Calculus II

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LSSU Math 152

1. Further Applications of the Integral and Taylor Polynomials
   - Arc Length and Surface Area
   - Fluid Pressure and Force
   - Center of Mass
   - Taylor Polynomials

Subsection 1

Arc Length and Surface Area

Arc Length Through Polygonal Approximation

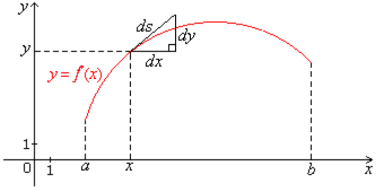
- We develop a formula for computing the arc length s of a given curve $y = f(x)$ from $x = a$ to $x = b$;
- The length of a small segment ds may be approximated by the length of the hypothenuse:

  $ds^2 = dx^2 + dy^2$

  $\Rightarrow ds^2 = (1 + (\frac{dy}{dx})^2)dx^2$

  $\Rightarrow ds = \sqrt{1 + [f'(x)]^2}dx;$

  Now, we integrate from $x = a$ to $x = b$ to obtain the entire length s:

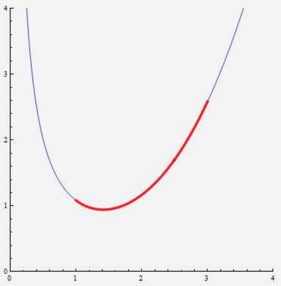


$$s = \int_a^b \sqrt{1 + [f'(x)]^2}dx;$$

Example I

- Find the arc length s of the graph $f(x) = \frac{1}{12}x^3 + x^{-1}$ over $[1,3]$;

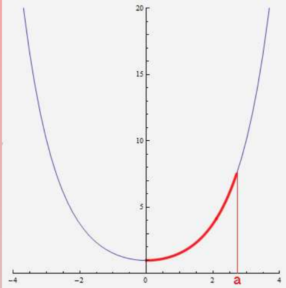
$$f'(x) = \frac{1}{4}x^2 - \frac{1}{x^2};$$
$$\sqrt{1+[f'(x)]^2} = \sqrt{1+(\frac{1}{4}x^2 - \frac{1}{x^2})^2} =$$
$$\sqrt{1+(\frac{1}{4}x^2)^2 - 2\cdot\frac{1}{4}x^2\cdot\frac{1}{x^2}+(\frac{1}{x^2})^2} =$$
$$\sqrt{(\frac{1}{4}x^2)^2 + 2\cdot\frac{1}{4}x^2\cdot\frac{1}{x^2}+(\frac{1}{x^2})^2} =$$
$$\sqrt{(\frac{1}{4}x^2+\frac{1}{x^2})^2} = \frac{1}{4}x^2 + \frac{1}{x^2};$$



$$s = \int_1^3 \sqrt{1+[f'(x)]^2}dx = \int_1^3 (\frac{1}{4}x^2 + \frac{1}{x^2})dx =$$
$$(\frac{1}{12}x^3 - \frac{1}{x})\Big|_1^3 = (\frac{9}{4} - \frac{1}{3}) - (\frac{1}{12} - 1) = \frac{17}{6};$$

Example II

- Find the arc length s of the graph $f(x) = \cosh x$ over $[0, a]$;



$$f'(x) = \sinh x;$$

$$\begin{aligned}\sqrt{1+[f'(x)]^2} &= \sqrt{1+\sinh^2 x}\\ &= \sqrt{\cosh^2 x}\\ &= \cosh x;\end{aligned}$$

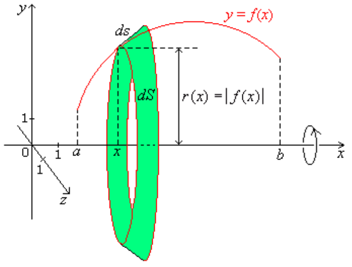
$$s = \int_0^a \sqrt{1+[f'(x)]^2}dx = \int_0^a \cosh x dx = \sinh x|_0^a = \sinh a - \sinh 0 = \sinh a;$$

Area of Surface of Revolution

- We develop a formula for the surface area S of the surface obtained by rotating the graph of $y = f(x)$ along the x-axis from $x = a$ to $x = b$;
- The amount of surface area dS of a thin truncated cone at x, with thickness (height) dx is

$$dS = 2\pi r(x) \cdot ds = 2\pi f(x) \cdot \sqrt{1 + [f'(x)]^2} dx;$$

Now, we integrate from $x = a$ to $x = b$ to obtain the entire area S:

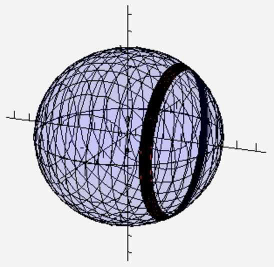


$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} dx;$$

Example I

- Find the surface area S of a sphere of radius R;

$$\begin{aligned} f(x) &= \sqrt{R^2-x^2}; \\ f'(x) &= -\frac{x}{\sqrt{R^2-x^2}}; \end{aligned}$$

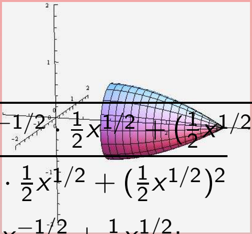


$$\begin{aligned} \sqrt{1+[f'(x)]^2} &= \sqrt{1+\frac{x^2}{R^2-x^2}} \\ &= \sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}} \\ &= \frac{R}{\sqrt{R^2-x^2}}; \end{aligned}$$

$$S = 2\pi\int_{-R}^{R} f(x)\sqrt{1+[f'(x)]^2}dx = 2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\frac{R}{\sqrt{R^2-x^2}}dx =$$
$$2\pi R\, x|_{-R}^{R} = 2\pi R\cdot 2R = 4\pi R^2;$$

Example II

- Find the surface area S of the surface obtained by rotating the graph of $f(x) = x^{1/2} - \frac{1}{3}x^{3/2}$ about the x-axis for $1 \leq x \leq 3$;



$$\begin{aligned} f(x) &= x^{1/2} - \tfrac{1}{3}x^{3/2}; \\ f'(x) &= \tfrac{1}{2}x^{-1/2} - \tfrac{1}{2}x^{1/2}; \end{aligned}$$

$$\begin{aligned} \sqrt{1 + [f'(x)]^2} &= \sqrt{1 + (\tfrac{1}{2}x^{-1/2})^2 - 2 \cdot \tfrac{1}{2}x^{-1/2} \cdot \tfrac{1}{2}x^{1/2} + (\tfrac{1}{2}x^{1/2})^2} \\ &= \sqrt{(\tfrac{1}{2}x^{-1/2})^2 + 2 \cdot \tfrac{1}{2}x^{-1/2} \cdot \tfrac{1}{2}x^{1/2} + (\tfrac{1}{2}x^{1/2})^2} \\ &= \sqrt{(\tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}x^{1/2})^2} = \tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}x^{1/2}; \end{aligned}$$

$$S = 2\pi \int_1^3 f(x)\sqrt{1 + [f'(x)]^2}dx =$$

$$2\pi \int_1^3 (x^{1/2} - \tfrac{1}{3}x^{3/2})(\tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}x^{1/2})dx =$$

$$2\pi \int_1^3 [\tfrac{1}{2} + \tfrac{1}{3}x - \tfrac{1}{6}x^2]dx = 2\pi \left(\tfrac{1}{2}x + \tfrac{1}{6}x^2 - \tfrac{1}{18}x^3\right)\Big|_1^3 = \tfrac{16\pi}{9};$$

Subsection 2

Fluid Pressure and Force

Fluid Pressure

Fluid Pressure

The pressure p at depth h in a fluid of mass density ρ is

$$p = \rho g h;$$

At each point of a certain object, pressure acts perpendicularly to the object's surface at that point;

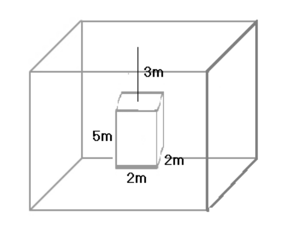
- If the pressure is constant throughout an entire surface of area A, then the total force exerted on the surface is

$$\text{Force} = \text{pressure} \times \text{area} = pA;$$

Example I

- Calculate the fluid force on the top and bottom of a box of dimensions $2 \times 2 \times 5$ m, submerged in a pool of water with its top 3 m below the water surface, given that density of water is $\rho = 10^3$ Kg/m^3;

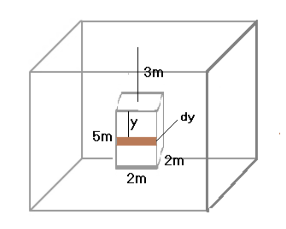
  The pressure p_t on top is $p_t = \rho g h_t = 10^3 \cdot 9.8 \cdot 3 = 29,400$ Pascals; Therefore, the downward force at the top is given by $F_t = p_t A_t = 29,400 \cdot 4 = 117,600$ Newtons;



  The pressure p_b on the bottom is $p_b = \rho g h_b = 10^3 \cdot 9.8 \cdot 8 = 78,400$ Pascals; Therefore, the upward force on the bottom is given by $F_b = p_b A_b = 78,400 \cdot 4 = 313,600$ Newtons;

Example II

- Calculate the fluid force on the side of the same box;
  The previous method cannot be applied since pressure varies with depth!
  We first compute the elementary pressure $p(y)$ on a narrow strip of thickness dy at (almost constant) depth y from the top of the box; $p(y) = \rho g(3+y)$; Then, the elementary force exerted on that narrow strip is $dF = p(y)dA = \rho g(3+y)2dy$;



  Now, we sum over all those elementary forces due to pressure by integrating:

$$F = \int_0^5 \rho g(3+y)2dy = 2\rho g \int_0^5 (3+y)dy =$$
$$2\rho g \left.\left(3y + \frac{1}{2}y^2\right)\right|_0^5 = 55\rho g = 55 \cdot 10^3 \cdot 9.8 = 539,000 \text{ Newtons};$$

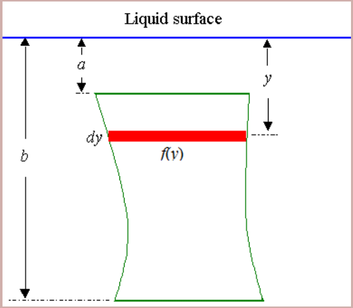
An Important Theorem

Fluid Force on Flat Surface Submerged Vertically

The force F on a flat side of an object submerged vertically in a fluid is

$$F = \rho g \int_a^b y f(y) dy,$$

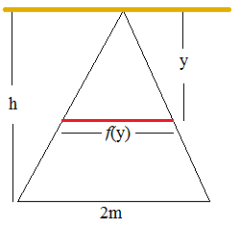
where $f(y)$ is the horizontal width of the side at depth y and the object extends from depth $y = a$ to depth $y = b$;



Example I

- What is the force F on one side of an equilateral triangular plate of side 2 m submerged vertically in a tank of oil of density $\rho = 900$ Kg/m^3?

Note that the height is $h^2 = 2^2 - 1^2 = 3$, i.e., $h = \sqrt{3}$; Thus, using similar triangles, we get $\frac{f(y)}{y} = \frac{2}{\sqrt{3}} \Rightarrow f(y) = \frac{2\sqrt{3}}{3}y$;

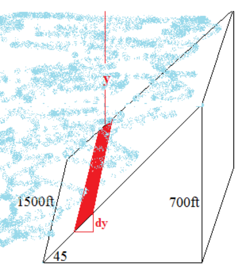


$$F = \rho g \int_0^{\sqrt{3}} y f(y) dy = \rho g \int_0^{\sqrt{3}} y \frac{2\sqrt{3}}{3} y dy = \frac{2\sqrt{3}}{3} \rho g \int_0^{\sqrt{3}} y^2 dy = \frac{2\sqrt{3}}{3} \rho g \left. \frac{y^3}{3} \right|_0^{\sqrt{3}} = 2\rho g = 2 \cdot 900 \cdot 9.8 = 17,640 \text{ Newtons;}$$

Example II

- What is the force F on the dam that is inclined at 45°, has height 700 ft and width 1500 ft, assuming that the reservoir is full of water whose density is 62.5 lb/ft^3?

A narrow strip at depth y from the top, whose vertical thickness is dy has area $dA = 1500\sqrt{2}dy$; Moreover, the product $\rho \cdot g$ gives the weight per unit volume of the fluid, which is $\rho g = 62.5$ lb/ft^3; Therefore, we get



$$F = \rho g \int_0^{700} y dA = \rho g \int_0^{700} y 1500\sqrt{2} dy = 1500\sqrt{2}\rho g \int_0^{700} y dy =$$

$$1500\sqrt{2} \cdot 62.5 \left. \frac{y^2}{2} \right|_0^{700} = 1500\sqrt{2} \cdot 62.5 \cdot \frac{700^2}{2} \approx 3.25 \times 10^{10} \text{ lb};$$

Subsection 3

Center of Mass

Moments and Center of Mass

- The **moments** of a system of n particles with coordinates (x_j, y_j) and mass m_j is the sum

$$\begin{aligned} M_x &= m_1 y_1 + m_2 y_2 + \cdots + m_n y_n; \\ M_y &= m_1 x_1 + m_2 x_2 + \cdots + m_n x_n; \end{aligned}$$

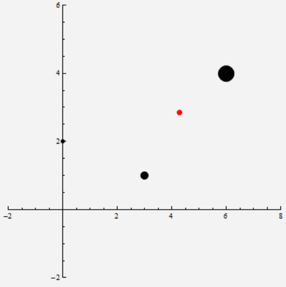
- The **center of mass** of the system is the point $(x_{\mathrm{CM}}, y_{\mathrm{CM}})$, with coordinates

$$x_{\mathrm{CM}} = \frac{M_y}{M}, \qquad y_{\mathrm{CM}} = \frac{M_x}{M},$$

where $M = m_1 + m_2 + \cdots + m_n$ is the total mass;

Example

- The center of mass of a system of three particles located at $(0,2),(3,1)$ and $(6,4)$ and having masses $2,4$ and 8 is found as follows:



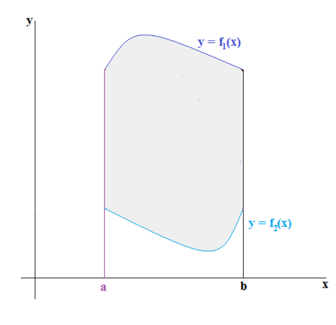
$$M = 2+4+8 = 14;$$
$$M_x = 2\cdot 2+4\cdot 1+8\cdot 4 = 40;$$
$$M_y = 2\cdot 0+4\cdot 3+8\cdot 6 = 60;$$

Thus, $x_{\text{CM}} = \frac{60}{14} = \frac{30}{7}$ and $y_{\text{CM}} = \frac{40}{14} = \frac{20}{7}$;

Moments of Laminas (Thin Plates) I

- Consider a lamina of constant mass density ρ occupying the region under the graph of $y = f(x)$ over $[a, b]$, where f is continuous with $f(x) \geq 0$ on $[a, b]$; Then, the y-**moment** of the lamina is given by

$$M_y = \rho \int_a^b x f(x) dx;$$



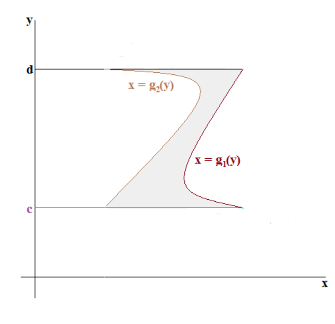
- If the lamina occupies the region between the graphs of $y = f_1(x)$ and $y = f_2(x)$ over $[a, b]$, with $f_1(x) \geq f_2(x)$, then

$$M_y = \rho \int_a^b x[f_1(x) - f_2(x)] dx;$$

Moments of Laminas (Thin Plates) II

- If the lamina occupies the region between the graphs of $x = g_1(y)$ and $x = g_2(y)$ over $[c, d]$, with $g_1(y) \geq g_2(y)$, then

$$M_x = \rho \int_c^d y[g_1(y) - g_2(y)]dy;$$



- The total mass of the lamina is
$$M = \rho A = \rho \int_a^b (f_1(x) - f_2(x))dx \text{ or } \rho \int_c^d (g_1(y) - g_2(y))dy;$$
- Finally, its center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}})$, with $x_{\mathrm{CM}} = \dfrac{M_y}{M}$ and $y_{\mathrm{CM}} = \dfrac{M_x}{M}$;

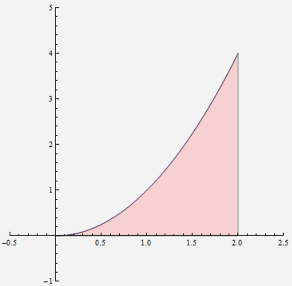
Example

Find the moments and center of mass of the lamina of uniform density ρ occupying the region under $y = x^2$, for $0 \leq x \leq 2$;

$$M = \rho \int_1^2 x^2 dx = \rho \left. \frac{x^3}{3} \right|_0^2 = \frac{8\rho}{3};$$

$$M_y = \rho \int_0^2 x f(x) dx = \rho \int_0^2 x^3 dx =$$

$$\rho \left. \frac{x^4}{4} \right|_0^2 = 4\rho;$$



$$M_x = \rho \int_0^4 y[2 - \sqrt{y}] dy = \rho \int_0^4 (2y - y^{3/2}) dy =$$

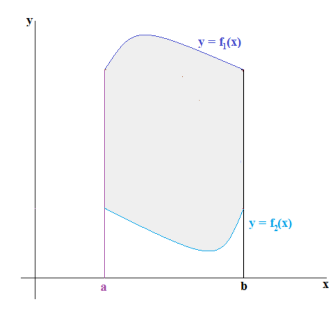
$$\rho \left(y^2 - \tfrac{2}{5} y^{5/2} \right) \Big|_0^4 = \rho(16 - \tfrac{2}{5} \cdot 32) = \tfrac{16\rho}{5};$$

$$x_{\mathrm{CM}} = \frac{M_y}{M} = \frac{4\rho}{8\rho/3} = \frac{3}{2}, \qquad y_{\mathrm{CM}} = \frac{M_x}{M} = \frac{16\rho/5}{8\rho/3} = \frac{6}{5};$$

Alternative Formula for M_x

- The formula $M_x = \rho \int_c^d y[g_1(y) - g_2(y)]dy$ requires expressing the boundaries as functions of x in terms of y;
- If that is not possible, we can still determine the x-moment;
- If the region has boundaries $f_1(x)$ and $f_2(x)$ over $[a, b]$, with $f_1(x) \geq f_2(x)$, then the x-moment is given by

$$M_x = \frac{1}{2}\rho \int_a^b (f_1(x)^2 - f_2(x)^2)dx;$$



Example

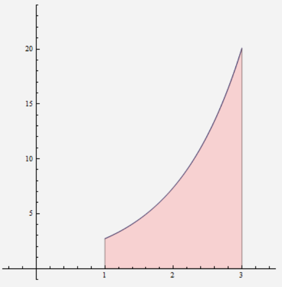
Find the moments and center of mass of the lamina of uniform density ρ occupying the region lying under $y = e^x$ for $1 \leq x \leq 3$;

$$M = \rho \int_1^3 e^x dx = \rho\ e^x|_1^3 = \rho(e^3 - e);$$

$$M_y = \rho \int_1^3 xf(x)dx = \rho \int_1^3 xe^x dx =$$

$$\rho \int_1^3 x(e^x)' dx = \rho[xe^x|_1^3 - \int_1^3 e^x dx] =$$

$$\rho[xe^x|_1^3 - e^x|_1^3] = 2\rho e^3;$$



$$M_x = \frac{1}{2}\rho \int_1^3 f(x)^2 dx = \frac{1}{2}\rho \int_1^3 e^{2x} dx = \frac{1}{2}\rho\ \frac{e^{2x}}{2}\Big|_1^3 = \frac{1}{2}\rho(\frac{1}{2}e^6 - \frac{1}{2}e^2) =$$

$$\frac{1}{4}\rho e^2(e^4 - 1);$$

$$x_{\rm CM} = \frac{M_y}{M} = \frac{2\rho e^3}{\rho e(e^2-1)} = \frac{2e^2}{e^2-1}, \qquad y_{\rm CM} = \frac{M_x}{M} = \frac{\frac{1}{4}\rho e^2(e^4-1)}{\rho e(e^2-1)} = \frac{e(e^2+1)}{4};$$

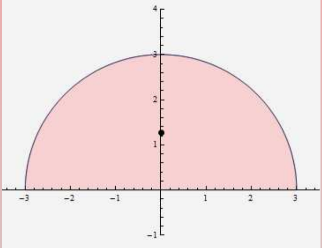
Using Symmetry

Symmetry Principle

If a lamina is symmetric with respect to a line, then its centroid lies on that line.

Example: Find the centroid of a semicircle of radius 3;

$$M = \rho\frac{\pi 3^2}{2} = \frac{9\rho\pi}{2};$$
$$M_y = 0;$$
$$M_x = \frac{1}{2}\rho\int_{-3}^{3}(\sqrt{9-x^2})^2 dx =$$
$$\frac{1}{2}\rho\int_{-3}^{3}(9-x^2)dx =$$
$$\frac{1}{2}\rho\left(9x - \tfrac{1}{3}x^3\right)\Big|_{-3}^{3} = \frac{1}{2}\rho[(27-9)-(-27+9)] = 18\rho$$
$$x_{\mathrm{CM}} = \frac{M_y}{M} = 0, \qquad y_{\mathrm{CM}} = \frac{M_x}{M} = \frac{18\rho}{9\rho\pi/2} = \frac{4}{\pi};$$



Subsection 4

Taylor Polynomials

Taylor Polynomials

- Recall that the **linearization** $L(x) = f(a) + f'(a)(x-a)$ **of a function** $f(x)$ **near** a is a linear function, such that $L(x) \approx f(x)$ for values of x close to a;
- In fact, this is a special case for $n = 1$ of the n**-th Taylor polynomial of** $f(x)$ **centered at** a:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n;$$

Taylor Theorem

The polynomial $T_n(x)$ centered at a agrees with $f(x)$ and all its derivatives up to order n at $x = a$ and it is the only polynomial of degree at most n having this property.

- Sometimes we use the notation $T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j$;
- Moreover, note that $T_n(x) = T_{n-1}(x) + \frac{f^{(n)}(a)}{n!}(x-a)^n$;

Maclaurin Polynomials

- The n-**th Maclaurin polynomial** is the special case of the n-th Taylor polynomial centered at $a = 0$:

$$T_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n;$$

- Example: Find the n-th Maclaurin polynomial for $f(x) = e^x$;
We have

$$\begin{array}{l} f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \ldots, f^{(n)}(x) = e^x; \\ f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \ldots, f^{(n)}(0) = 1; \end{array}$$

Therefore,

$$\begin{aligned} T_n(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n; \end{aligned}$$

Computing a Taylor Polynomial

- Compute the Taylor polynomial $T_4(x)$ centered at $a = 3$ for $f(x) = \sqrt{x+1}$;

$$\begin{aligned}
f(x) &= (x+1)^{1/2} \quad \Rightarrow \quad f(3) = 2 \\
f'(x) &= \tfrac{1}{2}(x+1)^{-1/2} \quad \Rightarrow \quad f'(3) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} \\
f''(x) &= -\tfrac{1}{4}(x+1)^{-3/2} \quad \Rightarrow \quad f''(3) = -\tfrac{1}{4} \cdot \tfrac{1}{8} = -\tfrac{1}{32} \\
f'''(x) &= \tfrac{3}{8}(x+1)^{-5/2} \quad \Rightarrow \quad f'''(3) = \tfrac{3}{8} \cdot \tfrac{1}{32} = \tfrac{3}{256} \\
f^{(4)}(x) &= -\tfrac{15}{16}(x+1)^{-7/2} \quad \Rightarrow \quad f^{(4)}(3) = -\tfrac{15}{16} \cdot \tfrac{1}{128} = -\tfrac{15}{2048}
\end{aligned}$$

$$T_4(x) =$$

$$f(3) + \frac{f'(3)}{1!}(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + \frac{f^{(4)}}{4!}(x-3)^4$$

$$= 2 + \tfrac{1}{4}(x-3) - \tfrac{1}{32 \cdot 2!}(x-3)^2 + \tfrac{3}{256 \cdot 3!}(x-3)^3 - \tfrac{15}{2048 \cdot 4!}(x-3)^4;$$

General Formula for $T_n(x)$

- Find $T_n(x)$ for $f(x) = \ln x$ centered at $a = 1$;

$$\begin{aligned}
f(x) &= \ln x \quad \Rightarrow \quad f(1) = 0;\\
f'(x) &= \tfrac{1}{x} \quad \Rightarrow \quad f'(1) = 1;\\
f''(x) &= -\tfrac{1}{x^2} \quad \Rightarrow \quad f''(1) = -1;\\
f'''(x) &= \tfrac{1\cdot 2}{x^3} \quad \Rightarrow \quad f'''(1) = 1\cdot 2;\\
f^{(4)}(x) &= -\tfrac{1\cdot 2\cdot 3}{x^4} \quad \Rightarrow \quad f^{(4)}(1) = -3!;\\
f^{(5)}(x) &= \tfrac{1\cdot 2\cdot 3\cdot 4}{x^5} \quad \Rightarrow \quad f^{(5)}(1) = 4!;\\
&\vdots\\
f^{(n)}(x) &= (-1)^{n-1}\tfrac{(n-1)!}{x^n} \quad \Rightarrow \quad f^{(n)}(1) = (-1)^{n-1}(n-1)!;
\end{aligned}$$

$$\begin{aligned}
T_n(x) &= \sum_{j=1}^{n} \frac{f^{(j)}(1)}{j!}(x-1)^j = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j}(x-1)^j\\
&= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 + \cdots + \tfrac{(-1)^{n-1}}{n}(x-1)^n;
\end{aligned}$$

Maclaurin Series for Cosine

- Find $T_n(x)$ for $f(x) = \cos x$ centered at $a = 0$;

$$
\begin{aligned}
f(x) &= \cos x \quad \Rightarrow \quad f(0) = 1; \\
f'(x) &= -\sin x \quad \Rightarrow \quad f'(0) = 0; \\
f''(x) &= -\cos x \quad \Rightarrow \quad f''(0) = -1; \\
f'''(x) &= \sin x \quad \Rightarrow \quad f'''(0) = 0; \\
f^{(4)}(x) &= \cos x \quad \Rightarrow \quad f^{(4)}(0) = 1; \\
f^{(5)}(x) &= -\sin x \quad \Rightarrow \quad f^{(5)}(0) = 0; \\
&\vdots \\
f^{(2n)}(x) &= (-1)^n \cos x \quad \Rightarrow \quad f^{(2n)}(0) = (-1)^n;
\end{aligned}
$$

$$
T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{f^{(2j)}(0)}{(2j)!} x^{2j} = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j}{(2j)!} x^{2j}
$$

$$
= 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots ;
$$

Error Bound

Error Bound of Taylor Approximation

If $f^{(n+1)}(x)$ exists, is continuous and $|f^{(n+1)}(u)| \leq K$, for u between a and x, then

$$|f(x) - T_n(x)| \leq K\frac{|x-a|^{n+1}}{(n+1)!}$$

where $T_n(x)$ is n-th Taylor polynomial centered at $x = a$;

- Example: Recall that for $f(x) = \ln x$ around $a = 1$,

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n};$$

$$T_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n-1}\frac{1}{n}(x-1)^n;$$

Note that for $1 \leq x \leq 1.2$, we have $|f^{(4)}(x)| = |\frac{(-1)^3 \cdot 3!}{x^4}| = \frac{6}{x^4} \leq 6$;
So $K = 6$; Let us find

$$|\ln(1.2) - T_3(1.2)| \leq K\frac{|1.2-1|^4}{4!} = 6\frac{0.2^4}{4!} = 0.0004;$$