Calculus I

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

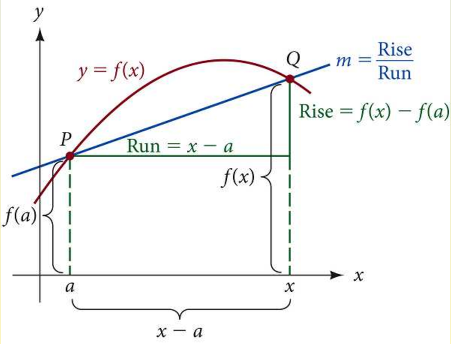
LSSU Math 151

Subsection 1

Definition of the Derivative

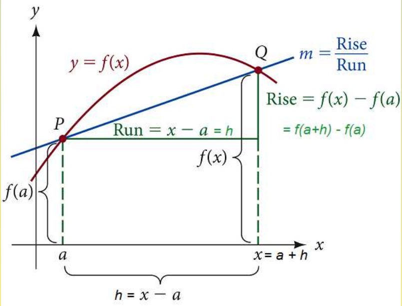
Difference Quotient

- Consider the graph of the function $f(x)$ and two points $P(a, f(a))$ and $Q(x, f(x))$;
- The **difference quotient** is the expression $\dfrac{f(x) - f(a)}{x - a}$;
- It represents the slope of the **secant line** to $y = f(x)$ through the points P and Q;



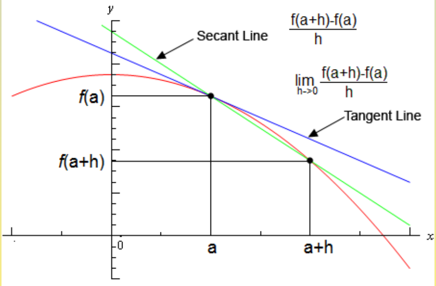
Alternative Expression for Difference Quotient

- The difference quotient of f from $P(a, f(a))$ to $Q(x, f(x))$ is
$$\frac{f(x) - f(a)}{x - a};$$
- If we set $h = x - a$, then $x = a + h$, and the difference quotient can be rewritten in the form: $$\frac{f(a+h) - f(a)}{h};$$



The Derivative $f'(a)$ of $f(x)$ at $x = a$

- The **derivative** of $f(x)$ at $x = a$ is the limit of the difference quotients (if it exists)
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h};$$
- When the limit exists, f is called **differentiable** at $x = a$;
- Another expression is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$;
- The derivative $f'(a)$ represents the slope of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$;



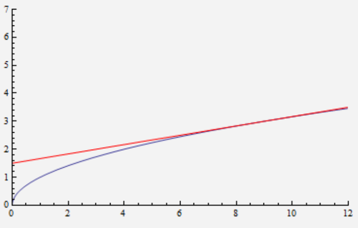
Computing a Derivative

- Compute $f'(3)$ if $f(x) = x^2 - 8x$;
  We have

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2-8(3+h)-(3^2-8\cdot 3)}{h} \\ &= \lim_{h\to 0} \frac{9+6h+h^2-24-8h+15}{h} \\ &= \lim_{h\to 0} \frac{h^2-2h}{h} \\ &= \lim_{h\to 0} \frac{h(h-2)}{h} \\ &= \lim_{h\to 0} (h-2) \\ &= -2. \end{aligned}$$

Computing the Slope of a Tangent Line

- Find the slope of the tangent line to the graph of $f(x) = \sqrt{x}$ at $x = 9$.



We have

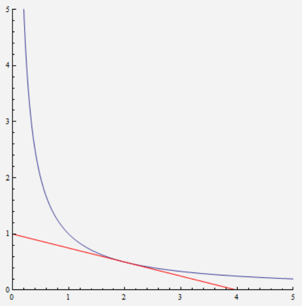
$$\begin{aligned} f'(9) &= \lim_{h\to 0} \frac{f(9+h)-f(9)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h\to 0} \frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)} \\ &= \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)} = \lim_{h\to 0} \frac{h}{h(\sqrt{9+h}+3)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6} \end{aligned}$$

Computing an Equation of a Tangent Line

- Compute the equation of the tangent line to the graph of $f(x) = \frac{1}{x}$ at $x = 2$.

First, compute the slope

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{2(2+h)}}{h} \\ &= \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}. \end{aligned}$$



Now, set up the equation for the tangent $y - f(2) = f'(2)(x-2)$, i.e., $y - \frac{1}{2} = -\frac{1}{4}(x-2)$ or $y = -\frac{1}{4}x + 1$.

Subsection 2

The Derivative as a Function

Differentiability

- To compute the derivative of a function at an arbitrary point x, we use
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$
- If the limit exists, then f is **differentiable at** x.
- Let us show that $f(x) = x^3 - 12x$ is differentiable at all $x \in \mathbb{R}$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3-12(x+h)-(x^3-12x)}{h} \\
&= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-12x-12h-x^3+12x}{h} \\
&= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-12h}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2-12)}{h} \\
&= \lim_{h\to 0} (3x^2+3xh+h^2-12) = 3x^2-12.
\end{aligned}$$

Another Example

- Show that $f(x) = x^{-2}$ is differentiable, for all $x \neq 0$ and find $f'(x)$; We compute the limit

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-h(2x+h)}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2} \\
&= \frac{-2x}{x^4} = -\frac{2}{x^3}.
\end{aligned}
$$

Power Rule

- For all exponents n, we have

$$\boxed{(x^n)' = \frac{d}{dx}x^n = nx^{n-1}.}$$

- The following are examples of applications of the Power Rule:
  - $(x^2)' = 2x$
  - $(x^{20})' = 20x^{19}$
  - $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$
  - $(\frac{1}{\sqrt[5]{x^3}})' = (x^{-3/5})' = -\frac{3}{5}x^{-8/5} = -\frac{3}{5\sqrt[5]{x^8}}$

Sum/Difference and Constant Factor Rules

- The Sum/Difference Rule:

$$\boxed{(f \pm g)' = f' \pm g'.}$$

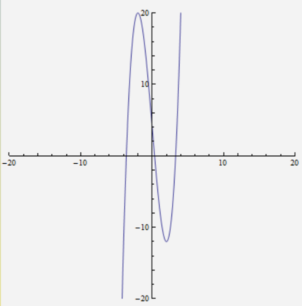
- The Constant Factor Rule:

$$\boxed{(cf)' = cf'.}$$

- The following are examples of applications of these rules:
  - $(2x^{32})' = 2(x^{32})' = 2 \cdot 32x^{31} = 64x^{31}$;
  - $(6x^5 - 3x^2)' = (6x^5)' - (3x^2)' = 6 \cdot 5x^4 - 3 \cdot 2x = 30x^4 - 6x$;
  - $(x^3 - 12x^2 + 36x - 16)' = (x^3)' - (12x^2)' + (36x)' - (16)' = 3x^2 - 24x + 36.$

A Geometric Application

- Determine all points on the graph of $f(x) = x^3 - 12x + 4$, where the tangent line to the graph is horizontal;

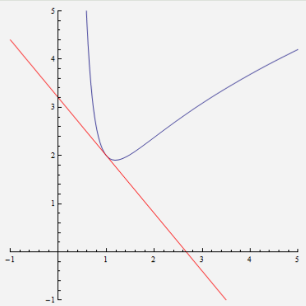


We compute the derivative and find the points where it zeros: $f'(x) = (x^3 - 12x + 4)' = 3x^2 - 12 = 0$; So $3(x^2 - 4) = 0$, i.e., $3(x-2)(x+2) = 0$, showing that $x = -2$ or $x = 2$; Thus the tangent lines are horizontal at $(-2, 20)$ and $(2, -12)$;

Another Geometric Application

- Find an equation of the tangent line to the graph of $f(x) = x^{-3} + 2\sqrt{x} - x^{-4/5}$ at $x = 1$;

First, compute the slope



$$
\begin{aligned}
f'(x) &= (x^{-3} + 2x^{1/2} - x^{-4/5})' \\
&= -3x^{-4} + 2\tfrac{1}{2}x^{-1/2} - (-\tfrac{4}{5})x^{-9/5} \\
&= -3x^{-4} + x^{-1/2} + \tfrac{4}{5}x^{-9/5}.
\end{aligned}
$$

Thus, $f'(1) = -3 + 1 + \frac{4}{5} = -\frac{6}{5}$.

Now, set up the equation for the tangent $y - f(1) = f'(1)(x-1)$, i.e., $y - 2 = -\frac{6}{5}(x-1)$ or $y = -\frac{6}{5}x + \frac{16}{5}$.

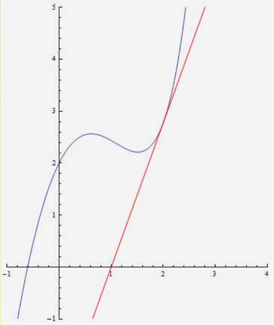
Derivatives of Exponentials

- Exponential Derivation Rules:

$$(b^x)' = b^x \ln b; \qquad \text{In particular } (e^x)' = e^x.$$

- Example: Find an equation for the tangent line to the graph of $f(x) = 2e^x - 3x^2$ at the point $x = 2$.

$$\begin{aligned} f'(x) &= (2e^x - 3x^2)' \\ &= 2e^x - 6x. \end{aligned}$$



Thus, $f'(2) = 2e^2 - 12$. So, an equation for the tangent line is

$$y - (2e^2 - 12) = (2e^2 - 12)(x - 2).$$

Differentiability Implies Continuity

Theorem

If f is differentiable at $x = c$, then f is continuous at $x = c$.

- The hypothesis that f is differentiable at $x = c$ means that the limit $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists, i.e., it is some specific real number. Therefore, we may compute

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} [(f(x) - f(c)) + f(c)] \\
&= \lim_{x \to c} [\tfrac{f(x)-f(c)}{x-c} \cdot (x - c) + f(c)] \\
&= \lim_{x \to c} \tfrac{f(x)-f(c)}{x-c} \cdot \lim_{x \to c} (x - c) + \lim_{x \to c} f(c) \\
&= f'(c) \cdot 0 + f(c) \\
&= f(c).
\end{aligned}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at $x = c$.

Continuity Does Not Imply Differentiability

- If $f(x)$ is continuous at $x = c$, this does not necessarily imply that $f(x)$ is differentiable at $x = c$.
- Consider, for instance $f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$ at $x = 0$;

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

and

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0,$$

whence $\lim_{x \to 0} |x| = 0 = |0|$, so $f(x) = |x|$ is continuous at $x = 0$;

$\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^-} \frac{|h|}{h} = \lim\limits_{h \to 0^-} \frac{-h}{h} = -1$

whereas $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^+} \frac{|h|}{h} = \lim\limits_{h \to 0^-} \frac{h}{h} = +1.$

So $\lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h}$ does not exist, i.e., $f(x) = |x|$ is not differentiable at $x = 0$.

Subsection 3

Product and Quotient Rules

Product Rule

- **Product Rule:** If f and g are differentiable, then fg is also differentiable and

$$\boxed{(f(x)g(x))' = f'(x)g(x) + f(x)g'(x);}$$

Example:

$$\begin{aligned}(x^2(9x+2))' &= (x^2)'(9x+2) + x^2(9x+2)' \\ &= 2x(9x+2) + x^2 \cdot 9 \\ &= 18x^2 + 4x + 9x^2 \\ &= 27x^2 + 4x.\end{aligned}$$

Some Examples

- Compute $f'(x)$ if $f(x) = (2 + x^{-1})(x^{3/2} + 1)$;

$$\begin{aligned}((2+x^{-1})(x^{3/2}+1))' &= (2+x^{-1})'(x^{3/2}+1) \\ &\qquad + (2+x^{-1})(x^{3/2}+1)' \\ &= -x^{-2}(x^{3/2}+1) + (2+x^{-1})\tfrac{3}{2}x^{1/2} \\ &= -x^{-1/2} - x^{-2} + 3x^{1/2} + \tfrac{3}{2}x^{-1/2} \\ &= \tfrac{1}{2}x^{-1/2} - x^{-2} + 3x^{1/2}.\end{aligned}$$

- Compute $f'(x)$ if $f(x) = x^2 e^x$;

$$\begin{aligned}(x^2e^x)' &= (x^2)'e^x + x^2(e^x)' \\ &= 2xe^x + x^2e^x \\ &= (x^2+2x)e^x.\end{aligned}$$

More Theoretical Examples

- Let $f(x) = xg(x)$, for some function g. Moreover, suppose that $g(3) = 5$ and that $g'(3) = 2$. What is then $f'(3)$?
  We apply the product rule to compute $f'(x)$:
  $$f'(x) = (xg(x))' = (x)'g(x) + xg'(x) = g(x) + xg'(x).$$
  Now substitute $x = 3$: $f'(3) = g(3) + 3 \cdot g'(3) = 5 + 3 \cdot 2 = 11.$
- Discover a formula for $(f(x)g(x)h(x))'$;

$$\begin{aligned}(f(x)g(x)h(x))' &= f'(x)(g(x)h(x)) + f(x)(g(x)h(x))' \\ &= f'(x)g(x)h(x) + f(x)(g'(x)h(x) + g(x)h'(x)) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).\end{aligned}$$

Therefore,

$$\boxed{(f(x)g(x)h(x))' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x);}$$

Quotient Rule

- **Quotient Rule:** If f and g are differentiable, with $g(x) \neq 0$, then $\frac{f}{g}$ is also differentiable and

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2};}$$

Example:

$$\begin{aligned} \left(\frac{x}{1+x^2}\right)' &= \frac{(x)'(1+x^2)-x(1+x^2)'}{(1+x^2)^2} \\ &= \frac{1+x^2-x\cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}. \end{aligned}$$

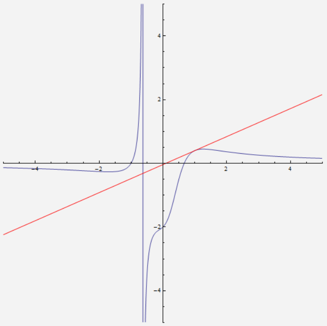
Example:

$$\begin{aligned} \left(\frac{e^x}{e^x+x}\right)' &= \frac{(e^x)'(e^x+x)-e^x(e^x+x)'}{(e^x+x)^2} \\ &= \frac{e^x(e^x+x)-e^x(e^x+1)}{(e^x+x)^2} \\ &= \frac{e^{2x}+xe^x-e^{2x}-e^x}{(e^x+x)^2} = \frac{(x-1)e^x}{(e^x+x)^2}. \end{aligned}$$

A Geometric Example

- Find an equation for the tangent line to $f(x) = \frac{3x^2+x-2}{4x^3+1}$ at $x=1$.

For the slope, we have



$$\begin{aligned} & f'(x) \\ = \quad & \frac{(3x^2+x-2)'(4x^3+1)-(3x^2+x-2)(4x^3+1)'}{(4x^3+1)^2} \\ = \quad & \frac{(6x+1)(4x^3+1)-(3x^2+x-2)12x^2}{(4x^3+1)^2} \\ = \quad & \frac{24x^4+4x^3+6x+1-36x^4-12x^3+24x^2}{(4x^3+1)^2} \\ = \quad & \frac{-12x^4-8x^3+24x^2+6x+1}{(4x^3+1)^2}. \end{aligned}$$

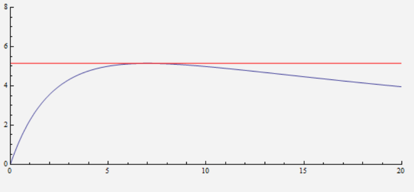
Thus, $f'(1) = \frac{11}{25}$, and, since $f(1) = \frac{2}{5}$, we get the equation $y - \frac{2}{5} = \frac{11}{25}(x-1)$.

Example from Physics

- For a battery of voltage $V = 12$ Volts and internal resistance $r = 7$ Ohms, the total power that the battery delivers to an apparatus of resistance R is $P(R) = \frac{V^2R}{(R+r)^2} = \frac{144R}{(R+7)^2}$. Compute $\frac{dP}{dR}$ and find the value of the resistance R for which the tangent to the graph P vs R is horizontal.

For the slope, we have $P'(R) =$

$$\begin{aligned}
&= \frac{(144R)'(R+7)^2 - 144R((R+7)^2)'}{((R+7)^2)^2} \\
&= \frac{144(R+7)^2 - 144R \cdot 2(R+7)}{(R+7)^4} \\
&= \frac{144(R+7)^2 - 288R(R+7)}{(R+7)^4} \\
&= \frac{(R+7)[144(R+7) - 288R]}{(R+7)^4} \\
&= \frac{144(7-R)}{(R+7)^3}
\end{aligned}$$



Thus, $P'(R) = 0$ when $R = 7$ Ohms.

Subsection 4

Rates of Change

Average and Instantaneous Rate of Change

- The **Average Rate of Change** of $y = f(x)$ per unit of x between x_0 and x_1 is given by

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0};}$$

- The **Instantaneous Rate of Change** of $y = f(x)$ at $x = x_0$ is given by

$$\boxed{f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0};}$$

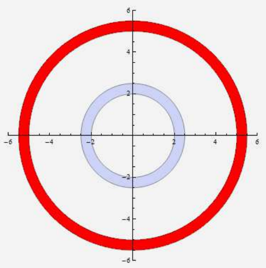
Example

- The area of a circle of radius r is given by $A(r) = \pi r^2$; Compute the rates of change $\frac{dA}{dr}$ at $r = 2$ and $r = 5$ and explain (in practical terms) why the second is larger than the first.

  We have
  $$\frac{dA}{dr} = (\pi r^2)' = 2\pi r.$$

  Therefore $\frac{dA}{dr} \mid_{r=2} = 4\pi$ and $\frac{dA}{dr} \mid_{r=5} = 10\pi$.

  $\frac{dA}{dr}$ is a measure of how much the area increases as r increases by a slight amount dr. Clearly, when the radius is larger the increase in the amount of area for the same increase in the radius is larger!



Approximating a One-Unit Change Using the Derivative

- For small values of h, we have that
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \approx \frac{f(x+h)-f(x)}{h}.$$
- For some applications, especially those in which x denotes number of items or units of a commodity, $h = 1$ may be already useful enough, whence $f(x+1)-f(x) \approx f'(x)$;

An Example: Change in Stopping Distance

- For speeds s between 30 and 75 mph, the stopping distance of a car after the brakes are applied is approximately $F(s) = 1.1s + 0.05s^2$ feet. Estimate the change in stopping distance for $s = 60$ mph, when the speed is increased by 1 mph. What is the exact change?
  For an estimate, we use the derivative:
  $F'(s) = (1.1s + 0.05s^2)' = 1.1 + 0.1s$, whence
  $$F(61) - F(60) \approx F'(60) = 1.1 + 0.1 \cdot 60 = 7.1.$$
  For the exact change, we compute
  $$\begin{aligned} & F(61) - F(60) \\ & \quad = 1.1 \cdot 61 + 0.05 \cdot 61^2 - (1.1 \cdot 60 + 0.05 \cdot 60^2) \\ & \quad = 7.15. \end{aligned}$$

Another Example: Marginal Cost

- If $C(x)$ is the cost function in terms of the number x of items produced, the **marginal cost** at production level x_0 is the additional cost for producing one additional unit;
- According to previous work: $C(x_0+1) - C(x_0) \approx C'(x_0)$; Thus, the derivative $C'(x)$ may be used to approximate the marginal cost;

  Example: If the total cost of a flight for an airline company is $C(x) = 0.0005x^3 - 0.38x^2 + 120x$, where x is the number of passengers, estimate the marginal cost when the flight has 150 passengers.

  Note that
  $C'(x) = (0.0005x^3 - 0.38x^2 + 120x)' = 0.0015x^2 - 0.76x + 120$,
  whence

$$
\begin{aligned}
C(151) - C(150) &\approx C'(150) \\
&= 0.0015 \cdot 150^2 - 0.76 \cdot 150 + 120 = 39.75.
\end{aligned}
$$

Linear Motion

- If $s(t)$ is the position of a moving object at time t, then its **average velocity** between t_0 and t_1 is $v_{\text{avg}}[t_0, t_1] = \frac{s(t_1)-s(t_0)}{t_1-t_0}$;
- Its **instantaneous velocity** at t_0 is given by $v(t_0) = \lim_{t_1 \to t_0} \frac{s(t_1)-s(t_0)}{t_1-t_0} = s'(t_0)$;
- Similarly, its **average acceleration** between t_0 and t_1 is $a_{\text{avg}}[t_0, t_1] = \frac{v(t_1)-v(t_0)}{t_1-t_0}$;
- Its **instantaneous acceleration** at t_0 is given by $a(t_0) = \lim_{t_1 \to t_0} \frac{v(t_1)-v(t_0)}{t_1-t_0} = v'(t_0)$;
- Example: The position function of a truck entering the off-ramp of a highway is $s(t) = 25t - 0.3t^3$ meters for $0 \leq t \leq 5$ in seconds after entering the ramp. How fast was the driver driving when he started on the ramp?
  $v(t) = s'(t) = (25t - 0.3t^3)' = 25 - 0.9t^2$; Therefore $v(0) = 25$ meters per second.

Motion Under the Influence of Gravity

- Suppose an object is tossed vertically upwards from an initial height s_0 with an initial velocity v_0;
- Since the only force acting on the object is that of gravity, the only acceleration applied on the object is that of gravity, which is taken to be constant and equal to $-g = -9.8 \text{ m/s}^2 = -32 \text{ ft/sec}^2$;
- Thus, since the acceleration is the derivative of its velocity function, we get
$$v(t) = -gt + v_0;$$
- Moreover, since its velocity function is the derivative of its distance function, we get
$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0;$$

An Example

- If an object is shot upward with an initial velocity of 20 m/sec from an initial height of 2 m, find its maximum height and the time at which it reaches the maximum height.

  We have $v(t) = -gt + v_0 = -9.8t + 20$;
  To find the max height, we set $v(t) = 0$ and solve for t;
  We get $v(t) = 0$ when $t = \frac{20}{9.8} = 2.04$ sec.
  Since $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$, i.e
  $s(t) = -\frac{1}{2}9.8t^2 + 20t + 2 = -4.9t^2 + 20t + 2$,
  we get maximum height $s(2.04) = 22.408$ meters.

Subsection 5

Higher Derivatives

The n-th Derivative

- The **second derivative** of a function $f(x)$ is the first derivative of its first derivative:
$$f''(x) = (f'(x))';$$
- The **third derivative** is the derivative of its second derivative
$$f'''(x) = (f''(x))';$$
- From the fourth derivative onwards the notation $f^{(4)}(x), f^{(5)}(x), \ldots$ is used instead of $f''''(x), f'''''(x), \ldots$ since we want to avoid piling up primes on the letter used to denote the function;
- Thus, the statement that the $(n+1)$-st derivative is the first derivative of the n-th derivative may be written symbolically
$$\boxed{f^{(n+1)}(x) = (f^{(n)}(x))'.}$$

Finding First Few Derivatives

- Find the three first derivatives of $f(x) = 3x^7 - 5x^2 + 7x^{-3}$;
  For the first derivative

$$\begin{aligned} f'(x) &= (3x^7 - 5x^2 + 7x^{-3})' \\ &= 21x^6 - 10x - 21x^{-4}. \end{aligned}$$

  For the second derivative

$$\begin{aligned} f''(x) &= (21x^6 - 10x - 21x^{-4})' \\ &= 126x^5 - 10 + 84x^{-5}. \end{aligned}$$

  Finally, for the third derivative

$$\begin{aligned} f'''(x) &= (126x^5 - 10 + 84x^{-5})' \\ &= 630x^4 - 420x^{-6}. \end{aligned}$$

Discovering a Pattern for the n-th Derivative

- Calculate the five first derivatives of $f(x) = x^{-1}$ and, then, find a pattern to determine $f^{(n)}(x)$ for an arbitrary n;
  Let $f(x) = x^{-1}$. We have

$$\begin{aligned}
f'(x) &= (x^{-1})' = \ -x^{-2};\\
f''(x) &= (-x^{-2})' = \ +1 \cdot 2x^{-3};\\
f'''(x) &= (1 \cdot 2x^{-3})' = \ -1 \cdot 2 \cdot 3x^{-4};\\
f^{(4)}(x) &= (-1 \cdot 2 \cdot 3x^{-4})' = \ +1 \cdot 2 \cdot 3 \cdot 4x^{-5};\\
f^{(5)}(x) &= (+1 \cdot 2 \cdot 3 \cdot 4x^{-5})' = \ -1 \cdot 2 \cdot 3 \cdot 4 \cdot 5x^{-6}.
\end{aligned}$$

  Thus, the pattern revealed is

$$f^{(n)}(x) = (-1)^n 1 \cdot 2 \cdot \cdots \cdot n x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}}.$$

One More Example

- Calculate the three first derivatives of $f(x) = xe^x$ and, then, find a pattern to determine $f^{(n)}(x)$ for an arbitrary n;
  Let $f(x) = xe^x$. We have

$$\begin{aligned} f'(x) &= (xe^x)' = (x)'e^x + x(e^x)' = e^x + xe^x = (1+x)e^x; \\ f''(x) &= ((1+x)e^x)' = (1+x)'e^x + (1+x)(e^x)' \\ &= e^x + (1+x)e^x = (2+x)e^x; \\ f'''(x) &= ((2+x)e^x)' = (2+x)'e^x + (2+x)(e^x)' \\ &= e^x + (2+x)e^x = (3+x)e^x; \end{aligned}$$

  Thus, the pattern revealed is

$$f^{(n)}(x) = (n+x)e^x.$$

Subsection 6

Trigonometric Functions

- **Basic Trigonometric Derivatives:**

$$\boxed{(\sin x)' = \cos x \quad \text{and} \quad (\cos x)' = -\sin x;}$$

- Let us see why the first holds:

$$\begin{aligned}
(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\sin h\cos x}{h}\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\cdot 0+\cos x\cdot 1\\
&= \cos x.
\end{aligned}$$

Example

- Find $f''(x)$ if $f(x) = x\cos x$;
  We have
  $$\begin{aligned} f'(x) &= (x\cos x)' \\ &= (x)'\cos x + x(\cos x)' \\ &= \cos x - x\sin x. \end{aligned}$$
  Therefore,
  $$\begin{aligned} f''(x) &= (\cos x - x\sin x)' \\ &= (\cos x)' - (x\sin x)' \\ &= -\sin x - ((x)'\sin x + x(\sin x)') \\ &= -\sin x - (\sin x + x\cos x) \\ &= -2\sin x - x\cos x. \end{aligned}$$

More Trigonometric Formulas

- For the other Trigonometric Functions we have

$$(\tan x)' = \sec^2 x, \quad (\sec x)' = \sec x \tan x,$$
$$(\cot x)' = -\csc^2 x, \quad (\csc x)' = -\csc x \cot x.$$

- Let us see why the third holds:

$$\begin{aligned}
(\cot x)' &= \left(\frac{\cos x}{\sin x}\right)' \\
&= \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x} \\
&= \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
&= -\left(\frac{1}{\sin x}\right)^2 \\
&= -\csc^2 x.
\end{aligned}$$

A Geometry Problem

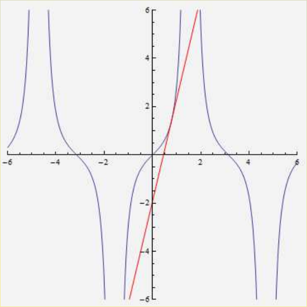
- Find an equation for the tangent line to the graph of $f(x) = \tan x \sec x$ at $x = \frac{\pi}{4}$;

For the slope, we have

$$\begin{aligned} f'(x) &= (\tan x \sec x)' = (\tan x)' \sec x + \tan x (\sec x)' \\ &= \sec^2 x \sec x + \tan x \sec x \tan x = \sec^3 x + \sec x \tan^2 x; \end{aligned}$$

Therefore, $f'(\frac{\pi}{4})$
$= \sec^3 \frac{\pi}{4} + \sec \frac{\pi}{4} \tan \frac{\pi}{4}$
$= \sqrt{2}^3 + \sqrt{2} = 3\sqrt{2}.$
Thus, an equation for the tangent line is

$$y - \sqrt{2} = 3\sqrt{2}(x - \frac{\pi}{4}).$$



Subsection 7

The Chain Rule

- **Chain Rule**: If f and g are differentiable, then $(f \circ g)(x) = f(g(x))$ is also differentiable and its derivative is given by

$$\boxed{(f(g(x)))' = f'(g(x))g'(x).}$$

Example: Calculate the derivative of $h(x) = \cos(x^3)$;
Note that $h(x) = f(g(x))$, where $f(x) = \cos x$ and $g(x) = x^3$. Thus, taking into account the chain rule, we obtain

$$\begin{aligned} h'(x) &= (f(g(x)))' \\ &= f'(g(x))g'(x) \\ &= -\sin(x^3)(x^3)' \\ &= -3x^2 \sin(x^3). \end{aligned}$$

More Examples

- Calculate the derivative of $h(x) = \sqrt{x^4+1}$;
  Note that $h(x) = f(g(x))$, where $f(x) = \sqrt{x}$ and $g(x) = x^4+1$.
  Thus, taking into account the chain rule, we obtain

$$\begin{aligned} h'(x) &= (f(g(x)))' = f'(g(x))g'(x) \\ &= \tfrac{1}{2}(x^4+1)^{-1/2}(x^4+1)' \\ &= \frac{4x^3}{2\sqrt{x^4+1}} = \frac{2x^3}{\sqrt{x^4+1}}. \end{aligned}$$

- Calculate the derivative of $h(x) = \tan\left(\frac{x}{x+1}\right)$;
  Note that $h(x) = f(g(x))$, where $f(x) = \tan x$ and $g(x) = \frac{x}{x+1}$.
  Thus, taking into account the chain rule, we obtain

$$\begin{aligned} h'(x) &= (f(g(x)))' = f'(g(x))g'(x) \\ &= \sec^2\left(\frac{x}{x+1}\right)\left(\frac{x}{x+1}\right)' \\ &= \sec^2\left(\frac{x}{x+1}\right)\frac{(x)'(x+1)-x(x+1)'}{(x+1)^2} \\ &= \sec^2\left(\frac{x}{x+1}\right)\frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}\sec^2\left(\frac{x}{x+1}\right). \end{aligned}$$

Example Involving Rates of Change

- Suppose a sphere is inflated so that its radius is increasing at the rate of 3 cm/sec. At what rate is the volume of the sphere increasing when its radius is 10 cm?
  We have
  $$V(r) = \frac{4}{3}\pi r^3.$$
  Therefore,
  $$\frac{dV}{dt} = \frac{d}{dt}(\frac{4}{3}\pi r^3) = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$$
  Thus, we get
  $$\frac{dV}{dt} = 4\pi \cdot 10^2 \cdot 3 = 1200\pi \text{ cm}^3/\text{sec}.$$

Power and Exponential Rules

- **Power and Exponential Rules**: If g is differentiable, then

$$\boxed{[g(x)^n]' = ng(x)^{n-1}g'(x) \quad \text{and} \quad (e^{g(x)})' = e^{g(x)}g'(x).}$$

Example: Compute $f'(x)$ for $f(x) = (x^2+7x+2)^{-1/3}$;

$$\begin{aligned} f'(x) &= [(x^2+7x+2)^{-1/3}]' \\ &= -\tfrac{1}{3}(x^2+7x+2)^{-4/3}(x^2+7x+2)' \\ &= -\tfrac{1}{3}(x^2+7x+2)^{-4/3}(2x+7). \end{aligned}$$

Example: Compute $f'(x)$ for $f(x) = e^{\cos x}$;

$$\begin{aligned} f'(x) &= (e^{\cos x})' \\ &= e^{\cos x}(\cos x)' \\ &= -(\sin x)e^{\cos x}. \end{aligned}$$

Using Chain Rule Twice

- Compute the derivatives:

$$\begin{aligned}
(\sqrt[3]{1+\sqrt{x^2+1}})' &= [(1+(x^2+1)^{1/2})^{1/3}]' \\
&= \tfrac{1}{3}(1+(x^2+1)^{1/2})^{-2/3}(1+(x^2+1)^{1/2})' \\
&= \tfrac{1}{3}(1+(x^2+1)^{1/2})^{-2/3}\cdot \\
&\qquad \tfrac{1}{2}(x^2+1)^{-1/2}(x^2+1)' \\
&= \tfrac{1}{3}(1+(x^2+1)^{1/2})^{-2/3}\tfrac{1}{2}(x^2+1)^{-1/2}2x \\
&= \tfrac{1}{3}x(x^2+1)^{-1/2}(1+(x^2+1)^{1/2})^{-2/3}.
\end{aligned}$$

$$\begin{aligned}
(e^{(x^2+7x)^3})' &= e^{(x^2+7x)^3}\cdot((x^2+7x)^3)' \\
&= e^{(x^2+7x)^3}\cdot 3(x^2+7x)^2\cdot(x^2+7x)' \\
&= 3(2x+7)(x^2+7x)^2e^{(x^2+7x)^3}.
\end{aligned}$$

Subsection 8

Derivatives of Inverse Functions

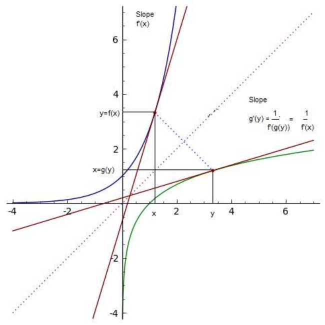
Derivative of the Inverse Function

- Suppose f is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If b belongs to the domain of $g(x)$ and $f'(g(b)) \neq 0$, then $g'(b)$ exists and

$$\boxed{g'(b) = \frac{1}{f'(g(b))}.}$$

Since $f(g(x)) = x$, by the chain rule, we get $f'(g(x))g'(x) = 1$. Therefore

$$g'(x) = \frac{1}{f'(g(x))}.$$



Two Examples

- If $f(x) = x^4 + 10$, with domain $\{x : x \geq 0\}$, compute $g'(x)$, where $g = f^{-1}$;
  First, note that $f'(x) = 4x^3$. Next, find a formula for $g(x)$ by solving $x = y^4 + 10$ for y: We have $y = \sqrt[4]{x - 10} = (x - 10)^{1/4}$; Hence $g(x) = (x - 10)^{1/4}$; Thus, using the formula for the derivative of the inverse, we get
  $$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{4((x-10)^{1/4})^3} = \frac{1}{4}(x-10)^{-3/4}.$$
- If $f(x) = x + e^x$, calculate $g'(1)$, where $g = f^{-1}$;
  Since $f(0) = 0 + e^0 = 1$, we must have $g(1) = f^{-1}(1) = 0$. Next, note that $f'(x) = (x + e^x)' = 1 + e^x$. Thus $f'(0) = 1 + e^0 = 2$. Now, we compute
  $$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

Derivatives of Inverse Trigonometric Functions

- Derivatives of $\sin^{-1} x, \cos^{-1} x$ and $\tan^{-1} x$:

$$\boxed{\begin{gathered}(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}, \\ (\tan^{-1} x)' = \frac{1}{x^2+1}, \quad (\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}},\end{gathered}}$$

Example: Find $f'(\frac{1}{2})$ if $f(x) = \sin^{-1}(x^2)$;

We have, using the chain rule,

$$f'(x) = [\sin^{-1}(x^2)]' = \frac{1}{\sqrt{1-(x^2)^2}}(x^2)' = \frac{2x}{\sqrt{1-x^4}}.$$

Therefore

$$f'\left(\frac{1}{2}\right) = \frac{2 \cdot (1/2)}{\sqrt{1-(1/2)^4}} = \frac{1}{\sqrt{15/16}} = \frac{4}{\sqrt{15}}.$$

A Geometric Application

- Find an equation for the tangent line to $f(x) = \csc^{-1}(e^x + 1)$ at $x = 0$;

  Compute

$$\begin{aligned} f'(x) &= (\csc^{-1}(e^x+1))' \\ &= -\frac{1}{|e^x+1|\sqrt{(e^x+1)^2-1}}(e^x+1)' \\ &= -\frac{e^x}{(e^x+1)\sqrt{e^{2x}+2e^x}}. \end{aligned}$$

  Therefore, the slope of the tangent at $(0, \frac{\pi}{6})$ is

$$f'(0) = -\frac{1}{2\sqrt{3}}.$$

  Hence, the equation is

$$y - \frac{\pi}{6} = -\frac{1}{2\sqrt{3}}x.$$

Subsection 9

Derivatives of Exponential and Logarithmic Functions

Derivatives of Exponential Functions

- Derivatives of Exponential:

$$\boxed{(b^x)' = b^x \ln b, \quad (b^{f(x)})' = b^{f(x)} f'(x) \ln b;}$$

Examples:

- $(4^{3x})' = 4^{3x} \ln 4 \cdot (3x)' = 3 \cdot 4^{3x} \ln 4.$
- $(5^{x^2+1})' = 5^{x^2+1} \ln 5 \cdot (x^2 + 1)' = 2x \cdot 5^{x^2+1} \ln 5.$

Derivatives of Logarithmic Functions

- Derivatives of Logarithmic Functions:

$$\boxed{(\log_b x)' = \frac{1}{x \ln b}, \quad (\log_b f(x))' = \frac{f'(x)}{f(x) \ln b};}$$

Examples:

- $(x \ln x)' = (x)' \ln x + x(\ln x)' = \ln x + x\frac{1}{x} = \ln x + 1;$
- $((\ln x)^2)' = 2 \ln x(\ln x)' = \frac{2 \ln x}{x};$
- $(\ln (x^3+1))' = \frac{1}{x^3+1}(x^3+1)' = \frac{3x^2}{x^3+1};$
- $(\ln \sqrt{\sin x})' = \frac{1}{\sqrt{\sin x}}(\sqrt{\sin x})' = \frac{1}{\sqrt{\sin x}}\frac{1}{2\sqrt{\sin x}}(\sin x)' = \frac{\cos x}{2 \sin x} = \frac{1}{2} \cot x.$

Logarithmic Differentiation

- Let us find the derivative $f'(x)$ of $f(x) = \frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}$;
  First, we use properties of logarithms to rewrite $\ln f(x)$ as a sum/difference of logs:

$$\begin{aligned} \ln f(x) &= \ln \frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}} \\ &= \ln[(x+1)^2(2x^2-3)] - \ln[(x^2+1)^{1/2}] \\ &= 2\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\ln(x^2+1). \end{aligned}$$

  Next, compute the derivative of $\ln f(x)$ using sum/difference and logarithmic rules:

$$\begin{aligned} (\ln f(x))' &= [2\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\ln(x^2+1)]' \\ &= (2\ln(x+1))' + (\ln(2x^2-3))' - (\frac{1}{2}\ln(x^2+1))' \\ &= \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}. \end{aligned}$$

  Thus, we get $\frac{f'(x)}{f(x)} = \frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}$, i.e., that $f'(x) = f(x)[\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}]$.

Logarithmic Differentiation for Exponential Functions

- Let us find the derivative $f'(x)$ of $f(x) = x^x$;
  We have
  $$\ln f(x) = \ln(x^x) = x \ln x.$$
  Therefore $\frac{f'(x)}{f(x)} = \ln x + 1$, i.e., $f'(x) = f(x)(\ln x + 1) = x^x(\ln x + 1)$;
- Let us find the derivative $f'(x)$ of $f(x) = x^{\sin x}$;
  We have
  $$\ln f(x) = \ln(x^{\sin x}) = \sin x \ln x.$$
  Therefore $\frac{f'(x)}{f(x)} = \cos x \ln x + \frac{\sin x}{x}$, i.e.,
  $f'(x) = f(x)(\cos x \ln x + \frac{\sin x}{x}) = x^{\sin x}(\cos x \ln x + \frac{\sin x}{x})$;

Hyperbolic Functions

- Basic Hyperbolic Functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2};$$

- Additional Definitions:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \text{sech} x = \frac{1}{\cosh x};$$

- Compute $(\sinh x)' = (\frac{e^x - e^{-x}}{2})' = \frac{e^x + e^{-x}}{2} = \cosh x$;
- Similarly, $(\cosh x)' = \sinh x$;
- $(\coth x)' = (\frac{\cosh x}{\sinh x})' = \frac{(\cosh x)' \sinh x - \cosh x (\sinh x)'}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\text{csch}^2 x$;

Some Examples

- $$\begin{aligned}(\cosh(3x^2+1))' &= \sinh(3x^2+1)(3x^2+1)' \\ &= 6x\sinh(3x^2+1).\end{aligned}$$
- $$\begin{aligned}(\sinh x\tanh x)' &= (\sinh x)'\tanh x+\sinh x(\tanh x)' \\ &= \cosh x\tanh x+\sinh x\mathrm{sech}^2 x \\ &= \sinh x+\tanh x\mathrm{sech} x.\end{aligned}$$

Subsection 10

Implicit Differentiation

Example I

- Compute $\frac{dy}{dx}$ if y is defined implicitly as a function of x by $x^2 + y^2 = 1$;
  Take derivatives of both sides with respect to x:
  $$\frac{d}{dx}(x^2 + y^2) = \frac{d1}{dx};$$
  Use sum rule:
  $$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0;$$
  Then $2x + 2y\frac{dy}{dx} = 0$; Now solve for $\frac{dy}{dx}$:
  $$\frac{dy}{dx} = -\frac{x}{y};$$

Example II

- Find an equation for the tangent line to the graph of $y^4 + xy = x^3 - x + 2$ at $(x, y) = (1, 1)$;
  Take derivatives of both sides with respect to x:
  $(y^4 + xy)' = (x^3 - x + 2)'$; Use carefully the required rules:
  $(y^4)' + (xy)' = (x^3)' - (x)' + (2)'$, whence
  $$4y^3y' + y + xy' = 3x^2 - 1;$$
  Thus, $(4y^3 + x)y' = 3x^2 - y - 1$; and, therefore
  $$y' = \frac{3x^2 - y - 1}{4y^3 + x};$$
  It follows that $y'(1, 1) = \frac{1}{5}$; Thus, an equation for the tangent is $y - 1 = \frac{1}{5}(x - 1)$;

Example III

- Find an equation for the tangent line to the graph of $e^{x-y} = 2x^2 - y^2$ at $(x, y) = (1, 1)$;
  Take derivatives of both sides with respect to x:
  $(e^{x-y})' = (2x^2 - y^2)'$; Use again the rules: $e^{x-y}(x-y)' = 4x - 2yy'$, whence
  $$e^{x-y}(1-y') = 4x - 2yy';$$
  Thus, $(e^{x-y} - 2y)y' = e^{x-y} - 4x$; and, therefore
  $$y' = \frac{e^{x-y} - 4x}{e^{x-y} - 2y};$$
  It follows that $y'(1,1) = 3$; Thus, an equation for the tangent is $y - 1 = 3(x-1)$;

Example IV

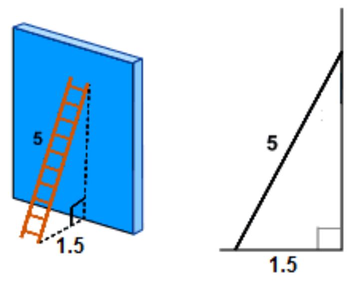
- Find an equation for the tangent line to the graph of $y\cos(y+t+t^2) = t^3$ at $(t,y) = (0, \frac{5\pi}{2})$;
  Take derivatives of both sides with respect to t:
  $(y\cos(y+t+t^2))' = (t^3)'$; Use again the rules:
  $(y)'\cos(y+t+t^2) + y(\cos(y+t+t^2))' = 3t^2$, whence
  $y'\cos(y+t+t^2) - y\sin(y+t+t^2)(y+t+t^2)' = 3t^2$;
  Hence, we have
  $$y'\cos(y+t+t^2) - y\sin(y+t+t^2)(y'+1+2t) = 3t^2;$$
  Thus, for $t=0$ and $y = \frac{5\pi}{2}$, $y'\cos\frac{5\pi}{2} - \frac{5\pi}{2}\sin\frac{5\pi}{2}(y'+1) = 0$; and, therefore
  $$-\frac{5\pi}{2}(y'+1) = 0;$$
  It follows that $y'(0, \frac{5\pi}{2}) = -1$; Thus, an equation for the tangent is $y - \frac{5\pi}{2} = -t$;

Subsection 11

Related Rates

Sliding Ladder

A 5 meter ladder leans against a wall. The bottom is 1.5 meters from the wall at time $t = 0$ and slides away at a rate of 0.8 meters per second. What is the velocity of the top of the ladder at $t = 1$?

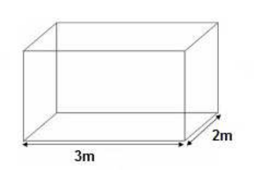


$x^2 + h^2 = 5^2$, whence $\frac{d}{dt}(x^2 + h^2) = 0$, and, therefore, $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$, yielding $\frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}$.

Note that, at $t = 1$, we have $x(1) = 2.3$, whence we obtain $h(1) = \sqrt{25 - 2.3^2} \approx 4.44$. Now, we substitute in these values: $\frac{dh}{dt} = -\frac{2.3}{4.44} \cdot 0.8 = -0.41$ meters per second.

Filling a Rectangular Tank

Water pours into a tank at a rate of 0.3 cubic meters per minute. How fast is water level rising if the base of the tank is a rectangle of dimensions 2×3 meters?

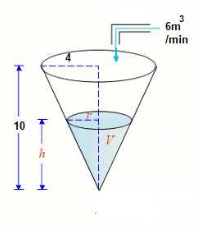


The volume V is related to the height h by the equation $V = 2 \cdot 3 \cdot h$, whence $\frac{dV}{dt} = 6\frac{dh}{dt}$, and, therefore, $\frac{dh}{dt} = \frac{1}{6}\frac{dV}{dt}$.
Now, we substitute the appropriate value:

$$\frac{dh}{dt} = \frac{1}{6} \cdot 0.3 = 0.05 \text{ meters per minute.}$$

Filling a Conical Tank

Water pours into a conical tank of height 10 m and radius 4 m at a rate of 6 m^3/min. At what rate is the water rising when the level is 5 m high?



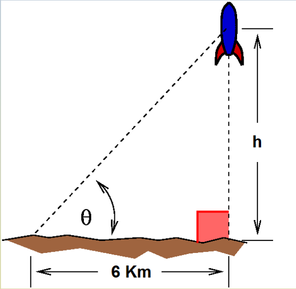
The volume V is related to the height h and the radius r by the equation $V = \frac{1}{3}\pi h r^2$. Moreover, by the two similar triangles of the figure, we have that $\frac{r}{h} = \frac{4}{10}$, whence $r = \frac{2}{5}h$. Therefore, $V = \frac{1}{3}\pi h \frac{4}{25} h^2$, yielding $V = \frac{4\pi}{75} h^3$. Computing derivatives of both sides with respect to t, we get $\frac{dV}{dt} = \frac{4\pi}{75} 3h^2 \frac{dh}{dt}$. So, we obtain

$$\frac{dh}{dt} = \frac{25}{4\pi h^2} \frac{dV}{dt}.$$

Substituting the appropriate values: $\frac{dh}{dt} = \frac{25}{4\pi 25} 6 = \frac{3}{2\pi}$ m/min.

Velocity of a Rocket

A rocket is launched vertically from a launching pad 6 Km away from a radar station. If at a certain moment the angle θ between the line of observation and the ground is $\frac{\pi}{3}$ and increasing at 0.9 rad/min, what is the rocket's velocity v at that time?



The angle θ is related to the height h and the distance from the launching pad by $\tan\theta = \frac{h}{6}$, whence we get $h = 6\tan\theta$. Computing derivatives of both sides with respect to t, we get

$$\frac{dh}{dt} = 6\sec^2\theta\frac{d\theta}{dt}.$$

So, we obtain $\frac{dh}{dt} = 6 \cdot 2^2 \cdot 0.9 = 21.6$ Km/min.