Calculus I

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LSSU Math 151

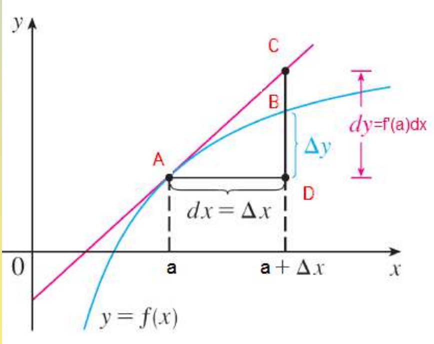
Subsection 1

Linear Approximations

Linear Approximation of Δf

- If f is differentiable at $x = a$ and Δx is small, then the difference $\Delta y = f(a + \Delta x) - f(a)$ can be approximated by

$$\boxed{\Delta y \approx f'(a)\Delta x.}$$



Examples

- Use a Linear Approximation to approximate $\frac{1}{10.2} - \frac{1}{10}$;
Consider $f(x) = \frac{1}{x}$; Note that $f'(x) = -\frac{1}{x^2}$; We would like to approximate $f(10+0.2) - f(10)$; According to the linear approximation scheme,
$$f(10.2) - f(10) \approx f'(10)\Delta x = -\frac{1}{10^2} \cdot 0.2 = -0.002.$$
- Approximate the value of $\sqrt[3]{8.1}$;
Consider $f(x) = \sqrt[3]{x}$; Note that $f'(x) = \frac{1}{3}x^{-2/3}$; We would like to approximate $f(8.1)$; According to the linear approximation scheme,
$$f(8.1) - f(8) \approx f'(8)\Delta x = \frac{1}{3}8^{-2/3} \cdot 0.1 = \frac{1}{120}.$$
Therefore $\sqrt[3]{8.1} \approx \sqrt[3]{8} + \frac{1}{120} = \frac{241}{120} \approx 2.0083.$

Applied Example: Thermal Expansion

- A thin metal cable has length $L = 12$ cm when the temperature is $T = 21°$C. Estimate the change in length when T rises to $24°$C, assuming that $\frac{dL}{dT} = kL$, where $k = 1.7 \times 10^{-5} {}^\circ\mathrm{C}^{-1}$ (k is the **coefficient of thermal expansion**);

  Consider the function $L(T)$; Note that $\frac{dL}{dT} = kL$; We would like to approximate $L(24) - L(21)$; According to the linear approximation scheme,

$$\begin{array}{rcl} L(24) - L(21) & \approx & \frac{dL}{dT}\,\vert_{T=21}\,\Delta T = kL(21)\Delta T = \\ & = & 1.7 \cdot 10^{-5} \cdot 12 \cdot 3 = 6.12 \cdot 10^{-4}. \end{array}$$

Applied Example: Pizza Gimmicks

- A pizza company at Corleone claims that its pizzas are circular with diameter 50 cm. Find the area of each pizza and estimate the quantity lost or gained if the diameter is off by at most 1.2 cm;

  Consider the function $A(d) = \pi(\frac{d}{2})^2 = \frac{\pi}{4}d^2$; Note that $A'(d) = \frac{\pi}{2}d$;
  Thus, $A(50) = \frac{\pi}{4}50^2 = 625\pi$ cm^2.
  Next, We would like to approximate ΔA, which we take to be $A(51.2) - A(50)$ or $A(50) - A(48.8)$; According to the linear approximation scheme,

$$|\Delta A| \approx A'(d)\Delta d = \frac{\pi}{2}d\Delta d = \frac{\pi}{2} \cdot 50 \cdot 1.2 = 30\pi \text{ cm}^2.$$

Linearization of a Function

- If f is differentiable at $x = a$ and x is a value close to a, then $f(x)$ may be approximated by

$$\boxed{f(x) \approx L(x) = f'(a)(x-a) + f(a).}$$

The straight line $L(x)$ is called the **linearization** of $f(x)$ at $x = a$;

Example: Compute the linearization of $f(x) = \sqrt{x}e^{x-1}$ at $a = 1$;

We have $f(1) = \sqrt{1}e^{0} = 1$; Moreover, $f'(x) = \frac{e^{x-1}}{2\sqrt{x}} + \sqrt{x}e^{x-1}$, whence $f'(1) = \frac{1}{2} + 1 = \frac{3}{2}$; Therefore,

$$\begin{aligned} L(x) &= f'(1)(x-1) + f(1) \\ &= \frac{3}{2}(x-1) + 1 = \frac{3}{2}x - \frac{1}{2}. \end{aligned}$$

Error Estimates

- If $y = f(x)$ is a function, then the **percentage error** (in decimal) incurred by estimating $f(a+\Delta x)$ by using the linearization of f is the absolute value of the actual error divided by the actual value:

$$\boxed{\mathrm{PErr} = \left|\frac{f(a+\Delta x) - f(a) - f'(a)\Delta x}{f(a+\Delta x)}\right|.}$$

Example: Estimate $\tan\left(\frac{\pi}{4} + 0.02\right)$ and compute the percentage error of the estimation;

Let $f(x) = \tan x$; Thus, $f'(x) = \sec^2 x$; For $a = \frac{\pi}{4}$ and $\Delta x = 0.02$, we have

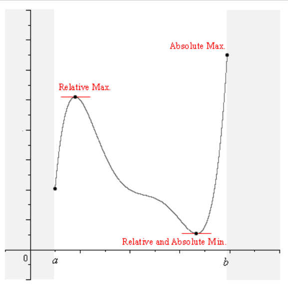
$$\begin{aligned}
\mathrm{PErr} &= \left|\frac{f(a+\Delta x)-f(a)-f'(a)\Delta x}{f(a+\Delta x)}\right| \\
&= \left|\frac{\tan\left(\frac{\pi}{4}+0.02\right)-\tan\frac{\pi}{4}-\sec^2\frac{\pi}{4}\cdot 0.02}{\tan\left(\frac{\pi}{4}+0.02\right)}\right| \\
&\approx \left|\frac{1.0408-1-2\cdot 0.02}{1.0408}\right| = \frac{0.0008}{1.0408} \approx 0.0008.
\end{aligned}$$

Subsection 2

Extreme Values

Absolute Extrema on an Interval

- Let $f(x)$ be a function on an interval I and $a \in I$; We say that $f(a)$ is the
  - **absolute minimum** of $f(x)$ on I if $f(a) \le f(x)$ for all $x \in I$;
  - **absolute maximum** of $f(x)$ on I if $f(x) \le f(a)$ for all $x \in I$;



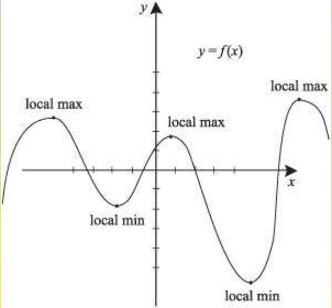
- The process of finding the max or min values, collectively referred to as **extreme values**, is called **optimization**;

Absolute Extrema on a Closed Interval

A continuous function $f(x)$ on a closed interval $I = [a, b]$ attains both a minimum and a maximum value on I.

Local Extrema

- We say that $f(x)$ has a
  - **local** (or **relative**) **minimum** at $x = c$ if $f(c)$ is the minimum value of f on some open interval containing c;
  - **local** (or **relative**) **maximum** at $x = c$ if $f(c)$ is the maximum value of f on some open interval containing c;



- Observe that at the local extrema, if the derivative (slope of the tangent line) exists, it has value 0.

Critical Points of a Function $f(x)$

- A number c in the domain of $f(x)$ is a **critical point** if $f'(c) = 0$ or $f'(c)$ does not exist;

  Example: Consider $f(x) = x^3 - 9x^2 + 24x - 10$;
  We have $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-4)(x-2)$.
  Therefore, the critical points of $f(x)$ are $x = 2$ and $x = 4$;

  Example: Consider $f(x) = |x|$;
  We saw that $f'(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$ Therefore, the only critical point of $f(x)$ is $x = 0$;

Fermat's Theorem for Local Extrema

If $f(c)$ is a local min or max, then c is a critical point of $f(x)$.

- Caution: The converse is not true!! Even though c might be a critical point, $f(c)$ might not be a local extremum.

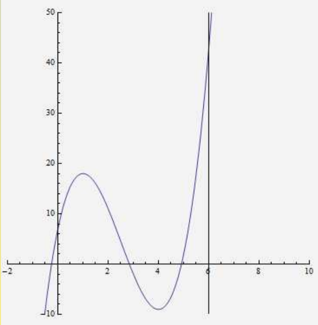
Optimization in a Closed Interval

Extreme Values on Closed Interval

If f is continuous on $[a,b]$ and $f(c)$ is an extreme value of f on $[a,b]$, then either c is a critical point or one of the endpoints a or b.

Example: Find the extrema of $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0,6]$;

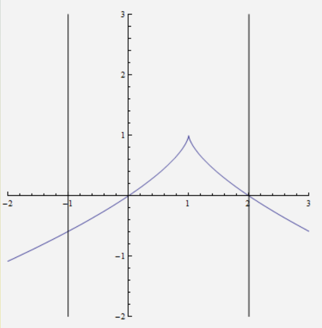
We have $f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-4)(x-1)$. Therefore, the critical points of f in $[0,6]$ are $x = 1$ and $x = 4$; Now, we compute: $f(0) = 7, f(1) = 18, f(4) = -9, f(6) = 43$; Thus, in $[0,6]$, $f(4)$ is the absolute min and $f(6)$ is the absolute max of f;



Another Example

- Find the max value of $f(x) = 1 - (x-1)^{2/3}$ on $[-1, 2]$;

  We have $f'(x) = \frac{2}{3}(x-1)^{-1/3} = \frac{2}{3\sqrt[3]{x-1}}$; Thus, in $[-1, 2]$, f has the critical point $x = 1$; We have $f(-1) = 1 - \sqrt[3]{4}$, $f(1) = 1$ and $f(2) = 0$; Therefore, $f(1)$ is the absolute max and $f(-1)$ the absolute min of f in $[-1, 2]$;

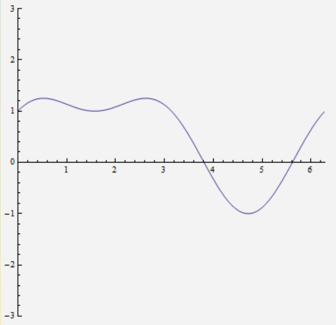
  

- How about the extreme values of $f(x) = x^2 - 8\ln x$ on $[1, 4]$?

A Third Example

- Find the max value of $f(x) = \sin x + \cos^2 x$ on $[0, 2\pi]$;

  We have $f'(x) = \cos x - 2\cos x \sin x = \cos x(1 - 2\sin x)$; Thus, in $[0, 2\pi]$, f has the critical points $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $x = \frac{\pi}{6}, \frac{5\pi}{6}$; We have $f(0) = 1$, $f(\frac{\pi}{6}) = \frac{5}{4}$, $f(\frac{\pi}{2}) = 1$, $f(\frac{5\pi}{6}) = \frac{5}{4}$, $f(\frac{3\pi}{2}) = -1$, $f(2\pi) = 1$; Therefore, $f(\frac{3\pi}{2})$ is the absolute min and $f(\frac{\pi}{6}) = f(\frac{5\pi}{6})$ the absolute max of f in $[0, 2\pi]$;



Rolle's Theorem

Rolle's Theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists a $c \in (a, b)$, such that $f'(c) = 0$.

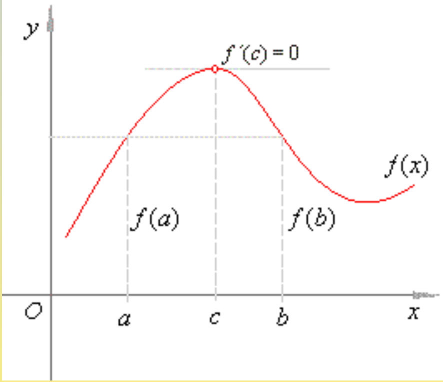
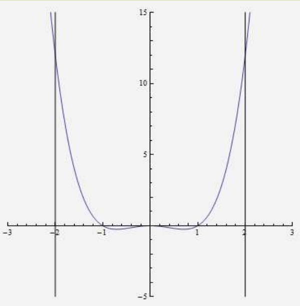


Illustration of Rolle's Theorem

- Verify Rolle's Theorem for $f(x) = x^4 - x^2$ on $[-2, 2]$;
  Clearly, $f(x)$ is continuous in $[-2, 2]$, since it is a polynomial function;
  It is differentiable on $(-2, 2)$, with derivative
  $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$; Therefore, by Rolle's Theorem, there exists a c between -2 and 2, such that $f'(c) = 2c(2c^2 - 1) = 0$.
  Actually, there are three solutions $c = 0$, or $c = \pm\frac{\sqrt{2}}{2}$;

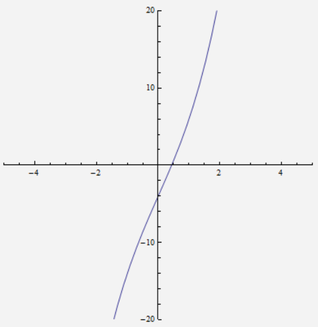


Using Rolle's Theorem to Prove Uniqueness of Roots

- Show that the equation $x^3 + 9x - 4$ has precisely one real root.

  Consider the function $f(x) = x^3 + 9x - 4$; Since $f(0) = -4 < 0$, whereas $f(1) = 6 > 0$, by the Intermediate Value Theorem, there exists a c in $(0, 1)$, such that $f(c) = c^3 + 9c - 4 = 0$; Therefore, the given equation has at least one real solution;

  Assume that it has two real solutions a and b, with $a < b$; This means that $f(a) = f(b) = 0$; Since f is continuous on $[a, b]$ and differentiable on (a, b), with derivative $f'(x) = 3x^2 + 9$, by Rolle's Theorem, there exists $c \in (a, b)$, such that $f'(c) = 3c^2 + 9 = 0$; But this is impossible!!



  Therefore, $f(x) = 0$ has at most one real solution, i.e., it has exactly one real solution!

Subsection 3

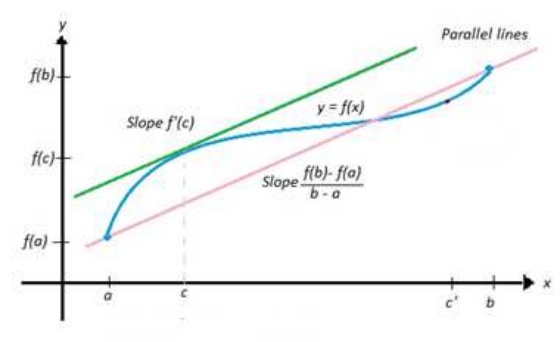
The Mean Value Theorem and Monotonicity

The Mean Value Theorem

The Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one value c in (a, b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a};$$



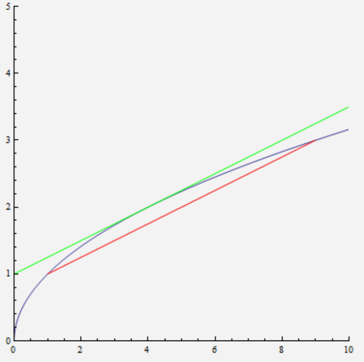
An Example

- Consider $f(x) = \sqrt{x}$;

  This function is continuous on $[1, 9]$ as a root function; Moreover, it is differentiable on $(1, 9)$, with derivative $f'(x) = \frac{1}{2\sqrt{x}}$; By the MVT, we may conclude that there exists at least one $c \in (1, 9)$, such that

$$f'(c) = \frac{f(9) - f(1)}{9 - 1};$$

  This equation says that $\frac{1}{2\sqrt{c}} = \frac{1}{4}$; Solving this, yields $c = 4$;



Sign of the Derivative and Monotonicity

- A function $f(x)$ is
  - **increasing on** (a, b) if $f(x_1) < f(x_2)$, for all $x_1 < x_2$ in (a, b);
  - **decreasing on** (a, b) if $f(x_1) > f(x_2)$, for all $x_1 < x_2$ in (a, b);
- $f(x)$ is **monotonic on** (a, b) if it is either increasing or decreasing on (a, b);

Sign of the Derivative

If f is differentiable on (a, b), then:

- If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b);
- If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b);

- Geometrically, the theorem says that the sign of the slopes of the tangent lines to $y = f(x)$ determines the kind of the monotonicity of f on (a, b);

First Derivative Test for Critical Points

First Derivative Test

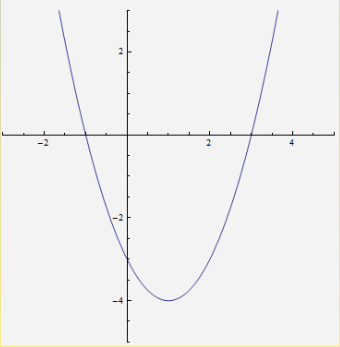
If $f(x)$ is differentiable and that c is a critical point of $f(x)$, then:

- if $f'(x)$ changes from $+$ to $-$ at c, then $f(c)$ is a local maximum;
- if $f'(x)$ changes from $-$ to $+$ at c, then $f(c)$ is a local minimum;

Example: Let $f(x) = x^2 - 2x - 3$; we have $f'(x) = 2x - 2 = 2(x-1)$; Thus, $x = 1$ is a critical point of f; The following is a sign table for $f'(x)$ and summarizes the behavior of $f(x)$ with respect to monotonicity;

Function	$x < 1$	$x > 1$
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$

It shows that $f(1)$ is a local minimum;

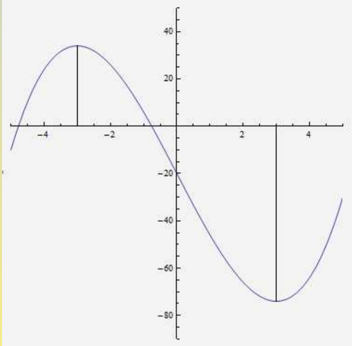


Second Example

- Consider the function $f(x) = x^3 - 27x - 20$;

We have $f'(x) = 3x^2 - 27 = 3(x^2 - 9) = 3(x+3)(x-3)$; Therefore, $f(x)$ has critical points at $x = \pm 3$; The following is a sign table for $f'(x)$ and summarizes the monotonicity of $f(x)$:

Function	$x < -3$	$-3 < x < 3$	$x > 3$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$



It shows that $f(-3)$ is a local maximum and $f(3)$ is a local minimum;

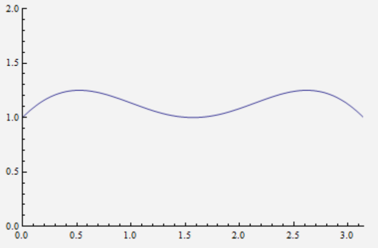
Third Example

- Consider the function $f(x) = \cos^2 x + \sin x$ in $(0, \pi)$;

We have $f'(x) = -2\cos x \sin x + \cos x = \cos x(1 - 2\sin x)$; Therefore, $f(x)$ has critical points at $x = \frac{\pi}{2}$ and $x = \frac{\pi}{6}, \frac{5\pi}{6}$. The following is a sign table for $f'(x)$ and summarizes the monotonicity of $f(x)$:

Function	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < \pi$
$f'(x)$	$+$	$-$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$

It shows that $f(\frac{\pi}{6}), f(\frac{5\pi}{6})$ are local maxima and $f(\frac{\pi}{2})$ is a local minimum;

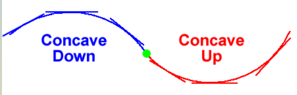


Subsection 4

The Shape of a Graph

Concavity and Inflection Points

- If $f(x)$ is differentiable in an open interval (a, b), then
  - f is **concave up** on (a, b) if $f'(x)$ is increasing on (a, b);
  - f is **concave down** on (a, b) if $f'(x)$ is decreasing on (a, b);



Test for Concavity

If $f''(x)$ exists for all $x \in (a, b)$, then:

- If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b);
- If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b);

- The points where concavity changes are called **inflection points**;

Test for Inflection Points

Assuming that $f''(c)$ exists, if $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then $f(x)$ has an inflection point at $x = c$.

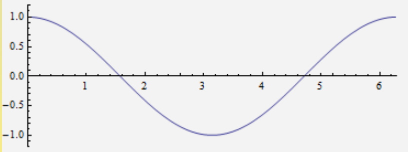
First Example

- Study the function $f(x) = \cos x$ on $[0, 2\pi]$ with respect to monotonicity and concavity;

We have $f'(x) = -\sin x$; Therefore, $f(x)$ has critical points at $x = 0, \pi, 2\pi$; We have $f''(x) = -\cos x$; Therefore, $f''(x)$ zeros at $x = \frac{\pi}{2}, \frac{3\pi}{2}$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \pi$	$\pi < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
$f'(x)$	$-$	$-$	$+$	$+$
$f''(x)$	$-$	$+$	$+$	$-$
$f(x)$	⤵	↳	⤴	↱

It shows that $f(0), f(2\pi)$ are local maxima, $f(\pi)$ is a local minimum and at $\frac{\pi}{2}, \frac{3\pi}{2}$ f has inflection points;



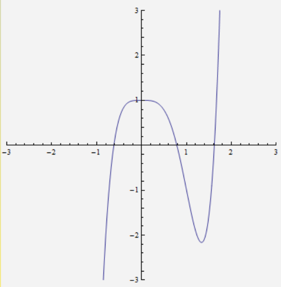
Second Example

- Study the function $f(x) = 3x^5 - 5x^4 + 1$ with respect to monotonicity and concavity;

We have $f'(x) = 15x^4 - 20x^3 = 5x^3(3x-4)$; Therefore, $f(x)$ has critical points at $x = 0, \frac{4}{3}$; We have $f''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$; Therefore, $f''(x)$ zeros at $x = 0, 1$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$x < 0$	$0 < x < 1$	$1 < x < \frac{4}{3}$	$\frac{4}{3} < x$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$
$f(x)$	↷	⤵	↳	⤴

It shows that $f(0)$ is a local maximum, $f(\frac{4}{3})$ is a local minimum and at $x = 1$ f has an inflection point;

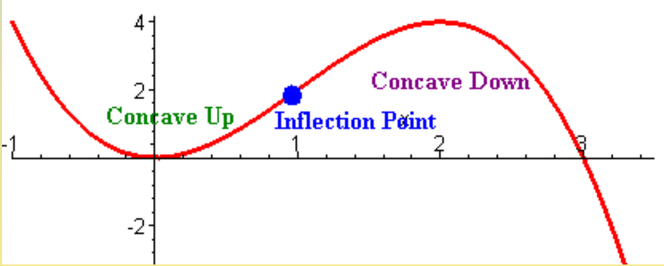


Second Derivative Test for Critical Points

Second Derivative Test

Let c be a critical point of $f(x)$. If $f''(c)$ exists, then:

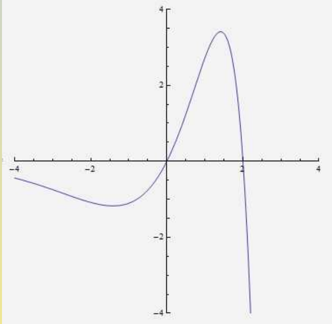
- if $f''(c) > 0$, then $f(c)$ is a local minimum;
- if $f''(c) < 0$, then $f(c)$ is a local maximum;
- if $f''(c) = 0$, the test is inconclusive.



Example of Second Derivative Test

- Use Second derivative test to analyze the critical points of $f(x) = (2x - x^2)e^x$;

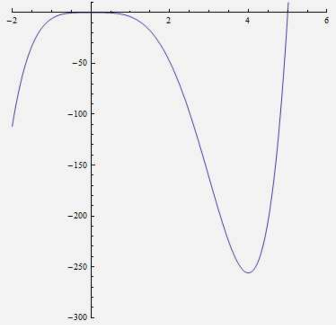
  $f'(x) = (2-2x)e^x + (2x - x^2)e^x = (2-x^2)e^x$; Therefore $x = \pm\sqrt{2}$ are the critical points; Moreover, $f''(x) = -2xe^x + (2-x^2)e^x = (2-2x-x^2)e^x$; Thus, $f''(-\sqrt{2}) > 0$ and $f''(\sqrt{2}) < 0$, showing that f has a local min $f(-\sqrt{2})$ and a local max $f(\sqrt{2})$;



One More Example

- Use Second derivative test to analyze the critical points of $f(x) = x^5 - 5x^4$;

  $f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$;
  Therefore $x = 0, 4$ are the critical points; Moreover,
  $f''(x) = 20x^3 - 60x^2 = 20x^2(x-3)$;
  Thus, $f''(0) = 0$ and $f''(4) > 0$, showing that f has a local min $f(4)$ but the Second Derivative Test for $x = 0$ is inconclusive; One needs to revert to the signs of the first derivative!



Subsection 5

L'Hôpital's Rule

L'Hôpital's Rule

- Assume that $f(x), g(x)$ are differentiable on an open interval containing a, that $f(a) = g(a) = 0$ and that $g'(x) \neq 0$ (except perhaps at a); Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$
assuming that the limit on the right exists or is infinite;
- The same conclusion holds if $f(x), g(x)$ are differentiable for x near, but not equal to, a and
$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty;$$
- The rule may also be applied for one-side limits;

Applying L'Hôpital's Rule I

- Compute $\lim_{x\to 2}\dfrac{x^3-8}{x^4+2x-20}$;

  Set $f(x)=x^3-8$ and $g(x)=x^4+2x-8$;
  We have $f(2)=0=g(2)$;
  Therefore, by L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to 2}\frac{f(x)}{g(x)} &= \lim_{x\to 2}\frac{f'(x)}{g'(x)}\\ &= \lim_{x\to 2}\frac{3x^2}{4x^3+2}\\ &= \frac{12}{34}\\ &= \frac{6}{17}.\end{aligned}$$

Applying L'Hôpital's Rule II

- Compute $\lim_{x \to 2} \frac{4 - x^2}{\sin(\pi x)}$;

  Set $f(x) = 4 - x^2$ and $g(x) = \sin(\pi x)$;
  We have $f(2) = 0 = g(2)$;
  Therefore, by L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \to 2} \frac{f(x)}{g(x)} &= \lim_{x \to 2} \frac{f'(x)}{g'(x)} \\ &= \lim_{x \to 2} \frac{-2x}{\pi \cos(\pi x)} \\ &= -\frac{4}{\pi}. \end{aligned}$$

Applying L'Hôpital's Rule III

- Compute $\lim_{x\to\frac{\pi}{2}} \frac{\cos^2 x}{1-\sin x}$;

  Set $f(x) = \cos^2 x$ and $g(x) = 1 - \sin x$;
  We have $f(\frac{\pi}{2}) = 0 = g(\frac{\pi}{2})$;
  Therefore, by L'Hôpital's Rule,

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)} &= \lim_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)} \\ &= \lim_{x\to\frac{\pi}{2}} \frac{-2\cos x \sin x}{-\cos x} \\ &= \lim_{x\to\frac{\pi}{2}} (2\sin x) \\ &= 2.\end{aligned}$$

The Form $0 \cdot \infty$

- Compute $\lim_{x \to 0^+} (x \ln x)$;

  Note that $\lim_{x \to 0^+} x = 0$ and $\lim_{x \to 0^+} \ln x = -\infty$; The form $0 \cdot \infty$ is indeterminate; The following method may be used to lift this indeterminacy:
  Rewrite $x \ln x = \frac{\ln x}{\frac{1}{x}}$; Set $f(x) = \ln x$ and $g(x) = \frac{1}{x}$;
  We have $\lim_{x \to 0^+} \ln x = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$;
  Therefore, by L'Hôpital's Rule,

$$\begin{aligned} \lim_{x \to 0^+} \frac{f(x)}{g(x)} &= \lim_{x \to 0^+} \frac{f'(x)}{g'(x)} \\ &= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \to 0^+} (-x) = 0. \end{aligned}$$

Using Rule Twice

- Evaluate $\displaystyle\lim_{x\to 0}\frac{e^x - x - 1}{\cos x - 1}$;

$$\begin{aligned}
\lim_{x\to 0}\tfrac{e^x - x - 1}{\cos x - 1} &= \left(\tfrac{0}{0}\right) \\
&= \lim_{x\to 0}\tfrac{(e^x - x - 1)'}{(\cos x - 1)'} \\
&= \lim_{x\to 0}\tfrac{e^x - 1}{-\sin x} \\
&= \left(\tfrac{0}{0}\right) \\
&= \lim_{x\to 0}\tfrac{(e^x - 1)'}{(-\sin x)'} \\
&= \lim_{x\to 0}\tfrac{e^x}{-\cos x} \\
&= -1.
\end{aligned}$$

Care Needed Before Applying L'Hôpital's Rule

- Evaluate $\lim_{x\to1}\frac{x^2+1}{2x+1}$;

$$\lim_{x\to1}\frac{x^2+1}{2x+1}=\frac{2}{3}.$$

- We do not want to apply L'Hôpital's Rule, when the form we are dealing with is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$;
- Careless application in the example above would yield the INCORRECT CONCLUSION

$$\lim_{x\to1}\frac{x^2+1}{2x+1}\underbrace{=}_{\text{Error!!}}\lim_{x\to1}\frac{2x}{2}=\lim_{x\to1}x=1\neq\frac{2}{3}.$$

The Form $\infty - \infty$

- Compute $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$;

  Note that $\lim_{x\to 0}\frac{1}{\sin x} = \infty$ and $\lim_{x\to 0}\frac{1}{x} = \infty$; The form $\infty - \infty$ is indeterminate; The following method may be used to lift this indeterminacy:
  Rewrite $\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$; Set $f(x) = x - \sin x$ and $g(x) = x \sin x$;
  We have $\lim_{x\to 0}(x - \sin x) = 0$ and $\lim_{x\to 0} x \sin x = 0$;
  Therefore, by L'Hôpital's Rule,

$$\begin{aligned} \lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\frac{f'(x)}{g'(x)} \\ &= \lim_{x\to 0}\frac{1 - cosx}{\sin x + x\cos x} = \left(\frac{0}{0}\right) \\ &= \lim_{x\to 0}\frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0. \end{aligned}$$

The Form 0^0

- Compute $\lim_{x \to 0^+} x^x$;

  Consider $f(x) = x^x$; Then

$$\begin{aligned} \lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \ln(x^x) \\ &= \lim_{x\to 0^+} (x \ln x) \\ &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0^+} (-x) = 0. \end{aligned}$$

  Therefore

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^{\lim_{x\to 0^+} \ln f(x)} = e^0 = 1.$$

L'Hôpital's Rule for Limits at Infinity

- Suppose that $f(x), g(x)$ are differentiable in an interval (b, ∞) and that $g'(x) \neq 0$ for $x > b$;
- If $\lim_{x \to \infty} f(x)$ and $\lim_{x \to \infty} g(x)$ exist and either both are zero or both infinite, then
$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$
assuming that the limit on the right exists;
- An analogous result holds for $x \to -\infty$;

Form $\frac{\infty}{\infty}$

- Suppose we want to discover which of the two functions $f(x) = x^2$ and $g(x) = x\ln x$ grows faster as $x \to \infty$; To this end, we compute $\lim\limits_{x\to\infty} \frac{x^2}{x\ln x}$;
$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{x\ln x} &= \lim_{x\to\infty} \frac{x}{\ln x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} \\ &= \lim_{x\to\infty} x = \infty;\end{aligned}$$
Thus $f(x)$ grows asymptotically faster than $g(x)$;
- Suppose we want to discover which of the two functions $f(x) = (\ln x)^2$ and $g(x) = \sqrt{x}$ grows faster as $x \to \infty$; To this end, we compute $\lim\limits_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2}$;
$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2} &= \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{2\ln x}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} \\ &= \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty;\end{aligned}$$
Thus $g(x)$ grows asymptotically faster than $f(x)$;

Super-polynomial Growth of e^x

- We show that $f(x) = e^x$ grows asymptotically faster than any power function $f(x) = x^n$, for fixed n;

  We calculate

$$
\begin{aligned}
\lim_{x\to\infty} \frac{e^x}{x^n} &= \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} \\
&= \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}} \\
&= \lim_{x\to\infty} \frac{e^x}{n(n-1)(n-2)x^{n-3}} \\
&= \cdots \\
&= \lim_{x\to\infty} \frac{e^x}{n!} \\
&= \frac{1}{n!} \lim_{x\to\infty} e^x = \infty.
\end{aligned}
$$

  Therefore $f(x)$ grows asymptotically faster than $g(x)$;

Subsection 6

Graph Sketching and Asymptotes

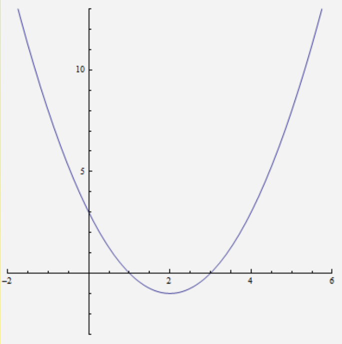
Graph Sketching I

- Study the function $f(x) = x^2 - 4x + 3$ and sketch its graph;

We have $f'(x) = 2x - 4 = 2(x-2)$; Therefore, $f(x)$ has a critical point at $x = 2$; We have $f''(x) = 2$; Therefore, $f''(x)$ does not have any zeros; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$x < 2$	$2 < x$
$f'(x)$	$-$	$+$
$f''(x)$	$+$	$+$
$f(x)$	$\searrow$	$\nearrow$

It shows that $f(2) = -1$ is a local minimum;

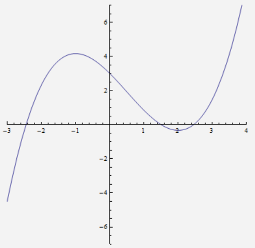


Graph Sketching II

- Study the function $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$ and sketch its graph;

We have $f'(x) = x^2 - x - 2 = (x+1)(x-2)$; Therefore, $f(x)$ has critical points at $x = -1, 2$; We have $f''(x) = 2x - 1$; Therefore, $f''(x)$ has a zero $x = \frac{1}{2}$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$x < -1$	$-1 < x < \frac{1}{2}$	$\frac{1}{2} < x < 2$	$2 < x$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$
$f(x)$	↗	↘	↘	↗



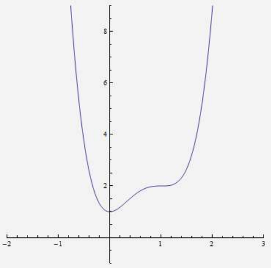
It shows that $f(2)$ is a local minimum, $f(-1)$ is a local maximum and at $x = \frac{1}{2}$ f has an inflection point;

Graph Sketching III

- Study the function $f(x) = 3x^4 - 8x^3 + 6x^2 + 1$ and sketch its graph;

We have $f'(x) = 12x^3 - 24x^2 + 12x = 12x(x-1)^2$; Therefore, $f(x)$ has critical points at $x = 0, 1$; We have $f''(x) = 36x^2 - 48x + 12 = 12(3x^2 - 4x + 1) = 12(x-1)(3x-1)$; Therefore, $f''(x)$ has zeros $x = \frac{1}{3}, 1$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$x < 0$	$0 < x < \frac{1}{3}$	$\frac{1}{3} < x < 1$	$1 < x$
$f'(x)$	$-$	$+$	$+$	$+$
$f''(x)$	$+$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\nearrow$	$\nearrow$



It shows that $f(0)$ is a local minimum, and at $x = \frac{1}{3}, 1$ f has inflection points;

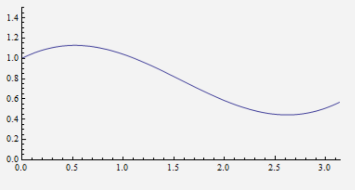
Graph Sketching with Trigonometric Functions

- Study the function $f(x) = \cos x + \frac{1}{2}x$ on $[0, \pi]$ and sketch its graph;

We have $f'(x) = -\sin x + \frac{1}{2}$; Therefore, $f(x)$ has critical points at $x = \frac{\pi}{6}, \frac{5\pi}{6}$; We have $f''(x) = -\cos x$; Therefore, $f''(x)$ has zero $x = \frac{\pi}{2}$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < \pi$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

It shows that $f(0), f(\frac{5\pi}{6})$ are local minima, $f(\frac{\pi}{6}), f(\pi)$ are local maxima and at $x = \frac{\pi}{2}$ f has inflection points;

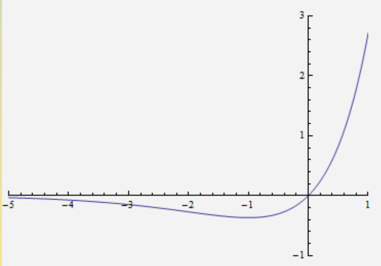


Graph Sketching with Exponential Functions

- Study the function $f(x) = xe^x$ and sketch its graph;

We have $f'(x) = e^x + xe^x = (1+x)e^x$; Therefore, $f(x)$ has critical points at $x = -1$; We have $f''(x) = e^x + e^x + xe^x = (2+x)e^x$; Therefore, $f''(x)$ has zero $x = -2$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Function	$x < -2$	$-2 < x < -1$	$-1 < x$
$f'(x)$	$-$	$-$	$+$
$f''(x)$	$-$	$+$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$



It shows that $f(-1)$ are local minima and at $x = -2$ f has inflection point; Since $\lim_{x\to-\infty} f(x) = 0$, $x = 0$ is a horizontal asymptote as $x \to -\infty$;

Graph Sketching with Rational Functions

- Study the function $f(x) = \frac{3x+2}{2x-4}$ and sketch its graph;

We have $\mathrm{Dom}(f) = \mathbb{R} - \{2\}$; Since $\lim_{x \to 2^-} f(x) = -\infty$ and $\lim_{x \to 2^+} f(x) = \infty$, the line $x = 2$ is a vertical asymptote to $y = f(x)$; Since $\lim_{x \to -\infty} f(x) = \frac{3}{2}$ and $\lim_{x \to \infty} f(x) = \frac{3}{2}$, the line $y = \frac{3}{2}$ is a horizontal asymptote to $y = f(x)$;

We have $f'(x) = \frac{3(2x-4)-2(3x+2)}{(2x-4)^2} = -\frac{4}{(x-2)^2}$; Therefore, $f(x)$ has critical point at $x = 2$; We have $f''(x) = \frac{8}{(x-2)^3}$; Therefore, $f''(x)$ is undefined at $x = 2$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

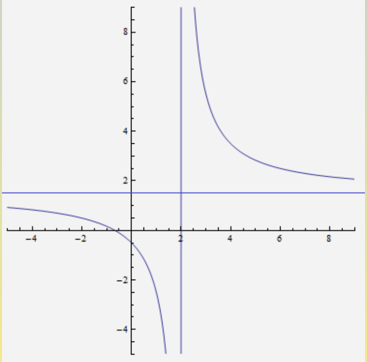
Study of $f(x) = \frac{3x+2}{2x-4}$ (Cont'd)

Recall

$$f'(x) = -\frac{4}{(x-2)^2} \quad \text{and} \quad f''(x) = \frac{8}{(x-2)^3}.$$

Function	$x < 2$	$2 < x$
$f'(x)$	$-$	$-$
$f''(x)$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$

$x = 2$ is outside the domain of f; so it cannot be either a local extremum or an inflection point; Taking into account the arrows of the table and the asymptotes, we sketch the graph:



Graphs of Rational Functions II

- Study the function $f(x) = \frac{1}{x^2-1}$ and sketch its graph;

We have $\mathrm{Dom}(f) = \mathbb{R} - \{-1, 1\}$; Since $\lim_{x\to-1^-} f(x) = \infty$, $\lim_{x\to-1^+} f(x) = -\infty$ and $\lim_{x\to1^-} f(x) = -\infty$, $\lim_{x\to1^+} f(x) = \infty$, the lines $x = -1$ and $x = 1$ are a vertical asymptotes to $y = f(x)$; Since $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote to $y = f(x)$;
We have $f'(x) = [(x^2-1)^{-1}]' = -\frac{2x}{(x^2-1)^2}$; Therefore, $f(x)$ has critical points at $x = 0, \pm1$; We also have

$$\begin{aligned} f''(x) &= -\frac{2(x^2-1)^2 - 2x2(x^2-1)2x}{(x^2-1)^4} = -\frac{2(x^2-1)-8x^2}{(x^2-1)^3} \\ &= -\frac{-6x^2-2}{(x^2-1)^3} = \frac{2(3x^2+1)}{(x^2-1)^3}; \end{aligned}$$

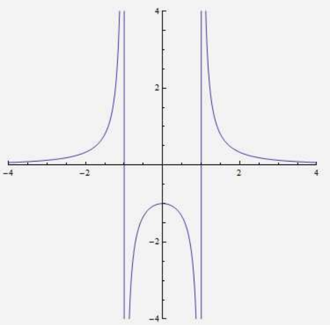
Therefore, $f''(x)$ is undefined at $x = \pm 1$; The following is a combined sign table for $f'(x), f''(x)$ and summarizes the monotonicity and concavity of $f(x)$:

Study of $f(x) = \frac{1}{x^2-1}$ (Cont'd)

Recall $f'(x) = -\frac{2x}{(x^2-1)^2}$ and $f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}$.

Function	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x$
$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$-$	$+$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$

$x = \pm 1$ are outside the domain of f; so they cannot be either local extrema or inflection points; Taking into account the arrows of the table and the asymptotes, we sketch the graph:

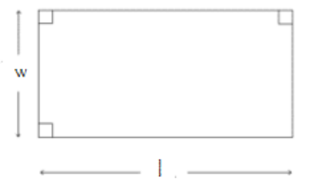


Subsection 7

Applied Optimization

Rectangle of Fixed Perimeter with Maximum Area

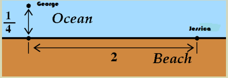
- A piece of wire of length L is bent into a rectangular shape. Which dimensions produce a rectangle with maximum possible area?

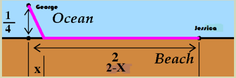


The objective function to be maximized is $A = \ell w$; The auxiliary condition $2\ell + 2w = L$ allows us to reduce w by solving for it: $w = \frac{1}{2}L - \ell$; Thus $A(\ell) = \ell(\frac{1}{2}L - \ell) = -\ell^2 + \frac{1}{2}L\ell$; Compute derivative $A'(\ell) = -2\ell + \frac{1}{2}L$; Find critical point: $-2\ell + \frac{1}{2}L = 0$ implies $\ell = \frac{1}{4}L$; Thus, the rectangle must be of dimensions $\frac{1}{4}L \times \frac{1}{4}L$ and it will have max area $A_{\max} = \frac{1}{16}L^2$;

Minimizing Travel Time

- George is swimming a quarter mile from shore and sees Jessica on the beach two miles down from his closest point to the beach. He must reach her as soon as possible in case she leaves. If he can run a mile in 10 minutes and swim a mile in 40 minutes how can he get to Jessica in the least time possible?





Suppose that George swims to a point x miles down the beach from the closest point and then runs the remaining distance on the beach.

The objective function to be maximized is time

$T(x) = \frac{\sqrt{x^2+\frac{1}{16}}}{\frac{1}{40}} + \frac{2-x}{\frac{1}{10}} = 40\sqrt{x^2+\frac{1}{16}} - 10(x-2)$; The derivative is

$T'(x) = 40\frac{2x}{2\sqrt{x^2+\frac{1}{16}}} - 10 = \frac{40x-10\sqrt{x^2+\frac{1}{16}}}{\sqrt{x^2+\frac{1}{16}}}$; Set that equal to zero and solve

$40x = 10\sqrt{x^2+\frac{1}{16}}$; so $16x^2 = x^2 + \frac{1}{16}$, i.e., $x = \frac{1}{4\sqrt{15}}$ miles;

Optimizing Price for Maximum Profit

- An apartment building has 30 units all of which are rented at \$ 1,000 per month. For each \$ 40 increase in rent an additional unit becomes vacant. Assume the maintenance cost per occupied unit is \$ 120 per month. What is the rental price that maximizes the monthly profit?

  To devise an objective function for the profit, set x be the number of \$ 40 increments and subtract revenue from cost:

$$\begin{aligned} P(x) &= R(x) - C(x) = \overbrace{(1000 + 40x)}^{\text{price/unit}} \overbrace{(30 - x)}^{\text{\# of units}} - \overbrace{120(30 - x)}^{\text{maintenace costs}} \\ &= -40x^2 + 200x + 30000 - 3600 + 120x \\ &= -40x^2 + 320x + 26400. \end{aligned}$$

  Thus, the maximum profit occurs when $P'(x) = 0$, i.e., when $-80x + 320 = 0$, giving $x = 4$. Thus, the most profitable rental price is $1000 + 40 \cdot 4 = \$1160$ per month.

Can of Fixed Volume Using Least Aluminum

- What are the dimensions of a cylindrical can of volume 900 cm^3 that uses the least amount of aluminum, i.e., that has the minimum surface area?

The objective function is $A = \overbrace{2\pi r^2}^{\text{top \& bottom}} + \overbrace{2\pi rh}^{\text{side}}$; The volume is $\pi r^2 h = 900$, whence, we get that $h = \frac{900}{\pi r^2}$; This allows us to express the objective function as

$$A(r) = 2\pi r^2 + \frac{1800}{r};$$

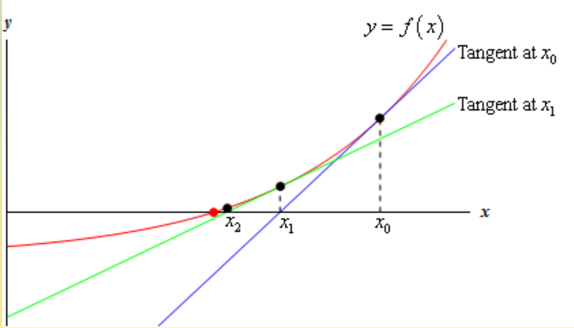
This has derivative $A'(r) = 4\pi r - \frac{1800}{r^2} = \frac{4\pi r^3 - 1800}{r^2}$ and has critical points $r = 0, \sqrt[3]{\frac{450}{\pi}}$; The second yields the min area; What is the required h?

Subsection 8

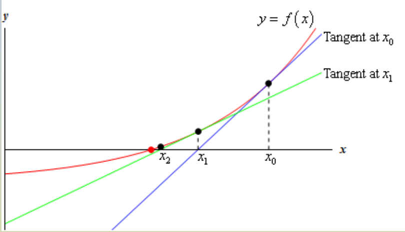
Newton's Method

Newton's Method I

- Suppose that $f(x)$ is a continuous function and that we would like to find a root of the equation $f(x) = 0$; This root is the x-intercept of the graph of $y = f(x)$;
- Assume, in addition, that we have an initial guess x_0 "close" to the real solution;

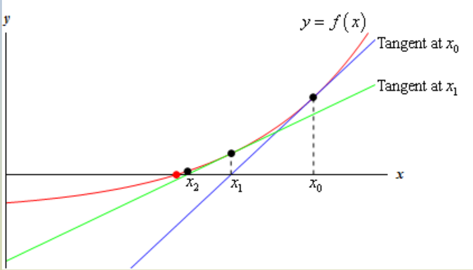


Newton's Method II



- The slope of the tangent line at x_0 is $f'(x_0)$; Thus, an equation of the tangent line at x_0 is $y - f(x_0) = f'(x_0)(x - x_0)$; Hence, its x-intercept x_1 is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$;
- The slope of the tangent line at x_1 is $f'(x_1)$; Thus, an equation of the tangent line at x_1 is $y - f(x_1) = f'(x_1)(x - x_1)$; Hence, its x-intercept x_2 is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$;
- We iterate this process, thus obtaining Newton's formula for approximating the roots of $f(x) = 0$;

Newton's Approximation Formula



Newton's Method

To compute a root of $f(x) = 0$:

- Choose an initial guess x_0 suspected to be close to root;
- Generate successive approximations using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};$$

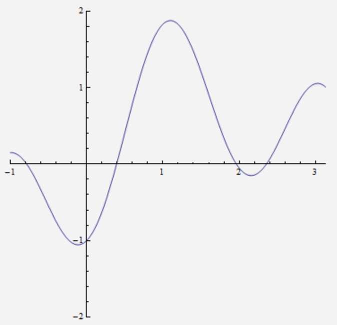
Approximating the $\sqrt{3}$

- To approximate the $\sqrt{3}$, we consider the function $f(x) = x^2 - 3$ and try to find its zero;
- We have $f'(x) = 2x$;
- Let us take $x_0 = 2$;
- Then we obtain successively:

$$\begin{aligned} x_1 &= 2 - \frac{f(2)}{f'(2)} = 2 - \frac{1}{4} = 1.75 \\ x_2 &= 1.75 - \frac{f(1.75)}{f'(1.75)} = 1.75 - \frac{0.0625}{3.5} = 1.732 \\ x_3 &= 1.732 - \frac{f(1.732)}{f'(1.732)} = 1.732 - \frac{-0.000176}{3.464} = 1.7320508. \end{aligned}$$

- Note that, using a calculator, we get $\sqrt{3} \approx 1.7320508$;

Approximating a root of $\cos 3x = \sin x$



- Consider the function $f(x) = \sin x - \cos 3x$ and try to find a zero;
- We have $f'(x) = \cos x + 3\sin x3x$;
- Let us take $x_0 = 1$;

Then we obtain successively:

$$\begin{array}{lcl}
x_1 & = & 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1.8315}{0.964} = -0.9 \\
x_2 & = & -0.9 - \frac{f(-0.9)}{f'(-0.9)} = -0.9 - \frac{0.121}{-0.661} = -0.717 \\
x_3 & = & -0.717 - \frac{f(-0.717)}{f'(-0.717)} = -0.717 - \frac{-0.109}{-1.755} = -0.779 \\
x_4 & = & -0.779 - \frac{f(-0.779)}{f'(-0.779)} = -0.779 - \frac{-0.009}{-1.45} = -0.785
\end{array}$$

Subsection 9

Antiderivatives

Antiderivatives

- A function $F(x)$ is an **antiderivative** of $f(x)$ on the interval (a, b) if $F'(x) = f(x)$ for all $x \in (a, b)$;
- What is an antiderivative of $f(x) = \cos x$?
  $F(x) = \sin x$;
  Is it the only one?
- What is an antiderivative of $f(x) = x^2$?
  $F(x) = \frac{1}{3}x^3$;
  Is it the only one?

The General Antiderivative

If $F(x)$ is an antiderivative of $f(x)$ on (a, b), then every other anti-derivative on (a, b) is of the form $F(x) + C$ for some constant C.

- What are the most general antiderivatives of $f(x) = x^6$ and $g(x) = \sin x$?
  $F(x) = \frac{1}{7}x^7 + C$; $G(x) = -\cos x + C$;

Integration and Indefinite Integrals

- The process of determining an antiderivative is called **integration**;

Indefinite Integral

If $F'(x) = f(x)$, we write

$$\int f(x)dx = F(x) + C;$$

The most general antiderivative $F(x) + C$ is called the **indefinite integral** of $f(x)$;

- Find the indefinite integral $\int x^n dx$, for $n \neq -1$.

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C;$$

Power Rule for Indefinite Integrals

Power Rule

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad \text{if } n \neq -1;$$

Example: Use the power rule to compute the indefinite integrals:

- $\int x^{12} dx = \frac{1}{13} x^{13} + C$
- $\int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{1}{-6} x^{-6} + C = -\frac{1}{6x^6} + C$
- $\int x^{4/7} dx = \frac{1}{11/7} x^{11/7} + C = \frac{7}{11} x^{11/7} + C$
- $\int \frac{1}{\sqrt[5]{x^3}} dx = \int x^{-3/5} dx = \frac{1}{2/5} x^{2/5} + C = \frac{5}{2} \sqrt[5]{x^2} + C$

$y = \frac{1}{x}$, Sum and Constant Factor Rules

Antiderivative of $y = \frac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C;$$

Linearity of the Indefinite Integral

- **Sum/Difference Rule:** $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
- **Constant Factor Rule:** $\int cf(x)dx = c \int f(x)dx$

Example: Use the rules to evaluate the indefinite integral:

- $\int (5x^2 - 3x^{4/5} + x^{-9})dx = \int 5x^2 dx - \int 3x^{4/5} dx + \int x^{-9} dx =$
$5\int x^2 dx - 3\int x^{4/5} dx + \int x^{-9} dx = 5\frac{1}{3}x^3 - 3\frac{5}{9}x^{9/5} - \frac{1}{8}x^{-8} + C =$
$\frac{5}{3}x^3 - \frac{5}{3}x^{9/5} - \frac{1}{8}x^{-8} + C;$

Basic Trigonometric Integrals

- The following formulas give the basic trigonometric integrals:

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$
$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C$$
$$\int \sec x \tan x dx = \sec x + C \qquad \int \csc x \cot x dx = -\csc x + C$$

- Based on the basic formulas, we obtain:

$$\int \cos(kx+b) dx = \frac{1}{k}\sin(kx+b) + C$$
$$\int \sin(kx+b) dx = -\frac{1}{k}\cos(kx+b) + C$$

Example: Evaluate: $\int (\sin(8x-3) + 20\cos 9x) dx =$
$\int \sin(8x-3) dx + 20\int \cos 9x dx = -\frac{1}{8}\cos(8x-3) + \frac{20}{9}\sin 9x + C;$

Integrals Involving e^x

Integrals Involving e^x

$$\int e^x dx = e^x + C \qquad \int e^{kx+b} dx = \frac{1}{k} e^{kx+b} + C$$

Example:

- $\int (7e^x - 4x^3) dx = 7\int e^x dx - 4\int x^3 dx = 7e^x - 4\frac{x^4}{4} + C = 7e^x - x^4 + C$
- $\int 12e^{7-3x} dx = 12\int e^{7-3x} dx = 12\frac{1}{-3}e^{7-3x} + C = -4e^{7-3x} + C$

Differential Equations and Initial Conditions

- An antiderivative of a function $f(x)$ is a solution to the **differential equation**
$$\frac{dy}{dx} = f(x);$$
- The most general antiderivative of the form $\int f(x)dx = F(x) + C$ involving C is called the **general solution** of the equation;
- By imposing an **initial condition**, i.e., some equation of the form $y(x_0) = y_0$, for some specific values x_0 and y_0, one may specify a **particular solution** of the differential equation, i.e., enable the specification of a value for C;
- The group $\begin{cases} \frac{dy}{dx} = f(x) \\ y(x_0) = y_0 \end{cases}$ (differential equation plus initial condition) is called an **initial value problem**;

Solving an Initial Value Problem

- Solve $\frac{dy}{dx} = 4x^7$ subject to the initial condition $y(0) = 4$;
  We have $y(x) = \int 4x^7 dx = 4\int x^7 dx = 4\frac{x^8}{8} + C = \frac{1}{2}x^8 + C$; Since $y(0) = 4$, we get $4 = 0 + C$, i.e., $C = 4$; Therefore, we obtain the particular solution
  $$y(x) = \frac{1}{2}x^8 + 4;$$
- Solve $\frac{dy}{dt} = \sin(\pi t)$ subject to the initial condition $y(2) = 2$;
  We have $y(t) = \int \sin(\pi t) dt = -\frac{1}{\pi}\cos(\pi t) + C$; Since $y(2) = 2$, we get $2 = -\frac{1}{\pi}\cos(2\pi) + C$, i.e., $C = 2 + \frac{1}{\pi}$; Therefore, we obtain the particular solution
  $$y(t) = -\frac{1}{\pi}\cos(\pi t) + 2 + \frac{1}{\pi}.$$

Applied Initial Value Problem

- A car traveling with velocity 50 m/s begins to slow down at $t = 0$ with a constant deceleration of $a = -10$ m/s^2. What is the velocity $v(t)$ at time t and what distance will be traveled before the car stops?

We have $v(t) = \int -10dt = -10t + C$; Since $v(0) = 50$, we get $C = 50$, i.e., $v(t) = -10t + 50$; Therefore, setting $v = 0$, we find that it will take $t = 5$ s for the car to stop;
Moreover, $s(t) = \int v(t)dt = \int (-10t + 50)dt = -5t^2 + 50t + C$;
Since $s(0) = 0$, we get $C = 0$, i.e., $s(t) = -5t^2 + 50t$; Therefore, setting $t = 5$, we find that the car will travel $s(5) = 125$ m before it comes to a halt;