16x^2 + 100x + 500}} = \lim_{x \to -\infty} \frac{12x(1 + \frac{25}{12x})}{\sqrt{16x^2(1 + \frac{100}{16x} + \frac{500}{16x^2})}}
\]

\[
= \lim_{x \to -\infty} \frac{12x(1 + \frac{25}{12x})}{-4x \sqrt{1 + \frac{100}{16x} + \frac{500}{16x^2}}}
\]

\[
= -3 \lim_{x \to -\infty} \frac{1 + \frac{25}{12x}}{\sqrt{1 + \frac{100}{16x} + \frac{500}{16x^2}}}
\]

\[
= -3.
\]

\[
\lim_{x \to \infty} \frac{12x + 25}{\sqrt{16x^2 + 100x + 500}} = \lim_{x \to \infty} \frac{12x(1 + \frac{25}{12x})}{\sqrt{16x^2(1 + \frac{100}{16x} + \frac{500}{16x^2})}}
\]

\[
= \lim_{x \to \infty} \frac{12x(1 + \frac{25}{12x})}{4x \sqrt{1 + \frac{100}{16x} + \frac{500}{16x^2}}}
\]

\[
= 3 \lim_{x \to \infty} \frac{1 + \frac{25}{12x}}{\sqrt{1 + \frac{100}{16x} + \frac{500}{16x^2}}} = 3.
\]
Subsection 8

Intermediate Value Theorem
The Intermediate Value Theorem

Intermediate Value Theorem

If \( f(x) \) is continuous on a closed interval \([a, b]\) and \( f(a) \neq f(b) \), then for every value \( M \) between \( f(a) \) and \( f(b) \), there exists at least one value \( c \in (a, b) \), such that \( f(c) = M \).

Example: Show that \( \sin x = \frac{1}{8} \) has at least one solution.

Consider \( f(x) = \sin x \) in the closed interval \([0, \frac{\pi}{2}]\). We have \( f(0) = 0 < \frac{1}{8} < 1 = f\left(\frac{\pi}{2}\right) \). Thus, by the Intermediate Value Theorem, there exists \( c \in (0, \frac{\pi}{2}) \), such that \( f(c) = \frac{1}{8} \), i.e., \( \sin c = \frac{1}{8} \). This \( c \) is a solution of the equation \( \sin x = \frac{1}{8} \).
Existence of Zeros

If $f(x)$ is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then $f(x)$ has a zero in $(a, b)$. 
Show that the equation $2^x + 3^x = 4^x$ has at least one zero.

Consider $f(x) = 2^x + 3^x - 4^x$ in the closed interval $[1, 2]$.

We have $f(1) = 1 > 0$, whereas $f(2) = -3 < 0$.

Thus, by the Existence of Zeros Theorem, there exists $c \in (1, 2)$, such that $f(c) = 0$, i.e., $2^c + 3^c - 4^c = 0$.

But, then, $c$ satisfies $2^c + 3^c = 4^c$, i.e., it is a zero of $2^x + 3^x = 4^x$. 

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The Bisection Method

- Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$;
- Consider the function $f(x) = x^7 + 3x - 10$ in $[1, 2]$. Since $f(1) = -6 < 0$ and $f(2) = 112 > 0$, by the Existence of Zeros Theorem, it has a root in $(1, 2)$.

Since $f(1) = -6 < 0$ and $f\left(\frac{3}{2}\right) = 11.586 > 0$, it has a root in the interval $(1, \frac{3}{2})$.

Finally, since $f\left(\frac{5}{4}\right) = -1.482 < 0$ and $f\left(\frac{3}{2}\right) = 11.586 > 0$, the root is in the interval $\left(\frac{5}{4}, \frac{3}{2}\right)$, which has length $\frac{1}{4}$. 