Calculus III

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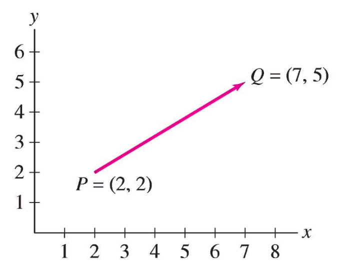
LSSU Math 251

Subsection 1

Vectors in the Plane

Vectors

- A two-dimensional **vector** $\boldsymbol{v}$ is determined by two points in the plane:
  - an **initial point** P (also called the **tail** or **basepoint**);
  - a **terminal point** Q (also called the **head**).

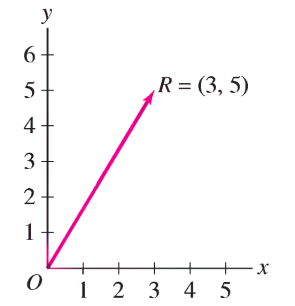


We write $\boldsymbol{v} = \overrightarrow{PQ}$ and we draw $\boldsymbol{v}$ as an arrow pointing from P to Q. This vector is said to be **based at** P.

- The **length** or **magnitude** of $\boldsymbol{v}$, denoted $\|\boldsymbol{v}\|$, is the distance from P to Q.

Position Vectors

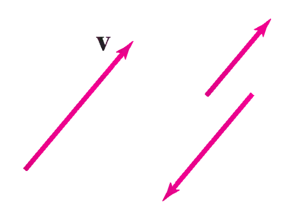
- The vector $\mathbf{v} = \overrightarrow{OR}$ pointing from the origin to a point R is called the **position vector** of R.

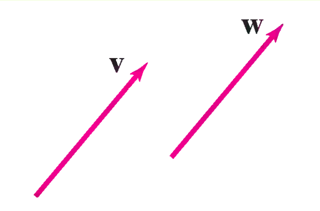


Example: The figure shows the position vector of $R = (3, 5)$.

Parallel and Translate Vectors

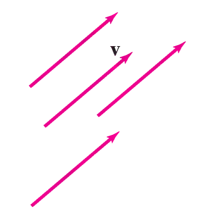
- Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ of nonzero length are called **parallel** if the lines through $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.
  Parallel vectors point either in the same or in opposite directions.
- A vector $\boldsymbol{v}$ is said to undergo a **translation** when it is moved parallel to itself without changing its length or direction.
  The resulting vector $\boldsymbol{w}$ is called a **translate** of $\boldsymbol{v}$.
  Translates have the same length and direction but different basepoints.

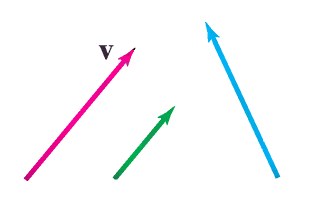




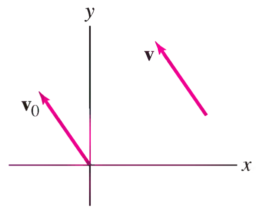
Equivalent Vectors

- Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **equivalent** if $\boldsymbol{w}$ is a translate of $\boldsymbol{v}$.





- Every vector can be translated so that its tail is at the origin. Therefore, every vector $\boldsymbol{v}$ is equivalent to a unique vector $\boldsymbol{v}_0$ based at the origin.

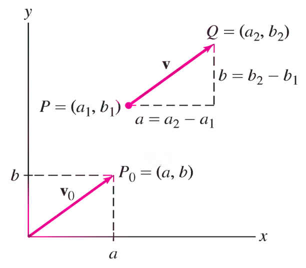


Components, Length and Zero Vector

- The **components** of $\boldsymbol{v} = \overrightarrow{PQ}$, where $P = (a_1, b_1)$ and $Q = (a_2, b_2)$, are the quantities

$$a = a_2 - a_1 \quad (x\text{-}\mathbf{component}),$$
$$b = b_2 - b_1 \quad (y\text{-}\mathbf{component}).$$

The pair of components is denoted $\langle a, b\rangle$.



- The length of a vector in terms of its components (by the distance formula) is

$$\|\overrightarrow{PQ}\| = \sqrt{a^2 + b^2}.$$

- The **zero vector** (whose head and tail coincide) is the vector $\mathbf{0} = \langle 0, 0\rangle$ of length zero.

Representation Convention

- The components $\langle a, b \rangle$ determine the length and direction of $\boldsymbol{v}$, but not its basepoint.

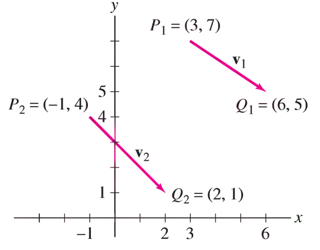
  Two vectors have the same components if and only if they are equivalent.
- Nevertheless, the standard practice is to describe a vector by its components, and thus we write $\boldsymbol{v} = \langle a, b \rangle$.

  Although this notation is ambiguous (because it does not specify the basepoint), it rarely causes confusion in practice.

  In the sequel we assume all vectors are based at the origin unless otherwise stated.

Example I

- Determine whether $\boldsymbol{v}_1 = \overrightarrow{P_1Q_1}$ and $\boldsymbol{v}_2 = \overrightarrow{P_2Q_2}$ are equivalent, where $P_1 = (3,7)$, $Q_1 = (6,5)$ and $P_2 = (-1,4)$, $Q_2 = (2,1)$. What is the magnitude of $\boldsymbol{v}_1$?

  We compare components:

  $$\boldsymbol{v}_1 = \langle 6-3, 5-7\rangle = \langle 3,-2\rangle$$

  and

  $$\boldsymbol{v}_2 = \langle 2-(-1), 1-4\rangle = \langle 3,-3\rangle.$$

  Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ have different components, they are not equivalent vectors.

  The magnitude of $\boldsymbol{v}_1$ is

  $$\|\boldsymbol{v}_1\| = \sqrt{3^2+(-2)^2} = \sqrt{13}.$$

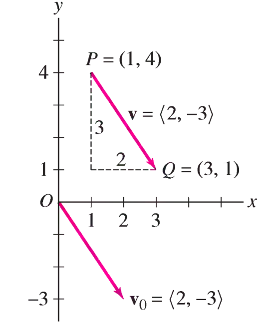
Example II

- Sketch the vector $\boldsymbol{v} = \langle 2, -3 \rangle$ based at $P = (1, 4)$ and the vector $\boldsymbol{v}_0$ equivalent to $\boldsymbol{v}$ based at the origin.

  The vector $\boldsymbol{v} = \langle 2, 3 \rangle$ based at $P = (1, 4)$ has terminal point

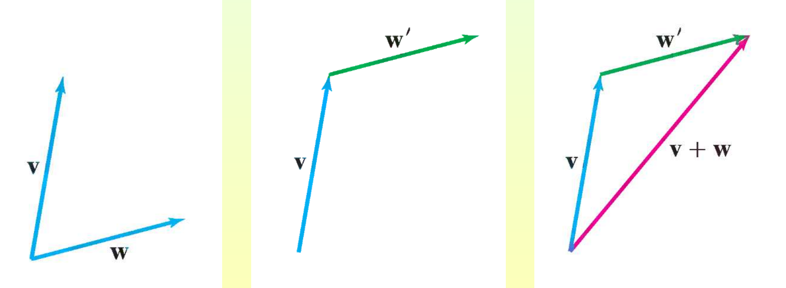
  $$Q = (1 + 2, 4 - 3) = (3, 1).$$

  The vector $\boldsymbol{v}_0$ equivalent to $\boldsymbol{v}$ based at O has terminal point $(2, -3)$.



Addition of Two Vectors: Tail-to-Head Method

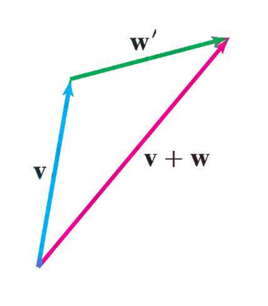
- The **vector sum** $\boldsymbol{v} + \boldsymbol{w}$ is defined when $\boldsymbol{v}$ and $\boldsymbol{w}$ have the same basepoint:
  - Translate $\boldsymbol{w}$ to the equivalent vector $\boldsymbol{w}'$ whose tail coincides with the head of $\boldsymbol{v}$.
  - The sum $\boldsymbol{v} + \boldsymbol{w}$ is the vector pointing from the tail of $\boldsymbol{v}$ to the head of $\boldsymbol{w}'$.

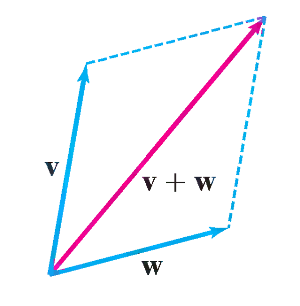


Addition of Two Vectors: Parallelogram Law

- Another way to add vectors is to use the **Parallelogram Law**:

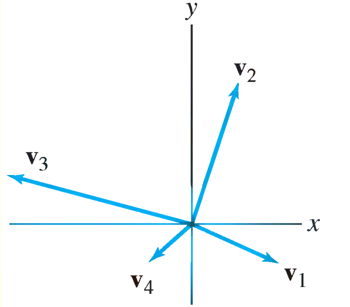
  $\boldsymbol{v} + \boldsymbol{w}$ is the vector pointing from the basepoint to the opposite vertex of the parallelogram formed by $\boldsymbol{v}$ and $\boldsymbol{w}$.

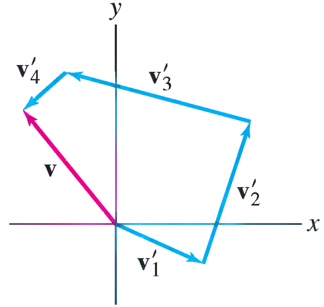




Addition of Multiple Vectors

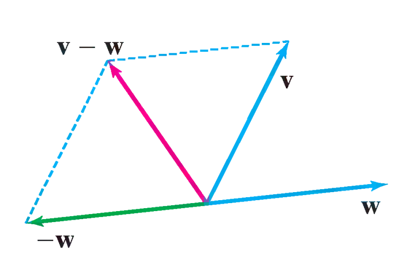
- To add several vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$:
  - translate the vectors to $\boldsymbol{v}_1 = \boldsymbol{v}_1', \boldsymbol{v}_2', \ldots, \boldsymbol{v}_n'$ so that they lie head to tail;
  - The vector sum $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n$ is the vector whose terminal point is the terminal point of $\boldsymbol{v}_n'$.

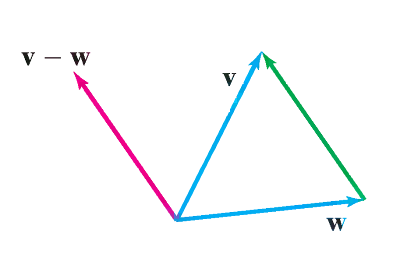




Subtraction of Vectors

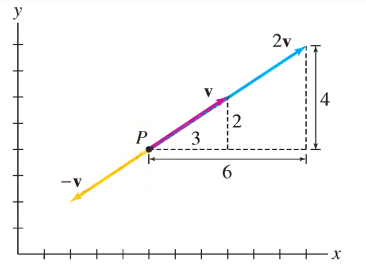
- **Vector subtraction** $\boldsymbol{v} - \boldsymbol{w}$ is carried out by adding $-\boldsymbol{w}$ to $\boldsymbol{v}$.
- More simply:
  - draw the vector pointing from $\boldsymbol{w}$ to $\boldsymbol{v}$;
  - translate it back to the basepoint to obtain $\boldsymbol{v} - \boldsymbol{w}$.





Scalar Multiplication

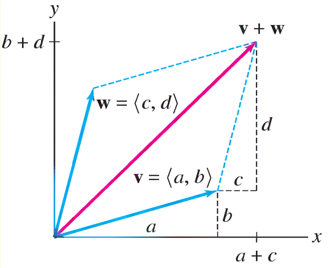
- The term **scalar** is another word for "real number".
- If λ is a scalar and $\boldsymbol{v}$ is a nonzero vector, **the scalar multiple** $\lambda\boldsymbol{v}$ is defined as follows:
  - $\lambda\boldsymbol{v}$ has length $|\lambda|\|\boldsymbol{v}\|$.
  - It points in the same direction as $\boldsymbol{v}$ if $\lambda > 0$.
  - It points in the opposite direction if $\lambda < 0$.

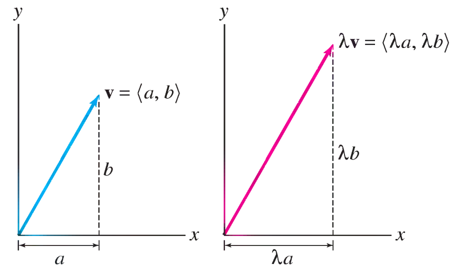


- Note that $0\boldsymbol{v} = \mathbf{0}$, for all $\boldsymbol{v}$;
- Also $\|\lambda\boldsymbol{v}\| = |\lambda|\|\boldsymbol{v}\|$.
- $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$ but points in the opposite direction.
- A vector $\boldsymbol{w}$ is parallel to $\boldsymbol{v}$ if and only if $\boldsymbol{w} = \lambda\boldsymbol{v}$, for some nonzero scalar λ.

Vector Operations Using Components

- To add or subtract two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we add or subtract their components.
- To multiply $\boldsymbol{v}$ by a scalar λ, we multiply the components of $\boldsymbol{v}$ by λ.





Vector Operations Using Components

- If $\boldsymbol{v} = \langle a, b \rangle$ and $\boldsymbol{w} = \langle c, d \rangle$, then:
  (i) $\boldsymbol{v} + \boldsymbol{w} = \langle a + c, b + d \rangle$;
  (ii) $\boldsymbol{v} - \boldsymbol{w} = \langle a - c, b - d \rangle$;
  (iii) $\lambda \boldsymbol{v} = \langle \lambda a, \lambda b \rangle$;
  (iv) $\boldsymbol{v} + \mathbf{0} = \mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$.
- Also note that if $P = (a_1, b_1)$ and $Q = (a_2, b_2)$, then the components of the vector $\boldsymbol{v} = \overrightarrow{PQ}$ are conveniently computed as the difference

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle a_2, b_2 \rangle - \langle a_1, b_1 \rangle = \langle a_2 - a_1, b_2 - b_1 \rangle.$$

Example

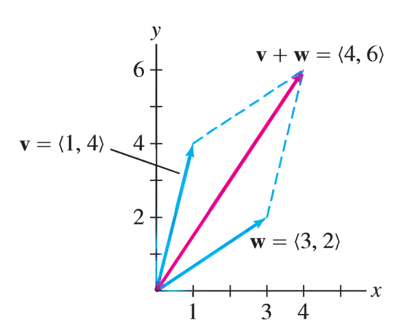
- For $\boldsymbol{v} = \langle 1, 4 \rangle$, $\boldsymbol{w} = \langle 3, 2 \rangle$, calculate

$$(a)\ \boldsymbol{v} + \boldsymbol{w} \qquad (b)\ 5\boldsymbol{v}$$

and sketch $\boldsymbol{v}$, $\boldsymbol{w}$ and their sum.

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= \langle 1, 4 \rangle + \langle 3, 2 \rangle \\ &= \langle 1 + 3, 4 + 2 \rangle \\ &= \langle 4, 6 \rangle. \end{aligned}$$

$$5\boldsymbol{v} = 5\langle 1, 4 \rangle = \langle 5, 20 \rangle.$$

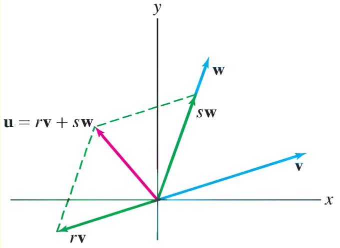


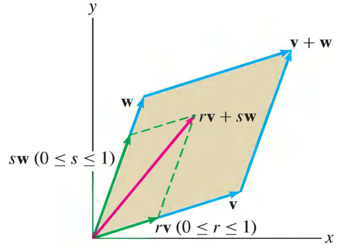
Basic Properties of Vector Algebra

- For all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and for all scalars λ,
  - $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$; **(Commutative Law)**
  - $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$; **(Associative Law)**
  - $\lambda(\boldsymbol{v} + \boldsymbol{w}) = \lambda\boldsymbol{v} + \lambda\boldsymbol{w}$. **(Distributive Law for Scalars)**
- These properties are easily checked using components:
  - $\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle = \langle c + a, d + b\rangle = \langle c, d\rangle + \langle a, b\rangle$;
  - $\langle a, b\rangle + (\langle c, d\rangle + \langle e, f\rangle) = \langle a, b\rangle + \langle c + e, d + f\rangle =$
    $\langle a + (c + e), b + (d + f)\rangle = \langle (a + c) + e, (b + d) + f\rangle =$
    $\langle a + c, b + d\rangle + \langle e, f\rangle = (\langle a, b\rangle + \langle c, d\rangle) + \langle e, f\rangle$.
  - $\lambda(\langle a, b\rangle + \langle c, d\rangle) = \lambda\langle a + c, b + d\rangle = \langle \lambda(a + c), \lambda(b + d)\rangle =$
    $\langle \lambda a + \lambda c, \lambda b + \lambda d\rangle = \langle \lambda a, \lambda b\rangle + \langle \lambda c, \lambda d\rangle = \lambda\langle a, b\rangle + \lambda\langle c, d\rangle$.

Linear Combinations

- A **linear combination** of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is a vector $r\boldsymbol{v} + s\boldsymbol{w}$, where r and s are scalars.
- If $\boldsymbol{v}$ and $\boldsymbol{w}$ are not parallel, then every vector $\boldsymbol{u}$ in the plane can be expressed as a linear combination $\boldsymbol{u} = r\boldsymbol{v} + s\boldsymbol{w}$.
- The parallelogram $\mathcal{P}$ whose vertices are the origin and the terminal points of $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{v} + \boldsymbol{w}$ is called the **parallelogram spanned** by $\boldsymbol{v}$ and $\boldsymbol{w}$. It consists of the linear combinations $r\boldsymbol{v} + s\boldsymbol{w}$, with $0 \le r \le 1$ and $0 \le s \le 1$.





Example

- Express the vector $\boldsymbol{u} = \langle 4, 4 \rangle$ as a linear combination of $\boldsymbol{v} = \langle 6, 2 \rangle$ and $\boldsymbol{w} = \langle 2, 4 \rangle$.

  We must find r and s, such that $r\boldsymbol{v} + s\boldsymbol{w} = \langle 4, 4 \rangle$. This gives $r\langle 6, 2 \rangle + s\langle 2, 4 \rangle = \langle 6r + 2s, 2r + 4s \rangle = \langle 4, 4 \rangle$. The components must be equal, so we have a system of two linear equations:

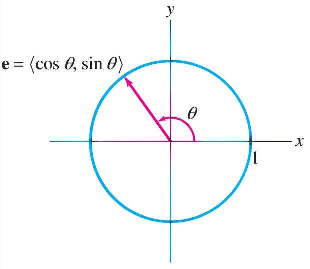
$$\left\{ \begin{array}{rcl} 6r + 2s & = & 4 \\ 2r + 4s & = & 4 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 6r + 2s & = & 4 \\ -r - 2s & = & -2 \end{array} \right\}$$
$$\Rightarrow \left\{ \begin{array}{rcl} s & = & 2 - 3r \\ 5r & = & 2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} s & = & 2 - 3 \cdot \frac{2}{5} = \frac{4}{5} \\ r & = & \frac{2}{5} \end{array} \right\}.$$

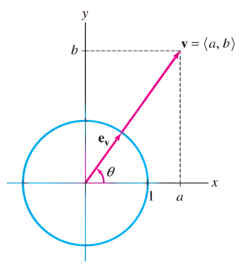
  Therefore, $\boldsymbol{u} = \langle 4, 4 \rangle = \frac{2}{5}\langle 6, 2 \rangle + \frac{4}{5}\langle 2, 4 \rangle$.

Unit Vectors

- A vector of length 1 is called a **unit vector**.
  The head of a unit vector $\boldsymbol{e}$ based at the origin lies on the unit circle and has components $\boldsymbol{e} = \langle \cos\theta, \sin\theta \rangle$, where θ is the angle between $\boldsymbol{e}$ and the positive x-axis.
- We can always scale a nonzero vector $\boldsymbol{v} = \langle a, b \rangle$ to obtain a unit vector pointing in the same direction: $\boldsymbol{e_v} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$.
  If $\boldsymbol{v} = \langle a, b \rangle$ makes an angle θ with the positive x-axis, then

$$\boldsymbol{v} = \langle a, b \rangle = \|\boldsymbol{v}\|\boldsymbol{e_v} = \|\boldsymbol{v}\|\langle \cos\theta, \sin\theta \rangle.$$





Example

- Find the unit vector in the direction of $\boldsymbol{v} = \langle 3, 5 \rangle$.

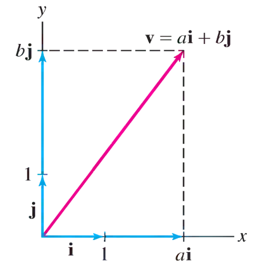
  Compute the magnitude
$$\|\boldsymbol{v}\| = \sqrt{3^2 + 5^2} = \sqrt{34}.$$
  Then, we get
$$\boldsymbol{e}_{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} = \frac{1}{\sqrt{34}}\langle 3, 5 \rangle = \langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle.$$

Standard Basis Vectors

- We introduce a special notation for the unit vectors in the direction of the positive x- and y-axes:

$$\boldsymbol{i} = \langle 1, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1 \rangle.$$

  The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are called the **standard basis vectors**.



- Every vector in the plane is a linear combination of $\boldsymbol{i}$ and $\boldsymbol{j}$:

$$\boldsymbol{v} = \langle a, b \rangle = \langle a, 0 \rangle + \langle 0, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle = a\boldsymbol{i} + b\boldsymbol{j}.$$

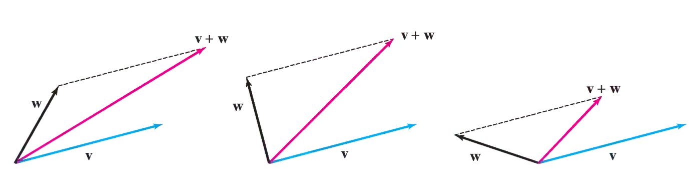
  Example: $\langle 4, -2 \rangle = 4\boldsymbol{i} - 2\boldsymbol{j}$.

- Moreover vector addition is performed by adding the $\boldsymbol{i}$ and $\boldsymbol{j}$ coefficients:

$$(4\boldsymbol{i} - 2\boldsymbol{j}) + (5\boldsymbol{i} + 7\boldsymbol{j}) = (4 + 5)\boldsymbol{i} + (-2 + 7)\boldsymbol{j} = 9\boldsymbol{i} + 5\boldsymbol{j}.$$

Triangle Inequality

- The vector sum $\boldsymbol{v}+\boldsymbol{w}$ for three different vectors $\boldsymbol{w}$ of the same length is shown below.
  Clearly, the length $\|\boldsymbol{v}+\boldsymbol{w}\|$ varies, depending on the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.
  So in general, $\|\boldsymbol{v}+\boldsymbol{w}\|$ is not equal to the sum $\|\boldsymbol{v}\|+\|\boldsymbol{w}\|$.



- **Triangle Inequality:** For any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$,

$$\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\|+\|\boldsymbol{w}\|.$$

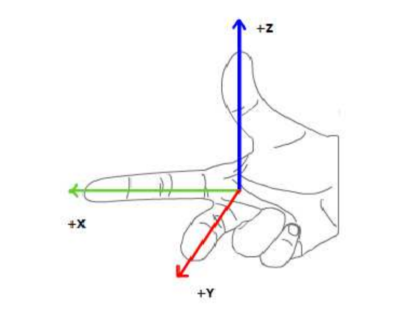
Equality holds only if $\boldsymbol{v}=\mathbf{0}$ or $\boldsymbol{w}=\mathbf{0}$, or if $\boldsymbol{w}=\lambda\boldsymbol{v}$, where $\lambda>0$.

Subsection 2

Vectors in Three Dimensions

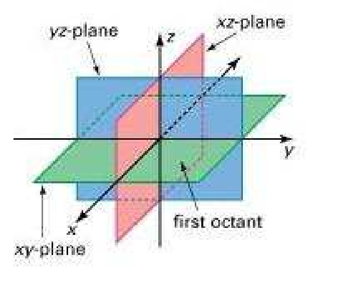
Three-Dimensional Coordinate System

- Fix a point O called the **origin**.
- Through O, fix three mutually perpendicular straight lines, called the **coordinate axes**. These are termed the x-, the y- and the z-**axis**.
- The positive direction on the z-axis is taken so that the **right-hand side rule is satisfied** with respect to the positive directions on the x- and the y-axis:



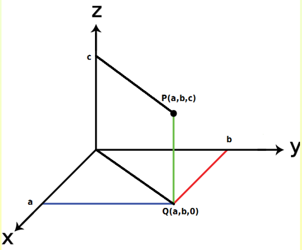
Coordinate Planes and Octants

- The planes formed by each pair of the axes are called the **coordinate planes**. These are the *xy*-, the *xz*- and the *yz*-**planes**.
- Each of the eight regions in which space is separated by the coordinate planes is called an **octant**.
- The **first octant** is the one enclosed by the positive *xy*-, the positive *yz*- and the positive *xz*-planes.



Coordinates of Points

- Let P be a point in space.
  Its **coordinates** are $P = (a, b, c)$ if
  - a is the signed distance from P to the yz-plane;
  - b is the signed distance from P to the xz-plane;
  - c is the signed distance from P to the xy-plane.
- $Q = (a, b, 0)$ is the **projection** of $P(a, b, c)$ on the xy-plane.
- We call the set of all triples $\mathbb{R}^3 = \{(x, y, z) : x, y, z \text{ in } \mathbb{R}\}$ the **three-dimensional space**.



The Plane $z = 3$

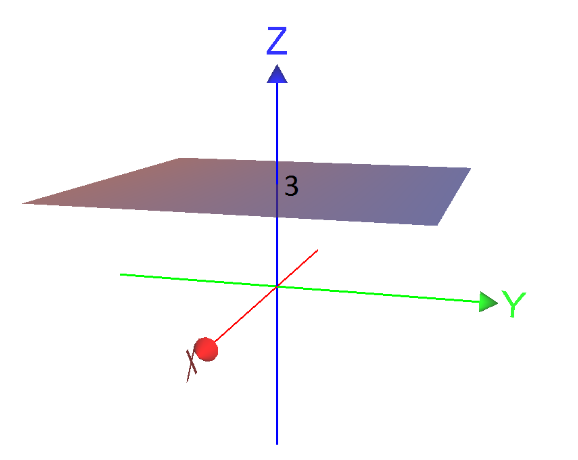


Figure: The Plane $z = 3$.

The Plane $y = 5$

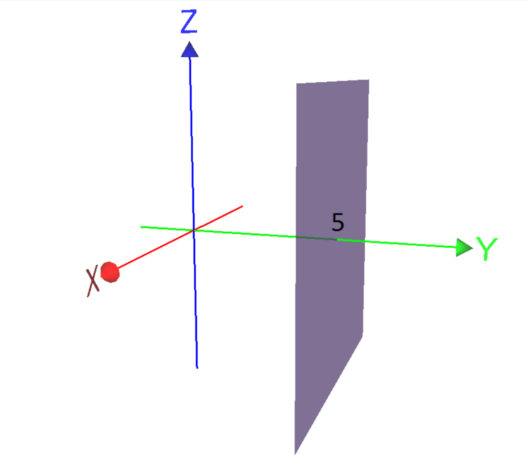


Figure: The Plane $y = 5$.

The Plane $y = x$

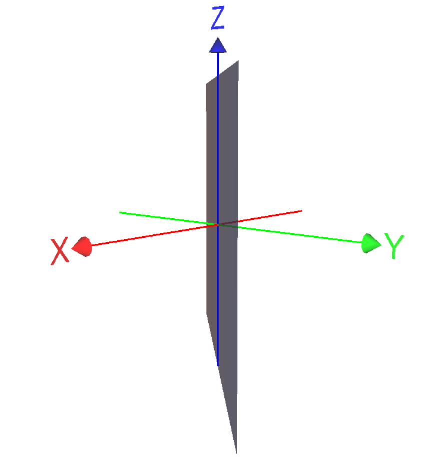
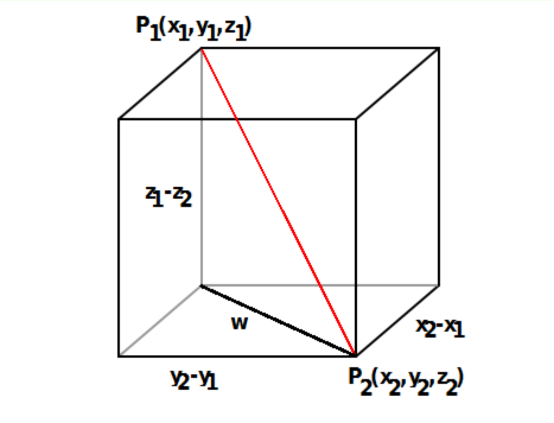


Figure: The Plane $y = x$.

Distance Formula in Three Dimensions: A Figure

- Suppose $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are points in space.



Distance Formula in Three Dimensions

- Suppose $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points in space.
- The distance $|P_1P_2|$ between P_1 and P_2 is calculated by

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example: If $P = (2, -1, 2)$ and $Q = (1, -3, 5)$,

$$\begin{array}{rcl} |PQ| & = & \sqrt{(1-2)^2 + (-3+1)^2 + (5-2)^2} \\ & = & \sqrt{1+4+9} \\ & = & \sqrt{14}. \end{array}$$

Equation of a Sphere

- An equation of a sphere with center $C = (a, b, c)$ and radius R is
$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2.$$
- In particular, if $C = (0, 0, 0)$, then
$$x^2 + y^2 + z^2 = R^2.$$

Example: Find the center and the radius of the sphere with equation
$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0.$$

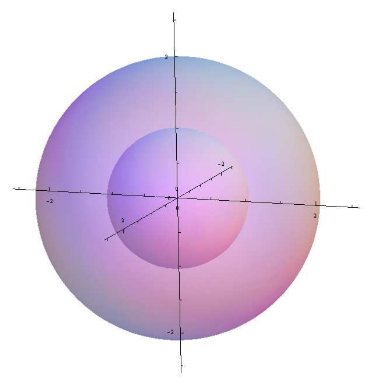
$$\begin{aligned} &x^2 + 4x + y^2 - 6y + z^2 + 2z = -6 \\ &(x^2 + 4x + 4) + (y^2 - 6y + 9) + (z^2 + 2z + 1) = -6 + 14 \\ &(x+2)^2 + (y-3)^2 + (z+1)^2 = (\sqrt{8})^2. \end{aligned}$$

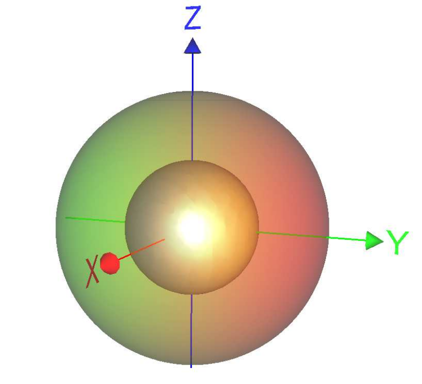
Thus the center is $C = (-2, 3, -1)$ and the radius $R = \sqrt{8}$.

Another Example

- What is the shape of the solid represented by

$$1 \leq x^2 + y^2 + z^2 \leq 4?$$

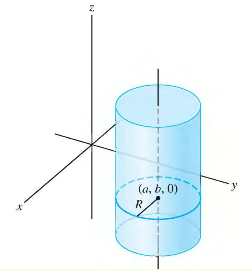




Equation of a Cylinder

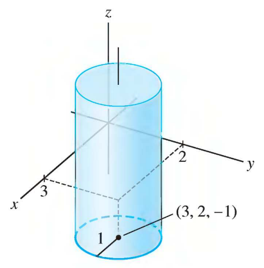
- An equation of a right circular cylinder of radius R, whose central axis is the vertical line through $(a, b, 0)$ is

$$(x-a)^2 + (y-b)^2 = R^2.$$



Example: Find the set of points defined by

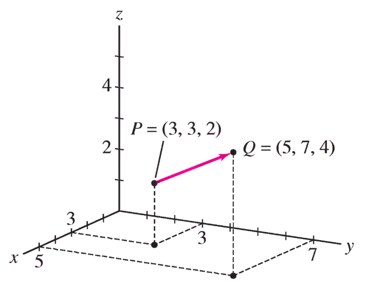
$$(x-3)^2 + (y-2)^2 = 1, \quad z \geq -1.$$

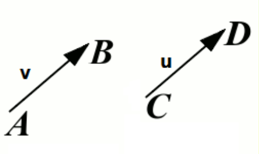


The given equation defines a right circular cylinder of radius 1 with central axis the vertical line through $(3, 2, 0)$. The condition $z \geq -1$ allows only the part of the cylinder on or above the plane $z = -1$.

Vectors in Three-Dimensional Space

- A **vector** has **magnitude** and **direction**.
- The vector $\boldsymbol{v} = \overrightarrow{PQ}$ has **initial point** P and **terminal point** Q.





- If $\boldsymbol{v} = \overrightarrow{AB}$ and $\boldsymbol{u} = \overrightarrow{CD}$, have the same length and direction, they are called **equivalent** (or **equal**), written $\boldsymbol{v} = \boldsymbol{u}$.
- The **zero vector 0** has 0 length and unspecified direction.

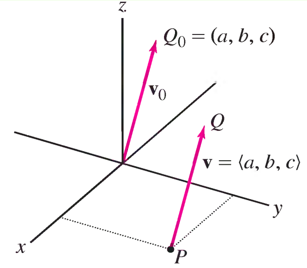
Components

- A vector $\boldsymbol{v} = \overrightarrow{OP}$ whose initial point is the origin and final point is $P = (a, b, c)$ is called the **position vector** of P.

  Then $\boldsymbol{v} = \overrightarrow{OP}$ will be denoted by $\langle a, b, c \rangle$ and the real numbers a, b, c are called the **components** of $\boldsymbol{v}$.

- Every vector $\overrightarrow{PQ}$, for $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$ has an equivalent **position vector** $\boldsymbol{v} = \overrightarrow{OQ_0}$. In that case

$$\boldsymbol{v} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle.$$



Example: Find the position vector equivalent to $\overrightarrow{PQ}$, if $P = (3, -4, -4)$ and $Q = (2, 5, -1)$.

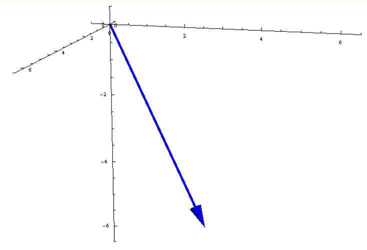
We get $\overrightarrow{OQ_0} = \langle 2 - 3, 5 - (-4), -1 - (-4) \rangle = \langle -1, 9, 3 \rangle$.

Length of a Vector

- Given a vector $\boldsymbol{v} = \overrightarrow{PQ}$, with $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$, its **length** $\|\boldsymbol{v}\| = \|\overrightarrow{PQ}\|$ is given by the formula:

$$\|\overrightarrow{PQ}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

Example: The position vector $\boldsymbol{v} = \langle 2, 3, -5 \rangle$



has length $\|\boldsymbol{v}\| = \sqrt{2^2 + 3^2 + (-5)^2} = \sqrt{38}$.

Remark on Lengths

- Consider $P = (a_1, b_1, c_1)$ and $Q = (a_2, b_2, c_2)$.

  Suppose that the vector $\boldsymbol{v} = \overrightarrow{PQ}$ has equivalent position vector $\overrightarrow{OQ_0}$.

  Then $Q_0 = (a_2 - a_1, b_2 - b_1, c_2 - c_1)$.

  We have

  $$\|\overrightarrow{PQ}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

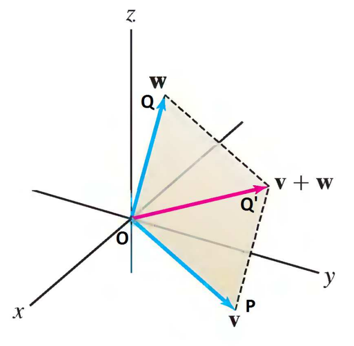
  Furthermore,

  $$\|\overrightarrow{OQ_0}\| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}.$$

- This was to be expected. After all $\overrightarrow{OQ_0} = \overrightarrow{PQ}$!

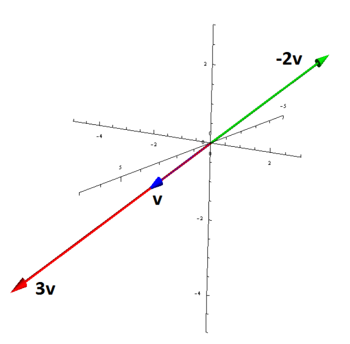
Sum of Vectors

- There are two ways to compute the **sum** $\boldsymbol{v}+\boldsymbol{w}$ of two vectors $\boldsymbol{v}=\overrightarrow{OP}$ and $\boldsymbol{w}=\overrightarrow{OQ}$:
  - **The Triangle Law**: Take $\boldsymbol{w}=\overrightarrow{PQ'}$; Then $\boldsymbol{v}+\boldsymbol{w}=\overrightarrow{OQ'}$.
  - **The Parallelogram Law**: Draw the parallelogram $OPQ'Q$ with sides $\overrightarrow{OP}$ and $\overrightarrow{OQ}$; Then $\boldsymbol{v}+\boldsymbol{w}=\overrightarrow{OQ'}$.



Scalar Multiplication

- The product of a real number λ times the vector $\boldsymbol{v} = \overrightarrow{PQ}$, written $\lambda\boldsymbol{v} = \lambda\overrightarrow{PQ}$, is the vector that has
  - length $|\lambda|\|\boldsymbol{v}\|$;
  - the same direction as $\boldsymbol{v}$, if $\lambda > 0$;
  - the opposite direction of $\boldsymbol{v}$, if $\lambda < 0$.



Addition, Subtraction and Scalar Multiplication

- We add two vectors $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$ component-wise
$$\boldsymbol{a} + \boldsymbol{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle.$$
- We multiply a vector $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ by a real constant λ
$$\lambda \boldsymbol{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle.$$
- In accordance with the definition of subtraction $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$, we then have
$$\boldsymbol{a} - \boldsymbol{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle.$$

Example: $\boldsymbol{a} = \langle 4, 0, 3 \rangle$ and $\boldsymbol{b} = \langle -2, 1, 5 \rangle$.
$\boldsymbol{a} + \boldsymbol{b} = \langle 4, 0, 3 \rangle + \langle -2, 1, 5 \rangle = \langle 2, 1, 8 \rangle$.
$\boldsymbol{a} - \boldsymbol{b} = \langle 4, 0, 3 \rangle - \langle -2, 1, 5 \rangle = \langle 6, -1, -2 \rangle$.
$3\boldsymbol{b} = 3\langle -2, 1, 5 \rangle = \langle -6, 3, 15 \rangle$.
$2\boldsymbol{a} + 5\boldsymbol{b} = 2\langle 4, 0, 3 \rangle + 5\langle -2, 1, 5 \rangle = \langle 8, 0, 6 \rangle + \langle -10, 5, 25 \rangle = \langle -2, 5, 31 \rangle$.

Basis Vectors

- The **basis vectors** are the three vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ of length 1 and pointing towards the positive direction on the x-, the y- and the z-axes, respectively.
- In component form these are:

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1, 0 \rangle, \quad \boldsymbol{k} = \langle 0, 0, 1 \rangle.$$

- Every vector $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ can be written in a unique way as a **linear combination** of the basis vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. We have

$$\begin{aligned} \boldsymbol{a} &= \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle \\ &= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}. \end{aligned}$$

Example: If $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} + 7\boldsymbol{k}$, then:
$2\boldsymbol{a}+3\boldsymbol{b} = 2(\boldsymbol{i}+2\boldsymbol{j}\text{-}3\boldsymbol{k})+3(4\boldsymbol{i}+7\boldsymbol{k}) = (2\boldsymbol{i}+4\boldsymbol{j}\text{-}6\boldsymbol{k})+(12\boldsymbol{i}+21\boldsymbol{k}) = 14\boldsymbol{i}+4\boldsymbol{j}+15\boldsymbol{k}$.

Unit Vectors

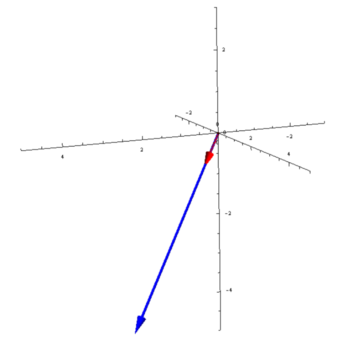
- Find the unit vector $\boldsymbol{e_v}$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$.

  First compute the magnitude:

$$\|\boldsymbol{v}\| = \sqrt{3^2 + 2^2 + (-4)^2} = \sqrt{29}.$$

Then set up

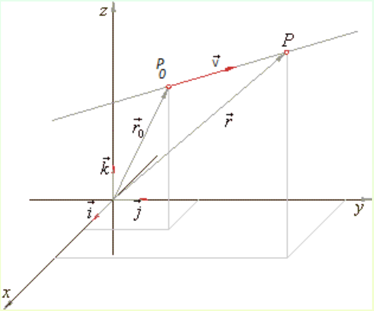
$$\begin{aligned} \boldsymbol{e_v} &= \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} \\ &= \frac{1}{\sqrt{29}}\langle 3, 2, -4\rangle \\ &= \langle \frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{-4}{\sqrt{29}}\rangle. \end{aligned}$$



Equations of Lines

- Given a point $P_0 = (x_0, y_0, z_0)$ in space, with position vector $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$, and a direction vector $\boldsymbol{v}$, the line passing through P_0 with direction $\mathbf{v}$ is given by the parametric vector equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}.$$



- If $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\boldsymbol{v} = \langle a, b, c \rangle$, then the vector equation $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$ gives $\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$.
So we get the three parametric equations:

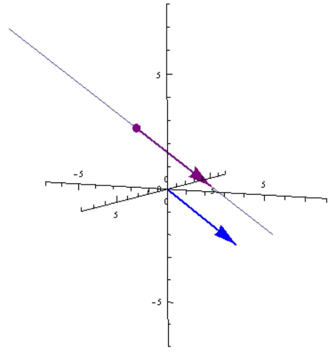
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

An Example of a Straight Line

- Find an equation for the line through $P_0 = (5, 1, 3)$ and parallel to the vector $\boldsymbol{v} = \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$.

  Its vector equation is

  $$\boldsymbol{r} = \langle 5, 1, 3 \rangle + t\langle 1, 4, -2 \rangle.$$

  The corresponding system of parametric equations is

  $$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t.$$

Different Parameterizations of the Same Line

- Show that $\boldsymbol{r}_1(t) = \langle 1,1,0\rangle + t\langle -2,1,3\rangle$ and $\boldsymbol{r}_2(t) = \langle -3,3,6\rangle + t\langle 4,-2,-6\rangle$ parametrize the same line.

  The line $\boldsymbol{r}_1$ has direction vector $\boldsymbol{v} = \langle -2,1,3\rangle$. The line $\boldsymbol{r}_2$ has direction vector $\boldsymbol{w} = \langle 4,-2,-6\rangle$. These vectors are parallel because $\boldsymbol{w} = -2\boldsymbol{v}$. Therefore, the lines described by $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are parallel.

  We must check that they have a point in common. Choose any point on $\boldsymbol{r}_1$, say $P = (1,1,0)$ (corresponding to $t = 0$). This point lies on $\boldsymbol{r}_2$ if there is a t such that $\langle 1,1,0\rangle = \langle -3,3,6\rangle + t\langle 4,-2,-6\rangle$. This yields three equations $\left\{\begin{array}{l} 1 = -3+4t \\ 1 = 3-2t \\ 0 = 6-6t \end{array}\right\}$. All three are satisfied when $t = 1$. Therefore P also lies on $\boldsymbol{r}_2$.

  We conclude that $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ parametrize the same line.

Intersection of Two Lines

- Determine whether the following two lines $\boldsymbol{r}_1(t) = \langle 1, 0, 1 \rangle + t\langle 3, 3, 5 \rangle$, $\boldsymbol{r}_2(t) = \langle 3, 6, 1 \rangle + t\langle 4, -2, 7 \rangle$ intersect.

  The two lines intersect if there exist parameter values t_1 and t_2 such that $\boldsymbol{r}_1(t_1) = \boldsymbol{r}_2(t_2)$. This gives

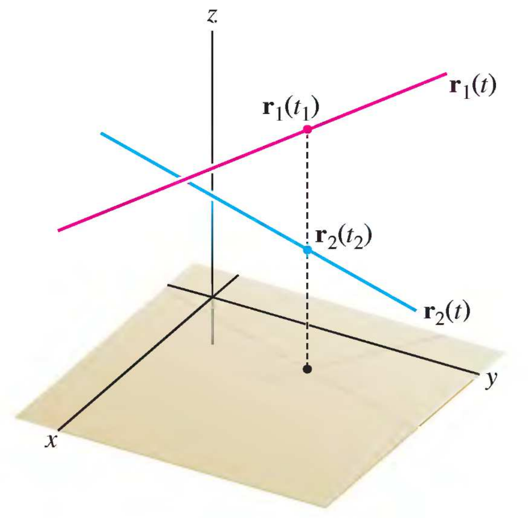
$$\langle 1, 0, 1 \rangle + t_1\langle 3, 3, 5 \rangle = \langle 3, 6, 1 \rangle + t_2\langle 4, -2, 7 \rangle.$$

  This is equivalent to three equations for the components:

$$x = 1 + 3t_1 = 3 + 4t_2, \ y = 3t_1 = 6 - 2t_2, \ z = 1 + 5t_1 = 1 + 7t_2.$$

  Solve the first two equations for t_1 and t_2. We get $t_1 = \frac{14}{9}, \ t_2 = \frac{2}{3}$. These values satisfy the first two equations. However, t_1 and t_2 do not satisfy the third equation $1 + 5 \cdot \frac{14}{9} \neq 1 + 7 \cdot \frac{2}{3}$. Therefore, the lines do not intersect.

The Two Non-Intersecting Lines



Skew Lines

- Two straight lines in space are called **skew lines** if they do not intersect and are not parallel.

  Example: Show that the lines with parametric equations

  $$x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t;$$
  $$x = 2s, \quad y = 3 + s, \quad z = -3 + 4s$$

  are skew lines.

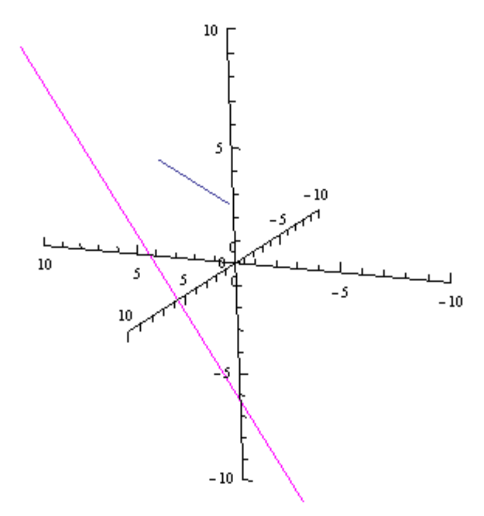
  To see that the lines do not intersect, we try to find a point of intersection by setting

  $$1 + t = 2s, \quad -2 + 3t = 3 + s, \quad 4 - t = -3 + 4s.$$

  The first two taken together give $(t, s) = (\frac{11}{5}, \frac{8}{5})$. These values do not satisfy the third equation! So there is no point of intersection.

  To see that they are not parallel, look at the direction vectors. The first has direction vector $\langle 1, 3, -1 \rangle$ and the second $\langle 2, 1, 4 \rangle$. These are not parallel vectors (Why?).

Plots of the Skew Lines



Line Segment Between Two Points

- Let $P_1 = (a_1, b_1, c_1)$ with position vector $\boldsymbol{r}_1 = \langle a_1, b_1, c_1 \rangle$ and $P_2 = (a_2, b_2, c_2)$ with position vector $\boldsymbol{r}_2 = \langle a_2, b_2, c_2 \rangle$.
  The vector equation of the line segment joining P_1 and P_2 is

$$\boldsymbol{r} = (1-t)\boldsymbol{r}_1 + t\boldsymbol{r}_2, \quad 0 \leq t \leq 1.$$

Example: Find the vector equation and the parametric equations for the line segment joining $P_1 = (10, 3, 1)$ with $P_2 = (5, 6, -3)$.
For the vector equation, we have

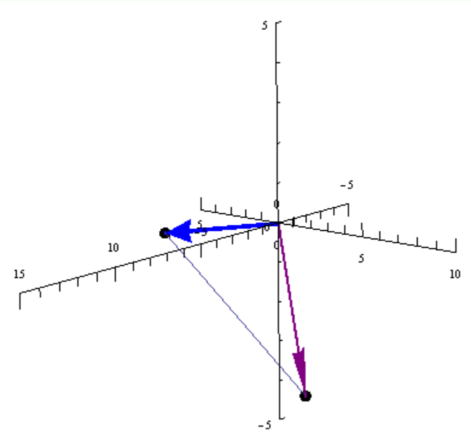
$$\boldsymbol{r} = (1-t)\langle 10, 3, 1 \rangle + t\langle 5, 6, -3 \rangle, \quad 0 \leq t \leq 1.$$

For the parametric equations, we have

$$x = 10 - 5t, \quad y = 3 + 3t, \quad z = 1 - 4t, \quad 0 \leq t \leq 1.$$

The Line Segment

- The line segment joining $(10, 3, 1)$ with $(5, 6, -3)$.



Example

- Parametrize the segment $\overrightarrow{PQ}$ where $P = (1, 0, 4)$ and $Q = (3, 2, 1)$. Find the midpoint of the segment.

  The line through $P = (1, 0, 4)$ and $Q = (3, 2, 1)$ has the parametrization

  $$\boldsymbol{r}(t) = (1 - t)\langle 1, 0, 4\rangle + t\langle 3, 2, 1\rangle = \langle 1 + 2t, 2t, 4 - 3t\rangle.$$

  The segment $\overline{PQ}$ is traced for $0 \leq t \leq 1$.

  The midpoint of $\overline{PQ}$ is the terminal point of the vector

  $$\boldsymbol{r}\left(\frac{1}{2}\right) = \frac{1}{2}\langle 1, 0, 4\rangle + \frac{1}{2}\langle 3, 2, 1\rangle = \langle 2, 1, \frac{5}{2}\rangle.$$

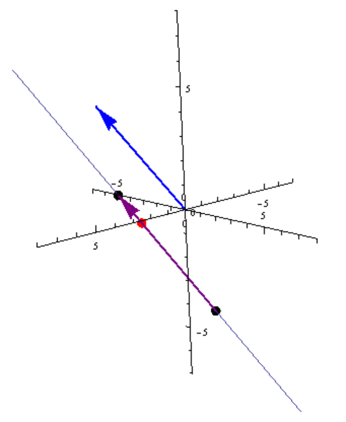
  In other words, the midpoint is $(2, 1, \frac{5}{2})$.

Symmetric Equations

- Given a point $P_0 = (x_0, y_0, z_0)$ in space, with position vector $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$, and a direction vector $\boldsymbol{v} = \langle a, b, c \rangle$, the **symmetric equations** for the straight line in space through P_0 with direction $\boldsymbol{v}$ are
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$
Example: Find the symmetric equations for the line passing through $A = (2, 4, -3)$ and $B = (3, -1, 1)$ and the point at which this line intersects the xy-plane.
The line has the direction of the vector $\overrightarrow{AB} = \langle 1, -5, 4 \rangle$. Therefore, the symmetric equations are:
$$\frac{x - 2}{1} = \frac{y - 4}{-5} = \frac{z + 3}{4}.$$
It intersects the xy-plane when $z = 0$. Therefore, we get $x - 2 = \frac{3}{4}$ and $-\frac{1}{5}(y - 4) = \frac{3}{4}$, which yield $(x, y, z) = (\frac{11}{4}, \frac{1}{4}, 0)$.

Picture of the Line

- The line passing through $A = (2, 4, -3)$ and $B = (3, -1, 1)$ and the point $(\frac{11}{4}, \frac{1}{4}, 0)$ at which this line intersects the xy-plane.



Subsection 3

Dot Product and Angle Between Vectors

The Dot Product

- Recall that the scalar product $\lambda \boldsymbol{v}$ of a real number λ times a vector $\boldsymbol{v}$ is a vector with length $|\lambda|\|\boldsymbol{v}\|$ and direction
  - the same as $\boldsymbol{v}$, if $\lambda > 0$;
  - opposite of $\boldsymbol{v}$, if $\lambda < 0$.
- In contrast, the **dot product** $\boldsymbol{v} \cdot \boldsymbol{w}$ of two vectors $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$ is a real number, defined by

$$\boldsymbol{v} \cdot \boldsymbol{w} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Examples:
$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2 \cdot 3 + 4 \cdot (-1) = 2.$
$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1) \cdot 6 + 7 \cdot 2 + 4 \cdot (-\frac{1}{2}) = -6 + 14 - 2 = 6.$
$(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}) \cdot (2\boldsymbol{i} - \boldsymbol{k}) = 1 \cdot 2 + 2 \cdot 0 + (-3) \cdot (-1) = 2 + 3 = 5.$

Properties of the Dot Product

- **Zero Property:** $\mathbf{0} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \mathbf{0} = 0$.
- **Commutativity:** $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.
- **Pulling out Scalars:** $(\lambda \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{v} \cdot (\lambda \boldsymbol{w}) = \lambda(\boldsymbol{v} \cdot \boldsymbol{w})$.
- **Distributive Law:** $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$.

$$\begin{aligned}
\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) &= \langle a_1, a_2, a_3 \rangle \cdot (\langle b_1, b_2, b_3 \rangle + \langle c_1, c_2, c_3 \rangle) \\
&= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= (a_1 b_1 + a_1 c_1) + (a_2 b_2 + a_2 c_2) + (a_3 b_3 + a_3 c_3) \\
&= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3) \\
&= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle + \langle a_1, a_2, a_3 \rangle \cdot \langle c_1, c_2, c_3 \rangle \\
&= \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}.
\end{aligned}$$

- **Relation with Length:** $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2$.

$$\begin{aligned}
\boldsymbol{v} \cdot \boldsymbol{v} &= \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 \\
&= \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = \|\boldsymbol{v}\|^2.
\end{aligned}$$

The Cosine Identity

- For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, we have
$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta,$$
where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.
- Law of cosines: $\|\boldsymbol{v} - \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$.
We get
$$\begin{aligned} (\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta \\ \boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta \\ \|\boldsymbol{v}\|^2 - 2(\boldsymbol{v} \cdot \boldsymbol{w}) + \|\boldsymbol{w}\|^2 &= \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 - 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta \\ -2(\boldsymbol{v} \cdot \boldsymbol{w}) &= -2\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta \\ \boldsymbol{v} \cdot \boldsymbol{w} &= \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta. \end{aligned}$$

Computing Angle Between Two Vectors

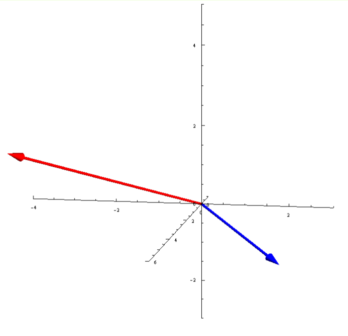
- One of the most useful applications of the Cosine Formula is finding the angle θ between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$:

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|}.$$

Example: Suppose $\boldsymbol{v} = \langle 2, 2, -1\rangle$ and $\boldsymbol{w} = \langle 5, -3, 2\rangle$.

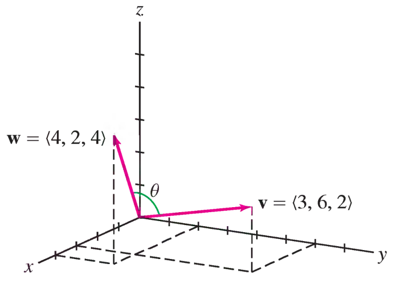
$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{2}{3\sqrt{38}}$. So

$$\theta = \cos^{-1}\frac{2}{3\sqrt{38}} \approx 1.46 \text{ rads}.$$



A Second Example

- Find the angle θ between $\boldsymbol{v} = \langle 3, 6, 2 \rangle$ and $\boldsymbol{w} = \langle 4, 2, 4 \rangle$.



$\|\boldsymbol{v}\| = \sqrt{3^2 + 6^2 + 2^2} = \sqrt{49} = 7$. $\|\boldsymbol{w}\| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6$.

$$\cos\theta = \frac{\langle 3, 6, 2 \rangle \cdot \langle 4, 2, 4 \rangle}{7 \cdot 6} = \frac{3 \cdot 4 + 6 \cdot 2 + 2 \cdot 4}{42} = \frac{16}{21}.$$

The angle is $\theta = \cos^{-1}\left(\frac{16}{21}\right) \approx 0.705$ rad.

Orthogonality

- Two nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are called **perpendicular** or **orthogonal** if the angle between them is $\frac{\pi}{2}$.
  In this case we write $\boldsymbol{v} \perp \boldsymbol{w}$.
- We can use the dot product to test whether $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal. Because an angle between 0 and π satisfies $\cos\theta = 0$ if and only if $\theta = \frac{\pi}{2}$, we see that

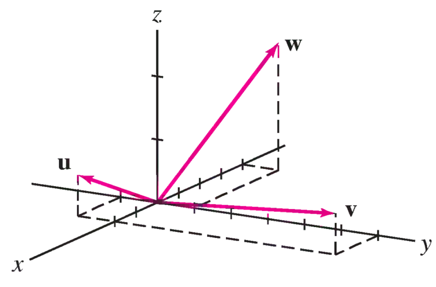
$$\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta = 0 \;\Leftrightarrow\; \theta = \frac{\pi}{2}.$$

  We conclude that $\boldsymbol{v} \perp \boldsymbol{w}$ if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.
  Example: The standard basis vectors are mutually orthogonal and have length 1.
  In terms of dot products, because $\boldsymbol{i} = \langle 1, 0, 0\rangle$, $\boldsymbol{j} = \langle 0, 1, 0\rangle$, and $\boldsymbol{k} = \langle 0, 0, 1\rangle$,

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0, \quad \boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1.$$

Testing for Orthogonality

- Determine whether $\boldsymbol{v} = \langle 2, 6, 1 \rangle$ is orthogonal to $\boldsymbol{u} = \langle 2, -1, 1 \rangle$ or $\boldsymbol{w} = \langle -4, 1, 2 \rangle$.



We test for orthogonality by computing the dot products:

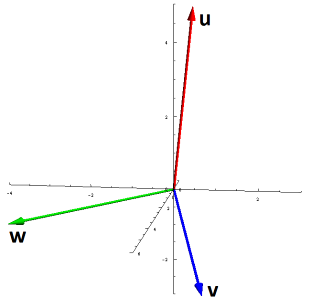
$$\boldsymbol{v} \cdot \boldsymbol{u} = \langle 2, 6, 1 \rangle \cdot \langle 2, -1, 1 \rangle = 2(2) + 6(-1) + 1(1) = -1;$$
$$\boldsymbol{v} \cdot \boldsymbol{w} = \langle 2, 6, 1 \rangle \cdot \langle -4, 1, 2 \rangle = 2(-4) + 6(1) + 1(2) = 0.$$

We conclude $\boldsymbol{v} \not\perp \boldsymbol{u}$, but $\boldsymbol{v} \perp \boldsymbol{w}$.

Testing for Obtuseness

- Determine whether the angles between the vector $\boldsymbol{v} = \langle 3, 1, -2\rangle$ and the vectors $\boldsymbol{u} = \langle \frac{1}{2}, \frac{1}{2}, 5\rangle$ and $\boldsymbol{w} = \langle 4, -3, 0\rangle$ are obtuse.

The angle θ between $\boldsymbol{v}$ and $\boldsymbol{u}$ is obtuse if $\frac{\pi}{2} < 0 \leq \pi$, and this is the case if $\cos\theta < 0$. Since $\boldsymbol{v} \cdot \boldsymbol{u} = \|\boldsymbol{v}\|\|\boldsymbol{u}\| \cos\theta$ and the lengths $\|\boldsymbol{v}\|$ and $\|\boldsymbol{u}\|$ are positive, we see that $\cos\theta$ is negative if and only if $\boldsymbol{v} \cdot \boldsymbol{u}$ is negative.



We have

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{u} &= \langle 3, 1, -2\rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{2}, 5\rangle = \tfrac{3}{2} + \tfrac{1}{2} - 10 = -8 < 0; \\ \boldsymbol{v} \cdot \boldsymbol{w} &= \langle 3, 1, -2\rangle \cdot \langle 4, -3, 0\rangle = 12 - 3 + 0 = 9 > 0. \end{aligned}$$

Thus, the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$ is obtuse, whereas the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is acute.

Using the Distributive Law

- Calculate the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$, where $\boldsymbol{v} = 4\boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{w} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

  Use the Distributive Law and the orthogonality of $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$:

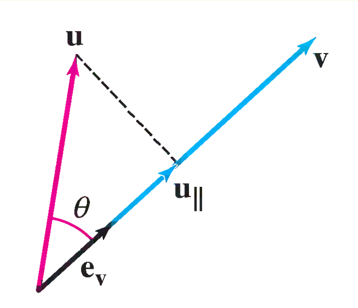
$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &= (4\boldsymbol{i} - 3\boldsymbol{j}) \cdot (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) \\ &= 4\boldsymbol{i} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) - 3\boldsymbol{j} \cdot (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}) \\ &= 4\boldsymbol{i} \cdot \boldsymbol{i} - 3\boldsymbol{j} \cdot (2\boldsymbol{j}) \\ &= 4 - 6 = -2. \end{aligned}$$

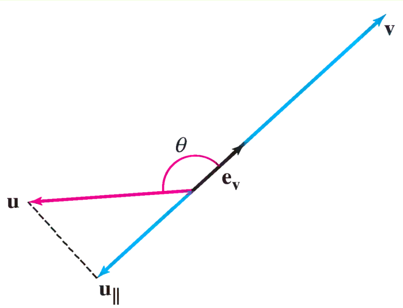
Projection

- Assume $\boldsymbol{v} \neq \mathbf{0}$. The **projection of $\boldsymbol{u}$ along $\boldsymbol{v}$** is the vector

$$\boldsymbol{u}_{\|} = (\boldsymbol{u} \cdot \boldsymbol{e_v})\boldsymbol{e_v} \quad \text{or} \quad \boldsymbol{u}_{\|} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\right)\boldsymbol{v}.$$

The scalar $\boldsymbol{u} \cdot \boldsymbol{e_v}$ is called the **component of $\boldsymbol{u}$ along $\boldsymbol{v}$**.

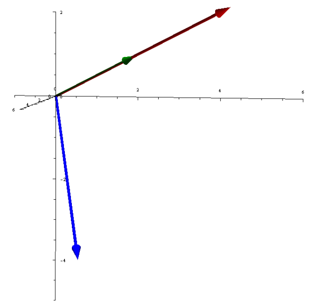




Example

- Find the projection of $\boldsymbol{u} = \langle 5, 1, -3 \rangle$ along $\boldsymbol{v} = \langle 4, 4, 2 \rangle$.

  We use the second formula:



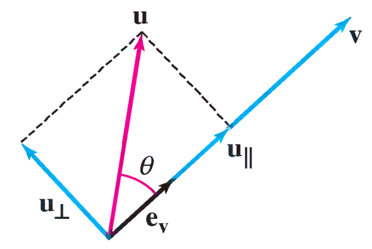
$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= \langle 5, 1, -3 \rangle \cdot \langle 4, 4, 2 \rangle \\ &= 20 + 4 - 6 = 18; \\ \boldsymbol{v} \cdot \boldsymbol{v} &= 4^2 + 4^2 + 2^2 = 36. \end{aligned}$$

  Therefore,

$$\boldsymbol{u}_{\parallel} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}\right)\boldsymbol{v} = \left(\frac{18}{36}\right)\langle 4, 4, 2 \rangle = \langle 2, 2, 1 \rangle.$$

Decomposition of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$

- If $\boldsymbol{v} \neq \mathbf{0}$, then every vector $\boldsymbol{u}$ can be written as the sum of the projection $\boldsymbol{u}_{\|}$ and a vector $\boldsymbol{u}_{\perp}$ that is orthogonal to $\boldsymbol{v}$.



  If we set $\boldsymbol{u}_{\perp} = \boldsymbol{u} - \boldsymbol{u}_{\|}$, then we have

$$\boldsymbol{u} = \boldsymbol{u}_{\|} + \boldsymbol{u}_{\perp}.$$

  This equation is called the **decomposition of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$**.

- We verify that $\boldsymbol{u}_{\perp}$ is orthogonal to $\boldsymbol{v}$:

$$\boldsymbol{u}_{\perp} \cdot \boldsymbol{v} = (\boldsymbol{u} - \boldsymbol{u}_{\|}) \cdot \boldsymbol{v} = (\boldsymbol{u} - (\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}})\boldsymbol{v}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} - (\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}})(\boldsymbol{v} \cdot \boldsymbol{v}) = 0.$$

Example

- Find the decomposition of $\boldsymbol{u} = \langle 5, 1, -3 \rangle$ with respect to $\boldsymbol{v} = \langle 4, 4, 2 \rangle$.
  We showed that $\boldsymbol{u}_{\|} = \langle 2, 2, 1 \rangle$. The orthogonal vector is

$$\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_{\|} = \langle 5, 1, -3 \rangle - \langle 2, 2, 1 \rangle = \langle 3, -1, -4 \rangle.$$

  The decomposition of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$ is

$$\boldsymbol{u} = \langle 5, 1, -3 \rangle = \boldsymbol{u}_{\|} + \boldsymbol{u}_{\perp} = \underbrace{\langle 2, 2, 1 \rangle}_{\text{Projection along } \boldsymbol{v}} + \underbrace{\langle 3, -1, -4 \rangle}_{\text{Orthogonal to } \boldsymbol{v}}.$$

Subsection 4

The Cross Product

The Cross-Product

- Recall that the dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ of two vectors is a scalar quantity, not a vector.
- The **cross-product** $\boldsymbol{v} \times \boldsymbol{w}$ of two vectors $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$, on the other hand, is a vector defined by
$$\boldsymbol{v} \times \boldsymbol{w} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$
- Matrices are useful when dealing with cross-products. Recall that, given a 2×2-matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ its **determinant** is computed by
$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}.$$
- Using this matrix notation, we have
$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}.$$

Cross-Products Using 3×3-Determinants

- In fact, there is a similar formula for determinants of 3×3-matrices:

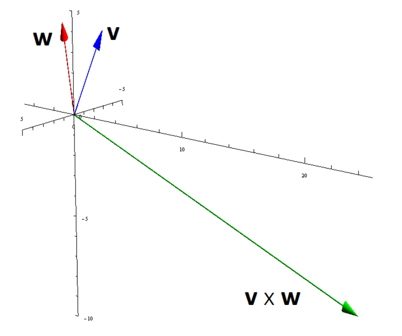
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- In that notation, the cross-product of $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$ can be written more succinctly as

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

Example

- Suppose $\boldsymbol{v} = \langle -2, 1, 4 \rangle$, $\boldsymbol{w} = \langle 3, 2, 5 \rangle$.
  Then
$$\begin{aligned} \boldsymbol{v} \times \boldsymbol{w} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} \boldsymbol{k} \\ &= -3\boldsymbol{i} + 22\boldsymbol{j} - 7\boldsymbol{k}. \end{aligned}$$



Direction of the Cross-Product

Orthogonality

For all vectors $\boldsymbol{v} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3 \rangle$, the vector $\boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$. Moreover, $\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v} \times \boldsymbol{w}\}$ forms a right-handed system.

- To see this compute the dot product:

$$\begin{aligned}
\boldsymbol{v} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= \langle a_1, a_2, a_3 \rangle \cdot \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \\
&= a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= 0.
\end{aligned}$$

Length of the Cross-Product

The length of $\boldsymbol{v} \times \boldsymbol{w}$

For all vectors $\boldsymbol{v}, \boldsymbol{w}$, $\|\boldsymbol{v} \times \boldsymbol{w}\| = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$, with $0 \leq \theta \leq \pi$.

- We have, for $\boldsymbol{v} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{w} = \langle b_1, b_2, b_3\rangle$,

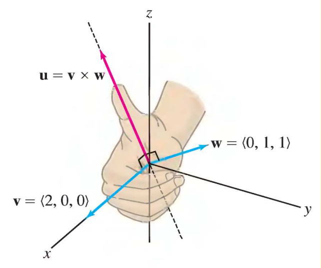
$$\begin{aligned}
\|\boldsymbol{v} \times \boldsymbol{w}\|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
&= \|\boldsymbol{v}\|^2 \|\boldsymbol{w}\|^2 - (\boldsymbol{v} \cdot \boldsymbol{w})^2 \\
&= \|\boldsymbol{v}\|^2 \|\boldsymbol{w}\|^2 - \|\boldsymbol{v}\|^2 \|\boldsymbol{w}\|^2 \cos^2\theta \\
&= \|\boldsymbol{v}\|^2 \|\boldsymbol{w}\|^2 (1 - \cos^2\theta) \\
&= \|\boldsymbol{v}\|^2 \|\boldsymbol{w}\|^2 \sin^2\theta.
\end{aligned}$$

- Thus, if $\boldsymbol{v} \| \boldsymbol{w}$, then $\boldsymbol{v} \times \boldsymbol{w} = \mathbf{0}$.

Using the Geometric Properties

- Let $\boldsymbol{v} = \langle 2,0,0\rangle$ and $\boldsymbol{w} = \langle 0,1,1\rangle$. Determine $\boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$ using the geometric properties of the cross product rather than its definition.
  First, $\boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$. Since $\boldsymbol{v}$ lies along the x-axis, $\boldsymbol{u}$ must lie in the yz-plane, i.e., $\boldsymbol{u} = \langle 0,b,c\rangle$. But $\boldsymbol{u}$ is also orthogonal to $\boldsymbol{w} = \langle 0,1,1\rangle$, so $\boldsymbol{u} \cdot \boldsymbol{w} = b + c = 0$. Thus, $\boldsymbol{u} = \langle 0,b,-b\rangle$.
  Next, we compute $\|\boldsymbol{v}\| = 2$ and $\|\boldsymbol{w}\| = \sqrt{2}$. Furthermore, the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\theta = \frac{\pi}{2}$ since $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

  Thus, $\|\boldsymbol{u}\| = \|\boldsymbol{v} \times \boldsymbol{w}\|$ yields $|b|\sqrt{2} = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\frac{\pi}{2} = 2\sqrt{2}$. So $|b| = 2$, i.e., $b = \pm 2$.
  By the right hand rule $\boldsymbol{u}$ points in the positive z-direction. So $b = -2$. We get $\boldsymbol{u} = \langle 0,-2,2\rangle$.



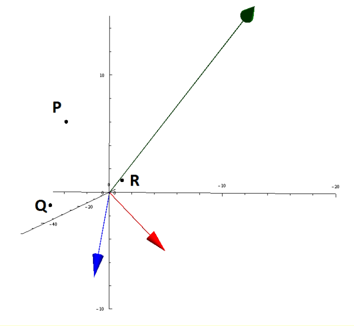
Determining A Vector Perpendicular to a Plane

- Determine a vector that is perpendicular to the plane passing through the points $P=(1,4,6)$, $Q=(-2,5,-1)$ and $R=(1,-1,1)$.

  Note that since P,Q,R are on the plane, the vectors $\overrightarrow{PQ}=\langle -3,1,-7\rangle$ and $\overrightarrow{PR}=\langle 0,-5,-5\rangle$ are also on the plane.

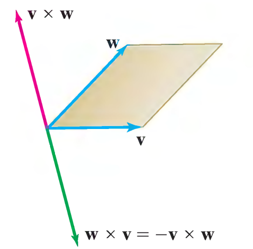
  Therefore, a vector perpendicular to the plane is given by the cross-product

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} \\ &= \langle -40,-15,15\rangle.\end{aligned}$$



Anticommutativity

- The cross product is **anticommutative**, i.e., reversing the order changes the sign:
$$\boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{w}.$$
- To verify this using the definition, note that when we interchange $\boldsymbol{v}$ and $\boldsymbol{w}$, each of the 2×2 determinants changes sign
$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = a_2 b_1 - a_1 b_2 = -(a_1 b_2 - a_2 b_1) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$
- Anticommutativity also follows from the geometric description of the cross product.
  - $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$ are both orthogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$ and have the same length.
  - However, $\boldsymbol{v} \times \boldsymbol{w}$ and $\boldsymbol{w} \times \boldsymbol{v}$ point in opposite directions by the right-hand rule, whence $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$.



Basic Properties of the Cross Product

(i) $\boldsymbol{w} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{w}$;

(ii) $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$;

(iii) $\boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{0}$ if and only if $\boldsymbol{w} = \lambda \boldsymbol{v}$, for some scalar λ, or $\boldsymbol{v} = \boldsymbol{0}$;

(iv) $(\lambda \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{v} \times (\lambda \boldsymbol{w}) = \lambda(\boldsymbol{v} \times \boldsymbol{w})$;

(v) $(\boldsymbol{u} + \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{w} + \boldsymbol{v} \times \boldsymbol{w}$;
$\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$.

- As a special case, we obtain that the cross product of any two of the standard basis vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ is equal to the third, possibly with a minus sign.

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j},$$
$$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}.$$

Using the $\boldsymbol{ijk}$ Relations

- Compute $(2\boldsymbol{i}+\boldsymbol{k})\times(3\boldsymbol{j}+5\boldsymbol{k})$.

  We use the Distributive Law for cross products:

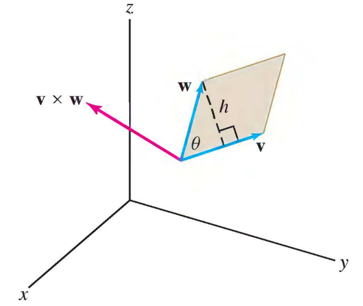
$$\begin{aligned}
&(2\boldsymbol{i}+\boldsymbol{k})\times(3\boldsymbol{j}+5\boldsymbol{k})\\
&=(2\boldsymbol{i})\times(3\boldsymbol{j})+(2\boldsymbol{i})\times(5\boldsymbol{k})+\boldsymbol{k}\times(3\boldsymbol{j})+\boldsymbol{k}\times(5\boldsymbol{k})\\
&=6(\boldsymbol{i}\times\boldsymbol{j})+10(\boldsymbol{i}\times\boldsymbol{k})+3(\boldsymbol{k}\times\boldsymbol{j})+5(\boldsymbol{k}\times\boldsymbol{k})\\
&=6\boldsymbol{k}-10\boldsymbol{j}-3\boldsymbol{i}+5(\boldsymbol{0})\\
&=-3\boldsymbol{i}-10\boldsymbol{j}+6\boldsymbol{k}.
\end{aligned}$$

Area of a Parallelogram

- Consider the parallelogram $\mathcal{P}$ spanned by nonzero vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ with a common basepoint.

  $\mathcal{P}$ has:

  - base $b = \|\boldsymbol{v}\|$;
  - height $h = \|\boldsymbol{w}\| \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

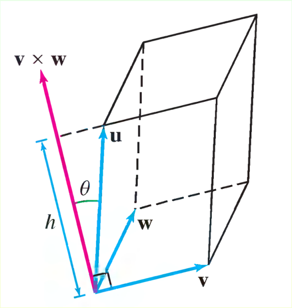
  Therefore, $\mathcal{P}$ has area

  $$A = bh = \|\boldsymbol{v}\|\|\boldsymbol{w}\| \sin\theta = \|\boldsymbol{v} \times \boldsymbol{w}\|.$$

Volume of a Parallelepiped

- Consider the parallelepiped $\boldsymbol{P}$ spanned by three nonzero vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

  The base of $\boldsymbol{P}$ is the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$. So the area of the base is $\|\boldsymbol{v} \times \boldsymbol{w}\|$.
  The height is $h = \|\boldsymbol{u}\| \cdot |\cos\theta|$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v} \times \boldsymbol{w}$.



  Therefore,

$$\text{Volume of } \boldsymbol{P} = (\text{area of base})(\text{height}) = \|\boldsymbol{v} \times \boldsymbol{w}\| \cdot \|\boldsymbol{u}\| \cdot |\cos\theta|.$$

  Thus,

$$\text{Volume of } \boldsymbol{P} = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|.$$

  The quantity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is called the **vector triple product**.

The Vector Triple Product

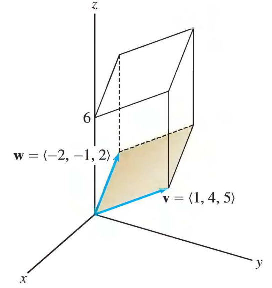
- The vector triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ can be expressed as a determinant.

  Suppose $\boldsymbol{u} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{v} = \langle b_1, b_2, b_3 \rangle$ and $\boldsymbol{w} = \langle c_1, c_2, c_3 \rangle$. Then we have:

$$\begin{aligned}
\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= \boldsymbol{u} \cdot \left(\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \boldsymbol{k} \right) \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \det \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}.
\end{aligned}$$

Example

- Let $\boldsymbol{v} = \langle 1,4,5\rangle$ and $\boldsymbol{w} = \langle -2,-1,2\rangle$. Calculate:
  (a) The area A of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$.
  (b) The volume V of the parallelepiped in the figure.



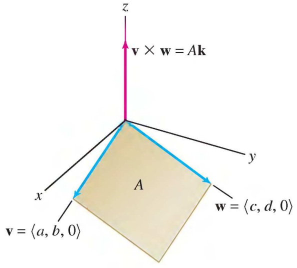
Both the area and the volume require computing the cross product

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 1 & 4 \\ -2 & -1 \end{vmatrix} \boldsymbol{k} = \langle 13,-12,7\rangle.$$

(a) The area of the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ is $A = \|\boldsymbol{v} \times \boldsymbol{w}\| = \sqrt{13^2 + (-12)^2 + 7^2} = \sqrt{362}$.
(b) The vertical leg of the parallelepiped is the vector $6\boldsymbol{k}$. So $V = |(6\boldsymbol{k}) \cdot (\boldsymbol{v} \times \boldsymbol{w})| = |\langle 0,0,6\rangle \cdot \langle 13,-12,7\rangle| = 6(7) = 42$.

Parallelograms on the Plane

- We can compute the area A of the parallelogram spanned by vectors $\boldsymbol{v} = \langle a, b \rangle$ and $\boldsymbol{w} = \langle c, d \rangle$ by regarding $\boldsymbol{v}$ and $\boldsymbol{w}$ as vectors in space with zero component in the z-direction.



We write $\boldsymbol{v} = \langle a, b, 0 \rangle$ and $\boldsymbol{w} = \langle c, d, 0 \rangle$. The cross product $\boldsymbol{v} \times \boldsymbol{w}$ is a vector pointing in the z-direction:

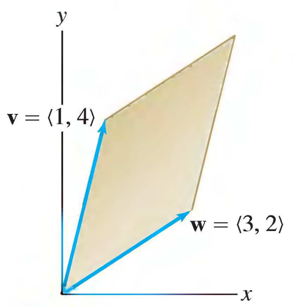
$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = \begin{vmatrix} b & 0 \\ d & 0 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} \boldsymbol{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \boldsymbol{k}.$$

Thus, the parallelogram spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$ has area

$$A = \|\boldsymbol{v} \times \boldsymbol{w}\| = \left| \det \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \right|.$$

Example

- Compute the area A of the parallelogram in the figure



We have

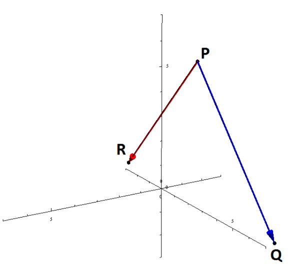
$$A = \left|\det\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}\right| = \left|\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}\right| = |1 \cdot 2 - 3 \cdot 4| = |-10| = 10.$$

Area of Triangle

- Find the area of a triangle with vertices

$$P = (1, 4, 6), Q = (-2, 5, -1), R = (1, -1, 1).$$

This triangle has sides $\overrightarrow{PQ} = \langle -3, 1, -7 \rangle$ and $\overrightarrow{PR} = \langle 0, -5, -5 \rangle$.



Its area is

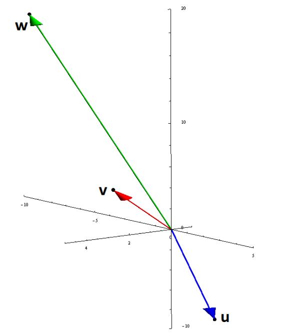
$$A = \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \frac{1}{2} \| \langle -40, -15, 15 \rangle \| = \frac{1}{2} 5\sqrt{82}.$$

An Example of Co-Planar Vectors

- Show that the vectors $\boldsymbol{u} = \langle 1, 4, -7 \rangle$, $\boldsymbol{v} = \langle 2, -1, 4 \rangle$ and $\boldsymbol{w} = \langle 0, -9, 18 \rangle$ are co-planar.

  We show that the vector triple product

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = 0.$$



$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

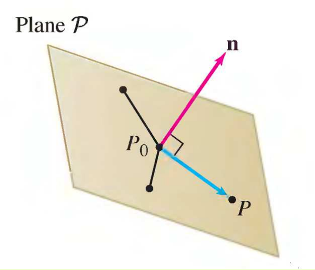
$$= 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} + (-7) \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 0.$$

Subsection 5

Planes in Three-Space

Equation of a Plane

- Consider a plane $\mathcal{P}$ that passes through a point $P_0 = (x_0, y_0, z_0)$ and is orthogonal to a nonzero vector $\boldsymbol{n} = \langle a, b, c\rangle$, called a **normal vector**.
  A point $P = (x, y, z)$ lies on $\mathcal{P}$ precisely when $\overrightarrow{P_0P}$ is orthogonal to $\boldsymbol{n}$.



Therefore, P lies on the plane if $\boldsymbol{n} \cdot \overrightarrow{P_0P} = 0$. In components, $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0\rangle$. So we get the **vector equation**

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

This gives us the following **scalar equation**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Alternative Form

- The plane $\mathcal{P}$ that passes through a point $P_0 = (x_0, y_0, z_0)$ and is orthogonal to a nonzero vector $\boldsymbol{n} = \langle a, b, c \rangle$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This can also be written

$$ax + by + cz = ax_0 + by_0 + cz_0$$

or

$$\boldsymbol{n} \cdot \overrightarrow{OP} = \boldsymbol{n} \cdot \overrightarrow{OP_0}.$$

When we set $d = ax_0 + by_0 + cz_0 = \boldsymbol{n} \cdot \overrightarrow{OP_0}$, the equations become

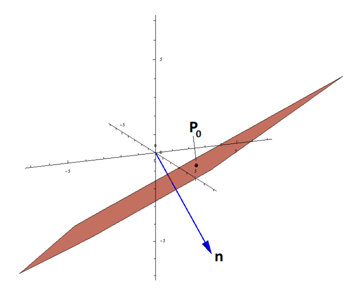
$$\boldsymbol{n} \cdot \langle x, y, z \rangle = d \quad \text{or} \quad ax + by + cz = d.$$

Example

- Find an equation of the plane through $P_0 = (3,1,0)$ with normal vector $\boldsymbol{n} = \langle 3,2,-5 \rangle$.

  We have
  $$3(x-3)+2(y-1)-5(z-0)=0.$$

  We may also compute $d = \boldsymbol{n} \cdot \overrightarrow{OP_0} = \langle 3,2,-5\rangle \cdot \langle 3,1,0\rangle = 11.$



  Then we have
  $$\langle 3,2,-5\rangle \cdot \langle x,y,z\rangle = 11 \quad \text{or} \quad 3x+2y-5z=11.$$

Parallel Planes

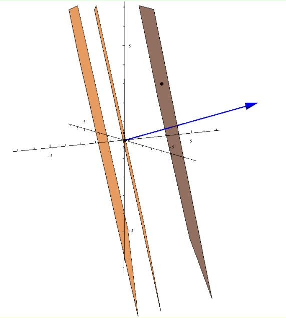
- Let $\mathcal{P}$ have equation $7x - 4y + 2z = -10$. Find equations of the plane parallel to $\mathcal{P}$ passing through (a) the origin and (b) $Q = (2, -1, 3)$.

The given plane has normal

$$\boldsymbol{n} = \langle 7, -4, 2 \rangle.$$

All parallel planes have the same normal. For the first plane, we get

$$7x - 4y + 2z = 0.$$



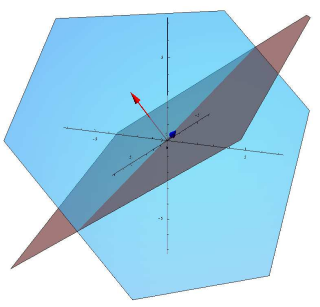
For the second plane, we get $7(x-2) - 4(y+1) + 2(z-3) = 0$ or equivalently

$$7x - 4y + 2z = 24.$$

Angle Between Planes

- To find the angle between two planes:
  - Find normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ of the planes;
  - Compute the angle between $\boldsymbol{n}_1, \boldsymbol{n}_2$ using dot product.

Example: Find the angle between $x+y+z=1$ and $x-2y+3z=1$.



The normals are

$$\begin{aligned} \boldsymbol{n}_1 &= \langle 1,1,1\rangle; \\ \boldsymbol{n}_2 &= \langle 1,-2,3\rangle. \end{aligned}$$

Therefore the angle θ between the normals has

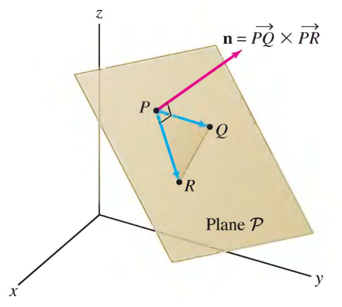
$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{\|\boldsymbol{n}_1\|\|\boldsymbol{n}_2\|} = \frac{1-2+3}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}}.$$

This gives $\theta = \cos^{-1}\frac{2}{\sqrt{42}} \approx 72^\circ$.

Plane Determined By Three Points

- Find an equation of the plane $\mathcal{P}$ determined by the points $P = (1, 0, -1)$, $Q = (2, 2, 1)$, $R = (4, 1, 2)$.

  Find a normal vector: The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane $\mathcal{P}$. So their cross product is normal to $\mathcal{P}$. We have



$$\begin{aligned} \boldsymbol{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 2 \\ 3 & 1 & 3 \end{vmatrix} \\ &= 4\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}. \end{aligned}$$

  Now set up the equation of the plane using any of the three given points:

$$4(x-1) + 3y - 5(z+1) = 0 \quad \text{or} \quad 4x + 3y - 5z = 9.$$

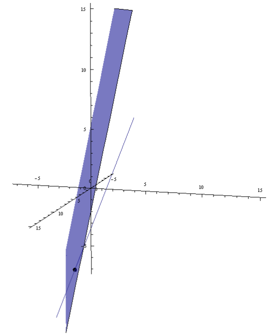
Intersection of a Plane and a Line

- Find the point P where the plane $3x - 9y + 2z = 7$ and the line $\boldsymbol{r}(t) = \langle 1, 2, 1\rangle + t\langle -2, 0, 1\rangle$ intersect.

  The line has parametric equations

  $$x = 1 - 2t,\ \ y = 2,\ \ z = 1 + t.$$

  Substitute in the equation of the plane and solve for t: $3x - 9y + 2z = 3(1 - 2t) - 9(2) + 2(1 + t) = 7$. Simplification yields $-4t - 13 = 7$ or $t = -5$.



  Therefore, P has coordinates

  $$x = 1 - 2(-5) = 11,\ \ y = 2,\ \ z = 1 + (-5) = -4.$$

  The plane and line intersect at the point $P = (11, 2, -4)$.

Equation of Line of Intersection of Two Planes

- To find a set of symmetric equations for the line of intersection between two planes, we need
  - a point on the line;
  - a vector in the direction of the line.

  Example: Find symmetric equations for the line of intersection of $x + y + z = 1$ and $x - 2y + 3z = 1$.

  Set $z = 0$ and solve for x, y to find a point on the line. This gives $(x, y, z) = (1, 0, 0)$. Since the line of intersection lies in both planes, it has a direction perpendicular to both normals $\boldsymbol{n}_1 = \langle 1, 1, 1 \rangle$ and $\boldsymbol{n}_2 = \langle 1, -2, 3 \rangle$. Such a vector is given by the cross-product $\boldsymbol{n}_1 \times \boldsymbol{n}_2$. So we compute

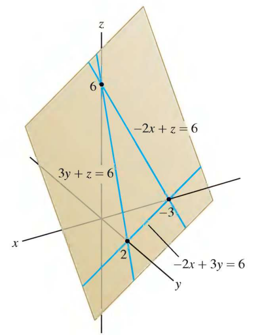
$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}.$$

  Thus, the symmetric equations are $\dfrac{x-1}{5} = \dfrac{y}{-2} = \dfrac{z}{-3}$.

The Traces of a Plane

- The intersection of a plane $\mathcal{P}$ with a coordinate plane or a plane parallel to a coordinate plane is called a **trace**.

  Example: Find the traces of the plane $-2x + 3y + z = 6$ in the coordinate planes.

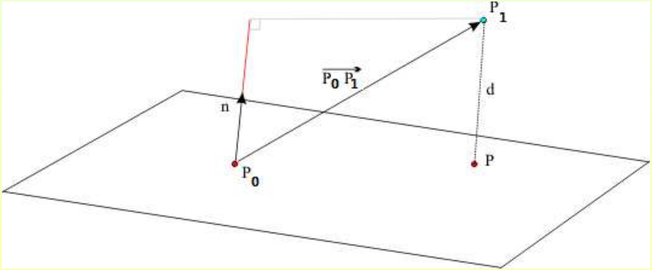
  We obtain the trace in the xy-plane by setting $z = 0$ in the equation of the plane. Thus, the trace is the line $-2x + 3y = 6$ in the xy-plane.

  Similarly, the trace in the xz-plane is obtained by setting $y = 0$, which gives the line $-2x + z = 6$ in the xz-plane.

  Finally, the trace in the yz-plane is $3y + z = 6$.

Distance of Point from a Plane

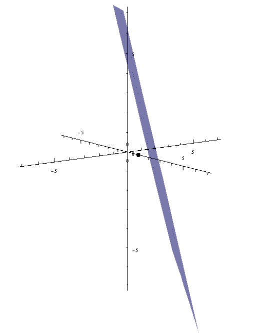
- To calculate the distance of a point $P_1 = (x_1, y_1, z_1)$ from a plane with linear equation $ax + by + cz = d$:
  - Take any point $P_0 = (x_0, y_0, z_0)$ on the plane;
  - Consider the vector $\overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ from P_0 to P_1;
  - Calculate the length $\|\overrightarrow{P_0P_1}_{\|}\|$ of the projection of $\overrightarrow{P_0P_1}$ onto the normal $\boldsymbol{n} = \langle a, b, c \rangle$ of the plane using the dot product.



Example

- Find the distance of $P_1 = (1,0,0)$ from the plane $2x + 3y + z - 5 = 0$.

Consider $P_0 = (0,0,5)$ on the plane. Then, $\overrightarrow{P_0P_1} = \langle 1,0,-5\rangle$. Thus, since $\mathbf{n} = \langle 2,3,1\rangle$, we have



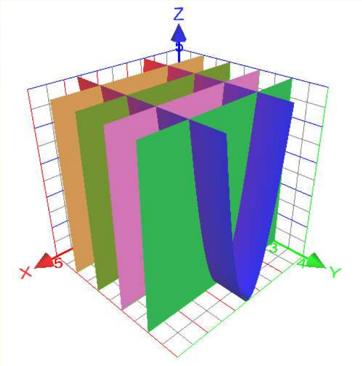
$$\begin{aligned} |\overrightarrow{P_0P_1}_{\|}| &= \frac{|\overrightarrow{P_0P_1}\cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|} \\ &= \frac{|\langle 1,0,-5\rangle \cdot \langle 2,3,1\rangle|}{\|\langle 2,3,1\rangle\|} \\ &= \frac{|2-5|}{\sqrt{4+9+1}} \\ &= \frac{3}{\sqrt{14}}. \end{aligned}$$

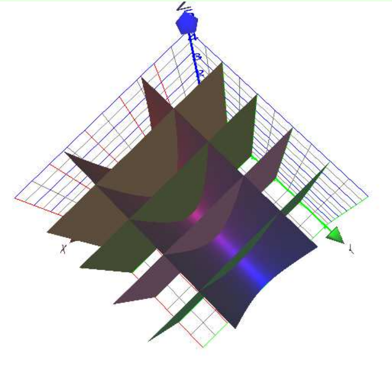
Subsection 6

A Survey of Quadratic Surfaces

Traces or Cross-Sections

- The curves of intersection of a given surface with planes parallel to the coordinate planes are called **traces** or **cross-sections** of the surface.

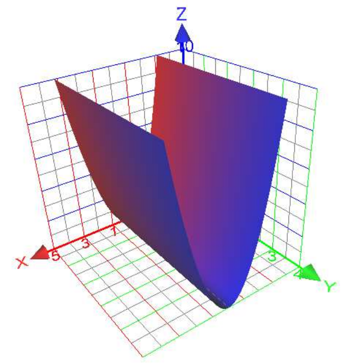


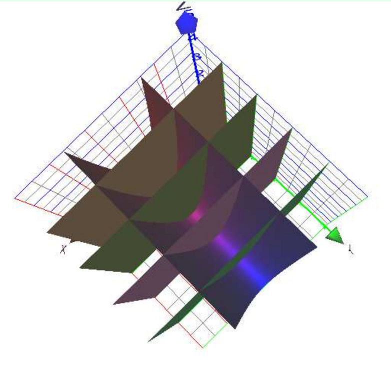


- Traces are very useful in sketching the graph of a 3-dimensional surface.

Parabolic Cylinders

- Consider the surface $z = x^2$.

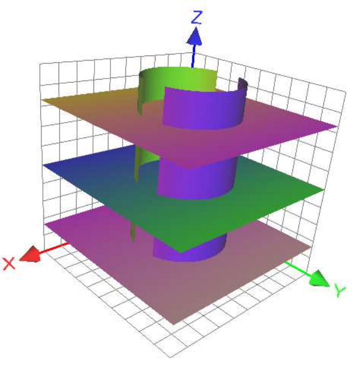




- For planes $y = k$ parallel to the coordinate xz-plane the traces are all curves with equations $z = x^2$, i.e., parabolas with vertex at the xz-origin and opening up.
- The surface $z = x^2$ is called a **parabolic cylinder**.

Cylinders

- Consider the surface $x^2 + y^2 = 1$.



- For planes $z = k$ parallel to the coordinate xy-plane the traces are all curves with equations $x^2 + y^2 = 1$, i.e., circles with center the xy-origin and radius 1.
- The surface $x^2 + y^2 = 1$ is called a **cylinder**.

Quadric Surfaces

- Quadric surfaces are the three dimensional analogs of the two dimensional conic sections, i.e., of parabolas, ellipses and hyperbolas.
- The general equation of a **quadric surface** is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0;$$

- If one translates and rotates the quadric surface, then its equation may be simplified to one of the forms

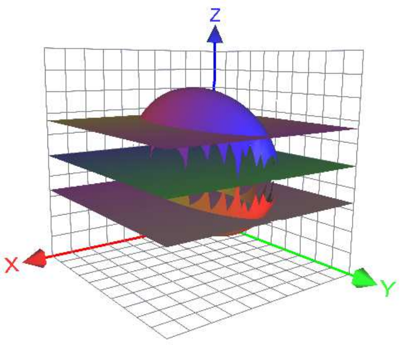
$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

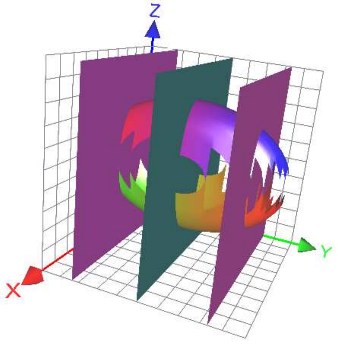
Example: What are the traces of the quadric $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ parallel to the coordinate planes?

On plane $z = k$, the trace is $x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$, which is the equation of an ellipse.

The quadric $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$ (Cont'd)

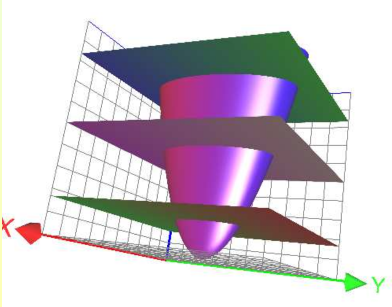
- On plane $y = k$, the trace is $x^2 + \dfrac{z^2}{4} = 1 - \dfrac{k^2}{9}$, which is the equation of an ellipse. On plane $x = k$, the trace is $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1 - k^2$, which is also the equation of an ellipse. Since all traces are ellipses, this surface is called an **ellipsoid**.

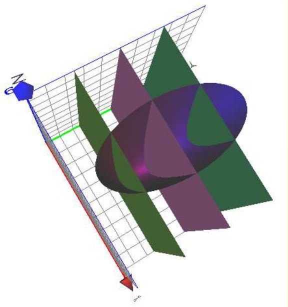




The Quadric Surface $z = 4x^2 + y^2$

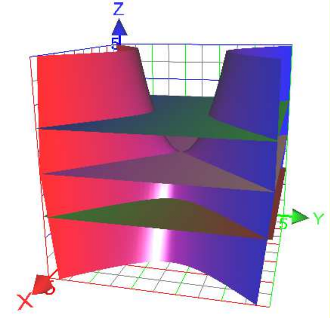
- What are the traces of the quadric $z = 4x^2 + y^2$ parallel to the coordinate planes?
  On plane $z = k$, the trace is $x^2 + \frac{y^2}{4} = \frac{k}{4}$, which is the equation of an ellipse. On plane $y = k$, the trace is $z = 4x^2 + k^2$, which is the equation of a parabola. On plane $x = k$, the trace is $z = y^2 + 4k^2$, which is also the equation of a parabola. This surface is called an **elliptic paraboloid**.

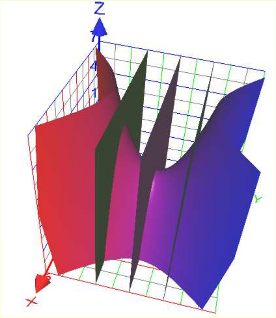




The Quadric Surface $z = y^2 - x^2$

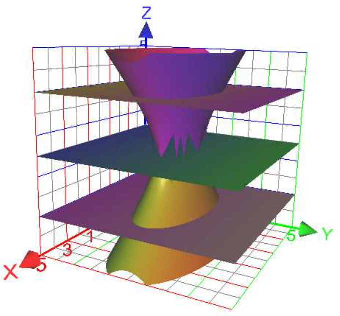
- What are the traces of the quadric $z = y^2 - x^2$ parallel to the coordinate planes?
  On plane $z = k$, the trace is $y^2 - x^2 = k$, which is the equation of a hyperbola. On plane $y = k$, the trace is $z = -x^2 + k^2$, which is the equation of a parabola. On plane $x = k$, the trace is $z = y^2 - k^2$, which is also the equation of a parabola; This surface is called an **hyperbolic paraboloid**.

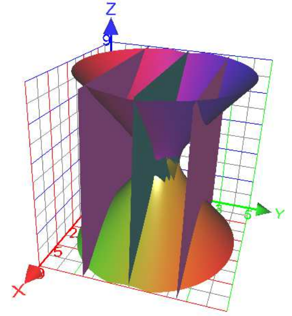




The Quadric Surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$

- What are the traces of the quadric $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$ parallel to the coordinate planes?
  On plane $z = k$, the trace is $\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4}$, which is the equation of an ellipse. On plane $y = k$, the trace is $\frac{x^2}{4} - \frac{z^2}{4} = 1 - k^2$, which is the equation of a hyperbola. On plane $x = k$, the trace is $y^2 - \frac{z^2}{4} = 1 - \frac{k^2}{4}$, which is also the equation of a hyperbola; This surface is called an **hyperboloid of one sheet**.



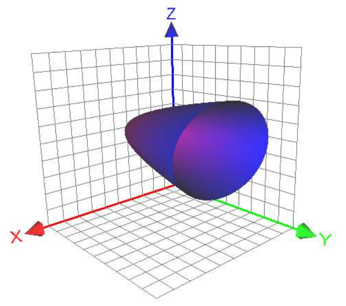


Types of Quadric Surfaces

- **Ellipsoids** with equations $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
- **Elliptic Paraboloids** with equations $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.
- **Hyperbolic Paraboloids** with equations $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.
- **Cones** with equations $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.
- **Hyperboloids of One Sheet** with equations $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
- **Hyperboloid of Two Sheets** with equations $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Identifying a Quadric Surface

- Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.
  Rewrite $x^2 - 6x - y + 2z^2 = -10$. Complete x-square
  $(x-3)^2 - y + 2z^2 = -1$. Separate square terms from linear terms
  $y - 1 = (x-3)^2 + 2z^2$. Divide by 2 and put in standard form
  $\frac{y-1}{(\sqrt{2})^2} = \frac{(x-3)^2}{(\sqrt{2})^2} + z^2$. This has form of an elliptic Paraboloid with vertex $(3,1,0)$ opening in the positive y-direction.



Subsection 7

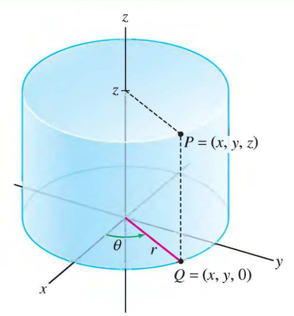
Cylindrical and Spherical Coordinates

Cylindrical Coordinates

- In cylindrical coordinates, we replace the x- and y-coordinates of a point $P = (x, y, z)$ by polar coordinates.
- The **cylindrical coordinates** of $P = (x, y, z)$ are

$$(r, \theta, z),$$

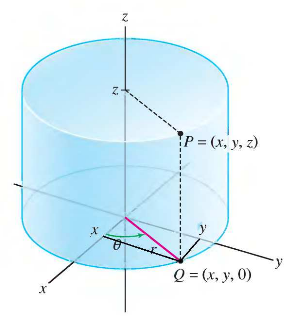
where (r, θ) are polar coordinates of the projection $Q = (x, y, 0)$ of P onto the xy-plane. We usually assume $r \geq 0$.



- Note that the points at fixed distance r from the z-axis make up a cylinder, hence the name "cylindrical coordinates".

Cylindrical and Rectangular

- We convert between rectangular and cylindrical coordinates using the familiar rectangular-polar formulas and we usually assume $r > 0$.



Cylindrical to Rectangular

$$\begin{aligned} x &= r\cos\theta; \\ y &= r\sin\theta; \\ z &= z. \end{aligned}$$

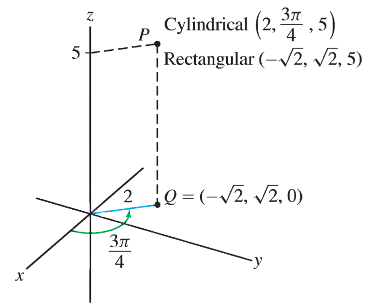
Rectangular to Cylindrical

$$\begin{aligned} r &= \sqrt{x^2+y^2}; \\ \tan\theta &= \frac{y}{x}; \\ z &= z. \end{aligned}$$

From Cylindrical to Rectangular

- Find the rectangular coordinates of the point P with cylindrical coordinates $(r,\theta,z) = (2, \frac{3\pi}{4}, 5)$.

$$\begin{aligned} x &= r\cos\theta = 2\cos\tfrac{3\pi}{4} \\ &= 2(-\tfrac{\sqrt{2}}{2}) = -\sqrt{2}; \\ y &= r\sin\theta = 2\sin\tfrac{3\pi}{4} \\ &= 2(\tfrac{\sqrt{2}}{2}) = \sqrt{2}. \end{aligned}$$



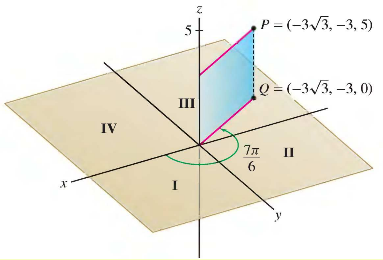
The z-coordinate is unchanged. So $(x,y,z) = (-\sqrt{2}, \sqrt{2}, 5)$.

From Rectangular to Cylindrical

- Find cylindrical coordinates for the point with rectangular coordinates $(x,y,z) = (-3\sqrt{3}, -3, 5)$.

  We have

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(-3\sqrt{3})^2+(-3)^2} = 6. \end{aligned}$$



The angle θ satisfies $\tan\theta = \frac{y}{x} = \frac{-3}{-3\sqrt{3}} = \frac{1}{\sqrt{3}}$. So $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$. The correct choice is $\theta = \frac{7\pi}{6}$ because the projection $Q = (-3\sqrt{3}, -3, 0)$ lies in the third quadrant.

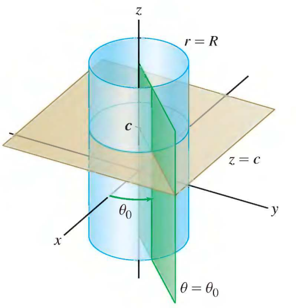
The cylindrical coordinates are $(r, \theta, z) = (6, \frac{7\pi}{6}, 5)$.

Level Surfaces of Cylindrical Coordinates

- The level surfaces of a coordinate system are the surfaces obtained by setting one of the coordinates equal to a constant.
  - In rectangular coordinates, the level surfaces are the planes $x = x_0$, $y = y_0$, and $z = z_0$.
  - In cylindrical coordinates, the level surfaces come in three types.

Level Surfaces in Cylindrical Coordinates:

- $r = R$: Cylinder of radius R with the z-axis as axis of symmetry;
- $\theta = \theta_0$: Half-plane through the z-axis making an angle θ_0 with the xz-plane;
- $z = c$: Horizontal plane at height c.



Equations in Cylindrical Coordinates

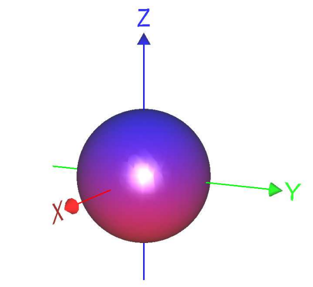
- Find an equation of the form $z = f(r, \theta)$ for the surfaces

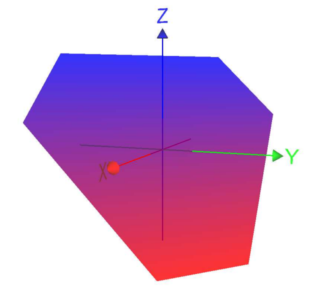
$$(a)\ x^2 + y^2 + z^2 = 9; \quad (b)\ x + y + z = 1.$$

We use the formulas $x^2 + y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$.

(a) The equation $x^2 + y^2 + z^2 = 9$ becomes $r^2 + z^2 = 9$, or $z = \pm\sqrt{9 - r^2}$. This is a sphere of radius 3.

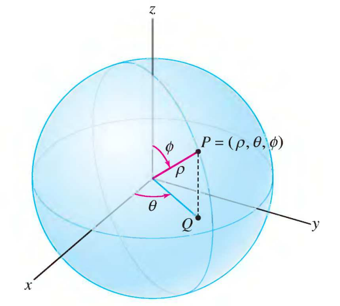
(b) The plane $x + y + z = 1$ becomes $z = 1 - x - y = 1 - r\cos\theta - r\sin\theta$ or $z = 1 - r(\cos\theta + \sin\theta)$.





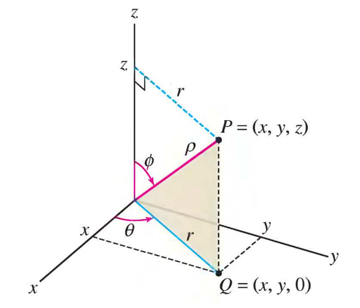
Spherical Coordinates

- Spherical coordinates make use of the fact that a point P on a sphere of radius ρ is determined by two angular coordinates θ and ϕ:
  - θ is the polar angle of the projection Q of P onto the xy-plane;
  - ϕ is the **angle of declination**, which measures how much the ray through P declines from the vertical.



Thus P is determined by the triple (ρ, θ, ϕ), which are called **spherical coordinates**.

Spherical and Rectangular



Spherical to Rectangular

$$\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta; \\ y &= r\sin\theta = \rho\sin\phi\sin\theta; \\ z &= \rho\cos\phi. \end{aligned}$$

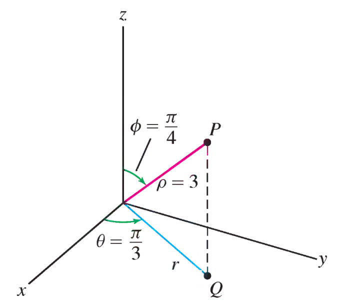
Rectangular to Spherical

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2}; \\ \tan\theta &= \frac{y}{x}; \\ \cos\phi &= \frac{z}{\rho}. \end{aligned}$$

From Spherical to Rectangular

- Find the rectangular coordinates of $P = (\rho, \theta, \phi) = (3, \frac{\pi}{3}, \frac{\pi}{4})$, and find the radial coordinate r of its projection Q onto the xy-plane.

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
&= 3 \sin\tfrac{\pi}{4} \cos\tfrac{\pi}{3} \\
&= 3\tfrac{\sqrt{2}}{2}\tfrac{1}{2} = \tfrac{3\sqrt{2}}{4}. \\
y &= \rho \sin\phi \sin\theta \\
&= 3 \sin\tfrac{\pi}{4} \sin\tfrac{\pi}{3} \\
&= 3\tfrac{\sqrt{2}}{2}\tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{6}}{4}. \\
z &= \rho \cos\phi \\
&= 3 \cos\tfrac{\pi}{4} = \tfrac{3\sqrt{2}}{2}.
\end{aligned}$$

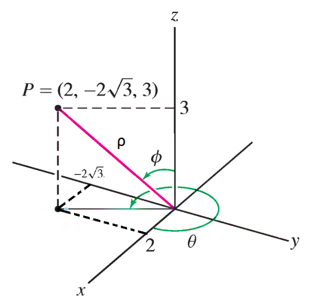


Now consider the projection $Q = (x, y, 0) = (\frac{3\sqrt{2}}{4}, \frac{3\sqrt{6}}{4}, 0)$. The radial coordinate r of Q is $r = \rho \sin\phi = 3 \sin\frac{\pi}{4} = \frac{3\sqrt{2}}{2}$.

From Rectangular to Spherical

- Find the spherical coordinates of the point $P = (x, y, z) = (2, -2\sqrt{3}, 3)$.

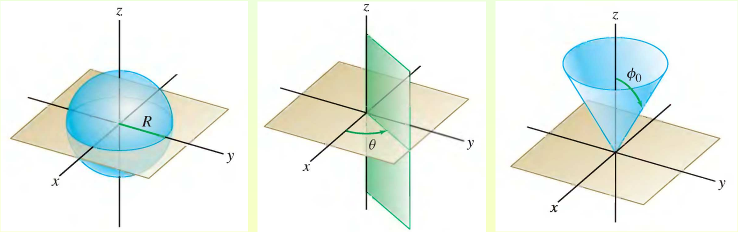
  The radial coordinate is $\rho = \sqrt{2^2 + (-2\sqrt{3})^2 + 3^2} = \sqrt{25} = 5$.
  The angular coordinate θ satisfies $\tan\theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$. Thus, $\theta = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$.
  Since the point $(x, y) = (2, -2\sqrt{3})$ lies in the fourth quadrant, the correct choice is $\theta = \frac{5\pi}{3}$.



  Finally, $\cos\phi = \frac{z}{\rho} = \frac{3}{5}$. Thus, $\phi = \cos^{-1}\frac{3}{5}$.
  Therefore, P has spherical coordinates $(5, \frac{5\pi}{3}, \cos^{-1}\frac{3}{5})$.

Level Surfaces of Cylindrical Coordinates

- There are three types of level surfaces in spherical coordinates.
  - $\rho = R$: Sphere of radius R;
  - $\theta = \theta_0$: Vertical half-plane at angle θ_0 from x-axis;
  - If $\phi \neq 0, \frac{\pi}{2}, \pi$, $\phi = \phi_0$ is the right circular cone consisting of points P such that $\overline{OP}$ makes an angle ϕ_0 with the z-axis.



There are three exceptional cases:

- $\phi = \frac{\pi}{2}$ defines the xy-plane;
- $\phi = 0$ is the positive z-axis;
- $\phi = \pi$ is the negative z-axis.

Equations in Spherical

- Find an equation of the form $\rho = f(\theta, \phi)$ for the following surfaces:

$$(a)\ x^2 + y^2 + z^2 = 9 \quad (b)\ z = x^2 - y^2.$$

(a) The equation $x^2 + y^2 + z^2 = 9$ defines the sphere of radius 3 centered at the origin. We know $\rho^2 = x^2 + y^2 + z^2$. So the equation in spherical coordinates is $\rho = 3$.

(b) To convert $z = x^2 - y^2$ to spherical coordinates, we substitute the formulas for x, y, and z in terms of ρ, θ, and ϕ:

$$\begin{aligned} & \rho\cos\phi = (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2 \\ \Rightarrow\ & \cos\phi = \rho\sin^2\phi(\cos^2\theta - \sin^2\theta) \\ \Rightarrow\ & \cos\phi = \rho\sin^2\phi\cos 2\theta. \end{aligned}$$

Solving for ρ, we obtain

$$\rho = \frac{\cos\phi}{\sin^2\phi\cos 2\theta}.$$