Calculus II

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LSSU Math 152

1. Introduction to Differential Equations
   - Solving Differential Equations
   - Models Involving $y' = k(y-b)$
   - The Logistic Equation
   - First-Order Linear Equations

Subsection 1

Solving Differential Equations

Differential Equations and Solutions

- A **differential equation** is one that involves a function $y = f(x)$ and its first or higher derivatives; An example is $y' = -2y$;
- A **solution** is a function that satisfies the given equation; For example, $y = Ce^{-2x}$ is a solution of the differential equation given above;
- Because it contains one or more unspecified constants, it is called a **general solution**;
- For each specific value of C we obtain a **particular solution**;
- The **order** of a differential equation is the order of the highest derivative appearing;
- A differential equation is **linear** if it is of form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x);$$

- Another example is $\frac{dy}{dt} = t$ with general solution $y = \frac{1}{2}t^2 + C$; Yet another example is $y'' + y = 0$ with general solution $y = A\sin x + B\cos x$;

Examples Regarding Order and Linearity

Diff. Equation	Order	Linear/Non-Linear
$x^2y' + e^xy = 4$	First-Order	Linear
$x(y')^2 = y + x$	First-Order	Non-Linear
$y'' = (\sin x)y'$	Second-Order	Linear
$y''' = x(\sin y)$	Third-Order	Non-Linear

Separation of Variables

- A simple differential equation of the form $y' = f(x)$ has as its general solution $y = \int f(x)dx$;
- A differential equation is called **separable** if it has the form

$$\frac{dy}{dx} = f(x)g(y);$$

  For instance $\frac{dy}{dx} = (\sin x)y$ is separable, but $\frac{dy}{dx} = x + y$ is not;
- Separable equations can be solved using the **method of separation of variables**:

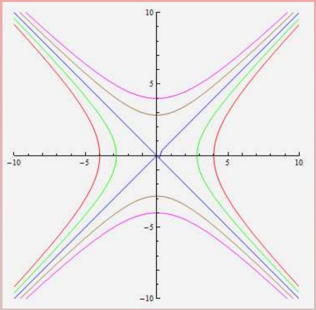
$$\frac{dy}{dx} = f(x)g(y) \;\Rightarrow\; \frac{1}{g(y)}dy = f(x)dx \;\Rightarrow\; \int \frac{1}{g(y)}dy = \int f(x)dx;$$

  Then, try to solve for y;

Applying Separation of Variables

- Find the general solution of the differential equation $y\frac{dy}{dx} - x = 0$;

$$\begin{aligned} & y\frac{dy}{dx} - x = 0 \\ \Rightarrow \quad & y\frac{dy}{dx} = x \\ \Rightarrow \quad & ydy = xdx \\ \Rightarrow \quad & \int ydy = \int xdx \\ \Rightarrow \quad & \frac{1}{2}y^2 = \frac{1}{2}x^2 + c \\ \Rightarrow \quad & y^2 = x^2 + C \qquad (C = 2c) \\ \Rightarrow \quad & y = \pm\sqrt{x^2 + C}; \end{aligned}$$



An Initial Value Problem

- Solve the **initial value problem**

$$y' = -ty, \quad y(0) = 3;$$

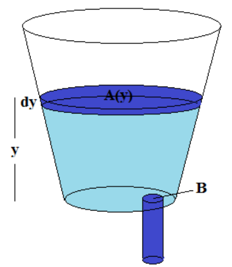
$$\begin{aligned}
\frac{dy}{dt} = -ty \ &\Rightarrow \ \frac{1}{y}dy = -tdt \\
&\Rightarrow \ \int \frac{1}{y}dy = \int -tdt \ \Rightarrow \ \ln|y| = -\tfrac{1}{2}t^2 + c \\
&\Rightarrow \ |y| = e^{-\frac{1}{2}t^2+c} = e^c e^{-\frac{1}{2}t^2} \\
&\Rightarrow \ y = \pm e^c e^{-\frac{1}{2}t^2} \\
&\Rightarrow \ y = Ce^{-\frac{1}{2}t^2};
\end{aligned}$$

Since $y(0) = 3$, we get $3 = Ce^0 = C$; Therefore, the particular solution of the initial value problem is

$$y(t) = 3e^{-\frac{1}{2}t^2};$$

Emptying Tank and Torricelli's Law

Consider a tank full of water that is slowly emptying through a hole of area B at its bottom; Suppose that at time t, the water level is y and $A(y)$ is the area of the horizontal cross section of the tank at height y; Let $v(y)$ be the velocity of the water flowing through the hole when the tank is filled to height y;



Then, the water lost between time t and time $t + dt$ is

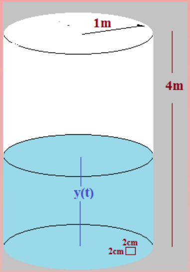
$$A(y)dy = Bv(y)dt \;\Rightarrow\; \frac{dy}{dt} = \frac{Bv(y)}{A(y)};$$

Torricelli's Law for Velocity $v(y)$

The velocity of the water leaving the tank when it is filled to height y is $v(y) = -\sqrt{2gy}$, where $g = 9.8$ m/sec^2;

Application: Torricelli's Law

A cylindrical tank of height 4 m and radius 1 m is filled with water; Water drains through a square hole of side 2 cm in the bottom; Determine the water level $y(t)$ at time t seconds;



$$\frac{dy}{dt} = \frac{Bv(y)}{A(y)} = -\frac{B\sqrt{2gy}}{A(y)}$$

$$\Rightarrow \frac{dy}{dt} = -\frac{(0.02)^2\sqrt{2 \cdot 9.8 \cdot y}}{\pi \cdot 1^2} = -\frac{4\sqrt{19.6} \cdot 10^{-4}}{\pi}\sqrt{y}$$

$$\Rightarrow \frac{dy}{dt} = K\sqrt{y}, \text{ where } K = -\frac{4\sqrt{19.6} \cdot 10^{-4}}{\pi};$$

$$\int \frac{1}{\sqrt{y}} dy = \int K dt \Rightarrow 2\sqrt{y} = Kt + c \Rightarrow \sqrt{y} = \frac{1}{2}Kt + C$$

$$\Rightarrow y(t) = (\tfrac{1}{2}Kt + C)^2;$$

Since $y(0) = 4$, we get $C = 2$; Therefore, $y(t) = (2 - \frac{2\sqrt{19.6} \cdot 10^{-4}}{\pi}t)^2$;

Subsection 2

Models Involving $y' = k(y - b)$

Rate of Change of y Proportional to Amount $y - b$

- Assume $\frac{dy}{dt} = k(y-b)$;

$$\begin{aligned}
&\frac{dy}{dt} = k(y-b) \\
&\quad \Rightarrow \ \frac{1}{y-b}dy = kdt \\
&\quad \Rightarrow \ \int \frac{1}{y-b}dy = \int kdt \\
&\quad \Rightarrow \ \ln(y-b) = kt + c \quad (\text{assuming } y \geq b) \\
&\quad \Rightarrow \ y - b = e^{kt+c} = e^c e^{kt} \\
&\quad \Rightarrow \ y = b + Ce^{kt};
\end{aligned}$$

Newton's Law of Cooling

Newton's Law of Cooling

The rate of cooling (change of temperature) $\frac{dy}{dt}$ is proportional to the temperature difference $y - T_0$ of y from the ambient temperature T_0: $\frac{dy}{dt} = -k(y - T_0)$;

Example: A hot metal bar with cooling constant $k = 2.1\ \text{min}^{-1}$ is submerged in a tank of water held at temperature $T_0 = 10°\text{C}$. Let $y(t)$ be the bar's temperature at time t minutes;

- Write a differential equation for y and find its general solution;

$$\frac{dy}{dt} = -2.1(y-10) \ \Rightarrow\ \frac{1}{y-10}dy = -2.1dt$$
$$\Rightarrow\ \int \frac{1}{y-10}dy = \int -2.1dt \ \Rightarrow\ \ln(y-10) = -2.1t + c$$
$$\Rightarrow\ y - 10 = e^{-2.1t+c} = e^c e^{-2.1t} \ \Rightarrow\ y = 10 + Ce^{-2.1t};$$

Newton's Law of Cooling (Cont'd)

We found that

$$y(t) = 10 + Ce^{-2.1t};$$

- What is $y(t)$ given that the initial temperature was 180°C?
  $y(0) = 180 \Rightarrow 10 + C \cdot 1 = 180 \Rightarrow C = 170$;
  Therefore, $y(t) = 10 + 170e^{-2.1t}$;
- What is $y(t)$ if the bar cooled to 80°C in 30 seconds? What was its initial temperature?
  $y(0.5) = 80 \Rightarrow 10 + Ce^{-2.1 \cdot 0.5} = 80 \Rightarrow Ce^{-1.05} = 70 \Rightarrow C = 70e^{1.05} \approx 200$;
  Therefore, $y(t) = 10 + 200e^{-2.1t}$;
  Thus, we get $y(0) = 10 + 200 \cdot e^{-2.1 \cdot 0} = 210^\circ$C.

Object in Free Fall

- The force F on an object of mass m in free fall with velocity $v(t)$ is
$$F = \underbrace{-mg}_{\text{gravity}} \underbrace{-kv}_{\text{friction}};$$
- By Newton's Law of Motion: $F = ma = m\frac{dv}{dt}$;
- So, we get
$$m\frac{dv}{dt} = -mg - kv \;\Rightarrow\; \frac{dv}{dt} = -\frac{k}{m}(v + \frac{mg}{k});$$
- If we set $K = \frac{k}{m}$ and $b = -\frac{mg}{k}$, we get
$$\frac{dv}{dt} = -K(v-b) \;\Rightarrow\; v = b + Ce^{-Kt};$$
- Thus, $$v(t) = -\frac{mg}{k} + Ce^{-\frac{k}{m}t}.$$

Skydiver in Free Fall

A 60 Kg skydiver steps out of an airplane; Her $k = 6$ Kg/sec;

- Find an equation for her velocity $v(t)$;

$$v(t) = -\frac{mg}{k} + Ce^{-\frac{k}{m}t} = -\frac{60 \cdot 9.8}{6} + Ce^{-\frac{6}{60}t} \Rightarrow v(t) = -98 + Ce^{-0.1t};$$

But $v(0) = 0$, whence $0 = -98 + C \Rightarrow C = 98$; Therefore

$$v(t) = 98e^{-0.1t} - 98;$$

- What is her terminal velocity?

$$v_{\mathrm{t}} = \lim_{t \to \infty} v(t) = \lim_{t \to \infty} (98e^{-0.1t} - 98) = -98 \text{ m/sec};$$

Annuities with Continuous Interest and Withdrawal

- Suppose $P(t)$ is the balance of an annuity which earns interest rate r compounded continuously and from which money is withdrawn continuously at rate N; Then

$$\underbrace{\frac{dP}{dt}}_{\text{rate of change}} = \underbrace{rP(t)}_{\text{growth rate}} - \underbrace{N;}_{\text{withdrawal rate}}$$

- Thus,

$$\frac{dP}{dt} = r(P - \frac{N}{r})$$
$$\Rightarrow \ P(t) = \frac{N}{r} + Ce^{rt};$$

Example: Annuities

- An annuity earns interest rate 0.07 and withdrawals are made continuously at a rate of \$500/year; If the initial deposit is \$5,000, when will the annuity run out of money?

$$P(t) = \frac{N}{r} + Ce^{rt} = \frac{5000}{0.07} + Ce^{0.07t} \approx 7143 + Ce^{0.07t};$$

Since $P(0) = 5000$, we get $5000 = 7143 + Ce^0 \Rightarrow C = -2143$; Thus,

$$P(t) = 7143 - 2143e^{0.07t}.$$

We set

$$P(t) = 0 \Rightarrow 2143e^{0.07t} = 7143$$
$$\Rightarrow e^{0.07t} = \tfrac{7143}{2143} \Rightarrow t = \tfrac{100}{7} \ln \tfrac{7143}{2143} \approx 17;$$

Subsection 3

The Logistic Equation

The Logistic Equation

- Population growth is sometimes modeled by the **logistic equation**

$$\frac{dy}{dt} = ky(1 - \frac{y}{A}), \quad A \text{ a constant } \textbf{capacity};$$

$$\begin{aligned}
&\frac{dy}{y(1-\frac{y}{A})} = kdt \ \Rightarrow \ \int\left(\frac{1}{y} - \frac{1}{y-A}\right)dy = \int kdt \\
&\Rightarrow \ \ln|y| - \ln|y-A| = kt + c \ \Rightarrow \ \ln\left|\frac{y}{y-A}\right| = kt + c \\
&\Rightarrow \ \left|\frac{y}{y-A}\right| = e^c e^{kt} \ \Rightarrow \ \frac{y}{y-A} = \pm e^c e^{kt} \\
&\Rightarrow \ \frac{y}{y-A} = Ce^{kt} \ \Rightarrow \ y = (y-A)Ce^{kt} \\
&\Rightarrow \ y(1 - Ce^{kt}) = -ACe^{kt} \ \Rightarrow \ y = \frac{ACe^{kt}}{Ce^{kt} - 1} = \frac{A}{1 - \frac{1}{C}e^{-kt}};
\end{aligned}$$

An Example of a Logistic Equation

- Solve $\frac{dy}{dt} = 0.3y(4-y)$ with initial condition $y(0) = 1$;

  Note that $\frac{dy}{dt} = 0.3y(4-y) \Rightarrow \frac{dy}{dt} = 1.2y(1-\frac{y}{4})$; Thus, $k = 1.2$ and $A = 4$; So the general solution is

  $$y(t) = \frac{A}{1-\frac{1}{C}e^{-kt}} = \frac{4}{1-\frac{1}{C}e^{-1.2t}};$$

  Since $y(0) = 1$, we get $1 = \frac{4}{1-\frac{1}{C}} \Rightarrow 1-\frac{1}{C} = 4 \Rightarrow C = -\frac{1}{3}$; Hence, the particular solution sought is

  $$y(t) = \frac{4}{1+3e^{-1.2t}};$$

An Application: Deer Population

- A deer population grows logistically with growth constant $k = 0.4$ year^{-1} in a forest with carrying capacity $A = 1000$ deer;
  - Find the population $P(t)$ if $P(0) = 100$;

$$P(t) = \frac{A}{1 - \frac{1}{C}e^{-kt}} = \frac{1000}{1 - \frac{1}{C}e^{-0.4t}};$$

Since $P(0) = 100$, we get $100 = \dfrac{1000}{1 - \frac{1}{C}} \Rightarrow 1 - \frac{1}{C} = 10 \Rightarrow C = -\frac{1}{9}$;

Therefore, $P(t) = \dfrac{1000}{1 + 9e^{-0.4t}}$;

  - How long does it take for the population to reach 500?

$$P(t) = 500 \;\Rightarrow\; \frac{1000}{1 + 9e^{-0.4t}} = 500 \;\Rightarrow\; 1 + 9e^{-0.4t} = 2$$
$$\Rightarrow\; e^{-0.4t} = \frac{1}{9} \;\Rightarrow\; -0.4t = \ln\frac{1}{9} \;\Rightarrow\; t = \frac{5}{2}\ln 9 \approx 5.5 \text{ years};$$

Subsection 4

First-Order Linear Equations

First-Order Linear Differential Equations

- A **first-order linear equation** has the form
$$a(x)y' + b(x)y = c(x), \quad a(x) \neq 0;$$
- By dividing by $a(x)$ we write it in **standard form**:
$$y' + A(x)y = B(x);$$
- To solve, multiply both sides by an **integrating factor** $e^{\int A(x)dx}$:
$$e^{\int A(x)dx}(y' + A(x)y) = e^{\int A(x)dx}B(x);$$
Note that, by the product rule,
$(e^{\int A(x)dx}y)' = (e^{\int A(x)dx})'y + e^{\int A(x)dx}y' = e^{\int A(x)dx}(\int A(x)dx)'y + e^{\int A(x)dx}y' = e^{\int A(x)dx}A(x)y + e^{\int A(x)dx}y' = e^{\int A(x)dx}(y' + A(x)y);$
Thus, we get
$$(e^{\int A(x)dx}y)' = e^{\int A(x)dx}B(x) \ \Rightarrow \ e^{\int A(x)dx}y = \int e^{\int A(x)dx}B(x)dx + C$$
$$\Rightarrow \ y = \frac{1}{e^{\int A(x)dx}}\left[\int e^{\int A(x)dx}B(x)dx + C\right];$$

Applying the Integrating Factor Method I

- Solve the linear differential equation

$$xy' - 3y = x^2, \quad y(1) = 2;$$

$$xy' - 3y = x^2 \ \Rightarrow \ y' - \frac{3}{x}y = x;$$
$$A(x) = -\frac{3}{x}, \quad B(x) = x;$$
$$\alpha(x) = e^{\int A(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln(x^{-3})} = \frac{1}{x^3};$$
$$y = \frac{1}{\alpha(x)}\left[\int \alpha(x)B(x)dx + C\right] = x^3\left[\int \frac{1}{x^3}xdx + C\right] =$$
$$x^3\left[\int x^{-2}dx + C\right] = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3;$$

Since $y(1) = 2$, $2 = -1^2 + C \cdot 1^3 \Rightarrow C = 3$;
Therefore, $y = -x^2 + 3x^3$;

Applying the Integrating Factor Method II

- Solve the linear differential equation

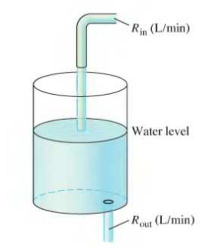
$$y' + (1 - x^{-1})y = x^2, \quad y(1) = 2;$$

$$A(x) = 1 - x^{-1}, \quad B(x) = x^2;$$
$$\alpha(x) = e^{\int (1-x^{-1})dx} = e^{x-\ln x} = \frac{e^x}{e^{\ln x}} = \tfrac{1}{x}e^x;$$
$$y = \frac{1}{\alpha(x)}\left[\int \alpha(x)B(x)dx + C\right] = xe^{-x}\left[\int \frac{1}{x}e^x x^2 dx + C\right] =$$
$$xe^{-x}\left[\int xe^x dx + C\right] \stackrel{\text{By-Parts}}{=} xe^{-x}(xe^x - e^x + C) =$$
$$x^2 - x + Cxe^{-x};$$

Since $y(1) = 2$, $2 = 1^2 - 1 + C \cdot 1 \cdot e^{-1} \Rightarrow C = 2e$;
Therefore, $y = x^2 - x + 2exe^{-x}$;

Application: Mixing

A tank contains 600 liters of water with a sucrose concentration of 0.2 kg/L. We begin adding water with concentration 0.1 kg/L at a rate of $R_{\text{in}} = 40$ L/min. The water mixes and exits the bottom of the tank at a rate of $R_{\text{out}} = 20$ L/min. If $y(t)$ is the quantity of sucrose in the tank at time t, set up a differential equation for $y(t)$ and solve it for $y(t)$;



$$\frac{dy}{dt} = \underbrace{(0.1 \text{ kg/L})(40 \text{ L/min})}_{\text{Rate In}} - \underbrace{\left(\frac{y}{600+20t} \text{ kg/L}\right)(20 \text{ L/min})}_{\text{Rate Out}}$$

Therefore, we get $\dfrac{dy}{dt} = 4 - \dfrac{y}{t+30} \Rightarrow \dfrac{dy}{dt} + \dfrac{1}{t+30}y = 4$, showing that we have a linear equation, with $A(t) = \dfrac{1}{t+30}$ and $B(t) = 4$;

Mixing (Cont'd)

$$\frac{dy}{dt} + \frac{1}{t+30}y = 4;$$

$$A(t) = \frac{1}{t+30}, \qquad B(t) = 4;$$

So $\alpha(t) = e^{\int \frac{1}{t+30}dt} = e^{\ln(t+30)} = t+30;$ Hence

$$\begin{aligned} y(t) &= \frac{1}{\alpha(t)}\left[\int \alpha(t)B(t)dt + C\right] \\ &= \frac{1}{t+30}\left[\int 4(t+30)dt + C\right] \\ &= \frac{1}{t+30}(2(t+30)^2 + C) = 2t + 60 + \frac{C}{t+30}; \end{aligned}$$

Since $y(0) = 120$, we get $120 = 60 + \frac{C}{30} \Rightarrow C = 1800;$

Therefore $y(t) = 2t + 60 + \dfrac{1800}{t+30};$