Calculus II

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LSSU Math 152

1. Techniques of Integration
    - Integration by Parts
    - Trigonometric Integrals
    - Trigonometric Substitution
    - Hyperbolic and Inverse Hyperbolic Functions
    - The Method of Partial Fractions
    - Improper Integrals
    - Numerical Integration

Subsection 1

Integration by Parts

Integration By Parts

- Recall the Product Rule for Derivatives:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x);$$

Integrate both sides with respect to x:

$$\int (f(x)g(x))' dx = \int [f'(x)g(x) + f(x)g'(x)]dx;$$

Since integration is the reverse operation of differentiation, we have

$$\int (f(x)g(x))' dx = f(x)g(x);$$

Moreover, because of the sum rule for integrals:

$$\int [f'(x)g(x) + f(x)g'(x)]dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx;$$

Putting all these together:

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx;$$

Finally, subtract to get the **Integration By Parts Formula**:

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

Alternative Form

- We came up with the formula

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

- Use two new variables u and v as follows: Set

$$\begin{array}{ll} u = g(x) & du = g'(x)dx \\ v = f(x) & dv = f'(x)dx \end{array}$$

- Now substitute into the formula above to get the uv-form of the **By Parts Rule**:

$$\int u dv = uv - \int v du;$$

Example I (fg-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int x\cos x dx &= \int x(\sin x)' dx \\
&= x\sin x - \int (x)'\sin x dx \\
&= x\sin x - \int \sin x dx \\
&= x\sin x - (-\cos x) + C; \\
&= x\sin x + \cos x + C;
\end{aligned}$$

Example I (uv-form)

$$\int udv = uv - \int vdu;$$

We want to compute $\int x \cos x dx$;

Set $u = x$ and $dv = \cos x dx$; Then $\frac{du}{dx} = 1 \Rightarrow du = dx$; Moreover $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$;

$$\begin{aligned} \int x \cos x dx &= \int udv \\ &= uv - \int vdu \\ &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C; \end{aligned}$$

Example II (fg-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}\int xe^x dx &= \int x(e^x)' dx \\ &= xe^x - \int (x)' e^x dx \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C;\end{aligned}$$

Example II (uv-form)

$$\int u dv = uv - \int v du;$$

We want to compute $\int xe^x dx$;

Set $u = x$ and $dv = e^x dx$; Then $\frac{du}{dx} = 1 \Rightarrow du = dx$; Moreover $\frac{dv}{dx} = e^x \Rightarrow v = e^x$;

$$\begin{aligned} \int xe^x dx &= \int u dv \\ &= uv - \int v du \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C; \end{aligned}$$

Example III (fg-form)

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int x^7 \ln x dx &= \int (\frac{1}{8}x^8)' \ln x dx \\
&= \frac{1}{8}x^8 \ln x - \int \frac{1}{8}x^8 (\ln x)' dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^8 \cdot \tfrac{1}{x} dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8}\int x^7 dx \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{8} \cdot \frac{1}{8}x^8 + C \\
&= \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C;
\end{aligned}$$

Example III (uv-form)

$$\int u dv = uv - \int v du;$$

We want to compute $\int x^7 \ln x dx$;

Set $u = \ln x$ and $dv = x^7 dx$; Then $\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$; Moreover $\frac{dv}{dx} = x^7 \Rightarrow v = \frac{1}{8} x^8$;

$$\begin{aligned} \int x^7 \ln x dx &= \int u dv = uv - \int v du \\ &= \frac{1}{8} x^8 \ln x - \int \frac{1}{8} x^8 \cdot \frac{1}{x} dx = \frac{1}{8} x^8 \ln x - \frac{1}{8} \int x^7 dx \\ &= \frac{1}{8} x^8 \ln x - \frac{1}{64} x^8 + C; \end{aligned}$$

Example IV: Applying By Parts Twice

$$\begin{aligned}
\int x^2 \cos x dx &= \int x^2 (\sin x)' dx \\
&= x^2 \sin x - \int (x^2)' \sin x dx \\
&= x^2 \sin x - \int 2x \sin x dx \\
&= x^2 \sin x - \int 2x(-\cos x)' dx \\
&= x^2 \sin x - [-2x \cos x - \int (2x)'(-\cos x) dx] \\
&= x^2 \sin x + 2x \cos x - \int 2 \cos x dx \\
&= x^2 \sin x + 2x \cos x - 2 \sin x + C;
\end{aligned}$$

Example V: Applying By Parts Twice

$$\begin{aligned}
\int x^2 e^{3x} dx &= \int x^2 (\tfrac{1}{3} e^{3x})' dx \\
&= x^2 \tfrac{1}{3} e^{3x} - \int (x^2)' \tfrac{1}{3} e^{3x} dx \\
&= x^2 \tfrac{1}{3} e^{3x} - \int 2x \tfrac{1}{3} e^{3x} dx \\
&= x^2 \tfrac{1}{3} e^{3x} - \int 2x (\tfrac{1}{9} e^{3x})' dx \\
&= x^2 \tfrac{1}{3} e^{3x} - [2x \tfrac{1}{9} e^{3x} - \int (2x)' \tfrac{1}{9} e^{3x} dx] \\
&= x^2 \tfrac{1}{3} e^{3x} - 2x \tfrac{1}{9} e^{3x} + \int 2 \tfrac{1}{9} e^{3x} dx \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{9} x e^{3x} + \tfrac{2}{27} e^{3x} + C;
\end{aligned}$$

Example VI: Integral of $\ln x$

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int \ln x dx &= \int (x)' \ln x dx \\
&= x\ln x - \int x(\ln x)' dx \\
&= x\ln x - \int x \cdot \tfrac{1}{x} dx \\
&= x\ln x - \int dx \\
&= x\ln x - x + C;
\end{aligned}$$

Example VII: Returning to the Original Form

$$\begin{aligned}\int e^x \cos x dx &= \int (e^x)' \cos x dx \\ &= e^x \cos x - \int e^x (\cos x)' dx \\ &= e^x \cos x + \int e^x \sin x dx \\ &= e^x \cos x + \int (e^x)' \sin x dx \\ &= e^x \cos x + e^x \sin x - \int e^x (\sin x)' dx \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x dx\end{aligned}$$

Thus, $\int e^x \cos x dx = e^x(\cos x + \sin x) - \int e^x \cos x dx$, and, hence,

$2\int e^x \cos x dx = e^x(\cos x + \sin x) \Rightarrow \int e^x \cos x dx = \frac{1}{2}e^x(\cos x + \sin x) + C;$

Example VIII

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx;$$

$$\begin{aligned}
\int (x-2)(x+4)^8 dx &= \int (x-2)[\frac{1}{9}(x+4)^9]' dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \int (x-2)' \frac{1}{9}(x+4)^9 dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{9}\int (x+4)^9 dx \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{9} \cdot \frac{1}{10}(x+4)^{10} + C \\
&= \frac{1}{9}(x-2)(x+4)^9 - \frac{1}{90}(x+4)^{10} + C;
\end{aligned}$$

Example IX: A Definite Integral By Parts

$$\begin{aligned}
\int_0^{\pi/4} x \sin 2x dx &= \int_0^{\pi/4} x(-\tfrac{1}{2}\cos 2x)' dx \\
&= -\tfrac{1}{2}x\cos 2x\Big|_0^{\pi/4} - \int_0^{\pi/4}(x)'(-\tfrac{1}{2}\cos 2x)dx \\
&= -\tfrac{1}{2}x\cos 2x\Big|_0^{\pi/4} + \int_0^{\pi/4}\tfrac{1}{2}\cos 2x dx \\
&= -\tfrac{1}{2}x\cos 2x\Big|_0^{\pi/4} + \tfrac{1}{4}\sin 2x\Big|_0^{\pi/4} \\
&= (0-0)+(\tfrac{1}{4}-0) \\
&= \tfrac{1}{4};
\end{aligned}$$

Subsection 2

Trigonometric Integrals

Odd Powers of $\sin x$

$$
\begin{aligned}
\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\
&= \int (1-\cos^2 x) \sin x dx \\
&\overset{u=\cos x}{=} \int (1-u^2)(-du) \\
&= \int (u^2-1) du \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}\cos^3 x - \cos x + C;
\end{aligned}
$$

Odd Power of $\sin x$ or $\cos x$

$$\begin{aligned}
\int \sin^4 x \cos^5 x dx &= \int \sin^4 x \cos^4 x \cos x dx \\
&= \int \sin^4 x (\cos^2 x)^2 \cos x dx \\
&= \int \sin^4 x (1 - \sin^2 x)^2 \cos x dx \\
&\overset{u=\sin x}{=} \int u^4 (1 - u^2)^2 du \\
&= \int u^4 (u^4 - 2u^2 + 1) du \\
&= \int (u^8 - 2u^6 + u^4) du \\
&= \frac{1}{9} u^9 - \frac{2}{7} u^7 + \frac{1}{5} u^5 + C \\
&= \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C
\end{aligned}$$

Double-Angle Identities

$$\cos^2 x = \frac{1+\cos 2x}{2}, \qquad \sin^2 x = \frac{1-\cos 2x}{2};$$

$$\begin{aligned}
\int \sin^4 x dx &= \int (\sin^2 x)^2 dx = \int (\frac{1-\cos 2x}{2})^2 dx \\
&= \int (\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x) dx \\
&= \int (\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}(\frac{1+\cos 4x}{2})) dx \\
&= \int (\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x) dx \\
&= \int (\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x) dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C;
\end{aligned}$$

Integrals of Tangent and Secant

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &\overset{u=\cos x}{=} \int \frac{1}{u}(-du) \\ &= -\int \frac{1}{u} du \\ &= -\ln|u| + c \\ &= -\ln|\cos x| + C \\ &= \ln\left|\frac{1}{\cos x}\right| + C \\ &= \ln|\sec x| + C;\end{aligned}$$

$$\begin{aligned}\int \sec x dx &= \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x} dx \\ &= \int \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} dx \\ &\overset{u=\tan x+\sec x}{=} \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \ln|\tan x + \sec x| + C;\end{aligned}$$

Tangent and Secant I

Recall the identity $1+\tan^2 x = \sec^2 x$;

$$\begin{aligned}
\int \tan^3 x \sec^4 x dx &= \int \tan^3 x \sec^2 x \sec^2 x dx \\
&= \int \tan^3 x (1+\tan^2 x) \sec^2 x dx \\
&\overset{u=\tan x}{=} \int u^3(1+u^2) du \\
&= \int (u^5+u^3) du \\
&= \frac{1}{6}u^6 + \frac{1}{4}u^4 + C \\
&= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C;
\end{aligned}$$

Tangent and Secant II

Again, we will use $1 + \tan^2 x = \sec^2 x$;

$$\begin{aligned}
\int \tan^3 x \sec^3 x dx &= \int \tan^2 x \sec^2 x \tan x \sec x dx \\
&= \int (\sec^2 x - 1) \sec^2 x \tan x \sec x dx \\
&\overset{u=\sec x}{=} \int (u^2 - 1) u^2 du \\
&= \int (u^4 - u^2) du \\
&= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\
&= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C;
\end{aligned}$$

Subsection 3

Trigonometric Substitution

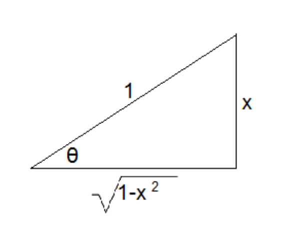
Integrals Involving $\sqrt{a^2 - x^2}$

Integrals Involving $\sqrt{a^2 - x^2}$

$$x = a\sin\theta, \qquad dx = a\cos\theta d\theta, \qquad \sqrt{a^2 - x^2} = a\cos\theta;$$

- Example: Evaluate $\int \sqrt{1-x^2}dx$;

  Set $x = \sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; Then $dx = \cos\theta d\theta$; Moreover, $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$; Note, also, that $\theta = \sin^{-1}x$ and $\cos\theta = \sqrt{1-x^2}$;



$$\int \sqrt{1-x^2}dx = \int \cos\theta\cos\theta d\theta = \int \frac{1+\cos 2\theta}{2}d\theta =$$
$$\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C =$$
$$\frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C;$$

A Trigonometric Identity

- Show that $\sin\theta\tan\theta+\cos\theta=\dfrac{1}{\cos\theta}$;

$$\begin{aligned}\sin\theta\tan\theta+\cos\theta &= \sin\theta\frac{\sin\theta}{\cos\theta}+\cos\theta\\ &= \frac{\sin^2\theta}{\cos\theta}+\cos\theta\\ &= \frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\cos\theta}\\ &= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta}\\ &= \frac{1}{\cos\theta};\end{aligned}$$

Integral of $\tan^2\theta$

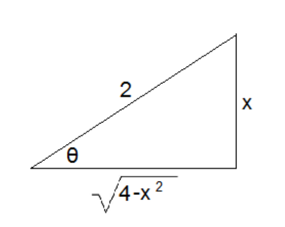
- Evaluate $\int \tan^2\theta d\theta$;

$$\int \tan^2\theta d\theta = \int \frac{\sin^2\theta}{\cos^2\theta} d\theta = \int \sin^2\theta \sec^2\theta d\theta =$$
$$\int \sin^2\theta (\tan\theta)' d\theta = \sin^2\theta \tan\theta - \int (\sin^2\theta)' \tan\theta d\theta =$$
$$\sin^2\theta \tan\theta - \int 2\sin\theta\cos\theta \frac{\sin\theta}{\cos\theta} d\theta =$$
$$\sin^2\theta \tan\theta - \int 2\sin^2\theta d\theta = \sin^2\theta\tan\theta - \int (1-\cos 2\theta) d\theta =$$
$$\sin^2\theta\tan\theta - (\theta - \frac{1}{2}\sin 2\theta) + C =$$
$$\sin^2\theta\tan\theta - \theta + \sin\theta\cos\theta + C =$$
$$\sin^2\theta\tan\theta + \sin\theta\cos\theta - \theta + C =$$
$$\sin\theta[\sin\theta\tan\theta + \cos\theta] - \theta + C \overset{\text{preceding slide}}{=}$$
$$\sin\theta \cdot \frac{1}{\cos\theta} - \theta + C = \tan\theta - \theta + C;$$

Integrals Involving $(a^2 - x^2)^{n/2}$

- Example: Evaluate $\int \frac{x^2}{(4-x^2)^{3/2}}dx$;

  Set $x = 2\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$; Then $dx = 2\cos\theta d\theta$; Moreover, $(4-x^2)^{3/2} = (4-4\sin^2\theta)^{3/2} = (4\cos^2\theta)^{3/2} = 8\cos^3\theta$; Note, also, that $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ and $\tan\theta = \frac{x}{\sqrt{4-x^2}}$;



$$\int \frac{x^2}{(4-x^2)^{3/2}}dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}2\cos\theta d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}d\theta = \int \tan^2\theta d\theta \overset{\text{preceding slide}}{=} \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C;$$

Integral of $\sec^3\theta$

- Recall that $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$; Evaluate $\int \sec^3\theta d\theta$;

  $\int \sec^3\theta d\theta = \int \sec\theta \sec^2\theta d\theta = \int \sec\theta(\tan\theta)' d\theta =$

  $\sec\theta\tan\theta - \int (\sec\theta)'\tan\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta =$

  $\sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) d\theta =$

  $\sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta) d\theta =$

  $\sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$;

  Therefore, $2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta \Rightarrow 2\int \sec^3\theta d\theta =$
  $\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$, i.e.,

  $$\int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C;$$

Integrals Involving $\sqrt{x^2+a^2}$

Integrals Involving $\sqrt{x^2+a^2}$

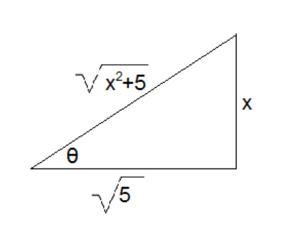
$$x = a\tan\theta, \qquad dx = a\sec^2\theta d\theta, \qquad \sqrt{x^2+a^2} = a\sec\theta;$$

- Example: Evaluate $\int \sqrt{4x^2+20}dx$;

  Note $\int \sqrt{4x^2+20}dx = \int \sqrt{4(x^2+5)}dx = 2\int \sqrt{x^2+5}dx$;

  Set $x = \sqrt{5}\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$; Then $dx = \sqrt{5}\sec^2\theta d\theta$; Moreover, $\sqrt{x^2+5} = \sqrt{5\tan^2\theta+5} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$; Note, also, that $\theta = \tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$ and

  $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{x^2+5}}{\sqrt{5}}$;



Integrals Involving $\sqrt{x^2+a^2}$ (Cont'd)

Recall

$$\sqrt{x^2+5} = \sqrt{5}\sec\theta, \qquad dx = \sqrt{5}\sec^2\theta d\theta$$
$$\theta = \tan^1\left(\frac{x}{\sqrt{5}}\right), \qquad \sec\theta = \frac{\sqrt{x^2+5}}{\sqrt{5}};$$

$$2\int\sqrt{x^2+5}dx = 2\int(\sqrt{5}\sec\theta)\sqrt{5}\sec^2\theta d\theta = 10\int\sec^3\theta d\theta \overset{\text{preceding problem}}{=}$$
$$10\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta+\tan\theta)\right] + C =$$
$$5\frac{x}{\sqrt{5}}\cdot\frac{\sqrt{x^2+5}}{\sqrt{5}} + 5\ln\left(\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right) + C =$$
$$x\sqrt{x^2+5} + 5\ln\left(\frac{\sqrt{x^2+5}+x}{\sqrt{5}}\right) + C;$$

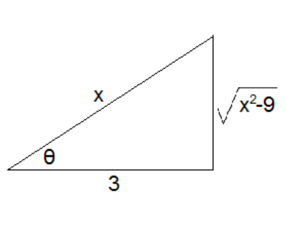
Integrals Involving $\sqrt{x^2 - a^2}$

Integrals Involving $\sqrt{x^2 - a^2}$

$$x = a\sec\theta, \qquad dx = a\sec\theta\tan\theta d\theta, \qquad \sqrt{x^2 - a^2} = a\tan\theta;$$

- Example: Evaluate $\int \frac{1}{x^2\sqrt{x^2-9}}dx$;

  Set $x = 3\sec\theta$; Then $dx = 3\sec\theta\tan\theta d\theta$; Moreover, $\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\tan^2\theta} = 3\tan\theta$; Note, also, that $\theta = \sec^{-1}\left(\frac{x}{3}\right)$ and $\sin\theta = \frac{\sqrt{x^2-9}}{x}$;



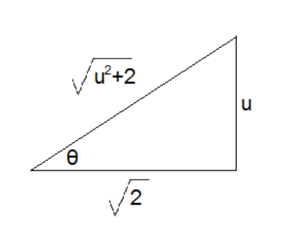
$$\int \frac{1}{x^2\sqrt{x^2-9}}dx = \int \frac{1}{9\sec^2\theta 3\tan\theta} 3\sec\theta\tan\theta d\theta =$$
$$\int \frac{1}{9\sec\theta}d\theta = \frac{1}{9}\int\cos\theta d\theta = \frac{1}{9}\sin\theta + C = \frac{\sqrt{x^2-9}}{9x} + C;$$

Completing the Square

- Example: Evaluate $\int \frac{1}{(x^2-6x+11)^2}dx$;

  Note $\int \frac{1}{(x^2-6x+11)^2}dx = \int \frac{1}{[(x^2-6x+9)+2]^2}dx =$
  $\int \frac{1}{[(x-3)^2+2]^2}dx \stackrel{u=x-3}{=} \int \frac{1}{(u^2+2)^2}du$;

  Set $u=\sqrt{2}\tan\theta$; Then $du=\sqrt{2}\sec^2\theta d\theta$;
  Moreover, $u^2+2=2\tan^2\theta+2=2\sec^2\theta$;
  Note, also, that $\theta = \tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$ and
  $\sin\theta = \frac{u}{\sqrt{u^2+2}}$, $\cos\theta = \frac{\sqrt{2}}{\sqrt{u^2+2}}$;



Completing the Square (Cont'd)

Recall

$$u = x - 3 \qquad u^2 + 2 = 2\sec^2\theta, \qquad du = \sqrt{2}\sec^2\theta d\theta$$
$$\sin\theta = \frac{u}{\sqrt{u^2+2}}, \qquad \cos\theta = \frac{\sqrt{2}}{\sqrt{u^2+2}};$$

$$\int \frac{1}{(u^2+2)^2} du = \int \frac{\sqrt{2}\sec^2\theta}{(2\sec^2\theta)^2} d\theta = \int \frac{\sqrt{2}\sec^2\theta}{4\sec^4\theta} d\theta =$$
$$\frac{\sqrt{2}}{4}\int \cos^2\theta d\theta = \frac{\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2\theta) d\theta =$$
$$\frac{\sqrt{2}}{8}(\theta + \frac{1}{2}\sin 2\theta) + C = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{8}\sin\theta\cos\theta + C =$$
$$\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{\sqrt{2}}{8}\frac{u}{\sqrt{u^2+2}}\frac{\sqrt{2}}{\sqrt{u^2+2}} + C =$$
$$\frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + \frac{x-3}{4(x^2-6x+11)} + C;$$

Subsection 4

Hyperbolic and Inverse Hyperbolic Functions

Hyperbolic Functions and Derivatives

Definition of Hyperbolics

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x} \quad \text{sech} x = \frac{1}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \coth x = \frac{\cosh x}{\sinh x} \quad \text{csch} x = \frac{1}{\sinh x}$$

Basic Derivatives

$$(\sinh x)' = \cosh x \quad (\tanh x)' = \text{sech}^2 x \quad (\text{sech} x)' = -\text{sech} x \tanh x$$

$$(\cosh x)' = \sinh x \quad (\coth x)' = -\text{csch}^2 x \quad (\text{csch} x)' = -\text{csch} x \coth x$$

Basic Integral Formulas

Basic Integrals

$$\int \sinh x dx = \cosh x + C \qquad \int \cosh x dx = \sinh x + C$$

$$\int \mathrm{sech}^2 x dx = \tanh x + C \qquad \int \mathrm{csch}^2 x dx = -\coth x + C$$

$$\int \mathrm{sech} x \tanh x dx = -\mathrm{sech} x + C \qquad \int \mathrm{csch} x \coth x dx = -\mathrm{csch} x + C$$

- Example: Calculate $\int x \cosh(x^2) dx$;

$$\int x \cosh(x^2) dx \stackrel{u=x^2}{=} \int \frac{1}{2} \cosh u du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh(x^2) + C;$$

Powers of $\sinh x$ and $\cosh x$

- Calculate $\int \sinh^4 x \cosh^5 x dx$;

$$\int \sinh^4 x \cosh^5 x dx = \int \sinh^4 x (\cosh^2 x)^2 \cosh x dx =$$
$$\int \sinh^4 x (1+\sinh^2 x)^2 \cosh x dx \overset{u=\sinh x}{=} \int u^4(1+u^2)^2 du =$$
$$\int u^4(u^4+2u^2+1)du = \int (u^8+2u^6+u^4)du =$$
$$\frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + C = \frac{1}{9}\sinh^9 x + \frac{2}{7}\sinh^7 x + \frac{1}{5}\sinh^5 x + C;$$

- Calculate $\int \cosh^2 x dx$;

$$\int \cosh^2 x dx = \int \frac{1}{2}(1+\cosh 2x)dx = \frac{1}{2}(x + \frac{1}{2}\sinh 2x) + C =$$
$$\frac{1}{2}x + \frac{1}{4}\sinh 2x + C;$$

Hyperbolic Substitutions (instead of Trig Substitutions)

- Instead of trigonometric substitutions, one may sometimes perform hyperbolic substitutions to calculate an integral:

The Method

- For expressions of the form $\sqrt{x^2 + a^2}$, instead of $x = a \tan \theta$, we may use $x = a \sinh u$; In that case
  - $dx = a \cosh u du$;
  - $\sqrt{x^2 + a^2} = a \cosh u$;
- For expressions of the form $\sqrt{x^2 - a^2}$, instead of $x = a \sec \theta$, we may use $x = a \cosh u$; In that case
  - $dx = a \sinh u du$;
  - $\sqrt{x^2 - a^2} = a \sinh u$;

Example of Hyperbolic Substitution

- Example: Calculate $\int \sqrt{x^2+16}dx$;
  We set $x = 4\sinh u$; Then $dx = 4\cosh u du$,
  $\sqrt{x^2+16} = \sqrt{16\sinh^2 u + 16} = 4\cosh u$; Moreover, $u = \sinh^{-1}\frac{x}{4}$,
  $\sinh u = \frac{x}{4}$ and $\cosh u = \sqrt{\sinh^2 u + 1} = \sqrt{\frac{x^2}{16}+1}$; Therefore,

$$\int \sqrt{x^2+16}dx = \int 4\cosh u 4\cosh u du = \int 16\cosh^2 u du =$$
$$\int 8(1+\cosh 2u)du = 8u + 4\sinh 2u + C =$$
$$8u + 8\sinh u \cosh u + C = 8\sinh^{-1}\frac{x}{4} + 8\frac{x}{4}\sqrt{\frac{x^2}{16}+1} + C;$$

Integrals of Inverse Hyperbolic Functions

Integrals Involving Inverse Trigonometric Functions

- $\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C;$
- $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C; \qquad (x > 1)$
- $\int \frac{dx}{1-x^2} = \tanh^{-1} x + C; \qquad (|x| < 1)$
- $\int \frac{dx}{1-x^2} = \coth^{-1} x + C; \qquad (|x| > 1)$
- $\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1} x + C; \qquad (0 < x < 1)$
- $\int \frac{dx}{|x|\sqrt{1+x^2}} = -\operatorname{csch}^{-1} x + C; \qquad (x \neq 0)$

Examples of Inverse Hyperbolic Integrals

- Evaluate the following integrals:
  - $\int_2^4 \frac{dx}{\sqrt{x^2-1}}$;
  $$\int_2^4 \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x\Big|_2^4 = \cosh^{-1} 4 - \cosh^{-1} 2;$$
  - $\int_{0.2}^{0.6} \frac{xdx}{1-x^4}$;
  $$\int_{0.2}^{0.6} \frac{xdx}{1-x^4} \stackrel{u=x^2}{=} \int_{0.04}^{0.36} \frac{\frac{1}{2}du}{1-u^2} =$$
  $$\frac{1}{2}\tanh^{-1} u\Big|_{0.04}^{0.36} = \frac{1}{2}(\tanh^{-1} 0.36 - \tanh^{-1} 0.04);$$

Subsection 5

The Method of Partial Fractions

Outline of Partial Fractions Method

- To integrate a rational function $f(x) = \frac{P(x)}{Q(x)}$, we write it as a sum of simpler rational functions that can be integrated directly;
- For example, to integrate $\int \frac{1}{x^2-1}dx$:
  1. We decompose the fraction into **partial fractions**:
  $$\frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1};$$
  2. Then, work as follows
  $$\begin{aligned} \int \frac{1}{x^2-1}dx &= \int \left[\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right] dx \\ &= \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{x+1}dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C; \end{aligned}$$

Distinct Linear Factors I

- Evaluate $\int \frac{1}{x^2-7x+10}dx$;
  Factor the denominator: $x^2-7x+10=(x-2)(x-5)$;
  Decompose into partial fractions:
  $\frac{1}{x^2-7x+10}=\frac{A}{x-2}+\frac{B}{x-5} \Rightarrow \frac{(x-2)(x-5)}{x^2-7x+10}=$
  $\frac{A(x-2)(x-5)}{x-2}+\frac{B(x-2)(x-5)}{x-5} \Rightarrow 1=$
  $A(x-5)+B(x-2) \Rightarrow 1=(A+B)x+(-5A-2B) \Rightarrow$
  $\left\{\begin{array}{rcl} A+B & = & 0 \\ -5A-2B & = & 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} A & = & -B \\ 5B-2B & = & 1 \end{array}\right\} \Rightarrow$
  $\left\{\begin{array}{rcl} A & = & -\frac{1}{3} \\ B & = & \frac{1}{3} \end{array}\right\}$; So we get

$$\frac{1}{x^2-7x+10}=\frac{-\frac{1}{3}}{x-2}+\frac{\frac{1}{3}}{x-5};$$

Distinct Linear Factors I (Cont'd)

We obtained

$$\frac{1}{x^2 - 7x + 10} = \frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5};$$

So, we have

$$\begin{aligned} \int \frac{1}{x^2 - 7x + 10} dx &= \int \left[\frac{-\frac{1}{3}}{x-2} + \frac{\frac{1}{3}}{x-5} \right] dx \\ &= -\frac{1}{3} \int \frac{1}{x-2} dx + \frac{1}{3} \int \frac{1}{x-5} dx \\ &= -\frac{1}{3} \ln|x-2| + \frac{1}{3} \ln|x-5| + C; \end{aligned}$$

Distinct Linear Factors II

- Evaluate $\int \frac{x^2+2}{(x-1)(2x-8)(x+2)}dx$;

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2} \quad \Rightarrow$$
$$\frac{(x-1)(2x-8)(x+2)(x^2+2)}{(x-1)(2x-8)(x+2)} = \frac{A(x-1)(2x-8)(x+2)}{x-1} +$$
$$\frac{B(x-1)(2x-8)(x+2)}{2x-8} + \frac{C(x-1)(2x-8)(x+2)}{x+2} \quad \Rightarrow$$
$x^2+2 = A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8)$;

Now, we get:
  - $x=1 \Rightarrow 3 = A \cdot (-6) \cdot 3 \Rightarrow A = -\frac{1}{6}$;
  - $x=4 \Rightarrow 18 = B \cdot 3 \cdot 6 \Rightarrow B = 1$;
  - $x=-2 \Rightarrow 6 = C \cdot (-3) \cdot (-12) \Rightarrow C = \frac{1}{6}$;

Therefore, we obtain
$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{-\frac{1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2};$$

Distinct Linear Factors II (Cont'd)

We obtained

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{-\frac{1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2};$$

Now, we integrate $\int \frac{x^2+2}{(x-1)(2x-8)(x+2)} dx$

$$\begin{aligned} &= \int \left[\frac{-\frac{1}{6}}{x-1} + \frac{1}{2x-8} + \frac{\frac{1}{6}}{x+2}\right] dx \\ &= -\frac{1}{6}\int \frac{1}{x-1} dx + \int \frac{1}{2x-8} dx + +\frac{1}{6}\int \frac{1}{x+2} dx \\ &= -\frac{1}{6}\ln|x-1| + \frac{1}{2}\ln|2x-8| + \frac{1}{6}\ln|x+2| + C; \end{aligned}$$

Long Division First...

- Evaluate $\int \frac{x^3+1}{x^2-4} dx$;
  Numerator has higher degree than denominator!
  Start by performing the long division $(x^3+1) \div (x^2-4)$;

$$\begin{array}{r|l} & x \\ \hline x^2-4 & x^3 \qquad\qquad +1 \\ & x^3 \quad -4x \\ \hline & \qquad 4x \quad +1 \end{array}$$

  It has quotient x and remainder $4x+1$; Thus,
  $\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4} = x + \frac{4x+1}{(x-2)(x+2)}$;

...Breaking Into Partial Fractions Next...

- We found $\frac{x^3+1}{x^2-4} = x + \frac{4x+1}{(x-2)(x+2)}$.

  Decompose the second fraction:
  $\frac{4x+1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \Rightarrow \frac{(4x+1)(x-2)(x+2)}{(x-2)(x+2)} = \frac{A(x-2)(x+2)}{x-2} + \frac{B(x-2)(x+2)}{x+2} \Rightarrow 4x+1 = A(x+2)+B(x-2)$;
  - $x = 2 \Rightarrow 9 = 4A \Rightarrow A = \frac{9}{4}$;
  - $x = -2 \Rightarrow -7 = -4B \Rightarrow B = \frac{7}{4}$;

  This gives $\frac{x^3+1}{x^2-4} = x + \frac{\frac{9}{4}}{x-2} + \frac{\frac{7}{4}}{x+2}$;

...and Integrating

We got

$$\frac{x^3+1}{x^2-4} = x + \frac{\frac{9}{4}}{x-2} + \frac{\frac{7}{4}}{x+2};$$

Hence, we have

$$\begin{aligned}\int \frac{x^3+1}{x^2-4}dx &= \int \left[x + \frac{\frac{9}{4}}{x-2} + \frac{\frac{7}{4}}{x+2}\right] dx \\ &= \int x dx + \frac{9}{4}\int \frac{1}{x-2}dx + \frac{7}{4}\int \frac{1}{x+2}dx \\ &= \frac{1}{2}x^2 + \frac{9}{4}\ln|x-2| + \frac{7}{4}\ln|x+2| + C;\end{aligned}$$

Repeated Linear Factors

- Evaluate $\int \frac{3x-9}{(x-1)(x+2)^2}dx$;
  Decompose into partial fractions:
  $\frac{3x-9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \quad \Rightarrow$
  $\frac{(3x-9)(x-1)(x+2)^2}{(x-1)(x+2)^2} =$
  $\frac{A(x-1)(x+2)^2}{x-1} + \frac{B(x-1)(x+2)^2}{x+2} + \frac{C(x-1)(x+2)^2}{(x+2)^2} \quad \Rightarrow$
  $3x-9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$;
  - $x = 1 \Rightarrow -6 = 9A \Rightarrow A = -\frac{2}{3}$;
  - $x = -2 \Rightarrow -15 = -3C \Rightarrow C = 5$;
  - $x = 0 \Rightarrow -9 = 4A - 2B - C \Rightarrow B = \frac{4A-C+9}{2} = \frac{2}{3}$;

  So we get
  $$\frac{3x-9}{(x-1)(x+2)^2} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2};$$

Repeated Linear Factors (Cont'd)

We just got

$$\frac{3x-9}{(x-1)(x+2)^2} = \frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2};$$

Now, we integrate $\int \frac{3x-9}{(x-1)(x+2)^2} dx$

$$\begin{aligned} &= \int \left[\frac{-\frac{2}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} + \frac{5}{(x+2)^2} \right] dx \\ &= -\frac{2}{3} \int \frac{1}{x-1} dx + \frac{2}{3} \int \frac{1}{x+2} dx + 5 \int \frac{1}{(x+2)^2} dx \\ &= -\frac{2}{3} \ln|x-1| + \frac{2}{3} \ln|x+2| - \frac{5}{x+2} + C; \end{aligned}$$

Irreducible Quadratic Factors

- Evaluate $\int \frac{18}{(x+3)(x^2+9)}dx$;
  Decompose the fraction
  $\frac{18}{(x+3)(x^2+9)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+9} \quad \Rightarrow \quad \frac{18(x+3)(x^2+9)}{(x+3)(x^2+9)} = \frac{A(x+3)(x^2+9)}{x+3} + \frac{(Bx+C)(x+3)(x^2+9)}{x^2+9} \quad \Rightarrow \quad 18 = A(x^2+9)+(Bx+C)(x+3)$;
  - $x=-3 \Rightarrow 18 = 18A \Rightarrow A = 1$;
  - $x=0 \Rightarrow 18 = 9+3C \Rightarrow 3C = 9 \Rightarrow C = 3$;
  - $x=1 \Rightarrow 18 = 10+(B+3)\cdot 4 \Rightarrow 8 = 4B+12 \Rightarrow B = -1$;

  Therefore $\frac{18}{(x+3)(x^2+9)} = \frac{1}{x+3} + \frac{-x+3}{x^2+9}$;

Irreducible Quadratic Factors (Cont'd)

We found that

$$\frac{18}{(x+3)(x^2+9)} = \frac{1}{x+3} + \frac{-x+3}{x^2+9};$$

Now we integrate: $\int \frac{18}{(x+3)(x^2+9)} dx$

$$\begin{aligned}
&= \int \left[\frac{1}{x+3} + \frac{-x+3}{x^2+9}\right] dx \\
&= \int \frac{1}{x+3} dx + \int \frac{-x+3}{x^2+9} dx \\
&= \int \frac{1}{x+3} dx - \int \frac{x}{x^2+9} dx + 3\int \frac{1}{x^2+9} dx \\
&= \ln|x+3| - \frac{1}{2}\ln(x^2+9) + 3 \cdot \frac{1}{3}\tan^{-1}\frac{x}{3} + C \\
&= \ln|x+3| - \frac{1}{2}\ln(x^2+9) + \tan^{-1}\frac{x}{3} + C;
\end{aligned}$$

Repeated Quadratic Factors

- Evaluate $\int \frac{4-x}{x(x^2+2)^2}dx$;

  Decompose the fraction $\frac{4-x}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2} \quad \Rightarrow$

  $A=1, B=-1, C=0, D=-2, E=-1$; So, we get

  $\frac{4-x}{x(x^2+2)^2} = \frac{1}{x} + \frac{-x}{x^2+2} + \frac{-2x-1}{(x^2+2)^2}$; Integrating, we get

$$\begin{aligned}
\int \frac{4-x}{x(x^2+2)^2}dx
&= \int \left[\frac{1}{x} + \frac{-x}{x^2+2} + \frac{-2x-1}{(x^2+2)^2}\right] dx \\
&= \int \frac{1}{x}dx - \int \frac{x}{x^2+2}dx - \int \frac{2x}{(x^2+2)^2}dx - \int \frac{1}{(x^2+2)^2}dx \\
&= \ln|x| - \frac{1}{2}\ln(x^2+2) - \frac{-1}{x^2+2} - \int \frac{1}{(x^2+2)^2}dx;
\end{aligned}$$

The Integral $\int \frac{1}{(x^2+2)^2} dx$

Set $x = \sqrt{2}\tan\theta$; Then $dx = \sqrt{2}\sec^2\theta d\theta,\ x^2+2 = 2\tan^2\theta + 2 = 2\sec^2\theta,\ \theta = \tan^{-1}\frac{x}{\sqrt{2}},\ \sin\theta = \frac{x}{\sqrt{x^2+2}},\ \cos\theta = \frac{\sqrt{2}}{\sqrt{x^2+2}}$;

$$\int \frac{1}{(x^2+2)^2} dx = \int \frac{1}{4\sec^4\theta}\sqrt{2}\sec^2\theta d\theta =$$
$$\frac{\sqrt{2}}{4}\int \cos^2\theta d\theta = \frac{\sqrt{2}}{8}\int (1+\cos 2\theta) d\theta =$$
$$\frac{\sqrt{2}}{8}(\theta + \frac{1}{2}\sin 2\theta) + C = \frac{\sqrt{2}}{8}\theta + \frac{\sqrt{2}}{8}\sin\theta\cos\theta + C =$$
$$\frac{\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{\sqrt{2}}{8}\frac{x}{\sqrt{x^2+2}}\frac{\sqrt{2}}{\sqrt{x^2+2}} + C =$$
$$\frac{\sqrt{2}}{8}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{x}{4(x^2+2)} + C;$$

Subsection 6

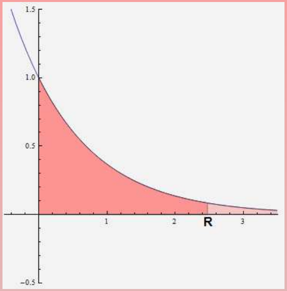
Improper Integrals

Overview of Improper Integrals

Suppose, we wanted to determine the amount of area under the graph of $f(x) = e^{-x}$ over the unbounded interval $[0, \infty)$; This is given by the **improper integral**

$$\int_0^\infty e^{-x} dx;$$

It is called improper because it represents the area of an unbounded region;



To compute such an integral, we first introduce an artificial bound $R > 0$ and we compute instead the proper integral

$$\int_0^R e^{-x} dx = -e^{-x}\Big|_0^R = (-e^{-R} - (-1)) = 1 - e^{-R};$$

Finally, we "push" R towards $+\infty$:

$$\int_0^\infty e^{-x} dx = \lim_{R\to\infty} \int_0^R e^{-x} dx = \lim_{R\to\infty} (1 - e^{-R}) = 1 - 0 = 1;$$

Formal Definitions

Definitions of Improper Integrals

- If, for some fixed a, the function $f(x)$ is integrable on $[a,b]$ for all $b > a$, then define the **improper integral of** $f(x)$ **over** $[a,\infty)$ by
$$\int_a^\infty f(x)dx = \lim_{R\to\infty}\int_a^R f(x)dx;$$
The integral **converges** if the limits exists and is finite, and it **diverges**, otherwise;
- If, for some fixed a, the function $f(x)$ is integrable on $[b,a]$ for all $b < a$, then define the **improper integral of** $f(x)$ **over** $(-\infty,a]$ by
$$\int_{-\infty}^a f(x)dx = \lim_{R\to-\infty}\int_R^a f(x)dx;$$
The integral **converges** if the limits exists and is finite, and it **diverges**, otherwise;

Formal Definitions (Cont'd)

Definition of Third Type of Improper Integral

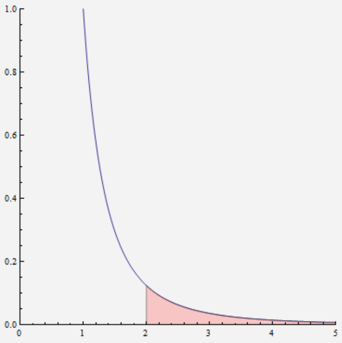
- If, for all $a < 0, b > 0$, the function $f(x)$ is integrable on $[a, 0], [0, b]$, then we define the **improper integral of** $f(x)$ **over** $(-\infty, \infty)$ by

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx;$$

  The integral **converges** if both integrals on the right converge and it **diverges**, otherwise;

Example of Improper Integral I

Show that $\int_2^\infty \frac{1}{x^3} dx$ converges and compute its value;



We first calculate

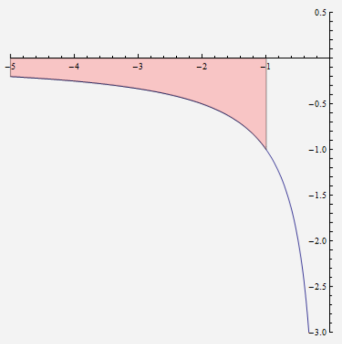
$$\int_2^R \frac{1}{x^3} dx = \left. \frac{-1}{2x^2} \right|_2^R = \frac{1}{8} - \frac{1}{2R^2};$$

Therefore, we obtain:

$$\int_2^\infty \frac{1}{x^3} dx = \lim_{R\to\infty} \int_2^R \frac{1}{x^3} dx = \lim_{R\to\infty} \left(\frac{1}{8} - \frac{1}{2R^2}\right) = \frac{1}{8} - 0 = \frac{1}{8};$$

Example of Improper Integral II

Determine whether $\int_{-\infty}^{-1} \frac{1}{x} dx$ converges; If so, compute its value;



We first calculate

$$\int_R^{-1} \frac{1}{x} dx = \ln|x| \Big|_R^{-1} = -\ln|R|;$$

Therefore, we obtain:

$$\int_{-\infty}^{-1} \frac{1}{x} dx = \lim_{R \to -\infty} \int_R^{-1} \frac{1}{x} dx = \lim_{R \to -\infty} [-\ln|R|] = \lim_{R \to \infty} [-\ln R] = -\infty;$$

The p-Integral

- Show that, for $a > 0$, $\int_a^\infty \frac{1}{x^p} dx = \begin{cases} \frac{a^{1-p}}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \leq 1 \end{cases}$;

$$\int_a^\infty \frac{1}{x^p} dx = \lim_{R\to\infty} \int_a^R x^{-p} dx = \lim_{R\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_a^R = \lim_{R\to\infty} \left(\frac{R^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right);$$

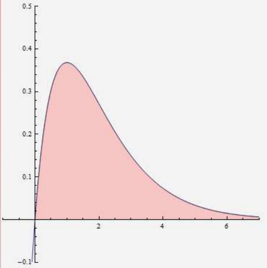
  - If $p > 1$, then $1 - p < 0$, so $\lim_{R\to\infty} R^{1-p} = 0$ and, therefore, $\int_a^\infty \frac{1}{x^p} dx = \frac{a^{1-p}}{p-1}$;
  - If $p < 1$, then $1 - p > 0$, so $\lim_{R\to\infty} R^{1-p} = \infty$; Therefore, the integral diverges;
  - If $p = 1$, then $\int_a^\infty \frac{1}{x} dx = \lim_{R\to\infty} \int_a^R \frac{1}{x} dx = \lim_{R\to\infty} (\ln R - \ln a) = \infty$;

Using L'Hôpital's Rule

- Recall how L'Hôpital's Rule is applied:

$$\lim_{x\to\infty}\frac{x+1}{e^x}\left(=\frac{\infty}{\infty}\right)\overset{\text{L'Hôpital}}{=}\lim_{x\to\infty}\frac{(x+1)'}{(e^x)'}=\lim_{x\to\infty}\frac{1}{e^x}=0;$$

- Calculate $\int_0^\infty xe^{-x}dx$;



$$\int xe^{-x}dx=\int x(-e^{-x})'dx\overset{\text{By Parts}}{=}$$
$$-xe^{-x}-\int -e^{-x}dx=$$
$$-xe^{-x}-e^{-x}=-\frac{x+1}{e^x};$$

$$\int_0^\infty xe^{-x}dx=\lim_{R\to\infty}\int_0^R xe^{-x}dx=\lim_{R\to\infty}(1-\frac{R+1}{e^R})=1-0=1;$$

Application: Escape Velocity

- The earth exerts a gravitational force of magnitude $F(r) = G\frac{M_e m}{r^2}$ on an object of mass m at distance r from its center;
  - Find the work required to move the object infinitely far from the earth;

$$W = \int_{r_e}^{\infty} F(r)dr = \int_{r_e}^{\infty} G\frac{M_e m}{r^2}dr = \lim_{R\to\infty}\int_{r_e}^{R} G\frac{M_e m}{r^2}dr =$$
$$GM_e m\lim_{R\to\infty}\int_{r_e}^{R}\frac{1}{r^2}dr = GM_e m\lim_{R\to\infty}\left(-\frac{1}{r}\right)\Big|_{r_e}^{R} =$$
$$GM_e m\lim_{R\to\infty}\left[\frac{1}{r_e}-\frac{1}{R}\right] = G\frac{M_e m}{r_e}\text{ J};$$

- Calculate the escape velocity v_{esc} on the earth's surface;

  The escape velocity must provide kinetic energy at least as big as the work required to move the object infinitely far from the earth;

$$\frac{1}{2}mv_{\mathrm{esc}}^2 \geq G\frac{M_e m}{r_e} \quad\Rightarrow\quad v_{\mathrm{esc}}^2 \geq \frac{2GM_e}{r_e} \quad\Rightarrow\quad v_{\mathrm{esc}} \geq \sqrt{\frac{2GM_e}{r_e}};$$

Application: Perpetual Annuity

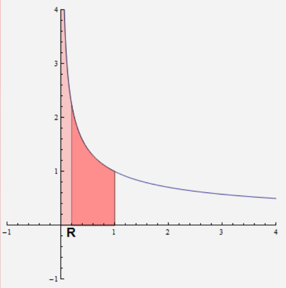
- An investment pays a dividend continuously at a rate of \$6,000 per year; Compute the present value of the income stream if the interest rate is 4% and the dividends continue forever.

$$\text{PV} = \int_0^\infty Pe^{-rt}dt = \lim_{T\to\infty}\int_0^T 6000e^{-0.04t}dt =$$

$$\lim_{T\to\infty} -\frac{6000}{0.04}e^{-0.04t}\Big|_0^T = -150000\lim_{T\to\infty}(e^{-0.04T}-1) =$$

$$-15000\cdot(-1) = \$150,000;$$

Improper Integrals for Infinite Discontinuities at Endpoints



We determine the amount of area under the graph of $f(x) = \frac{1}{\sqrt{x}}$ over the interval $[0, 1]$;

This is given by the **improper integral**

$$\int_0^1 \frac{1}{\sqrt{x}} dx;$$

It is improper because it represents the area of an unbounded region;

To compute such an integral, we first introduce an artificial bound $0 < R < 1$ and we compute instead the proper integral

$$\int_R^1 x^{-1/2} dx = 2\sqrt{x}\Big|_R^1 = 2\sqrt{1} - 2\sqrt{R} = 2 - 2\sqrt{R};$$

Finally, we "push" R towards 0 from the right:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{R \to 0^+} \int_R^1 \frac{1}{\sqrt{x}} dx = \lim_{R \to 0^+} (2 - 2\sqrt{R}) = 2 - 0 = 2;$$

Definitions of Integrals with Infinite Discontinuities

Integrants with Infinite Discontinuities

- If $f(x)$ is continuous on $[a, b)$ but discontinuous at $x = b$, we define

$$\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx;$$

- If $f(x)$ is continuous on $(a, b]$ but discontinuous at $x = a$, we define

$$\int_a^b f(x)dx = \lim_{R \to a^+} \int_R^b f(x)dx;$$

- In both cases the improper integral **converges** if the limit exists and it **diverges** otherwise;

Examples of Improper Integrals

- Calculate $\int_0^9 \frac{1}{\sqrt{x}} dx$;

$$\int_0^9 \frac{1}{\sqrt{x}} dx = \lim_{R \to 0^+} \int_R^9 x^{-1/2} dx =$$
$$\lim_{R \to 0^+} [2\sqrt{x}\Big|_R^9] = \lim_{R \to 0^+} (6 - 2\sqrt{R}) = 6 - 0 = 6;$$

- Calculate $\int_0^{1/2} \frac{1}{x} dx$;

$$\int_0^{1/2} \frac{1}{x} dx = \lim_{R \to 0^+} \int_R^{1/2} \frac{1}{x} dx =$$
$$\lim_{R \to 0^+} [\ln x\Big|_R^{1/2}] = \lim_{R \to 0^+} (\ln \frac{1}{2} - \ln R) = \infty;$$

p-Integral Revisited

- Show that, for $a > 0$, $\int_0^a \frac{1}{x^p} dx = \begin{cases} \frac{a^{1-p}}{1-p}, & \text{if } p < 1 \\ \text{diverges}, & \text{if } p \geq 1 \end{cases}$;

$$\int_0^a \frac{1}{x^p} dx = \lim_{R \to 0^+} \int_R^a x^{-p} dx = \lim_{R \to 0^+} \left. \frac{x^{1-p}}{1-p} \right|_R^a = \lim_{R \to 0^+} \left(\frac{a^{1-p}}{1-p} - \frac{R^{1-p}}{1-p} \right);$$

  - If $p < 1$, then $1 - p > 0$, so $\lim_{R \to 0^+} R^{1-p} = 0$; Therefore, $\int_0^a \frac{1}{x^p} dx = \frac{a^{1-p}}{1-p}$;
  - If $p > 1$, then $1 - p < 0$, so $\lim_{R \to 0^+} R^{1-p} = \infty$ and, therefore, the integral diverges;
  - If $p = 1$, then $\int_0^a \frac{1}{x} dx = \lim_{R \to 0^+} \int_R^a \frac{1}{x} dx = \lim_{R \to 0^+} (\ln a - \ln R) = \infty$;

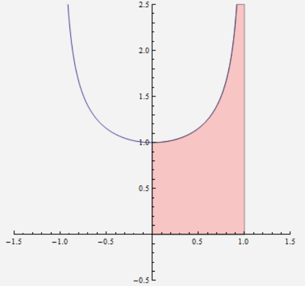
An Additional Example

- Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$;

First, recall the formula $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$;

We compute

$$\int_0^R \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x\Big|_0^R = \sin^{-1} R - \sin^{-1} 0 = \sin^{-1} R;$$



Therefore, we get

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{R\to 1^-}\int_0^R \frac{1}{\sqrt{1-x^2}}dx = \lim_{R\to 1^-}\sin^{-1} R = \frac{\pi}{2};$$

The Comparison Test for Improper Integrals

Comparison Test for Improper Integrals

Assume that $f(x) \geq g(x) \geq 0$ for $x \geq a$;

- If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ also converges;
- If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ also diverges;

The Comparison Test may also be applied for improper integrals with infinite discontinuities at the endpoints;

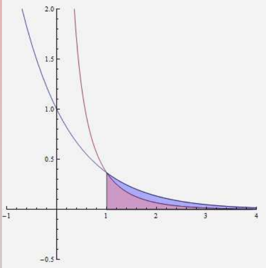
Applying the Comparison Test I

- Show that $\int_1^\infty \frac{e^{-x}}{x} dx$ converges;

  Note that for $x \geq 1$, we get $0 \leq \frac{1}{x} \leq 1 \quad \Rightarrow \quad 0 \leq \frac{e^{-x}}{x} \leq e^{-x}$;

  Therefore, by the comparison test, to show that $\int_1^\infty \frac{e^{-x}}{x} dx$ converges, it suffices to show that $\int_1^\infty e^{-x} dx$ converges; Here is the computation:

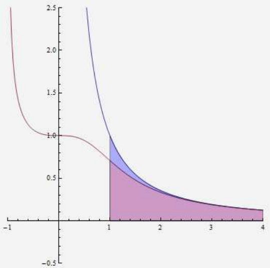
$$\int_1^\infty e^{-x} dx = \lim_{R\to\infty} \int_1^R e^{-x} dx =$$
$$\lim_{R\to\infty} [(-e^{-x})|_1^R] =$$
$$\lim_{R\to\infty} (e^{-1} - e^{-R}) = \frac{1}{e};$$



Applying the Comparison Test II

- Show that $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$ converges;
  Note that for $x \geq 1$, we get
  $x^3 \leq x^3+1 \Rightarrow \sqrt{x^3} \leq \sqrt{x^3+1} \Rightarrow 0 \leq \frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}}$; By the comparison test, to show that $\int_1^\infty \frac{1}{\sqrt{x^3+1}}dx$ converges, it suffices to show that $\int_1^\infty \frac{1}{\sqrt{x^3}}dx$ converges;

  This is, however, true, since this is a p-integral, with $p = \frac{3}{2} > 1$;



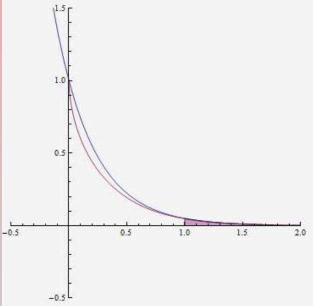
Applying the Comparison Test III

- Does $\int_1^\infty \frac{1}{\sqrt{x}+e^{3x}}dx$ converge?

  Note that for $x \geq 1$, we get $e^{3x} \leq \sqrt{x}+e^{3x} \Rightarrow 0 \leq \frac{1}{\sqrt{x}+e^{3x}} \leq \frac{1}{e^{3x}}$;

  By the comparison test, to show $\int_1^\infty \frac{1}{\sqrt{x}+e^{3x}}dx$ converges, it suffices to show $\int_1^\infty \frac{1}{e^{3x}}dx$ converges;

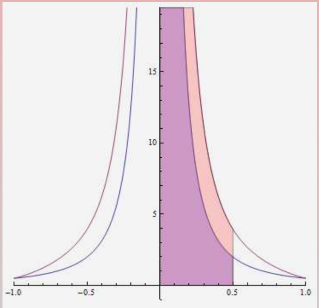
  $$\int_1^\infty \frac{1}{e^{3x}}dx = \lim_{R\to\infty}[-\tfrac{1}{3}e^{-3x}|_1^R] =$$
  $$\lim_{R\to\infty}\left[\frac{1}{3e^3}-\frac{1}{3e^{3R}}\right] = \frac{1}{3e^3};$$



Applying the Comparison Test IV

- Does $\int_0^{1/2} \frac{1}{x^8+x^2}dx$ converge?
  Note that for $0 < x \leq \frac{1}{2}$, we get
  $x^8 \leq x^2 \Rightarrow x^8 + x^2 \leq x^2 + x^2 = 2x^2 \Rightarrow 0 \leq \frac{1}{2x^2} \leq \frac{1}{x^8+x^2}$; By the comparison test, to show $\int_0^{1/2} \frac{1}{x^8+x^2}dx$ diverges, it suffices to show $\frac{1}{2}\int_0^{1/2} \frac{1}{x^2}dx$ diverges;

  This is, however, true, since this is a p-integral, with $p = 2 > 1$;



Subsection 7

Numerical Integration

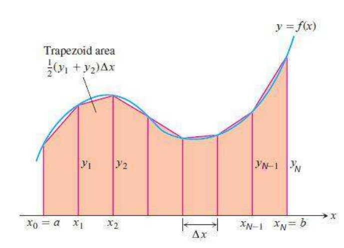
Trapezoidal Rule

Trapezoidal Rule

The N-th trapezoidal approximation to $\int_a^b f(x)dx$ is $T_N =$

$$\frac{1}{2}\Delta x(y_0 + 2y_1 + \cdots + 2y_{N-1} + y_N),$$

where $\Delta x = \frac{b-a}{N}$ and $y_j = f(a + j\Delta x)$.



Error Bound for T_N

If $f''(x)$ exists and is continuous, and, for all x in $[a, b]$, $|f''(x)| \leq K_2$, then

$$\text{Error}(T_N) = \left| \int_a^b f(x)dx - T_N \right| \leq \frac{K_2(b-a)^3}{12N^2};$$

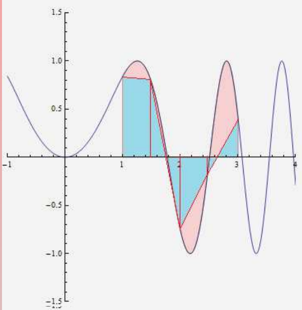
Applying the Trapezoidal Rule

- Consider the integral $\int_1^3 \sin(x^2)dx$; Calculate T_4;

  We divide $[1,3]$ into four subintervals of length $\Delta x = \frac{3-1}{4} = \frac{1}{2}$;

  Then, we apply the formula for T_4:

  $T_4 = \frac{1}{2}\Delta x(y_0+2y_1+2y_2+2y_3+y_4) =$



$$\frac{1}{2}\cdot\frac{1}{2}(f(1)+2f(1.5)+2f(2)+2f(2.5)+f(3)) =$$
$$\frac{1}{4}(\sin 1 + 2\sin(1.5^2) + 2\sin 4 + 2\sin(2.5^2) + \sin 9)$$
$$\approx 0.30744;$$

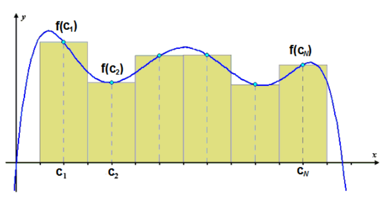
Midpoint Rule

Midpoint Rule

The N-th midpoint approximation to $\int_a^b f(x)dx$ is $M_N =$

$$\Delta x(f(c_1) + f(c_2) + \cdots + f(c_N)),$$

where $\Delta x = \frac{b-a}{N}$ and $c_j = a + (j - \frac{1}{2})\Delta x$ is the midpoint of $[x_{j-1}, x_j]$.



Error Bound for M_N

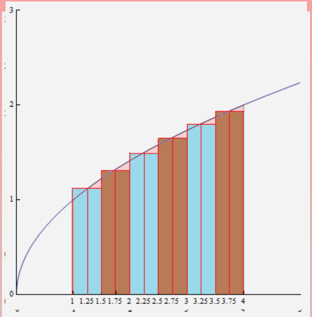
If $f''(x)$ exists and is continuous, and, for all x in $[a,b]$, $|f''(x)| \leq K_2$, then

$$\text{Error}(M_N) = \left| \int_a^b f(x)dx - M_N \right| \leq \frac{K_2(b-a)^3}{24N^2};$$

Applying the Midpoint Rule

- Consider the integral $\int_1^4 \sqrt{x}\,dx$;
  - Calculate its exact value;

$$\int_1^4 x^{1/2}dx = \left.\frac{2}{3}\sqrt{x}^3\right|_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.667;$$



  - Find M_6;

$$M_6 = \Delta x[f(c_1)+f(c_2)+f(c_3)+f(c_4)+f(c_5)+f(c_6)] = \frac{1}{2}[f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25)+f(3.75)] = \frac{1}{2}(\sqrt{1.25}+\sqrt{1.75}+\sqrt{2.25}+\sqrt{2.75}+\sqrt{3.25}+\sqrt{3.75}) \approx 4.669;$$

Error and Bound of the Midpoint Rule

- Consider again the integral $\int_1^4 \sqrt{x}dx$; We found that

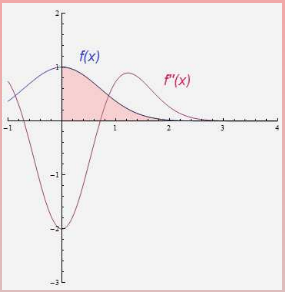
$$\int_1^4 x^{1/2}dx \approx 4.667 \qquad M_6 \approx 4.669;$$

  - Find the exact value of Error(M_6);
    $\text{Error}(M_6) = \left| \int_1^4 \sqrt{x}dx - M_6 \right| \approx |4.667 - 4.669| = 0.002;$
  - Find the theoretical error bound and verify that the actual error is less than that bound;
    Note that $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt{x^3}}$;
    Therefore, for $1 \leq x \leq 4$, $|f''(x)| \leq \frac{1}{4} = K_2$; This shows that
    $\frac{K_2(b-a)^3}{24N^2} = \frac{\frac{1}{4} \cdot 3^3}{24 \cdot 6^2} = \frac{1}{128} \approx 0.009$; Therefore,
    $\text{Error}(M_6) \approx 0.002 \leq 0.009 \approx \frac{K_2(b-a)^3}{24N^2}$;

Example (Accuracy of Trapezoidal)

- Find how large N should be so that T_N approximates $\int_0^3 e^{-x^2}dx$ with an error of at most 10^{-4};

  We are asked to determine N, so that $\text{Error}(T_N) \leq 10^{-4}$; Since $\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2}$, it suffices to find N, such that $\frac{K_2(b-a)^3}{12N^2} \leq 10^{-4}$; Note that $f'(x) = -2xe^{-x^2}$ and $f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2-2)e^{-x^2}$; Its graph over the interval $[0,3]$ is given in the figure; Observe that $|f''(x)| \leq 2 = K_2$;



$$\frac{K_2(b-a)^3}{12N^2} \leq 10^{-4} \Rightarrow \frac{2(3-0)^3}{12N^2} \leq 10^{-4} \Rightarrow \frac{54 \cdot 10^4}{12} \leq N^2$$

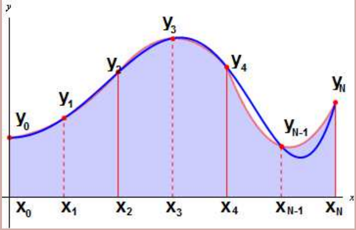
$$\Rightarrow \quad N \geq \sqrt{\frac{54 \cdot 10^4}{12}} \quad \Rightarrow \quad N \geq \sqrt{4.5} \cdot 100 \approx 212.1;$$

Simpson's Rule

Simpson's Rule

For N even, the N-th midpoint approximation to $\int_a^b f(x)dx$ is $S_N =$

$$\frac{1}{3}\Delta x[y_0 + 4y_1 + 2y_2 + \cdots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N],$$



where $\Delta x = \frac{b-a}{N}$ and $y_j = f(a + j\Delta x)$;

Error Bound for M_N

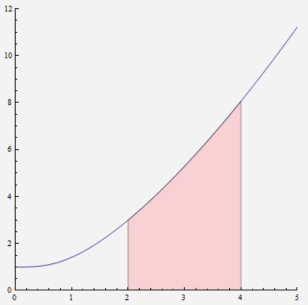
If $f^{(4)}(x)$ exists and is continuous, and, for all x in $[a,b]$, $|f^{(4)}(x)| \leq K_4$, then

$$\text{Error}(S_N) = \left|\int_a^b f(x)dx - S_N\right| \leq \frac{K_4(b-a)^5}{180N^4};$$

Applying Simpson's Rule

- Consider the integral $\int_2^4 \sqrt{1+x^3}dx$; Calculate S_8;

  We divide $[2,4]$ into eight subintervals of length $\Delta x = \frac{4-2}{8} = \frac{1}{4}$;

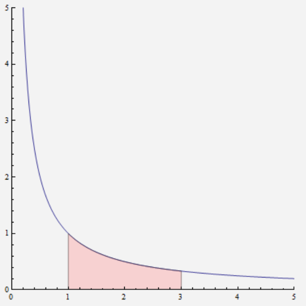
  

  Then, we apply the formula for S_8:
  $S_8 = \frac{1}{3}\Delta x(y_0+4y_1+2y_2+\cdots+2y_6+4y_7+y_8) =$

  $\frac{1}{3}\cdot\frac{1}{4}(f(2)+4f(2.25)+2f(2.5)+\cdots+2f(3.5)+4f(3.75)+f(4)) = \frac{1}{12}(\sqrt{1+2^2}+4\sqrt{1+2.25^2}+2\sqrt{1+2.5^2}+\cdots+2\sqrt{1+3.5^2}+4\sqrt{1+3.75^2}+\sqrt{1+4^2}) \approx 10.74;$

Applying Simpson's Rule

- Consider the integral $\int_1^3 \frac{1}{x} dx$;
  - Calculate S_8;
    $S_8 = \frac{1}{3}\frac{1}{4}(f(1)+4f(1.25)+2f(1.5)+ \cdots + 2f(2.5) + 4f(2.75) + f(3)) = \frac{1}{12}(\frac{1}{1} + \frac{4}{1.25} + \frac{2}{1.5} + \cdots + \frac{2}{2.5} + \frac{4}{2.75} + \frac{1}{3}) \approx 1.099$;



  - Find a bound for Error(S_8);
    Compute carefully: $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{6}{x^4}$, $f^{(4)}(x) = \frac{24}{x^5}$; Thus, on $[1,3]$, we have $|f^{(4)}(x)| \leq 24 = K_4$;
    $$\text{Error}(S_8) \leq \frac{K_4(b-a)^5}{180N^4} = \frac{24 \cdot 2^5}{180 \cdot 8^4} \approx 0.001;$$