Calculus II

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Subsection 1

Sequences

Sequences

- A **sequence** is an ordered collection of numbers defined by a function $f(n)$ on a set of integers;
- The values $a_n = f(n)$ are the **terms** of the sequence and n the **index**;
- We think of $\{a_n\}$ as a list $a_1,\ a_2,\ a_3,\ a_4,\ \ldots$
- The sequence may not start at $n=1$; It may start at $n=0, n=2$ or any other integer;
- When a_n is given by a formula, then it is referred to as the **general term** of the sequence;
- Examples:

General Term	Domain	Sequence
$a_n = 1 - \frac{1}{n}$	$n \geq 1$	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
$a_n = (-1)^n n$	$n \geq 0$	$0, -1, 2, -3, 4, \ldots$
$a_n = \frac{n^2}{n^2-4}$	$n \geq 3$	$\frac{9}{5}, \frac{16}{12}, \frac{25}{21}, \frac{36}{32}, \frac{49}{45}, \ldots$

Recursively Defined Sequences

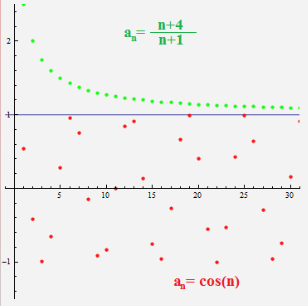
- A sequence is defined **recursively** if one or more of its first few terms are given and the n-th term a_n is computed in terms of one or more of the preceding terms $a_{n-1}, a_{n-2}, \ldots$;
- Example: Compute a_2, a_3, a_4 for the sequence defined recursively by

$$a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right);$$

$$\begin{aligned} a_2 &= \frac{1}{2}\left(a_1 + \frac{2}{a_1}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2}; \\ a_3 &= \frac{1}{2}\left(a_2 + \frac{2}{a_2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right) = \frac{1}{2} \cdot \frac{17}{6} = \frac{17}{12}; \\ a_4 &= \frac{1}{2}\left(a_3 + \frac{2}{a_3}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right) = \frac{1}{2} \cdot \frac{577}{204} = \frac{577}{408}; \end{aligned}$$

Limit of a Sequence

- We say that the sequence $\{a_n\}$ **converges** to a limit L, written $\lim_{n\to\infty} a_n = L$ or $a_n \to L$, if the values of a_n get arbitrarily close to the value L when n is taken sufficiently large;
- If a sequence does not converge, we day it **diverges**;
- If the terms increase without bound, $\{a_n\}$ **diverges to infinity**;



Sequence Defined by a Function

Theorem (Limit of a Sequence Defined by a Function)

If $\lim_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit, i.e., $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$;

- Example: Show that $\lim_{n\to\infty} a_n = 1$, where $a_n = \dfrac{n+4}{n+1}$;

  We consider the function $f(x) = \dfrac{x+4}{x+1}$; Clearly, $a_n = f(n)$;

  Therefore, by the Theorem, it suffices to show that $\lim_{x\to\infty} f(x) = 1$;

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+4}{x+1} = \lim_{x\to\infty} \frac{1+\frac{4}{x}}{1+\frac{1}{x}} = \frac{1+0}{1+0} = 1;$$

Example I

- Find the limit of the sequence $\frac{2^2-2}{2^2}, \frac{3^2-2}{3^2}, \frac{4^2-2}{4^2}, \frac{5^2-2}{5^2}, \ldots$;

  The general term of the given sequence is $a_n = \frac{n^2-2}{n^2}$; We consider the function $f(x) = \frac{x^2-2}{x^2} = 1 - \frac{2}{x^2}$; Clearly, $a_n = f(n)$; Therefore, it suffices to find the limit $\lim_{x\to\infty} f(x)$;

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (1 - \tfrac{2}{x^2}) = 1 - 0 = 1;$$

  Thus, $\lim_{n\to\infty} a_n = 1$;

Example II

- Find the limit $\lim_{n\to\infty}\frac{n+\ln n}{n^2}$;

  We consider the function $f(x)=\frac{x+\ln x}{x^2}$; Clearly, $a_n=f(n)$;
  Therefore, it suffices to find the limit $\lim_{x\to\infty}f(x)$;

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{x+\ln x}{x^2}=$$
$$\left(\frac{\infty}{\infty}\right)\overset{\text{L'Hôpital}}{=}\lim_{x\to\infty}\frac{(x+\ln x)'}{(x^2)'}=\lim_{x\to\infty}\frac{1+(1/x)}{2x}=0;$$

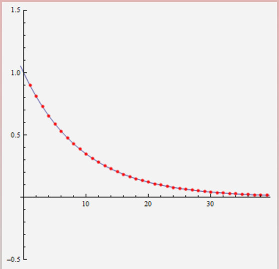
  Thus, $\lim_{n\to\infty}\frac{n+\ln n}{n^2}=0$;

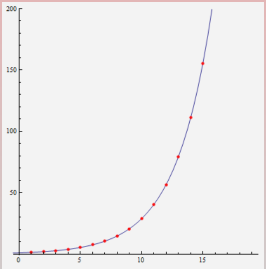
Geometric Sequences

- For $r \geq 0$ and $c > 0$,

$$\lim_{n\to\infty} cr^n = \begin{cases} 0, & \text{if } 0 \leq r < 1 \\ c, & \text{if } r = 1 \\ \infty, & \text{if } r > 1 \end{cases}$$

To see this, one considers the corresponding function $f(x) = cr^x$; If $r < 1$, then, $\lim_{x\to\infty} cr^x = 0$, and, if $r > 1$, then, $\lim_{x\to\infty} cr^x = \infty$;





Limits Laws for Sequences

Limit Laws for Sequences

Assume $\{a_n\}$ and $\{b_n\}$ are convergent sequences with

$$\lim_{n\to\infty} a_n = L, \qquad \lim_{n\to\infty} b_n = M;$$

Then, we have:

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$;
2. $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM$;
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$, if $M \neq 0$;
4. $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$, ($c$ a constant;)

Squeeze Theorem for Sequences

Squeeze Theorem for Sequences

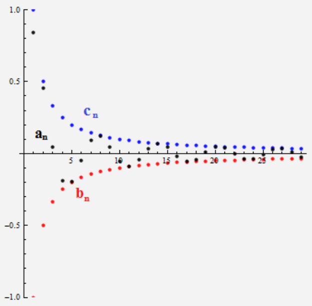
Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences, such that, for some number M,

$$b_n \leq a_n \leq c_n, \text{ for all } n > M$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L;$$

Then $\lim_{n\to\infty} a_n = L$;



- Example: Show that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.
  Note that $-|a_n| \leq a_n \leq |a_n|$; By hypothesis $\lim_{n\to\infty} |a_n| = 0$; This also implies $\lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty} |a_n| = 0$; Now, by the Squeeze Theorem for Sequences, $\lim_{n\to\infty} a_n = 0$;

Geometric Sequences with $r < 0$

- For $c \neq 0$,

$$\lim_{n\to\infty} cr^n = \begin{cases} 0, & \text{if } -1 < r < 0 \\ \text{diverges}, & \text{if } r \leq -1 \end{cases}$$

  - If $-1 < r < 0$, then $0 < |r| < 1$ and, therefore $\lim_{n\to\infty} |cr^n| = \lim_{n\to\infty} |c| \cdot |r|^n = 0$; Thus, since $-|cr^n| \leq cr^n \leq |cr^n|$, by the Squeeze Theorem, we get $\lim_{n\to\infty} cr^n = 0$;
  - If $r = -1$, then $\lim_{n\to\infty} (-1)^n c$ diverges, since $|(-1)^n c| = |c|$ and its sign keeps alternating;
  - If $r < -1$, then $|r| > 1$, whence $|cr^n| = |c| \cdot |r|^n \to \infty$, whence $\lim_{n\to\infty} cr^n$ diverges in this case also;

Exploiting Continuity

Theorem

If $f(x)$ is a continuous function and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L);$$

This says, informally speaking, that if f is continuous, we can "push the limit in";

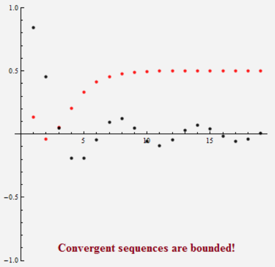
- Example: Since $f(x) = e^x$ and $g(x) = x^2$ are both continuous, we may use this theorem to compute:
  - $\lim_{n\to\infty} e^{\frac{3n}{n+1}} = \lim_{n\to\infty} f(\frac{3n}{n+1}) = f(\lim_{n\to\infty} \frac{3n}{n+1}) = f(3) = e^3;$
  - $\lim_{n\to\infty} (\frac{3n}{n+1})^2 = \lim_{n\to\infty} g(\frac{3n}{n+1}) = g(\lim_{n\to\infty} \frac{3n}{n+1}) = g(3) = 9;$

Bounded Sequences

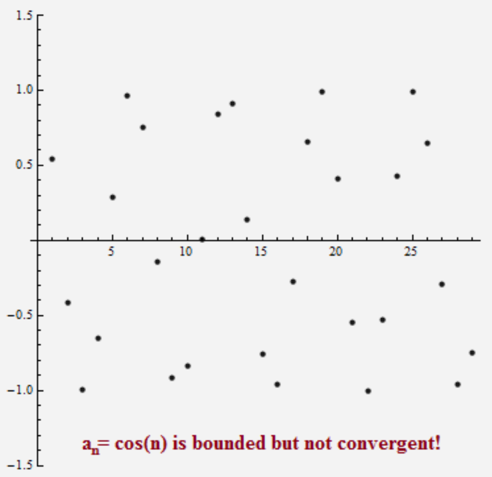
- A sequence $\{a_n\}$ is
  - **bounded from above** if there is a number M, such that $a_n \leq M$, for all n; In this case M is called an **upper bound**;
  - **bounded from below** if there is a number m, such that $a_n \geq m$, for all n; In this case m is called a **lower bound**;
- $\{a_n\}$ is **bounded** if it is bounded from above and from below; A sequence is **unbounded** if it is not bounded;

Theorem

If $\{a_n\}$ converges, then $\{a_n\}$ is bounded;



Is Every Bounded Sequence Convergent?



Bounded Monotonic Sequences

- A sequence $\{a_n\}$ is
  - **increasing** if $a_n < a_{n+1}$, for all n;
  - **decreasing** if $a_n > a_{n+1}$, for all n;
  - **monotonic** if it is either increasing or decreasing;

Theorem (Bounded Monotonic Sequences Converge)

- If $\{a_n\}$ is increasing and $a_n \leq M$, then a_n converges and $\lim_{n\to\infty} a_n \leq M$;
- If $\{a_n\}$ is decreasing and $a_n \geq m$, then a_n converges and $\lim_{n\to\infty} a_n \geq m$;

Example I

- Show that $a_n = \sqrt{n+1} - \sqrt{n}$ is decreasing and bounded from below; Does $\lim\limits_{n\to\infty} a_n$ exist?

  We show that a_n is decreasing by two different methods; The first uses the sequence itself, the second uses the corresponding function;

  - Method 1: Rewrite $a_n = \sqrt{n+1} - \sqrt{n} =$
    $$\frac{(\sqrt{n+1}+\sqrt{n})(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}};$$
    Now we see
    $$a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{\sqrt{(n+1)+1}+\sqrt{n+1}} = a_{n+1};$$
    So $\{a_n\}$ is decreasing;
  - Method 2: Consider $f(x) = \sqrt{x+1} - \sqrt{x}$ and compute
    $f'(x) = \dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{2\sqrt{x}} < 0,$ for $x > 0$; Thus, since $f' < 0$, we get that $f \searrow [0,\infty)$, showing that $\{a_n\}$ is a decreasing sequence;

  Clearly $a_n = \sqrt{n+1} - \sqrt{n} > 0$, which shows that $\{a_n\}$ is bounded from below;

Example II

- Show that the following sequence is bounded and increasing; Then find its limit:

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \quad \ldots$$

The key here is to realize that $a_{n+1} = \sqrt{2a_n}$, for all n;
We show $\{a_n\}$ is bounded: Clearly, $a_1 = \sqrt{2} < 2$; If $a_n < 2$, then $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$; Therefore, $a_n < 2$, for every $n \geq 1$;
Next, we show that $\{a_n\}$ is increasing:

$$a_n = \sqrt{a_n \cdot a_n} < \sqrt{2 \cdot a_n} = a_{n+1};$$

Since $\{a_n\}$ is increasing and bounded from above, the theorem asserts that it converges; Let $\lim_{n\to\infty} a_n = L$; Then
$a_{n+1} = \sqrt{2a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \sqrt{2 \lim_{n\to\infty} a_n} \Rightarrow L = \sqrt{2L} \Rightarrow L^2 = 2L \Rightarrow$
$L^2 - 2L = 0 \Rightarrow L(L-2) = 0 \Rightarrow L = 0$ or $L = 2$; So $\lim_{n\to\infty} a_n = 2$;

Subsection 2

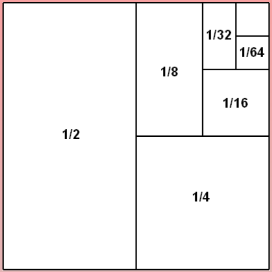
Summing an Infinite Series

Introducing Infinite Series and Partial Sums

- If we look carefully at the figure on the right we realize that

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots;$$

Infinite sums of this type are called **infinite series**;



- The **partial sum** S_N of an infinite series is the sum of the terms up to and including the N-th term:

$$\begin{array}{rcl} S_1 & = & \frac{1}{2}; \\ S_2 & = & \frac{1}{2} + \frac{1}{4}; \\ S_3 & = & \frac{1}{2} + \frac{1}{4} + \frac{1}{8}; \\ S_4 & = & \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}; \\ & \vdots & \end{array}$$

Definition of Infinite Series and Partial Sums

- An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots,$$

where $\{a_n\}$ is any *sequence*;

- Example:

Sequence	General Term	Infinite Series
$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$	$a_n = \frac{1}{3^n}$	$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$
$\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$	$a_n = \frac{1}{n^2}$	$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots$

- The ***N*-th partial sum** S_N is defined as the finite sum of the terms up to and including a_N:

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N;$$

Convergence of an Infinite Series

Convergence of an Infinite Series

An infinite series $\sum_{n=k}^{\infty} a_n$ **converges** to the sum S if its partial sums converge to S:

$$\lim_{N \to \infty} S_N = S;$$

In this case, we write $S = \sum_{n=k}^{\infty} a_n$;

- If the limit $\lim_{N \to \infty} S_N$ does not exist, then we say the infinite series **diverges**;
- If $\lim_{N \to \infty} S_N = \infty$, then we say that the infinite series **diverges to infinity**;

Telescoping Series

- Compute the sum S of the infinite series

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \frac{1}{4(5)} + \cdots ;$$

Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$; Therefore, we have

$$\frac{1}{1 \cdot 2} = 1 - \frac{1}{2}, \quad \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{3 \cdot 4} = \frac{1}{3} - \frac{1}{4}, \quad \ldots$$

Now, we compute the N-th partial sum:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n(n+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \\ &\quad (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{N} - \frac{1}{N+1}) = 1 - \frac{1}{N+1}; \end{aligned}$$

Therefore, $S = \lim_{N \to \infty} S_N = \lim_{N \to \infty} (1 - \frac{1}{N+1}) = 1 - 0 = 1$;

Sequence $\{a_n\}$ versus Series $\sum a_n$

- The previous example provides an opportunity to discuss the difference between the sequence $\{a_n\}$ and the infinite series
$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots;$$
- The sequence $a_n = \frac{1}{n(n+1)}$ is the list of numbers
$$\frac{1}{1 \cdot 2}, \quad \frac{1}{2 \cdot 3}, \quad \frac{1}{3 \cdot 4}, \quad \ldots$$
Clearly $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n(n+1)} = 0$;
- On the other hand, for the *sum of the infinite series* $S = \sum_{n=1}^{\infty} a_n$, we look **not** at $\lim_{n \to \infty} a_n$, but rather at $\lim_{N \to \infty} S_N$, where
$$S_N = \sum_{n=1}^{N} a_n = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{N(N+1)};$$
We saw that this limit is 1, not 0!

Linearity of Infinite Series

Linearity of Infinite Series

If the infinite series $\sum a_n$ and $\sum b_n$ converge, then the series $\sum (a_n \pm b_n)$ and $\sum c a_n$ also converge and we have

- $\sum a_n + \sum b_n = \sum (a_n + b_n)$;
- $\sum a_n - \sum b_n = \sum (a_n - b_n)$;
- $\sum c a_n = c \sum a_n$;

- In the sequel, we will be interested in establishing techniques for determining whether an infinite series converges or diverges;

Geometric Series

- A **geometric series** with **ratio** $r \neq 0$ is a series defined by the geometric sequence cr^n, where $c \neq 0$;
- The series looks like
$$S = \sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \cdots ;$$
- The following work determines the N-th partial sum S_N of the geometric series:
$$\begin{aligned} S_N &= c + cr + cr^2 + cr^3 + \cdots + cr^N \\ rS_N &= \phantom{c + {}} cr + cr^2 + cr^3 + \cdots + cr^N + cr^{N+1} \\ S_N - rS_N &= c - cr^{N+1} \\ S_N(1-r) &= c(1-r^{N+1}) \\ S_N &= \frac{c(1-r^{N+1})}{1-r}; \end{aligned}$$
    - If $|r| < 1$, the the Geometric Series converges and $S = \dfrac{c}{1-r}$;
    - If $|r| \geq 1$, it diverges;

Examples I

- Evaluate $\sum_{n=0}^{\infty} 5^{-n}$;

$$\sum_{n=0}^{\infty} 5^{-n} = \sum_{n=0}^{\infty} (\frac{1}{5})^n \overset{c=1,\, r=\frac{1}{5}<1}{=} \frac{1}{1-\frac{1}{5}} = \frac{5}{4};$$

- Evaluate $\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n$;

$$\begin{aligned} \sum_{n=3}^{\infty} 7(-\tfrac{3}{4})^n &= 7(-\tfrac{3}{4})^3 + 7(-\tfrac{3}{4})^4 + 7(-\tfrac{3}{4})^5 + \cdots \\ &= 7(-\tfrac{3}{4})^3[1 + (-\tfrac{3}{4}) + (-\tfrac{3}{4})^2 + \cdots] \\ &\overset{c=1, r=-\frac{3}{4}}{=} 7(-\tfrac{3}{4})^3 \frac{1}{1-(-\frac{3}{4})} \\ &= -\frac{189}{64} \cdot \frac{4}{7} = -\frac{27}{16}; \end{aligned}$$

Examples II

- Evaluate $S = \sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$;

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{2+3^n}{5^n} \\ &= \sum_{n=0}^{\infty} \frac{2}{5^n} + \sum_{n=0}^{\infty} \frac{3^n}{5^n} \\ &= 2\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n \\ &= 2 \cdot \frac{1}{1-\frac{1}{5}} + \frac{1}{1-\frac{3}{5}} \\ &= 2 \cdot \frac{5}{4} + \frac{5}{2} \\ &= 5; \end{aligned}$$

Divergence Test

Divergence Test

If the n-th term a_n does not converge to 0, i.e., if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges;

- Example: Prove the divergence of $S = \sum_{n=1}^{\infty} \frac{n}{4n+1}$;

  Clearly, $\lim_{n\to\infty} \frac{n}{4n+1} = \frac{1}{4} \neq 0$; Thus, by the Divergence Test, S diverges;

Another Example

- Example: Determine the convergence or divergence of

$$S=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+1}=\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\cdots;$$

The n-th term $a_n=(-1)^{n-1}\frac{n}{n+1}$ does not approach a limit; To see this, note that:

  - for even indices,
  $$\lim_{n\to\infty}a_{2n}=\lim_{n\to\infty}(-1)^{2n-1}\frac{2n}{2n+1}=\lim_{n\to\infty}\frac{-2n}{2n+1}=-1;$$
  - for odd indices,
  $$\lim_{n\to\infty}a_{2n+1}=\lim_{n\to\infty}(-1)^{2n+1-1}\frac{2n+1}{2n+1+1}=\lim_{n\to\infty}\frac{2n+1}{2n+2}=1;$$

Since $\lim_{n\to\infty}a_n\neq 0$, by the Divergence Test, S diverges;

If $\lim_{n\to\infty} a_n = 0$, Cannot Apply Divergence Test

- Prove the divergence of $S = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$;

  Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; Therefore, the Divergence Test cannot be applied; We must find another way to prove that the series diverges; We will use comparison instead!

$$\begin{aligned} S_N &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}} \\ &\geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}} \\ &= N\frac{1}{\sqrt{N}} = \sqrt{N}; \end{aligned}$$

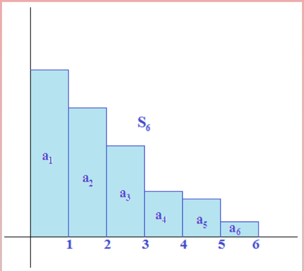
  Now note that $\lim_{N\to\infty} \sqrt{N} = \infty$; Therefore, since $S_N \geq \sqrt{N}$, we also have $\lim_{N\to\infty} S_N = \infty$, showing that S diverges to infinity;

Subsection 3

Convergence of Series with Positive Terms

Positive Series

- A **positive series** $\sum a_n$ is one with $a_n > 0$, for all n;
- The terms can be thought of as areas of rectangles with width 1 and height a_n;
  The partial sum
  $$S_N = a_1 + \cdots + a_N$$
  is equal to the area of the first N rectangles;



- Clearly, the partial sums form an *increasing sequence* $S_N < S_{N+1}$;

Dichotomy and Integral Test

Dichotomy for Positive Series

If $S = \sum_{n=1}^{\infty} a_n$ is a positive series, then either

1. The partial sums S_N are bounded above, in which case S converges, or
2. The partial sums S_N are not bounded above, in which case S diverges.

The Integral Test

Let $a_n = f(n)$, where the function $f(x)$ is positive, decreasing and continuous for $x \geq 1$;

1. If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges;
2. If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges;

Applying the Integral Test on the Harmonic Series

- **The Harmonic Series Diverges**: Show that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges;

  Consider the function $f(x)=\frac{1}{x}$; For $x\geq 1$, it is positive, decreasing and continuous, and, moreover, $f(n)=\frac{1}{n}=a_n$; So we check

$$\int_1^{\infty}\frac{dx}{x}=\lim_{R\to\infty}\int_1^R\frac{dx}{x}=\lim_{R\to\infty}\ln R=\infty;$$

  Therefore, by the Integral Test, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges;

Another Application of the Integral Test

- Does $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}=\frac{1}{2^2}+\frac{2}{5^2}+\frac{3}{10^2}+\cdots$ converge?

  Consider the function $f(x)=\dfrac{x}{(x^2+1)^2}$; It is positive and continuous for $x\geq 1$; Is it also decreasing for $x\geq 1$? Let us compute its first derivative

  $$f'(x)=\frac{(x)'(x^2+1)^2-x[(x^2+1)^2]'}{[(x^2+1)^2]^2}=$$
  $$\frac{(x^2+1)^2-x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}=\frac{(x^2+1)-4x^2}{(x^2+1)^3}=\frac{1-3x^2}{(x^2+1)^3}<0;$$

  Thus, the Integral Test is applicable and we get

  $$\int_1^{\infty}\frac{x}{(x^2+1)^2}dx=\lim_{R\to\infty}\int_1^R\frac{x}{(x^2+1)^2}dx\overset{u=x^2+1}{=}\lim_{R\to\infty}\int_2^R\frac{1}{2u^2}du=$$
  $$\lim_{R\to\infty}\left.\frac{-1}{2u}\right|_2^R=\lim_{R\to\infty}\left(\frac{1}{4}-\frac{1}{2R}\right)=\frac{1}{4};$$ So, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2}$ converges;

The p-Series

Convergence of the p-Series

The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, if $p > 1$, and diverges, otherwise.

- If $p \leq 0$, $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} \neq 0$; By Divergence Test, p-series diverges;
- If $p > 0$, $f(x) = \dfrac{1}{x^p}$ is positive, decreasing and continuous on $[1, \infty)$; Thus, the Integral Test applies and

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \infty, & \text{if } p \leq 1 \end{cases}$$

- Example: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges;

Comparison Test

Comparison Test

Assume that for some $M > 0$, $0 \leq a_n \leq b_n$, for all $n \geq M$;

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges;
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges;

- Example: Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ converge?

  Clearly, for all $n \geq 1$, we have $0 \leq \frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}$; Moreover, $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges since it is a geometric series with ration $\frac{1}{3} < 1$; Therefore, by Comparison $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$ also converges;

Example

- Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{1/3}}$ converge?

  Consider the function $f(x) = x^3 - x^2 - 3$; We show that for $x \geq 2$, $f(x) > 0$; Note $f(2) = 2^3 - 2^2 - 3 = 1 > 0$; Moreover, for $x \geq 2$ $f'(x) = 3x^2 - 2x = x(3x-2) > 0$, so f is increasing; Thus $f > 0$, all $x \geq 2$;

  We have shown, for $n \geq 2$, $f(n) = n^3 - n^2 - 3 > 0 \Rightarrow n^3 > n^2 + 3 \Rightarrow n > (n^2+3)^{1/3} \Rightarrow \frac{1}{n} < \frac{1}{(n^2+3)^{1/3}}$; But $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ is the harmonic series that diverges; therefore, by Comparison $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n^2+3)^{1/3}}$ also diverges;

Limit Comparison Test

Limit Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ be positive sequences and assume that $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists;

1. If $L > 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges;
2. If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges;
3. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges;

Example I

- Show that $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges;

Pick $a_n = \frac{n^2}{n^4 - n - 1}$ and $b_n = \frac{1}{n^2}$; Then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^4 - n - 1} \cdot \frac{n^2}{1} =$$

$$\lim_{n\to\infty} \frac{1}{1 - \frac{1}{n^3} - \frac{1}{n^4}} = 1;$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ also converges by the Limit Comparison Test;

Example II

- Show that $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$ diverges;

  Pick $a_n = \frac{1}{\sqrt{n^2+4}}$ and $b_n = \frac{1}{n}$; Then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+4}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{4}{n^2}}} = 1;$$

  Since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$ also diverges by the Limit Comparison Test;

Subsection 4

Absolute and Conditional Convergence

Absolute Convergence

Absolute Convergence

The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

- Example: Verify that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ converges absolutely;
  We check
  $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$
  which converges as a p-series with $p > 1$;

Absolute Convergence Implies Convergence

Theorem (Absolute Convergence Implies Convergence)

If $\sum |a_n|$ converges, then $\sum a_n$ also converges.

- Example: Verify that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges;

  It was shown in the previous slide that $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right|$ converges;

  Therefore, by the Theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ also converges;

Another Example

- Does $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$ converge absolutely?

  We have
  $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}},$$
  which is a p-series, with $p = \frac{1}{2} \leq 1$, and so diverges; Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not absolutely convergent;

Conditional Convergence

- We saw than absolute convergence implies convergence:

$$\text{If } \sum |a_n| \text{ converges, then } \sum a_n \text{ also converges;}$$

- The converse is not true in general! I.e., the convergence of a series does not necessarily imply its absolute convergence;

Conditional Convergence

An infinite series $\sum a_n$ **converges conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Alternating Series

- An **alternating series** is an infinite series of the form
$$S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots,$$
where $a_n > 0$ and decrease to 0;

Leibniz Test for Alternating Series

Suppose $\{a_n\}$ is a positive sequence that is decreasing and converges to 0:
$$a_1 > a_2 > a_3 > \cdots > 0, \qquad \lim_{n \to \infty} a_n = 0;$$
Then the alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges; Moreover, we have
$$0 < S < a_1 \qquad \text{and} \qquad S_{2N} < S < S_{2N+1}, \; N \geq 1;$$

Example

- Show that $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots$ converges conditionally and that $0 \leq S \leq 1$;
  - We already saw that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series;
  - On the other hand, S converges by the Leibniz Test, since $a_n = \frac{1}{\sqrt{n}}$ is a positive decreasing sequence converging to 0;
  - Therefore, S is conditionally convergent;
  - By the last part of the Leibniz Test, $0 < S < a_1 = 1$;

Error of Approximation of Alternating Series

Theorem

Let $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where a_n is a positive decreasing sequence that converges to 0; Then

$$|S - S_N| < a_{N+1};$$

I.e., the error committed when we approximate S by S_N is less than the size of the first omitted term a_{N+1};

Alternating Harmonic Series

- Show that $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally;

  Since $a_n = \frac{1}{n}$ is positive, decreasing and has limit 0, we get by the Leibniz Test that S converges;

  Moreover $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic series);

  Thus, S is conditionally convergent;
- Show that $|S - S_6| < \frac{1}{7}$;

  By the approximation error theorem, we get that $|S - S_6| < a_{6+1} = a_7 = \frac{1}{7}$;
- Find an N, such that S_N approximates S with an error less than 10^{-3};

  We know that $|S - S_N| < a_{N+1}$; To make the error $|S - S_N| < 10^{-3}$ it suffices to arrange N so that

  $$a_{N+1} \leq 10^{-3} \Rightarrow \frac{1}{N+1} \leq 10^{-3} \Rightarrow N + 1 \geq 1000 \Rightarrow N \geq 999;$$

Subsection 5

The Ratio and Root Tests

The Ratio Test

Theorem (Ratio Test)

Assume that $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists;

1. If $\rho < 1$, then $\sum a_n$ converges absolutely;
2. If $\rho > 1$, then $\sum a_n$ diverges;
3. If $\rho = 1$, then test is inconclusive.

Applying the Ratio Test I

- Prove that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges;

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \frac{2}{n+1} = 0;$$

Since $\rho < 1$, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by the Ratio Test;

- Does the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converge?

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right| = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2};$$

Since $\rho < 1$, the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges by the Ratio Test;

Applying the Ratio Test II

- Does the series $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{1000^n}$ converge?

$$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{(-1)^n n!} \right| =$$

$$\lim_{n \to \infty} \frac{n+1}{1000} = +\infty;$$

Since $\rho > 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{1000^n}$ diverges by the Ratio Test;

If Ratio Test is Inconclusive Anything Can Happen

- Consider $\sum_{n=1}^{\infty} n^2$;
$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\frac{n^2+2n+1}{n^2} = 1;$$
So Ratio Test is inconclusive; However, $\lim_{n\to\infty} a_n \neq 0$, so the series $\sum_{n=1}^{\infty} n^2$ diverges by Divergence Test;
- Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$;
$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1} = 1;$$
So Ratio Test is again inconclusive; However, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ and, hence, it converges!

The Root Test

Theorem (Root Test)

Assume that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists;

1. If $L < 1$, then $\sum a_n$ converges absolutely;
2. If $L > 1$, then $\sum a_n$ diverges;
3. If $L = 1$, the test is inconclusive.

- Example: Does $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converge?

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2};$$

Since $L < 1$, the series $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ converges by the Root Test;

Subsection 6

Power Series

Power Series Centered at c

- A **power series with center** c is an infinite series

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_n(x-c)^n \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ; \end{aligned}$$

- Example: The following is a power series centered at $c = 2$:

$$F(x) = 1 + (x-2) + 2(x-2)^2 + 3(x-2)^3 + \cdots ;$$

- A power series may converge for some values of x and diverge for some other values of x;
- Take a look again at $F(x) = 1 + (x-2) + 2(x-2)^2 + 3(x-2)^3 + \cdots ;$
  - $F(\frac{5}{2}) = 1 + \frac{1}{2} + 2(\frac{1}{2})^2 + 3(\frac{1}{2})^3 + \cdots = \sum_{n=0}^{\infty} \frac{n}{2^n}$; This series converges by the Ratio Test!
  - $F(3) = 1 + 1 + 2 + 3 + 4 + \cdots$; This series diverges by the Divergence Test!

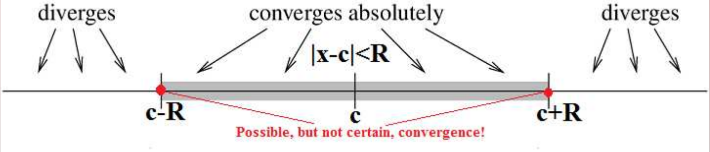
Radius and Interval of Convergence

Theorem (Radius of Convergence)

Every power series $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ has a **radius of convergence** R, which is either a nonnegative number ($R \geq 0$) or infinity ($R = \infty$).

- If R is finite, $F(x)$ converges absolutely when $|x - c| < R$ (i.e., in $(c - R, c + R)$) and diverges when $|x - c| > R$;
- If $R = \infty$, then $F(x)$ converges absolutely for all x.

- According to the Theorem, $F(x)$ converges in an **interval of convergence** consisting of the open $(c - R, c + R)$ and possibly one or both of the endpoints $c - R$ and $c + R$;



Using the Ratio Test I

- Find the interval of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n}$;

  Let $a_n = \dfrac{x^n}{2^n}$ and compute the ratio ρ of the Ratio Test:

  $$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{2} = \frac{|x|}{2};$$

  Therefore, we get $\rho < 1 \Rightarrow \dfrac{|x|}{2} < 1 \Rightarrow |x| < 2$; This shows that, if $|x| < 2$ the series converges absolutely; If $|x| > 2$ the series diverges;

  - If $x = -2$, then $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n$, which diverges!
  - If $x = 2$, then $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{2^n} = \sum_{n=0}^{\infty}\frac{2^n}{2^n} = \sum_{n=0}^{\infty}1$, which also diverges!

  Thus, the interval of convergence is $(-2, 2)$;

Using the Ratio Test II

- Find the interval of convergence of $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(x-5)^n$;

  Let $a_n = \frac{(-1)^n}{4^n n}(x-5)^n$ and compute the ratio ρ of the Ratio Test:

  $$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-5)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(-1)^n(x-5)^n}\right| =$$
  $$|x-5| \lim_{n\to\infty}\left|\frac{n}{4(n+1)}\right| = \frac{1}{4}|x-5|;$$

  Therefore, we get $\rho < 1 \Rightarrow \frac{|x-5|}{4} < 1 \Rightarrow |x-5| < 4$; This shows that, if $|x-5| < 4$ the series converges absolutely; If $|x-5| > 4$ the series diverges;
  - If $x-5=-4$, then $F(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(-4)^n = \sum_{n=0}^{\infty} \frac{1}{n}$, which diverges!
  - If $x-5=4$, then $F(9) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n}(4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which converges! Thus, interval of convergence is $(1,9]$;

An Even Power Series

- Where does $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ converge?

  Let $a_n = \frac{x^{2n}}{(2n)!}$ and compute the ratio ρ of the Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right| =$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0;$$

  Therefore, we get $\rho < 1$, for all x; This shows that the series is absolutely convergent everywhere;

Geometric Power Series

- Recall that the geometric infinite series $S = a + ar + ar^2 + \cdots$ converges when $|r| < 1$ and has sum $S = \frac{a}{1-r}$;
- As a special case, when $a = 1$ and $r = x$, we get the geometric series with center 0: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$; We have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1;$$

- Example: Show that $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$, for $|x| < \frac{1}{2}$;

  If $|x| < \frac{1}{2}$, then $2|x| < 1$ and, therefore $|2x| < 1$; Thus, the geometric series with ratio $2x$ converges; We have

$$\frac{1}{1-2x} \overset{\text{Geometric Sum}}{=} \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n;$$

Another Example of a Geometric Power Series

- Find a power series expansion with center $c = 0$ for $f(x) = \dfrac{1}{5+4x^2}$ and find the interval of convergence;

$$\frac{1}{5+4x^2} = \frac{1}{5} \cdot \frac{1}{1+\frac{4}{5}x^2} = \frac{1}{5} \cdot \frac{1}{1-(-\frac{4}{5}x^2)};$$

Therefore, if $|-\frac{4}{5}x^2| = \frac{4}{5}x^2 < 1 \Rightarrow x^2 \leq \frac{5}{4} \Rightarrow |x| \leq \frac{\sqrt{5}}{2}$, we have

$$\frac{1}{5+4x^2} = \frac{1}{5} \cdot \frac{1}{1-(-\frac{4}{5}x^2)} \overset{\text{Geometric}}{=} \frac{1}{5}\sum_{n=0}^{\infty}(-\frac{4}{5}x^2)^n =$$

$$\frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\frac{4^n}{5^n}x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{4^n}{5^{n+1}}x^{2n};$$

Term-by-Term Differentiation and Integration

Term-by-Term Differentiation and Integration

Assume that $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence $R > 0$;

Then $F(x)$ is differentiable on $(c-R, c+R)$ (or for all x, if $R = \infty$);

Moreover, we can integrate and differentiate term-by-term, i.e.,

1. $F'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$;
2. $\int F(x)dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$;

Both series for $F'(x)$ and $\int F(x)dx$ have the same radius of convergence R as $F(x)$;

Example of Differentiation of Power Series

- Prove that for $-1 < x < 1$,
  $\dfrac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$;
  We know that, for $|x| < 1$, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots ;$$

  Therefore, by Term-by-Term Differentiation, we get, for $|x| < 1$:

$$\begin{aligned} \frac{1}{(1-x)^2} &= \left(\frac{1}{1-x}\right)' \\ &= (1 + x + x^2 + x^3 + x^4 + x^5 + \cdots)' \\ &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots ; \end{aligned}$$

Example of Integration of Power Series

- Prove that for $|x| < 1$, we have

  $$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots;$$

  Since for $|x| < 1$, we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$, we obtain, also for $|x| < 1$,

  $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots ;$$

  Therefore, by Term-by-Term Integration we get

  $$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2} dx \\ &= \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots ; \end{aligned}$$

Power Series Solution of Differential Equations

- Consider $y' = y$ and $y(0) = 1$;
  Assume that the power series $F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ is a solution of the given initial value problem; Compute
  $F'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$; Since $F(x) = F'(x)$, we must have $a_0 = a_1, a_1 = 2a_2, a_2 = 3a_3, a_3 = 4a_4, \ldots$; Looking at these carefully, we obtain $a_n = \frac{a_{n-1}}{n}$, for all n; Thus,

$$a_n = \frac{1}{n} a_{n-1} = \frac{1}{n} \frac{1}{n-1} a_{n-2} = \frac{1}{n} \frac{1}{n-1} \frac{1}{n-2} a_{n-3} = \cdots = \frac{1}{n(n-1)(n-2) \cdot \cdots \cdot 1} a_0 = \frac{1}{n!} a_0;$$

Example I (Cont'd)

- We were solving $y' = y$ and $y(0) = 1$;
  We assumed $F(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution; We found $a_n = \frac{1}{n!} a_0$; This yields $F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = a_0 + a_0 \frac{1}{1!} x + a_0 \frac{1}{2!} x^2 + a_0 \frac{1}{3!} x^3 + \cdots = a_0 (1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$; Since $F(0) = 1 = a_0$, we get $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$;
- Since e^x is also a solution, we get

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots;$$

Example II

- Find a series solution to $x^2y'' + xy' + (x^2 - 1)y = 0$, with $y'(0) = 1$;
  Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$; Then $y' = F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and
  $y'' = F''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$; Plug those in equation:
  $$\begin{aligned} & x^2y'' + xy' + (x^2 - 1)y = \\ & x^2 \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + (x^2 - 1) \sum_{n=0}^{\infty} a_n x^n = \\ & \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_{n+2} x^n = \\ & \sum_{n=0}^{\infty} (n^2 - 1)a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0; \end{aligned}$$
  Thus,
  $$\sum_{n=0}^{\infty} (n^2 - 1)a_n x^n = -\sum_{n=2}^{\infty} a_{n-2} x^n \Rightarrow a_n = -\frac{a_{n-2}}{n^2 - 1};$$

Example II (Cont'd)

- We were solving $x^2y'' + xy' + (x^2-1)y = 0$, with $y'(0) = 1$;
  We assumed $F(x) = \sum_{n=0}^{\infty} a_n x^n$ is a solution; We found $a_n = -\frac{a_{n-2}}{n^2-1}$;
  Now, note $a_0 = 0$; Thus, $a_2 = -\frac{a_0}{2^2-1} = 0$; Then $a_4 = -\frac{a_2}{4^2-1} = 0$;
  We see that $a_{2n} = 0$, for all n;
  Moreover, $a_1 = 1$; Thus, $a_3 = -\frac{a_1}{3^2-1} = -\frac{1}{2\cdot 4}$; Then
  $a_5 = -\frac{a_3}{5^2-1} = +\frac{1}{2\cdot 4\cdot 4\cdot 6}$; Also $a_7 = -\frac{a_5}{7^2-1} = -\frac{1}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}$; In general
  $a_{2n+1} = \frac{(-1)^n}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot\cdots\cdot(2n)(2n+2)} = \frac{(-1)^n}{2^n(1\cdot 2\cdot 3\cdot\cdots\cdot n)2^n(2\cdot 3\cdot 4\cdot\cdots\cdot(n+1))} = \frac{(-1)^n}{4^n n!(n+1)!}$;
  So we get $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+1)!} x^{2n+1}$;

Subsection 7

Taylor Series

Taylor Series

- Assume that a function $f(x)$ is represented by a power series centered at $x = c$ on $(c - R, c + R)$ with $R > 0$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ;$$

- Then, for the derivatives of f on $(c - R, c + R)$, we have

$$\begin{array}{lcl} f(x) & = & a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots ; \\ f'(x) & = & a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots ; \\ f''(x) & = & 2a_2 + 2 \cdot 3a_3(x-c) + 3 \cdot 4a_4(x-c)^2 + 4 \cdot 5(x-c)^3 \cdots ; \\ f'''(x) & = & 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-c) + 3 \cdot 4 \cdot 5(x-c)^2 + \cdots ; \end{array}$$

- Plug in $x = c$ to get

$$f(c) = a_0, f'(c) = a_1, f''(c) = 2!a_2, f'''(c) = 3!a_3, f^{(4)}(c) = 4!a_4, \ldots ;$$

- In general, we get $a_n = \dfrac{f^{(n)}(c)}{n!}$;

Taylor and Maclaurin Series

Taylor Series Expansion

If f is represented as a power series centered at $x = c$ in an interval $|x - c| < R, R > 0$, then the power series is the **Taylor series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n;$$

Maclaurin Series

The special case of the Taylor series for $c = 0$ is the **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots ;$$

Finding a Taylor Series

- Find the Taylor series for $f(x) = x^{-3}$ centered at $c = 1$;

$$f(x) = x^{-3}, \quad f(1) = 1;$$
$$f'(x) = (-3)x^{-4}, \quad f'(1) = -3;$$
$$f''(x) = (-3)(-4)x^{-5}, \quad f''(1) = +3 \cdot 4;$$
$$f'''(x) = (-3)(-4)(-5)x^{-6}, \quad f'''(1) = -3 \cdot 4 \cdot 5;$$
$$\vdots$$
$$f^{(n)}(x) = (-3)(-4) \cdot \cdots \cdot (-n-2)x^{-n-3},$$
$$f^{(n)}(1) = (-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n+2) = \frac{(-1)^n}{2}(n+2)!;$$

Now we get by the Taylor series formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!}{2 \cdot n!}(x-1)^n =$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{2}(x-1)^n;$$

Convergence Issues

- We know that if $f(x)$ can be represented by a power series centered at $x = c$, then that power series will be the Taylor series
$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n;$$
- However, there is no guarantee that $T(x)$ converges; Moreover, there is no guarantee that, even if it converges, it will converge to $f(x)$!
- Let
$$T_k(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(c-x)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k;$$
Define the **remainder**
$$R_k(x) = f(x) - T_k(x);$$
The Taylor series converges to $f(x)$ if and only if $\lim_{k\to\infty} R_k(x) = 0$;

Convergence Theorem

Theorem

Let $I=(c-R,c+R)$, $R>0$; If there exists a $K>0$, such that all derivatives of f are bounded by K on I, i.e.,

$$|f^{(k)}(x)|\leq K,\ \text{for all } k\geq 0, x\in I,$$

then, for all $x\in I$,

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}(x-c)^n.$$

Sine and Cosine

- Show that
$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$
Let $f(x) = \sin x$;

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$\cdots$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cdots$
0	1	0	-1	0	$\cdots$

Note, also that for all x, $|f^{(k)}(x)| \leq 1$; Therefore, we have convergence of the Taylor series of f centered at $x = 0$ to $f(x) = \sin x$ everywhere and
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots;$$
One either works similarly from scratch for $g(x) = \cos x$ or notices that $\cos x = (\sin x)'$ and appeals to term-by-term differentiation of the series for $\sin x$;

Infinite Series for e^x

- The Maclaurin series for $f(x) = e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

- Example: Find a Maclaurin series for $f(x) = x^2 e^x$;

$$\begin{aligned} f(x) &= x^2 e^x = x^2 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right] \\ &= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!}; \end{aligned}$$

- Example: Find the Maclaurin series for $f(x) = e^{-x^2}$;

$$\begin{aligned} f(x) &= e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}; \end{aligned}$$

Using Integration

- Find the Maclaurin series for $f(x) = \ln(1+x)$;

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots \\
\frac{1}{1+x} &= \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots; \\
\ln(1+x) &= \int \frac{1}{1+x} dx \\
&= \int (1 - x + x^2 - x^3 + x^4 - \cdots) dx \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n};
\end{aligned}$$

Binomial Coefficients

- For any number a (integer or not) and any integer $n \geq 0$, we define the **binomial coefficient**

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!}, \quad \binom{a}{0} = 1;$$

- Example:

$$\begin{aligned} \binom{6}{3} &= \frac{6 \cdot 5 \cdot 4}{3!} = 20; \\ \binom{\frac{4}{3}}{3} &= \frac{\frac{4}{3} \cdot \frac{1}{3} \cdot (-\frac{2}{3})}{3!} = \frac{-\frac{8}{27}}{6} = -\frac{4}{81}; \end{aligned}$$

Binomial Series

The Binomial Series

For any exponent a and for $|x| < 1$,

$$(1+x)^a = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots ;$$

- Example: Find the terms through degree four of the Maclaurin expansion of $f(x) = (1+x)^{4/3}$;

$$\begin{aligned} T_4(x) &= 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \\ &\quad \frac{a(a-1)(a-2)(a-3)}{4!}x^4 \\ &= 1 + \frac{\frac{4}{3}}{1!}x + \frac{\frac{4}{3}\cdot\frac{1}{3}}{2!}x^2 + \frac{\frac{4}{3}\cdot\frac{1}{3}\cdot(-\frac{2}{3})}{3!}x^3 + \\ &\quad \frac{\frac{4}{3}\cdot\frac{1}{3}\cdot(-\frac{2}{3})\cdot(-\frac{5}{3})}{4!}x^4 \\ &= 1 + \frac{4}{3}x + \frac{2}{9}x^2 - \frac{4}{81}x^3 + \frac{5}{243}x^4; \end{aligned}$$

Applying the Binomial Series Expansion

- Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$; Recall that $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$; Hence, for $a = -\frac{1}{2}$, we get
$$(1+x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n;$$
Therefore, we obtain
$$\begin{aligned} f(x) &= \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = (1+(-x^2))^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+1)}{n!} (-1)^n x^{2n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} (-1)^n x^{2n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n n!} x^{2n}; \end{aligned}$$