Calculus III

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LSSU Math 251

1. Parametric Equations, Polar Coordinates, Conic Sections
   - Parametric Curves
   - Calculus With Parametric Curves
   - Polar Coordinates
   - Area and Arc Length in Polar Coordinates
   - Conic Sections

Subsection 1

Parametric Curves

Parametric Equations and Parametric Curves

- A **system of parametric equations** has the form

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad a \le t \le b.$$

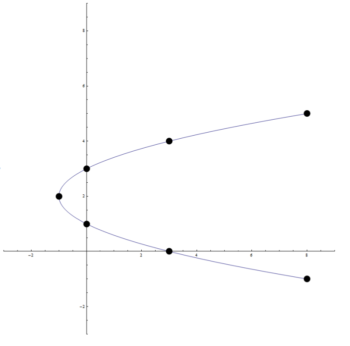
- The variable t is called the **parameter**.
- The set of points $(x, y) = (f(t), g(t))$, for $a \le t \le b$, is called the **parametric curve**.
- $(f(a), g(a))$ is the **initial point** and $(f(b), g(b))$ the **terminal point**.
- We imagine "traveling" along the parametric curve as the parameter t increases from a to b.

Example I

- Consider the parametric curve

$$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}, \quad -2 \leq t \leq 4.$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



Example I (Eliminating the Parameter)

- Consider again the curve

$$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}, \quad -2 \leq t \leq 4.$$

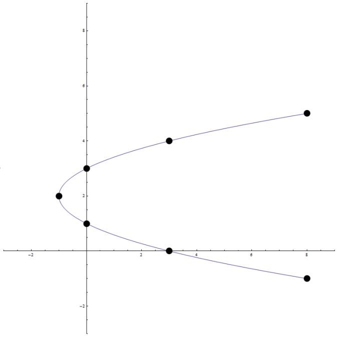
Since $t = y - 1$, we get

$$\begin{aligned} x &= (y-1)^2 - 2(y-1) \\ &= y^2 - 2y + 1 - 2y + 2 \\ &= y^2 - 4y + 3. \end{aligned}$$

The Cartesian representation

$$x = y^2 - 4y + 3$$

reminds us of a parabola opening "right".

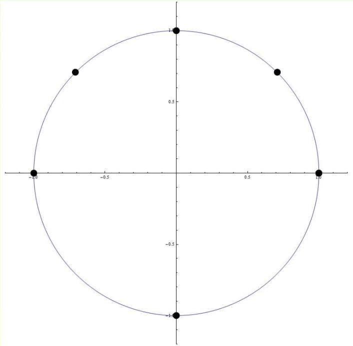


Example II

- Consider the parametric curve

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad 0 \leq t \leq 2\pi.$$

t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

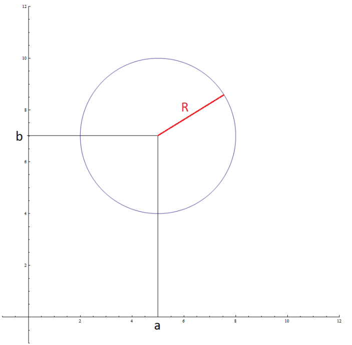


Note $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$.

Example III

- Consider the parametric curve

$$\begin{cases} x = a + R\cos t \\ y = b + R\sin t \end{cases}, \quad 0 \le t \le 2\pi.$$



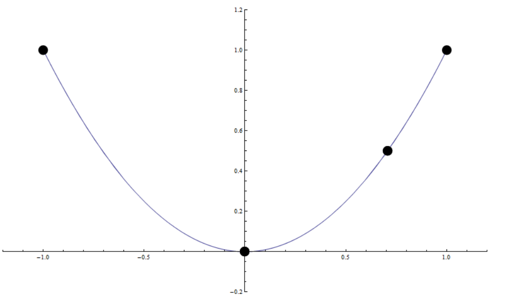
Note
$(x-a)^2 + (y-b)^2 = R^2\cos^2 t + R^2\sin^2 t = R^2(\cos^2 t + \sin^2 t) = R^2.$
This is the equation of a circle with center (a, b) and radius R.

Example IV

- Consider the parametric curve

$$\begin{cases} x = \sin t \\ y = \sin^2 t \end{cases}, \quad 0 \leq t \leq 2\pi.$$

t	x	y
0	0	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	1
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
π	0	0
$\frac{3\pi}{2}$	-1	1
2π	0	0

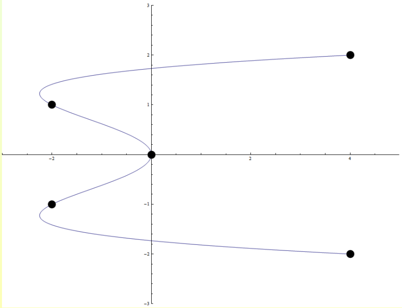


Graphing through Parametrization

- Consider the equation $x = y^4 - 3y^2$.
- It cannot be graphed directly using a calculator. (Why?)
- Introduce a parameter t and write:

$$\begin{cases} x = t^4 - 3t^2 \\ y = t \end{cases}, \quad \text{say } -2 \leq t \leq 2.$$

t	x	y
-2	4	-2
-1	-2	-1
0	0	0
1	-2	1
2	4	2

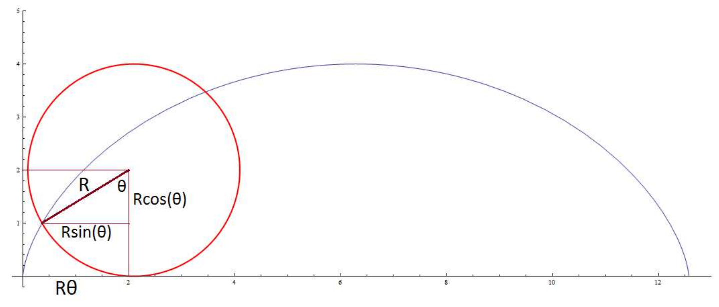


The Cycloid

- **Cycloid** is the curve traced by a point P on the circumference of a circle as the circle rolls along a straight line.

  In parametric form it is given by

$$\left\{ \begin{array}{l} x = R(\theta - \sin\theta) \\ y = R(1 - \cos\theta) \end{array} \right. \text{ with } 0 \leq \theta \leq 2\pi.$$



Subsection 2

Calculus With Parametric Curves

Slopes of Tangent Lines

- Consider the parametric curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}, a \leq t \leq b$.

  Applying the chain rule, we obtain $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$.

  Therefore, we get

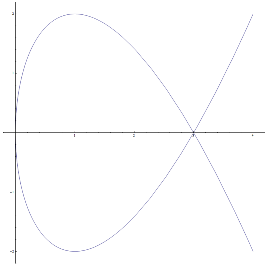
Derivative of Parametric System

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}, \quad \text{subject to } \frac{dx}{dt} \neq 0.$$

Example

- Consider the parametric curve

$$\begin{cases} x = t^2 \\ y = t^3 - 3t \end{cases}, \quad -2 \le t \le 2.$$



Find the equations of the tangent lines at $(x, y) = (3, 0)$.

We have $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t^2 - 3}{2t}$.

Note that $(3, 0)$ corresponds to $t = \pm\sqrt{3}$. Hence, the slope is $\dfrac{dy}{dx}\Big|_{t=\pm\sqrt{3}} = \frac{6}{\pm 2\sqrt{3}} = \pm\sqrt{3}$. Therefore, the tangent lines at $(3, 0)$ have equations

$$y = \sqrt{3}(x - 3) \quad \text{and} \quad y = -\sqrt{3}(x - 3).$$

Cycloid Revisited

- Recall the parametrization of the cycloid

$$\left\{\begin{array}{l} x = R(\theta - \sin\theta) \\ y = R(1 - \cos\theta) \end{array}\right., \ 0 \leq \theta \leq 2\pi.$$

Find its tangent line at $\theta = \frac{\pi}{3}$.

We have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{R\sin\theta}{R - R\cos\theta} = \frac{\sin\theta}{1-\cos\theta}.$$

Thus $\frac{dy}{dx}|_{\theta=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\sqrt{3}}{2-1} = \sqrt{3}$.

The tangent line has equation $y - \frac{R}{2} = \sqrt{3}(x - R(\frac{\pi}{3} - \frac{\sqrt{3}}{2}))$.

Cycloid (Cont'd)

- For the cycloid $\begin{cases} x = R(\theta - \sin\theta) \\ y = R(1 - \cos\theta) \end{cases}$, $0 \leq \theta \leq 2\pi$, show that at $\theta = 2k\pi$ (k any integer), the cycloid has a vertical tangent line.

  This requires showing that $\lim_{\theta \to 2k\pi} \frac{dy}{dx} = \pm\infty$.

  We have

$$\lim_{\theta \to 2k\pi} \frac{dy}{dx} = \lim_{\theta \to 2k\pi} \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \lim_{\theta \to 2k\pi} \frac{\sin\theta}{1 - \cos\theta} = (\frac{0}{0})$$

$$\overset{\text{L'Hôpital}}{=} \lim_{\theta \to 2k\pi} \frac{\cos\theta}{\sin\theta} = \pm\infty.$$

Areas under Parametric Curves

- Consider $x = f(t), y = g(t), a \le t \le b$.
  The area under this parametric curve from $f(a)$ to $f(b)$, assuming that it is "traveled" once, is given by
  $$A = \int_{f(a)}^{f(b)} y\ dx = \int_a^b g(t)\ df(t) = \int_a^b g(t) f'(t)\ dt.$$
  Example (The Area under the Cycloid):
  $x = R(\theta - \sin\theta), y = R(1 - \cos\theta), \quad 0 \le \theta \le 2\pi$.
  $$\begin{aligned} A &= \int_0^{2\pi} y(\theta) x'(\theta)\ d\theta = \int_0^{2\pi} R(1-\cos\theta)(R - R\cos\theta)\ d\theta \\ &= R^2 \int_0^{2\pi} (1-\cos\theta)^2\ d\theta = R^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\ d\theta \\ &= R^2 \int_0^{2\pi} (1 - 2\cos\theta + \tfrac{1+\cos 2\theta}{2})\ d\theta \\ &= R^2 \left[\theta - 2\sin\theta + \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)\right]_0^{2\pi} = 3\pi R^2. \end{aligned}$$

Arc Lengths of Parametric Curves

- Consider $x = f(t), y = g(t), a \leq t \leq b$.

  The length of this parametric curve from $f(a)$ to $f(b)$, assuming that it is "traveled" once, is given by

$$\begin{aligned} L &= \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt}\, dt \\ &= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt. \end{aligned}$$

  Example (The Length of a Circle of Radius R):
  $x = R\cos t, y = R\sin t,\ 0 \leq t \leq 2\pi$.

$$L = \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2}\, dt = \int_0^{2\pi} R\, dt = Rt|_0^{2\pi} = 2\pi R.$$

Arc Length of the Cycloid

- $x = R(\theta - \sin\theta), y = R(1 - \cos\theta),\ 0 \leq \theta \leq 2\pi.$

  We have

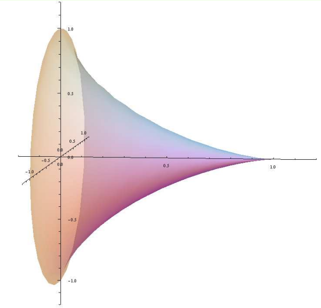
$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta \\ &= \int_0^{2\pi} \sqrt{(R - R\cos\theta)^2 + (R\sin\theta)^2}\ d\theta \\ &= \int_0^{2\pi} \sqrt{R^2 - 2R^2\cos\theta + R^2\cos^2\theta + R^2\sin^2\theta}\ d\theta \\ &= \int_0^{2\pi} \sqrt{2R^2(1 - \cos\theta)}\ d\theta = \int_0^{2\pi} \sqrt{2R^2 2\sin^2\frac{\theta}{2}}\ d\theta \\ &= 2R\int_0^{2\pi} \sin\frac{\theta}{2}\ d\theta = 2R\left(-2\cos\frac{\theta}{2}\right)\Big|_0^{2\pi} = 8R. \end{aligned}$$

Surface Area

- Let $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$, where $g(t) > 0$, $f(t)$ is increasing, and $f'(t)$ and $g'(t)$ are continuous.

  Then the surface obtained by rotating the curve $c(t) = (f(t), g(t))$ about the x-axis for $a \leq t \leq b$ has surface area

$$\begin{aligned} S &= 2\pi \int_a^b y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt \\ &= 2\pi \int_a^b g(t) \sqrt{f'(t)^2 + g'(t)^2} dt. \end{aligned}$$



Computing a Surface Area

- Calculate the surface area of the surface obtained by rotating $\left\{\begin{array}{l} x=\cos^3\theta \\ y=\sin^3\theta \end{array}\right.$, $0\leq\theta\leq\frac{\pi}{2}$, about the x-axis.

$$
\begin{aligned}
S &= 2\pi\int_0^{\pi/2} y\sqrt{(\tfrac{dx}{d\theta})^2+(\tfrac{dy}{d\theta})^2}d\theta \\
&= 2\pi\int_0^{\pi/2}\sin^3\theta\sqrt{(-3\cos^2\theta\sin\theta)^2+(3\sin^2\theta\cos\theta)^2}d\theta \\
&= 2\pi\int_0^{\pi/2}\sin^3\theta\sqrt{9\cos^4\theta\sin^2\theta+9\sin^4\theta\cos^2\theta}d\theta \\
&= 2\pi\int_0^{\pi/2}\sin^3\theta\sqrt{9\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)}d\theta \\
&= 2\pi\int_0^{\pi/2}\sin^3\theta 3\cos\theta\sin\theta d\theta \\
&= 6\pi\int_0^{\pi/2}\sin^4\theta\cos\theta d\theta \\
&\overset{u=\sin\theta}{=} 6\pi\int_0^1 u^4du \\
&= 6\pi\frac{u^5}{5}\Big|_0^1=\frac{6\pi}{5}.
\end{aligned}
$$

Summary

- Parametric curve $\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \ a \leq t \leq b.$

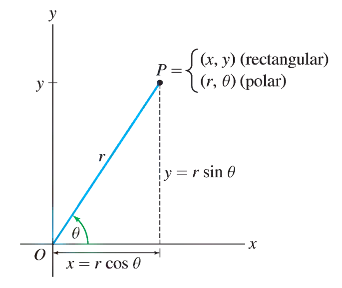
  Slope of Tangent:
  $$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$
  Area Under the Parametric Curve:
  $$A = \int_a^b g(t) f'(t) dt.$$
  Arc Length of the Parametric Curve:
  $$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2} dt.$$
  Surface of the Solid of Revolution:
  $$S = 2\pi \int_a^b g(t) \sqrt{f'(t)^2 + g'(t)^2} dt.$$

Subsection 3

Polar Coordinates

Polar Coordinates

- In **polar coordinates**, we label a point P by coordinates (r, θ), where:
  - r is the distance to the origin O;
  - θ is the angle between $\overline{OP}$ and the positive x-axis.



- An angle is positive if the corresponding rotation is counterclockwise.
- We call r the **radial coordinate** and θ the **angular coordinate**.

Polar Coordinates and Rectangular Coordinates

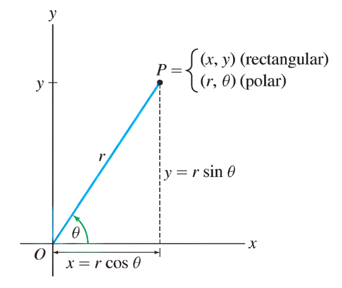
- The figure shows that polar and rectangular coordinates are related by the equations:

From Polar to Rectangular:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

From Rectangular to Polar:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, x \neq 0.$$

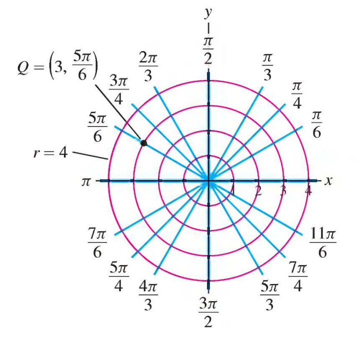


From Polar To Rectangular

- Find the rectangular coordinates of $Q = (r, \theta) = (3, \frac{5\pi}{6})$.

$$\begin{aligned} x &= r\cos\theta = 3\cos\left(\frac{5\pi}{6}\right) \\ &= 3\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}. \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta = 3\sin\left(\frac{5\pi}{6}\right) \\ &= 3 \cdot \frac{1}{2} = \frac{3}{2}. \end{aligned}$$



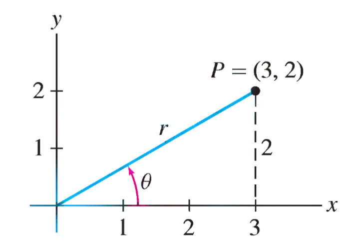
From Rectangular to Polar

- Find the polar coordinates of the point $P = (x, y) = (3, 2)$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 3^2 + 2^2 = 13. \end{aligned}$$

So $r = \sqrt{13}$.

$$\tan\theta = \frac{y}{x} = \frac{2}{3}.$$

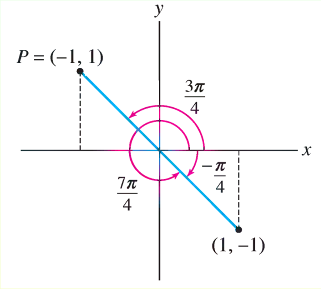


Since P is in Quadrant I, $\theta = \tan^{-1}(\frac{2}{3})$.

Choosing θ Correctly

- Find two polar representations of $P = (-1, 1)$, one with $r > 0$ and one with $r < 0$.

  We have
  $$r^2 = (-1)^2 + 1^2 = 2.$$
  So $r = \sqrt{2}$. Moreover,
  $$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1.$$



  However, $\theta \neq \tan^{-1}(-1) = -\frac{\pi}{4}$, because P is in Quadrant II. The correct angle is $\theta = \tan^{-1}(\frac{y}{x}) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.
  So, with $r > 0$, we have $P = (\sqrt{2}, \frac{3\pi}{4})$.
  With $r < 0$, we have $P = (-\sqrt{2}, -\frac{\pi}{4}) = (-\sqrt{2}, \frac{7\pi}{4})$.

Line Through the Origin

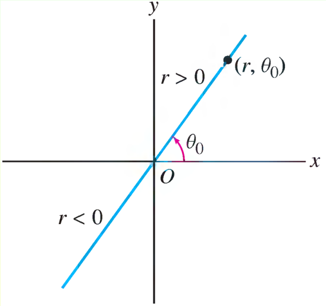
- Find the polar equation of the line through the origin of slope $\sqrt{3}$.

  We find the angle θ_0, such that

  $$\tan\theta_0 = \frac{y}{x} = \text{slope} = \sqrt{3}.$$

  We get $\theta_0 = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.
  Thus the equation of the line is

  $$\theta = \frac{\pi}{3}.$$



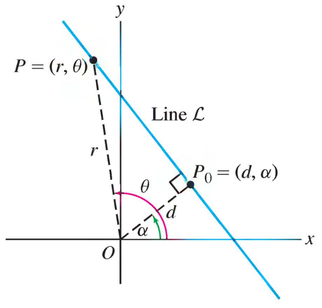
Line Not Through the Origin

- Find the polar equation of the line $\mathcal{L}$ whose point closest to the origin (in polar coordinates) is (d, α).

$P_0 = (d, \alpha)$ is the point of intersection of $\mathcal{L}$ with a perpendicular from O to $\mathcal{L}$. Let $P = (r, \theta)$ be any point on $\mathcal{L}$ other than P_0. From the right triangle $\triangle OPP_0$, we get

$$\cos(\theta - \alpha) = \frac{d}{r}$$

$$\Rightarrow \quad r = d \sec(\theta - \alpha).$$



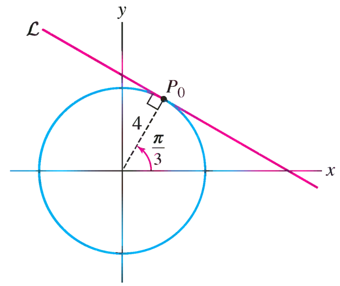
Example

- Find the polar equation of the line $\mathcal{L}$ tangent to the circle $r = 4$ at the point with polar coordinates $P_0 = (4, \frac{\pi}{3})$.

  The point of tangency has polar coordinates $(d, \alpha) = (4, \frac{\pi}{3})$.
  From the preceding slide, the polar equation of the tangent line is:

$$r = d \sec(\theta - \alpha)$$

$$\Rightarrow \quad r = 4 \sec(\theta - \tfrac{\pi}{3}).$$

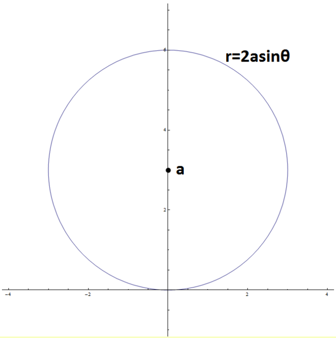


Converting to Rectangular Coordinates

- Identify the curve with polar equation $r = 2a\sin\theta$.

  We have:

$$\begin{aligned} r &= 2a\sin\theta \\ r^2 &= 2ar\sin\theta \\ x^2 + y^2 &= 2ay \\ x^2 + (y^2 - 2ay) &= 0 \\ x^2 + (y^2 - 2ay + a^2) &= a^2 \\ x^2 + (y-a)^2 &= a^2. \end{aligned}$$



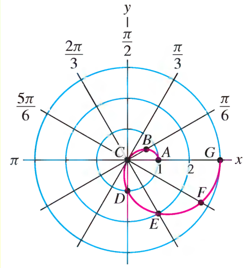
We get a circle with center $(0, a)$ and radius a.

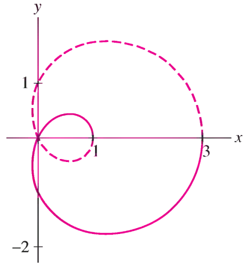
Graphing a Polar Curve Using Symmetry

- Sketch the limaçon curve $r = 2\cos\theta - 1$.
  1. $\cos\theta$ has period 2π. We study the graph for $-\pi \leq \theta < \pi$.
  2. Create a small table of values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$r = 2\cos\theta - 1$	1	$\sqrt{3}-1$	0	-1	-2	$-\sqrt{3}-1$	-3

     Plot the various points.
  3. Since $\cos(-\theta) = \cos\theta$ we have symmetry with respect to the x-axis.





Tangent Lines to Polar Curves

- Suppose that $r = f(\theta)$.
- Then $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$.
- These give, using the product rule,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

- Therefore, for the slope of the tangent at (r, θ),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Equation of Tangent to Cardioid $r = 1 + \sin\theta$

- Find the equation of the tangent to the cardioid $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}.\end{aligned}$$

So

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\frac{\sqrt{3}}{2})(1-\sqrt{3})} = \frac{\frac{1}{2}(1+\sqrt{3})^2}{(1+\frac{\sqrt{3}}{2})(1-3)} = -1.$$

For $\theta = \frac{\pi}{3}$, $r(\frac{\pi}{3}) = 1 + \frac{\sqrt{3}}{2}$. So $x = r\cos\theta = (1+\frac{\sqrt{3}}{2})\frac{1}{2} = \frac{2+\sqrt{3}}{4}$ and $y = r\sin\theta = (1+\frac{\sqrt{3}}{2})\frac{\sqrt{3}}{2} = \frac{3+2\sqrt{3}}{4}$.
Thus, the equation of the tangent line when $\theta = \frac{\pi}{3}$ is
$y - \frac{3+2\sqrt{3}}{4} = -(x - \frac{2+\sqrt{3}}{4})$.

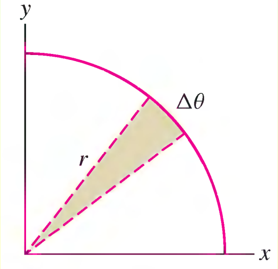
Subsection 4

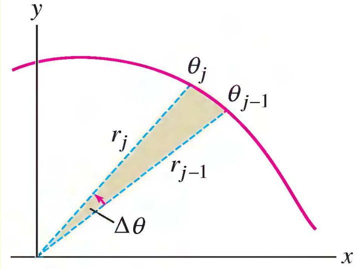
Area and Arc Length in Polar Coordinates

Areas in Polar Coordinates

- The area of a disk segment with central angle $\Delta\theta$ is $A = \frac{1}{2}(\Delta\theta)r^2$ (since total area of disk is $\pi r^2 = \frac{1}{2}(2\pi)r^2$).
- Thus, if a polar curve is given by $r = f(\theta)$, then for a small $\Delta\theta$, taking r constant at $f(\theta_j)$, we get

$$\Delta A_j \approx \frac{1}{2}[f(\theta_j)]^2\Delta\theta.$$





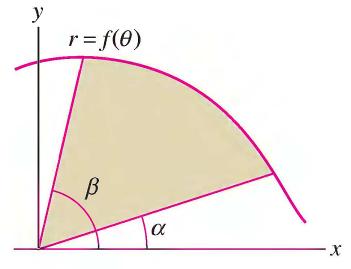
Areas in Polar Coordinates (Cont'd)

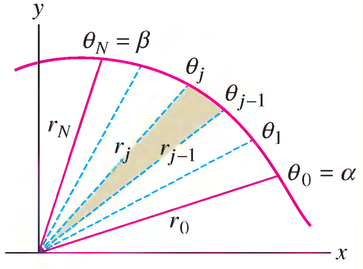
- Summing over those j's partitioning $\alpha \leq \theta \leq \beta$, we get

$$A \approx \sum_{j=1}^{N} \frac{1}{2}[f(\theta_j)]^2 \Delta\theta.$$

- Finally, passing to the limit, we end up with the integral

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta \quad \left(= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta\right).$$

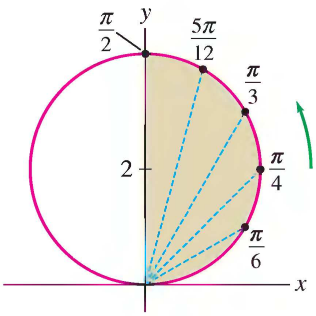




Area of Semicircle

- Compute the area of the right semicircle with equation $r = 4\sin\theta$.

  The right semicircle is plotted for $0 \leq \theta \leq \frac{\pi}{2}$. Therefore, we have



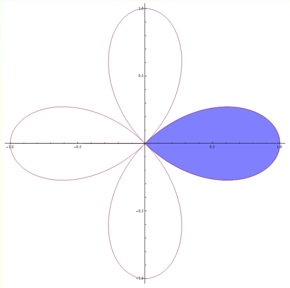
$$\begin{aligned}
A &= \tfrac{1}{2}\int_0^{\pi/2} r^2 d\theta \\
&= \tfrac{1}{2}\int_0^{\pi/2} (4\sin\theta)^2 d\theta \\
&= 8\int_0^{\pi/2} \sin^2\theta d\theta \\
&= 8\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta) d\theta \\
&= 4(\theta - \tfrac{1}{2}\sin 2\theta)\ \Big|_0^{\pi/2} \\
&= 4(\tfrac{\pi}{2} - 0) \\
&= 2\pi.
\end{aligned}$$

One Loop of the Four-Leaved Rose

- Compute the area of one loop of the four-leaved rose $r = \cos 2\theta$.

  One loop is traced for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.

  Therefore, we get



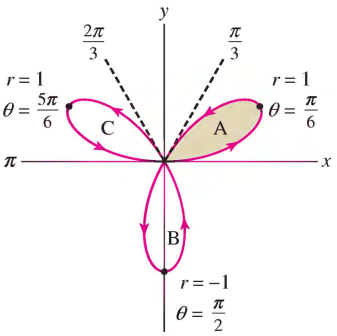
$$
\begin{aligned}
A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2 2\theta \; d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\frac{1+\cos 4\theta}{2} \; d\theta \\
&= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta) \; d\theta \\
&= \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right) \Big|_{-\pi/4}^{\pi/4} \\
&= \frac{1}{4}\Big(\frac{\pi}{4} + \frac{1}{4}\sin \pi - \\
&\qquad\qquad \left(-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi)\right)\Big) \\
&= \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{8}.
\end{aligned}
$$

Area of a Petal of a Rose

- Compute the area of one petal of the rose $r = \sin 3\theta$.

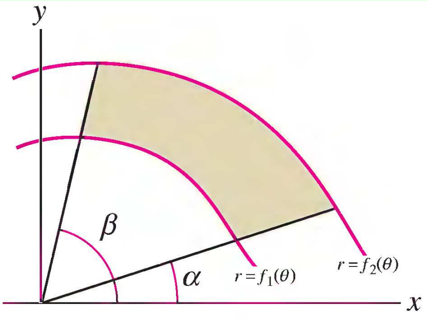
  One petal is traced when $0 \le \theta \le \frac{\pi}{3}$. So we have

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi/3}(\sin 3\theta)^2 d\theta \\ &= \frac{1}{2}\int_0^{\pi/3}\frac{1}{2}(1-\cos 6\theta)d\theta \\ &= \frac{1}{4}(\theta - \frac{1}{6}\sin 6\theta)\Big|_0^{\pi/3} \\ &= \frac{1}{4}[(\frac{\pi}{3}-0)-0] = \frac{\pi}{12}. \end{aligned}$$



Area Between Two Curves

- Consider the area A between two polar curves $r = f_1(\theta)$ and $r = f_2(\theta)$, with $f_1(\theta) \leq f_2(\theta)$, for $\alpha \leq \theta \leq \beta$.



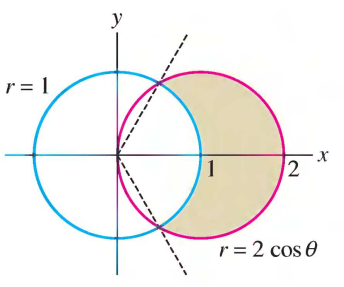
It is given by

$$A = \frac{1}{2}\int_\alpha^\beta f_2(\theta)^2 d\theta - \frac{1}{2}\int_\alpha^\beta f_1(\theta)^2 d\theta = \frac{1}{2}\int_\alpha^\beta [f_2(\theta)^2 - f_1(\theta)^2] d\theta.$$

Computing the Area Between Two Curves

- Find the area of the region inside the circle $r = 2\cos\theta$ but outside the circle $r = 1$.
  Set the equations equal to find the angle θ for the points of intersection: $2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos\theta)^2 d\theta \\
&\qquad - \frac{1}{2}\int_{-\pi/3}^{\pi/3}(1)^2 d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\frac{1+\cos 2\theta}{2} - 1)d\theta \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3}(2\cos 2\theta + 1)d\theta \\
&= \frac{1}{2}(\sin 2\theta + \theta)\,\Big|_{-\pi/3}^{\pi/3} \\
&= \frac{1}{2}[(\sin\frac{2\pi}{3} + \frac{\pi}{3}) - (\sin(-\frac{2\pi}{3}) - \frac{\pi}{3})] \\
&= \frac{\sqrt{3}}{2} + \frac{\pi}{3}.
\end{aligned}$$



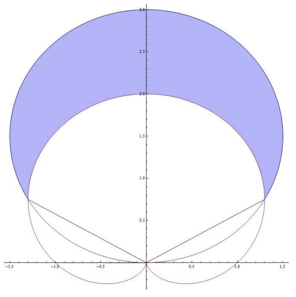
Region Between Cardioid and Circle

- Find the area of the region inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

  Find angles θ of intersection.

  $3\sin\theta = 1 + \sin\theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$.

$$\begin{aligned}
A &= \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}[(3\sin\theta)^2 - (1+\sin\theta)^2]\, d\theta \\
&= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (8\sin^2\theta - 2\sin\theta - 1)\, d\theta \\
&= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (4(1-\cos 2\theta) - 2\sin\theta - 1) d\theta \\
&= \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 2\sin\theta - 4\cos 2\theta) d\theta \\
&= \tfrac{1}{2}(3\theta + 2\cos\theta - 2\sin 2\theta)\,\Big|_{\pi/6}^{5\pi/6} \\
&= \tfrac{1}{2}[(\tfrac{5\pi}{2} - \sqrt{3} + \sqrt{3}) - (\tfrac{\pi}{2} + \sqrt{3} - \sqrt{3})] \\
&= \pi.
\end{aligned}$$



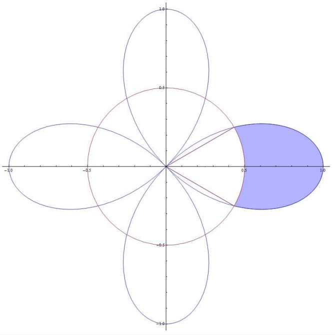
Area Between Circle and Petal of the Four-Leaved Rose

- Find the area of the region inside a petal of $r = \cos 2\theta$ and outside the circle $r = \frac{1}{2}$.

  Find angles θ of intersection.

  $\cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = -\frac{\pi}{3}$ or $2\theta = \frac{\pi}{3} \Rightarrow \theta = -\frac{\pi}{6}$ or $\theta = \frac{\pi}{6}$.

$$\begin{aligned} A &= \int_{-\pi/6}^{\pi/6} \frac{1}{2}[(\cos 2\theta)^2 - (\tfrac{1}{2})^2] d\theta \\ &= \frac{1}{2}\int_{-\pi/6}^{\pi/6} (\cos^2 2\theta - \tfrac{1}{4}) d\theta \\ &= \frac{1}{2}\int_{-\pi/6}^{\pi/6} (\tfrac{1+\cos 4\theta}{2} - \tfrac{1}{4}) d\theta \\ &= \frac{1}{8}\int_{-\pi/6}^{\pi/6} (1 + 2\cos 4\theta) d\theta \\ &= \frac{1}{8}(\theta + \tfrac{1}{2}\sin 4\theta) \Big|_{-\pi/6}^{\pi/6} \\ &= \frac{1}{8}(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}). \end{aligned}$$



Length of Polar Curves

- Recall from rectangular coordinates, using the Pythagorean Theorem, $L = \int_a^b \sqrt{dx^2 + dy^2}$.
- Multiplying and dividing by $d\theta$,

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

- Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta.$$

- Therefore $(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 \stackrel{\text{algebra}}{=} (\frac{dr}{d\theta})^2 + r^2$.
- This gives

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

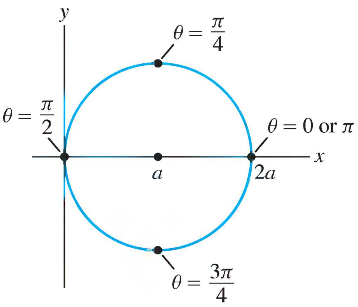
Arc Length of a Circle

- Find the total length of the circle $r = 2a\cos\theta$, for $a > 0$.

  We have $r = f(\theta) = 2a\cos\theta$. So we get

$$r^2 + (\frac{dr}{d\theta})^2 = (2a\cos\theta)^2 + (-2a\sin\theta)^2 = 4a^2.$$

$$\begin{aligned} L &= \int_0^\pi \sqrt{r^2 + (\tfrac{dr}{d\theta})^2} d\theta \\ &= \int_0^\pi \sqrt{4a^2} d\theta \\ &= \int_0^\pi (2a) d\theta \\ &= 2a\ \theta\ \big|_0^\pi \\ &= 2\pi a. \end{aligned}$$

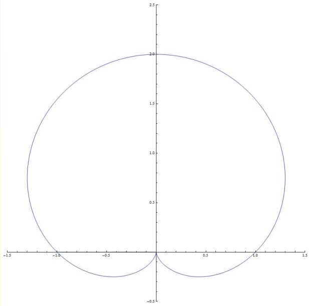


Length of the Cardioid $r = 1 + \sin\theta$

- Finf the length of the cardiod $r = 1 + \sin\theta$.

  We have $\dfrac{dr}{d\theta} = \cos\theta$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta \\ &= \int_0^{2\pi} \sqrt{2+2\sin\theta} d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{(2+2\sin\theta)(2-2\sin\theta)}}{\sqrt{2-2\sin\theta}} d\theta \\ &= 2 \cdot \frac{2}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta}{\sqrt{1-\sin\theta}} d\theta \\ & \quad (\text{set } u = 1 - \sin\theta) \\ &= 2\sqrt{2} \int_2^0 -\frac{1}{\sqrt{u}} du \\ &= 2\sqrt{2}(2\sqrt{u}) \Big|_0^2 \\ &= 2\sqrt{2} \cdot 2\sqrt{2} = 8. \end{aligned}$$

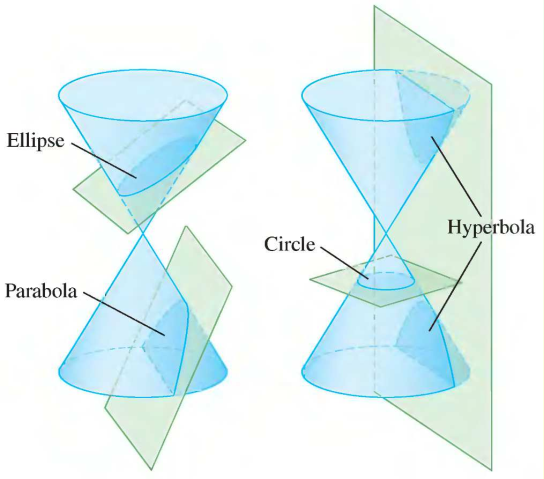


Subsection 5

Conic Sections

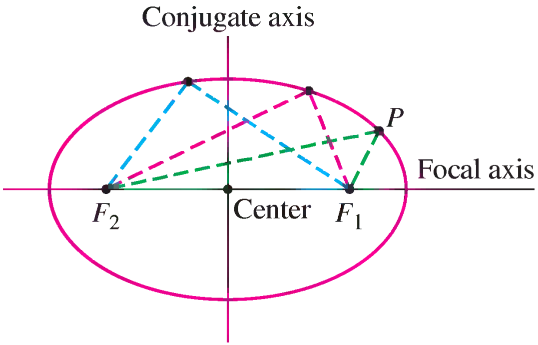
Conic Sections

- **Conic sections** are obtained as the intersection of a cone with a plane.



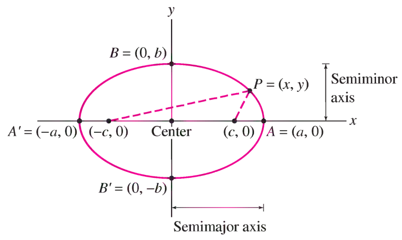
Ellipses

- An **ellipse** is the locus of all points P such that the sum of the distances to two fixed points F_1 and F_2 is a constant K.
  - The midpoint of $\overline{F_1F_2}$ is the **center** of the ellipse;
  - The line through the foci is the **focal axis**;
  - The line through the center and perpendicular to the focal axis is the **conjugate axis**.



Ellipse in Standard Position

- An ellipse is in **standard position** if the focal and conjugate axes are the x- and y-axes.

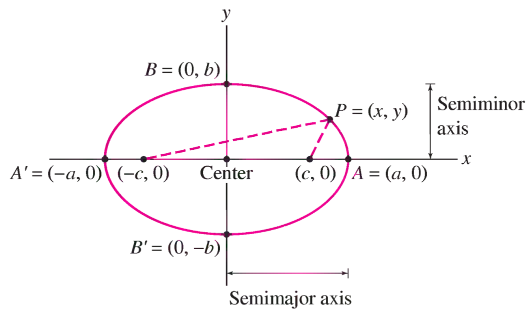


The foci have coordinates $F_1 = (c, 0)$, $F_2 = (-c, 0)$, for some $c > 0$. The equation of this ellipse has the simple form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a = \frac{K}{2}$ and $b = \sqrt{a^2 - c^2}$.

Additional Terminology



- The points A, A', B, B' of intersection with the axes are called **vertices**;
- The vertices A, A' on the focal axis are called **focal vertices**;
- The number a is the **semimajor axis**;
- The number b is the **semininor axis**.

Equation of Ellipse in Standard Position

- Let $a > b > 0$ and set $c = \sqrt{a^2 - b^2}$.

  The ellipse $PF_1 + PF_2 = 2a$, with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ has equation

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

  Furthermore, the ellipse has:

  - Semimajor axis a and semiminor axis b;
  - Focal vertices $(\pm a, 0)$ and minor vertices $(0, \pm b)$.

- If $b > a > 0$, the same equation defines an ellipse with foci $(0, \pm c)$, where $c = \sqrt{b^2 - a^2}$.

Finding an Equation and Sketching the Graph

- Find an equation of the ellipse with foci $(\pm\sqrt{11}, 0)$ and semimajor axis $a = 6$. Then sketch its graph.

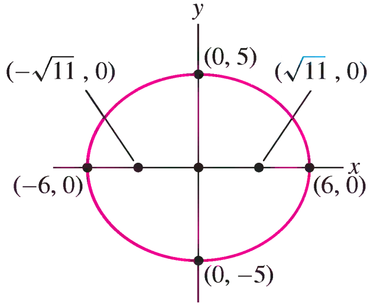
  Since the foci are at $(\pm c, 0)$, we get that $c = \sqrt{11}$.

  Since $b = \sqrt{a^2 - c^2}$, we get $b = \sqrt{36 - 11} = 5$.

  Thus, the equation is

  $$\frac{x^2}{36} + \frac{y^2}{25} = 1.$$

  Finally, we sketch the graph:



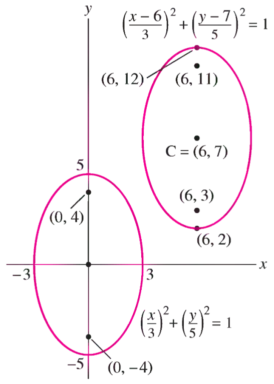
Transforming an Ellipse

- Find an equation of the ellipse with center $C = (6,7)$, vertical focal axis, semimajor axis 5 and semiminor axis 3. Then find the location of the foci and sketch its graph.

  We have $a = 3$ and $b = 5$. At standard position the equation would have been $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Translating to center C, we get

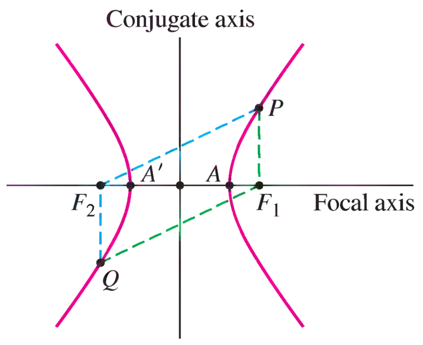
  $$\frac{(x-6)^2}{9} + \frac{(x-7)^2}{25} = 1.$$

  Now we compute $c = \sqrt{b^2 - a^2} = \sqrt{25-9} = 4$. Thus, the foci are at $(6, 7 \pm 4)$ or $(6,11)$ and $(6,3)$.



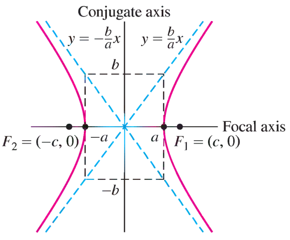
Hyperbolas

- A **hyperbola** is the locus of all points P such that the difference of the distances from P to two foci F_1 and F_2 is $\pm K$.
  - The midpoint of $\overline{F_1 F_2}$ is the **center** of the hyperbola;
  - The line through the foci is the **focal axis**;
  - The line through the center and perpendicular to the focal axis is the **conjugate axis**.



Hyperbola in Standard Position

- A hyperbola is in **standard position** if the focal and conjugate axes are the x- and y-axes.

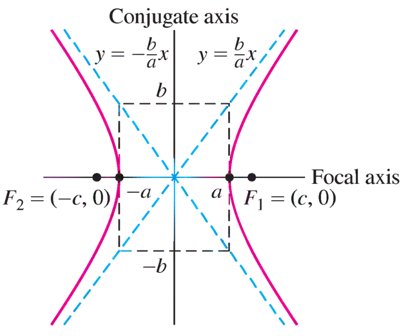


The foci have coordinates $F_1 = (c, 0)$, $F_2 = (-c, 0)$, for some $c > 0$. The equation of this hyperbola has the simple form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a = \frac{K}{2}$ and $b = \sqrt{c^2 - a^2}$.

Additional Terminology



- The points A, A' of intersection with the focal axis are called **vertices**;
- A hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has two **asymptotes** $y = \pm\frac{b}{a}x$, which are diagonals of the rectangle whose sides pass through $(\pm a, 0)$ and $(0, \pm b)$.

Equation of Hyperbola in Standard Position

- Let $a > 0$ and $b > 0$ and set $c = \sqrt{a^2 + b^2}$.

  The hyperbola $PF_1 - PF_2 = \pm 2a$, with foci $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ has equation

  $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

  Furthermore, the hyperbola has vertices $(\pm a, 0)$.

Finding an Equation and Sketching the Graph

- Find an equation of the hyperbola with foci $(\pm\sqrt{5}, 0)$ and vertices $(\pm 1, 0)$. Then sketch its asymptotes and its graph.

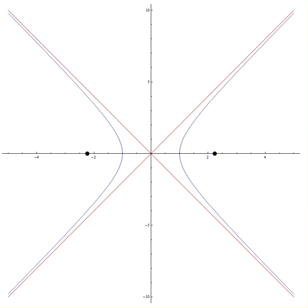
  Since the foci are at $(\pm c, 0)$, we get that $c = \sqrt{5}$.

  Since $b = \sqrt{c^2 - a^2}$, we get $b = \sqrt{5 - 1} = 2$.

  Thus, the equation is

  $$\frac{x^2}{1} - \frac{y^2}{4} = 1.$$

  Moreover the asymptotes are $y = \pm \frac{b}{a} x$, i.e., $y = \pm 2x$.



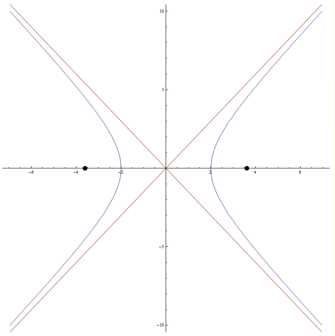
Finding the Foci and Sketching the Graph

- Find the foci of the hyperbola $9x^2 - 4y^2 = 36$. Then find the equations of the asymptotes and sketch the graph of the hyperbola using the asymptotes.

  Put the equation in the standard form:

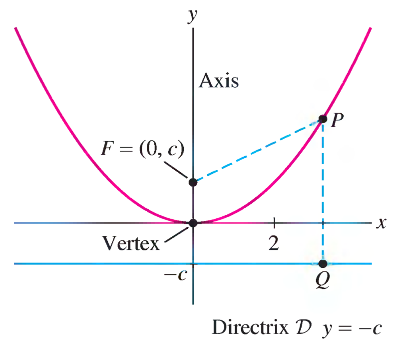
  $$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

  Thus, we get $a = 2$ and $b = 3$. This gives $c = \sqrt{a^2 + b^2} = \sqrt{4+9} = \sqrt{13}$. We conclude that the foci are at $(\pm\sqrt{13}, 0)$. Moreover, the asymptotes have equations $y = \pm\frac{3}{2}x$.



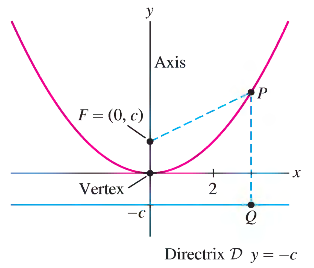
Parabolas

- A **parabola** is the locus of all points P that are equidistant from a **focus** F and a line $\mathcal{D}$ called the **directrix**: $PF = P\mathcal{D}$.
  - The line through F and perpendicular to $\mathcal{D}$ is called the **axis** of the parabola;
  - The **vertex** is the point of intersection of the parabola with its axis.



Parabola in Standard Position

- A parabola is in **standard position** if for some c, the focus is $F = (0, c)$ and the directrix is $y = -c$.



The equation of this parabola has the simple form

$$y = \frac{1}{4c}x^2.$$

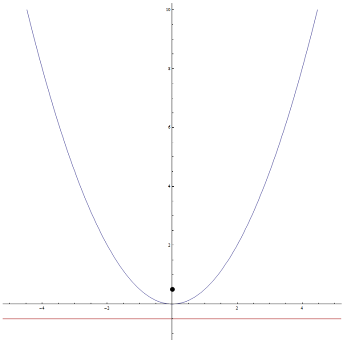
- The vertex is then located at the origin.
- The parabola opens upward if $c > 0$ and downward if $c < 0$.

Finding an Equation and Sketching the Graph

- A parabola is in the standard position and has directrix $y = -\frac{1}{2}$. Find the focus, the equation and sketch the parabola.

  Since the directrix is at $y = -c$, we get $c = \frac{1}{2}$. Therefore the focus is at $(0, c) = (0, \frac{1}{2})$.
  The equation of the parabola is

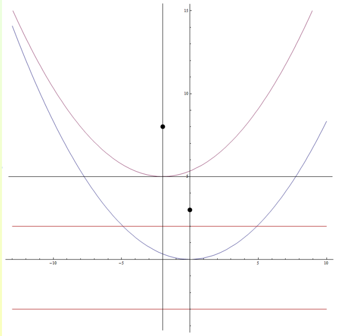
$$y = \frac{1}{4 \cdot \frac{1}{2}} x^2 \text{ or } y = \frac{1}{2} x^2.$$



Transforming a Parabola

- The standard parabola with directrix $y = -3$ is translated so that its vertex is located at $(-2, 5)$. Find its equation, directrix and focus.

  Consider, first, the standard parabola: It has $c = 3$. Thus, its focus is $(0, 3)$. It has equation $y = \frac{1}{12}x^2$.
  So the transformed parabola has equation
  $$y - 5 = \frac{1}{12}(x + 2)^2.$$
  Its directrix is $y = 2$ and its focus $(-2, 8)$.



Eccentricity

- The shape of a conic section is measured by a number e called the **eccentricity**:

$$e = \frac{\text{distance between foci}}{\text{distance between vertices on the focal axis}}.$$

A parabola is defined to have eccentricity 1.

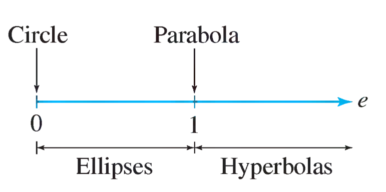
Theorem

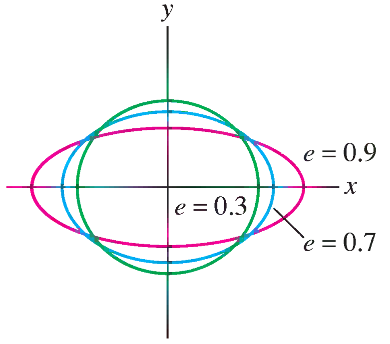
For ellipses and hyperbolas in the standard position

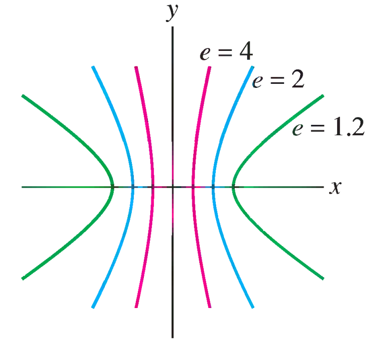
$$e = \frac{c}{a}.$$

1. An ellipse has eccentricity $0 \leq e < 1$;
2. A hyperbola has eccentricity $e > 1$.

Eccentricity and Shapes







Eccentricity as a Unification Tool

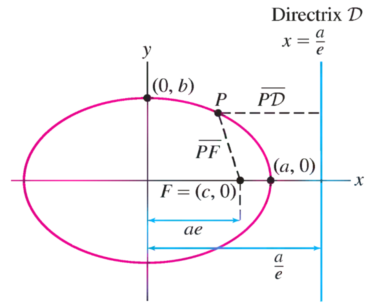
- Given a point F (the focus), a line $\mathcal{D}$ (the directrix) and a number $e > 0$, consider the locus of all points P, such that

$$PF = e \cdot P\mathcal{D}.$$

- For all $e > 0$, this locus is a conic of eccentricity e.
  - **Ellipse**: Let $a > b > 0$ and $c = \sqrt{a^2 - b^2}$. The ellipse

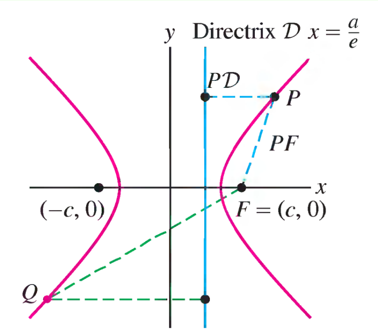
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

satisfies $PF = eP\mathcal{D}$, with $F = (c, 0)$, $e = \frac{c}{a}$ and vertical directrix $x = \frac{a}{e}$.



Eccentricity as a Unification Tool (Cont'd)

- The Second Case:
  - **Hyperbola**: Let $a, b > 0$ and $c = \sqrt{a^2 + b^2}$. The hyperbola
    $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
    satisfies $PF = eP\mathcal{D}$, with $F = (c, 0)$, $e = \frac{c}{a}$ and vertical directrix $x = \frac{a}{e}$.



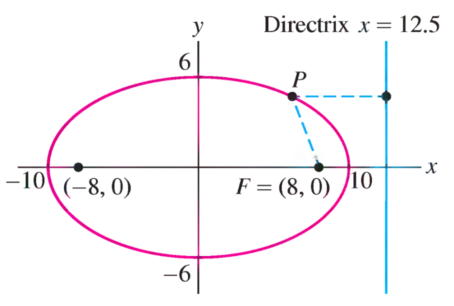
Example

- Find the equation, foci and directrix of the standard ellipse with eccentricity $e = 0.8$ and focal vertices $(\pm 10, 0)$.
  We have $\frac{c}{a} = e = 0.8$. Moreover $a = 10$. Therefore, $c = ea = 0.8 \cdot 10 = 8$. This shows that the foci are at $(\pm 8, 0)$. Moreover, we get $b^2 = a^2 - c^2 = 100 - 64 = 36$.

  Therefore, the equation of the ellipse is

  $$\frac{x^2}{100} + \frac{y^2}{36} = 1.$$

  Finally, the ellipse has directrix $x = \frac{a}{e} = \frac{10}{0.8} = 12.5$.

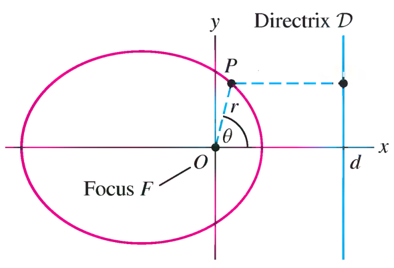


Polar Equation of a Conic Section

- We assume focus $F = (0,0)$ and directrix $\mathcal{D} : x = d$.

  We have

$$\begin{array}{rcl} PF & = & r \\ P\mathcal{D} & = & d - r\cos\theta. \end{array}$$



  So, if the conic section has equation $PF = eP\mathcal{D}$, we get $r = e(d - r\cos\theta)$.

  We solve for r:

$$\begin{aligned} & r = ed - er\cos\theta \\ \Rightarrow\ & r + er\cos\theta = ed \\ \Rightarrow\ & r(1 + e\cos\theta) = ed \\ \Rightarrow\ & r = \frac{ed}{1 + e\cos\theta}. \end{aligned}$$

Example

- Find the eccentricity, directrix and focus of the conic section

$$r = \frac{24}{4 + 3\cos\theta}.$$

We need to convert into standard form $r = \frac{ed}{1+e\cos\theta}$.

$$r = \frac{24}{4 + 3\cos\theta} \Rightarrow r = \frac{6}{1 + \frac{3}{4}\cos\theta}.$$

Now we get:

$$ed = 6, \quad e = \frac{3}{4}, \quad d = \frac{6}{3/4} = 8.$$

We conclude that the eccentricity is $e = \frac{3}{4}$, the directrix is $\mathcal{D} : x = 8$ and the focus is $F = (0, 0)$.

The General Quadratic Equation

- The equations of the standard conic sections are special cases of the general equation of degree 2 in x and y:
$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$
with a, b, e, d, e, f constants with a, b, c not all zero.
- Apart from "degenerate cases", this equation defines a conic section that is not necessarily in standard position:
  - It need not be centered at the origin;
  - Its focal and conjugate axes may be rotated relative to coordinate axes.
- We say that the equation is **degenerate** if the set of solutions is a pair of intersecting lines, a pair of parallel lines, a single line, a point, or the empty set. Some examples include:
  - $x^2 - y^2 = 0$ defines a pair of intersecting lines $y = x$ and $y = -x$.
  - $x^2 - x = 0$ defines a pair of parallel lines $x = 0$ and $x = 1$.
  - $x^2 = 0$ defines a single line (the y-axis).
  - $x^2 + y^2 = 0$ has just one solution $(0, 0)$.
  - $x^2 + y^2 = -1$ has no solutions.

General Quadratic Equation: Zero Cross Term

- Suppose that $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is nondegenerate. The term bxy is called the **cross term**.
  When the cross term is zero (that is, when $b = 0$), we can *complete the square* to show that the equation defines a translate of the conic in standard position.
  Example: Show that $4x^2 + 9y^2 + 24x - 72y + 144 = 0$ defines a translate of a conic section in standard position. Identify the conic section and find its focus, directrix and eccentricity.

$$\begin{aligned} & 4x^2 + 9y^2 + 24x - 72y + 144 = 0 \\ \Rightarrow\ & 4(x^2 + 6x) + 9(y^2 - 8y) = -144 \\ \Rightarrow\ & 4(x^2 + 6x + 9) + 9(y^2 - 8y + 16) = 36 + 144 - 144 \\ \Rightarrow\ & 4(x+3)^2 + 9(y-4)^2 = 36 \\ \Rightarrow\ & \frac{(x+3)^2}{9} + \frac{(y-4)^2}{4} = 1. \end{aligned}$$

  Ellipse with center $(-3, 4)$, focus $(-3 + \sqrt{5}, 4)$, eccentricity $e = \frac{\sqrt{5}}{3}$, directrix $x = -3 + \frac{9}{\sqrt{5}}$.

The Discriminant Test for Classification

- Suppose that the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

is nondegenerate and thus defines a conic section.

According to the Discriminant Test, the type of conic is determined by the **discriminant** D:

$$D = b^2 - 4ac.$$

We have the following cases:

  - $D < 0$: Ellipse or circle;
  - $D > 0$: Hyperbola;
  - $D = 0$: Parabola.

Example: Determine the conic section with equation $2xy = 1$.

The discriminant of $2xy = 1$ is $D = b^2 - 4ac = 2^2 - 0 = 4 > 0$.

According to the Discriminant Test, $2xy = 1$ defines a hyperbola.