Calculus III

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LSSU Math 251

Subsection 1

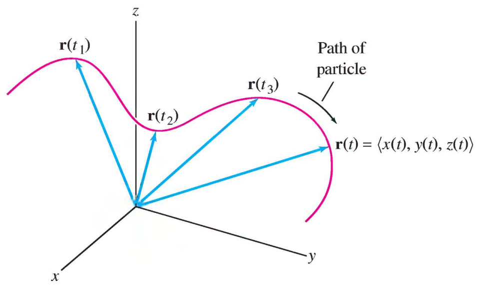
Vector-Valued Functions

Vector Functions

- A **vector-valued function** or **vector function**

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

has domain a set of real numbers $\mathcal{D}$ and range a set of vectors.



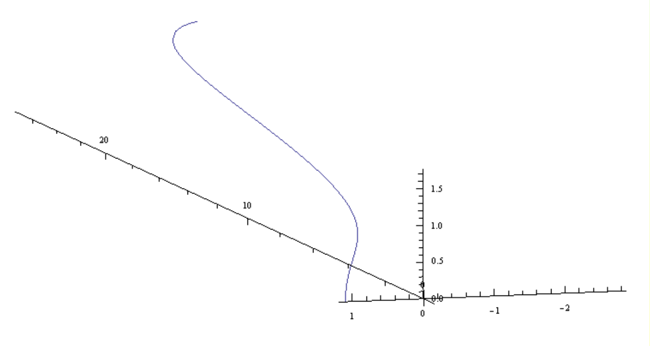
- t is called the **parameter**.
- The functions $x(t), y(t)$ and $z(t)$ giving the components of $\boldsymbol{r}(t)$ are called the **component** or **coordinate functions** of $\boldsymbol{r}(t)$.

Example of a Vector Function

- For the vector function $\boldsymbol{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$, the component functions are

$$x(t) = t^3, \quad y(t) = \ln(3-t), \quad z(t) = \sqrt{t}.$$

$\boldsymbol{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$ has domain $[0,3)$.



Domains of Vector Functions

- Find the domains of the following vector functions:

$$(a)\ \boldsymbol{r}(t) = \langle t^2, e^t, 4 - 7t \rangle; \quad (b)\ \boldsymbol{r}(s) = \langle \sqrt{s}, e^s, \frac{1}{s} \rangle.$$

(a) All three component functions have domain $\mathbb{R}$.
Therefore $\boldsymbol{r}(t)$ has domain $\mathcal{D} = \mathbb{R}$.

(b) $x(s)$ has domain $[0, \infty)$.
$y(s)$ has domain $\mathbb{R}$.
$z(s)$ has domain $\mathbb{R} - \{0\}$.
Therefore $\boldsymbol{r}(s)$ has domain $\mathcal{D} = (0, \infty)$.

Vector Functions and Space Curves

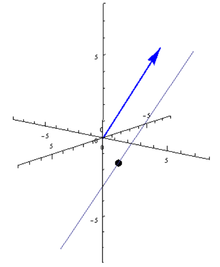
- A vector function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$ may also be viewed as providing **parametric equations**

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

  with **parameter** t, defining a **parametric curve in space**.

- If this point of view is taken, then the vector $\boldsymbol{r}(t)$ is the position vector of a particle moving on the space curve defined by the system of the corresponding parametric equations.

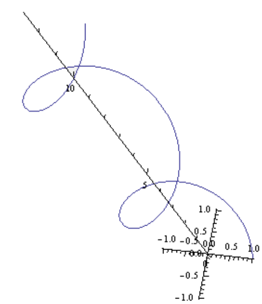
  Example: What is the curve defined by the vector function $\boldsymbol{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle$?
  This is the equation of the straight line passing through the point $(1, 2, -1)$ and having direction vector $\mathbf{v} = \langle 1, 5, 6\rangle$.



The Helix $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle$

- The curve $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle$ represents the orbit of a particle moving counterclockwise on the surface of a cylinder with base the unit circle.

  This is shown in the following figure, drawn from $t = 0$ to $t = 4\pi$. This curve is called a **helix**.



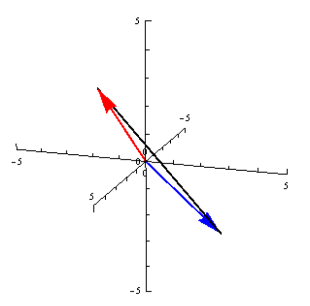
Vector Equations of Line Segments

- Find a vector equation and parametric equations for the line segment joining the point $P = (1, 3, -2)$ to the point $Q = (2, -1, 3)$.
- The two points have position vectors $\boldsymbol{r}_0 = \langle 1, 3, -2 \rangle$ and $\boldsymbol{r}_1 = \langle 2, -1, 3 \rangle$, respectively. Thus, the vector equation of the line segment joining them is $\boldsymbol{r} = (1 - t)\boldsymbol{r}_0 + t\boldsymbol{r}_1$, $0 \leq t \leq 1$.

i.e., $\boldsymbol{r} = (1 - t)\langle 1, 3, -2 \rangle + t\langle 2, -1, 3 \rangle$
$= \langle 1 + t, 3 - 4t, -2 + 5t \rangle$.
The corresponding parametric equations are

$$x = 1 + t, y = 3 - 4t, z = -2 + 5t,$$

for $0 \leq t \leq 1$.

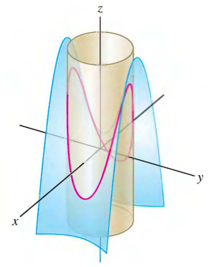


Parametrizing a Curve (Using a Variable for t)

- Parametrize the curve $\mathcal{C}$ obtained as the intersection of the surfaces $x^2 - y^2 = z - 1$ and $x^2 + y^2 = 4$.

  Method 1: Solve the given equations for y and z in terms of x.
  First, solve for y: $x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2 \Rightarrow y = \pm\sqrt{4 - x^2}$.
  The equation $x^2 - y^2 = z - 1$ can be written $z = x^2 - y^2 + 1$. Thus, we can substitute $y^2 = 4 - x^2$ to solve for z:
  $z = x^2 - y^2 + 1 = x^2 - (4 - x^2) + 1 = 2x^2 - 3$.
  Now use $t = x$ as the parameter. Then $y = \pm\sqrt{4 - t^2}$, $z = 2t^2 - 3$.
  The two signs of the square root correspond to the two halves of the curve where $y > 0$ and $y < 0$. Therefore, we need two vector-valued functions:

  $$\boldsymbol{r}_1(t) = \langle t, \sqrt{4 - t^2}, 2t^2 - 3\rangle,$$
  $$\boldsymbol{r}_2(t) = \langle t, -\sqrt{4 - t^2}, 2t^2 - 3\rangle,$$
  $$-2 \leq t \leq 2.$$



Parametrizing a Curve (Using Trigonometry)

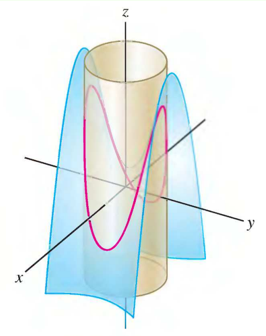
- Parametrize the curve $\mathcal{C}$ obtained as the intersection of the surfaces $x^2 - y^2 = z - 1$ and $x^2 + y^2 = 4$.

  Method 2: Note that $x^2 + y^2 = 4$ has a trigonometric parametrization:

  $$x = 2\cos t, \; y = 2\sin t, \; 0 \leq t < 2\pi.$$

  The equation $x^2 - y^2 = z - 1$ gives us $z = x^2 - y^2 + 1 = 4\cos^2 t - 4\sin^2 t + 1 = 4\cos 2t + 1$. Thus, we may parametrize the entire curve by a single vector-valued function:

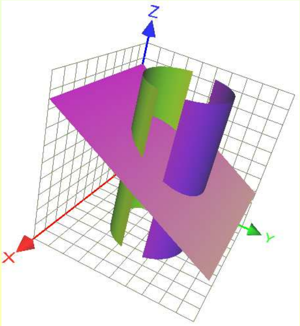
  $$\boldsymbol{r}(t) = \langle 2\cos t, 2\sin t, 4\cos 2t + 1\rangle, \quad 0 \leq t < 2\pi.$$

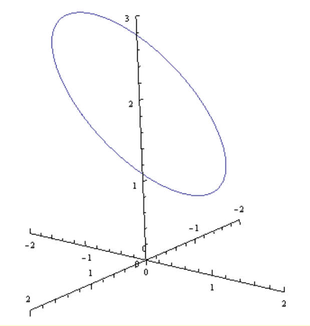


Parametrizing a Curve

- Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.
  If we set $x = \cos t$, $y = \sin t$, then, automatically, $x^2 + y^2 = 1$. Also, since $y + z = 2$, we get that $z = 2 - y = 2 - \sin t$. Therefore the required vector function is

$$\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + (2 - \sin t)\boldsymbol{k}, \ 0 \leq t \leq 2\pi.$$

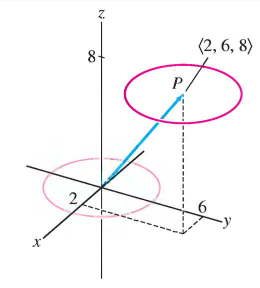




Additional Example

- Parametrize the circle of radius 3 with center $P = (2, 6, 8)$ located in a plane parallel to the xy-plane.

  A circle of radius R in the xy-plane centered at the origin has parametrization $\langle R\cos t, R\sin t\rangle$. We place it in 3 dimensions $\langle R\cos t, R\sin t, 0\rangle$. The circle of radius 3 centered at $(0, 0, 0)$ has parametrization $\langle 3\cos t, 3\sin t, 0\rangle$.



  We move the center to $P = (2, 6, 8)$ by translating by the vector $\langle 2, 6, 8\rangle$:

$$\boldsymbol{r}(t) = \langle 2, 6, 8\rangle + \langle 3\cos t, 3\sin t, 0\rangle = \langle 2 + 3\cos t, 6 + 3\sin t, 8\rangle.$$

Subsection 2

Calculus of Vector-Valued Functions

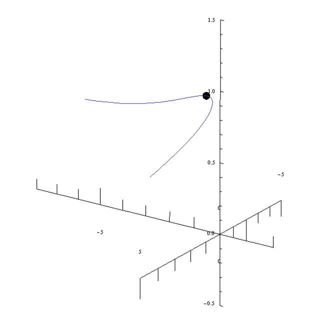
Limits of Vector Functions

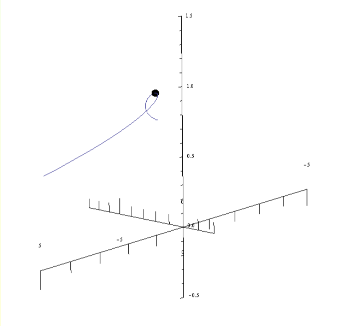
- A vector valued function $\boldsymbol{r}(t)$ approaches the **limit** $\boldsymbol{u}$ as t approaches t_0 if
$$\lim_{t \to t_0} \|\boldsymbol{r}(t) - \boldsymbol{u}\| = 0.$$
In this case, we write $\lim_{t \to t_0} \boldsymbol{r}(t) = \boldsymbol{u}$.
- The limit of a vector function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$ as $t \to t_0$ is given by
$$\lim_{t \to t_0} \boldsymbol{r}(t) = \langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \rangle.$$

Example

- What is the limit $\lim_{t\to 0} \boldsymbol{r}(t)$, if $\boldsymbol{r}(t) = (1+t^3)\boldsymbol{i} + te^{-t}\boldsymbol{j} + \frac{\sin t}{t}\boldsymbol{k}$?

  Since
  $$\lim_{t\to 0}(1+t^3) = 1, \quad \lim_{t\to 0} te^{-t} = 0, \quad \lim_{t\to 0} \frac{\sin t}{t} = 1,$$
  we have that $\lim_{t\to 0} \boldsymbol{r}(t) = \boldsymbol{i} + \boldsymbol{k}$.





Continuity of Vector Functions

- A vector function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$ is **continuous at** $t = t_0$ if
$$\lim_{t\to t_0} \boldsymbol{r}(t) = \boldsymbol{r}(t_0).$$
- Since $\lim_{t\to t_0} \boldsymbol{r}(t) = \langle \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\rangle$ and $\boldsymbol{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$, we have that $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$ is continuous at $t = t_0$ if and only if
$$\lim_{t\to t_0} x(t) = x(t_0), \quad \lim_{t\to t_0} y(t) = y(t_0), \quad \lim_{t\to t_0} z(t) = z(t_0).$$
This shows that $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$ is continuous at $t = t_0$ if and only if all three component functions $x(t), y(t)$ and $z(t)$ are continuous at $t = t_0$.

Derivatives of Vector Functions

- The derivative $\boldsymbol{r}'(t)$ of a vector function $\boldsymbol{r}(t)$ is defined similarly to the derivative of ordinary functions:

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{r}'(t) = \lim_{h \to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}.$$

- The geometric interpretation of $\boldsymbol{r}'(t_0)$ is also similar: It is a vector tangent to the curve at the point determined by $\boldsymbol{r}(t_0)$.
- For this reason, $\boldsymbol{r}'(t_0)$ is called the **tangent vector** to the curve at the point with position vector $\boldsymbol{r}(t_0)$, provided, of course, that $\boldsymbol{r}'(t_0)$ exists and $\boldsymbol{r}'(t_0) \neq \mathbf{0}$.
- The **tangent line** to the curve at $t = t_0$ goes through $\boldsymbol{r}(t_0)$ and has direction $\boldsymbol{r}'(t_0)$. Thus, it has equation $\ell(t) = \boldsymbol{r}(t_0) + t\boldsymbol{r}'(t_0)$.
- Finally, the **unit tangent vector** $\mathbf{T}(t)$ is defined by

$$\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}.$$

Calculating the Derivative of a Vector Function

Coordinate-Wise Calculation of $\boldsymbol{r}(t)$

If $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, and x, y and z are differentiable, then $\boldsymbol{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$.

Example: Find the derivative of $\boldsymbol{r}(t) = (1+t^3)\boldsymbol{i} + te^{-t}\boldsymbol{j} + \sin 2t\boldsymbol{k}$ and the unit tangent vector at the point where $t = 0$.
We have

$$\begin{aligned} \boldsymbol{r}'(t) &= (1+t^3)'\boldsymbol{i} + (te^{-t})'\boldsymbol{j} + (\sin 2t)'\boldsymbol{k} \\ &= 3t^2\boldsymbol{i} + (e^{-t} - te^{-t})\boldsymbol{j} + 2\cos 2t\boldsymbol{k}. \end{aligned}$$

Therefore, at $t = 0$, $\boldsymbol{r}'(0) = \boldsymbol{j} + 2\boldsymbol{k}$ and $\|\boldsymbol{r}'(0)\| = \sqrt{5}$.
This shows that $\mathbf{T}(0) = \frac{\boldsymbol{r}'(0)}{\|\boldsymbol{r}'(0)\|} = \frac{\boldsymbol{j} + 2\boldsymbol{k}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\boldsymbol{j} + \frac{2}{\sqrt{5}}\boldsymbol{k}$.

More Derivatives of Vector Functions

- Calculate:
  - (a) $\boldsymbol{v}'(t)$ if $\boldsymbol{v}(t) = \langle t^2, t^3, \sin t\rangle$;
  - (b) $\boldsymbol{u}'(\frac{\pi}{2})$ if $\boldsymbol{u}(t) = \langle \cos t, -1, e^{2t}\rangle$;
  - (c) $\boldsymbol{w}'(3)$ if $\boldsymbol{w}(t) = \langle \ln t, t, t^2\rangle$.

(a) We have $\boldsymbol{v}'(t) = \langle (t^2)', (t^3)', (\sin t)'\rangle = \langle 2t, 3t^2, \cos t\rangle$.

(b) We have $\boldsymbol{u}'(t) = \langle (\cos t)', (-1)', (e^{2t})'\rangle = \langle -\sin t, 0, 2e^{2t}\rangle$.
Therefore, $\boldsymbol{u}'(\frac{\pi}{2}) = \langle -\sin\frac{\pi}{2}, 0, 2e^{2(\pi/2)}\rangle = \langle -1, 0, 2e^{\pi}\rangle$.

(c) We have $\boldsymbol{w}'(t) = \langle (\ln t)', (t)', (t^2)'\rangle = \langle \frac{1}{t}, 1, 2t\rangle$.
Therefore, $\boldsymbol{w}'(3) = \langle \frac{1}{3}, 1, 6\rangle$.

Calculating Tangent Vectors

- Find the position vector $\boldsymbol{r}(1)$ and the tangent vector $\boldsymbol{r}'(1)$ for the curve $\boldsymbol{r}(t) = \sqrt{t}\boldsymbol{i} + (2-t)\boldsymbol{j}$.

  We have
  $$\boldsymbol{r}(1) = \sqrt{1}\boldsymbol{i} + (2-1)\boldsymbol{j} = \boldsymbol{i} + \boldsymbol{j}.$$
  Moreover,
  $$\boldsymbol{r}'(t) = (\sqrt{t})'\boldsymbol{i} + (2-t)'\boldsymbol{j} = \frac{1}{2\sqrt{t}}\boldsymbol{i} - \boldsymbol{j}.$$
  Therefore, $\boldsymbol{r}'(1) = \frac{1}{2}\boldsymbol{i} - \boldsymbol{j}$.

Calculating Tangent Lines

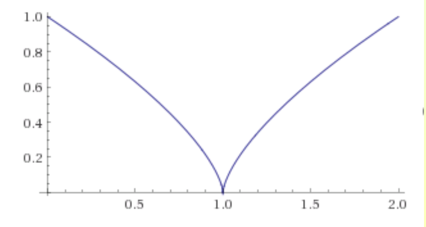
- Find parametric equations for the tangent line to the helix with parametric equations
$$x = 2\cos t,\ y = \sin t,\ z = t$$
at the point $(0, 1, \frac{\pi}{2})$.
The given point is the point corresponding to the position vector $\boldsymbol{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle$. The tangent vector is $\boldsymbol{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$, whence at the same point the tangent vector is $\boldsymbol{r}'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$. The line passing through $\boldsymbol{r}(\frac{\pi}{2})$ with direction $\boldsymbol{r}'(\frac{\pi}{2})$ is given by the vector equation
$$\ell(t) = \boldsymbol{r}(\frac{\pi}{2}) + t\boldsymbol{r}'(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2} \rangle + t\langle -2, 0, 1 \rangle = \langle -2t, 1, \frac{\pi}{2} + t \rangle.$$
Its parametric equations are
$$x = -2t,\ y = 1,\ z = \frac{\pi}{2} + t.$$

Second Derivatives and Smooth Curves

- The **second derivative** of the vector function $\boldsymbol{r}(t)$ is the first derivative of its first derivative $\boldsymbol{r}''(t) = (\boldsymbol{r}'(t))'$.
- The curve $\boldsymbol{r}(t)$ is called **smooth** on an interval I if
  - $\boldsymbol{r}'(t)$ is continuous;
  - $\boldsymbol{r}'(t) \neq \mathbf{0}$ except possibly at the endpoints of I.
- $\boldsymbol{r}(t)$ is **piece-wise smooth** if it is made up of a finite number of smooth pieces.

Example: The semicubical parabola $\boldsymbol{r}(t) = \langle 1 + t^3, t^2 \rangle$ is not smooth. Why?

Is it piece-wise smooth?



Some Rules for Computing Derivatives

Theorem

Assume that $\boldsymbol{u}, \boldsymbol{v}$ are differentiable vector functions, c is a scalar and f is a real-valued function. Then

1. $(\boldsymbol{u}(t)+\boldsymbol{v}(t))' = \boldsymbol{u}'(t)+\boldsymbol{v}'(t)$ (**Sum Rule**);
2. $(c\boldsymbol{u}(t))' = c\boldsymbol{u}'(t)$ (**Constant Factor Rule**);
3. $(f(t)\boldsymbol{u}(t))' = f'(t)\boldsymbol{u}(t)+f(t)\boldsymbol{u}'(t)$ (**Scalar Product Rule**);
4. $(\boldsymbol{u}(t)\cdot\boldsymbol{v}(t))' = \boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)+\boldsymbol{u}(t)\cdot\boldsymbol{v}'(t)$ (**Dot Product Rule**);
5. $(\boldsymbol{u}(t)\times\boldsymbol{v}(t))' = \boldsymbol{u}'(t)\times\boldsymbol{v}(t)+\boldsymbol{u}(t)\times\boldsymbol{v}'(t)$ (**Cross Product Rule**);
6. $(\boldsymbol{u}(f(t)))' = f'(t)\boldsymbol{u}'(f(t))$ (**Chain Rule**).

Example: Let $\boldsymbol{r}(t)=\langle t^2,5t,1\rangle$ and $f(t)=e^{3t}$. Compute:

(a) $(f(t)\boldsymbol{r}(t))' = f'(t)\boldsymbol{r}(t)+f(t)\boldsymbol{r}'(t) = 3e^{3t}\langle t^2,5t,1\rangle + e^{3t}\langle 2t,5,0\rangle = \langle (3t^2+2t)e^{3t},(15t+5)e^{3t},3e^{3t}\rangle$.

(b) $[\boldsymbol{r}(f(t))]' = f'(t)\boldsymbol{r}'(f(t)) = 3e^{3t}\langle 2(e^{3t}),5,0\rangle = \langle 6e^{6t},15e^{3t},0\rangle$.

Proving a Formula

- Prove the formula

$$\frac{d}{dt}(\boldsymbol{r}(t) \times \boldsymbol{r}'(t)) = \boldsymbol{r}(t) \times \boldsymbol{r}''(t).$$

$$\begin{array}{rcl} \frac{d}{dt}(\boldsymbol{r}(t) \times \boldsymbol{r}'(t)) & = & \frac{d}{dt}\boldsymbol{r}(t) \times \boldsymbol{r}'(t) + \boldsymbol{r}(t) \times \frac{d}{dt}\boldsymbol{r}'(t) \\ & = & \boldsymbol{r}'(t) \times \boldsymbol{r}'(t) + \boldsymbol{r}(t) \times \boldsymbol{r}''(t) \\ & = & \boldsymbol{0} + \boldsymbol{r}(t) \times \boldsymbol{r}''(t) \\ & = & \boldsymbol{r}(t) \times \boldsymbol{r}''(t). \end{array}$$

Proving a General Property

- Example: Suppose that $\|\boldsymbol{r}(t)\| = c$, a constant, for all t. Show that $\boldsymbol{r}'(t)$ is orthogonal to $\boldsymbol{r}(t)$, for all t.

  We have

  $$(\boldsymbol{r}(t) \cdot \boldsymbol{r}(t))' = \boldsymbol{r}'(t) \cdot \boldsymbol{r}(t) + \boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 2\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t).$$

  Therefore, we get

  $$\begin{aligned} \boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) &= \tfrac{1}{2}(\boldsymbol{r}(t) \cdot \boldsymbol{r}(t))' \\ &= \tfrac{1}{2}(\|\boldsymbol{r}(t)\|^2)' \\ &= \tfrac{1}{2}(c^2)' \\ &= 0. \end{aligned}$$

  Therefore, $\boldsymbol{r}'(t) \cdot \boldsymbol{r}(t) = 0$, showing that $\boldsymbol{r}'(t) \perp \boldsymbol{r}(t)$.

Definite Integrals of Vector Functions

- We define the **definite integral** of a continuous vector function $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$ by

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b x(t)dt\right)\boldsymbol{i} + \left(\int_a^b y(t)dt\right)\boldsymbol{j} + \left(\int_a^b z(t)dt\right)\boldsymbol{k}.$$

- If $\boldsymbol{R}(t)$ is an antiderivative of $\boldsymbol{r}(t)$, i.e., if $\boldsymbol{R}'(t) = \boldsymbol{r}(t)$, then

$$\int_a^b \boldsymbol{r}(t)dt = \boldsymbol{R}(t)\Big|_a^b = \boldsymbol{R}(b) - \boldsymbol{R}(a).$$

  We write $\int \boldsymbol{r}(t)dt = \boldsymbol{R}(t) + \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector, in this case.

Example I

- Compute the following:
  (a) $\int \langle 1, t, \sin t \rangle dt$;
  (b) $\int_0^\pi \langle 1, t, \sin t \rangle dt$.

(a)

$$\begin{aligned} \int \langle 1, t, \sin t \rangle dt &= \langle \int dt, \int t dt, \int \sin t dt \rangle \\ &= \langle t + c_1, \frac{1}{2}t^2 + c_2, -\cos t + c_3 \rangle \\ &= \langle t, \frac{1}{2}t^2, -\cos t \rangle + \boldsymbol{c}. \end{aligned}$$

(b)

$$\begin{aligned} \int_0^\pi \langle 1, t, \sin t \rangle dt &= \langle t, \frac{1}{2}t^2, -\cos t \rangle \Big|_0^\pi \\ &= \langle \pi, \frac{1}{2}\pi^2, -\cos \pi \rangle - \langle 0, 0, -\cos 0 \rangle \\ &= \langle \pi, \frac{1}{2}\pi^2, 2 \rangle. \end{aligned}$$

Example II

- Suppose $\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + \sin t\boldsymbol{j} + 2t\boldsymbol{k}$. Calculate
  (a) $\int \boldsymbol{r}(t)dt$;
  (b) $\int_0^{\pi/2} \boldsymbol{r}(t)dt$.

(a)

$$\int \boldsymbol{r}(t)dt = 2\sin t\mathbf{i} - \cos t\mathbf{j} + t^2\mathbf{k} + \mathbf{c}.$$

(b)

$$\begin{aligned}\int_0^{\pi/2} \boldsymbol{r}(t)dt &= (2\sin t\boldsymbol{i} - \cos t\boldsymbol{j} + t^2\boldsymbol{k})\,\Big|_0^{\pi/2} \\ &= (2\sin\tfrac{\pi}{2} - 2\sin 0)\boldsymbol{i} - (\cos\tfrac{\pi}{2} - \cos 0)\boldsymbol{j} \\ &\quad + ((\tfrac{\pi}{2})^2 - 0^2)\boldsymbol{k} \\ &= 2\boldsymbol{i} + \boldsymbol{j} + \tfrac{\pi^2}{4}\boldsymbol{k}.\end{aligned}$$

Finding a Position Vector

- The path of a particle satisfies $\frac{d\boldsymbol{r}}{dt} = \langle 1 - 6\sin 3t, \frac{1}{5}t \rangle$. Find the particle's location at $t = 4$ if $\boldsymbol{r}(0) = \langle 4, 1 \rangle$.

  The general solution is obtained by integration:

  $$\boldsymbol{r}(t) = \int \langle 1 - 6\sin 3t, \frac{1}{5}t \rangle dt = \langle t + 2\cos 3t, \frac{1}{10}t^2 \rangle + \boldsymbol{c}.$$

  The initial condition $\boldsymbol{r}(0) = \langle 4, 1 \rangle$ gives us

  $$\boldsymbol{r}(0) = \langle 2, 0 \rangle + \boldsymbol{c} = \langle 4, 1 \rangle \quad \Rightarrow \quad \boldsymbol{c} = \langle 2, 1 \rangle.$$

  This now yields

  $$\boldsymbol{r}(t) = \langle t + 2\cos 3t, \frac{1}{10}t^2 \rangle + \langle 2, 1 \rangle = \langle t + 2\cos 3t + 2, \frac{1}{10}t^2 + 1 \rangle.$$

  The particle's position at $t = 4$ is

  $$\boldsymbol{r}(4) = \langle 4 + 2\cos 12 + 2, \frac{16}{10} + 1 \rangle = \langle 6 + 2\cos 12, \frac{13}{5} \rangle.$$

Subsection 3

Arc Length and Speed

Arc Length

- Suppose that $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$. Then, the length of the arc traversed as t increases from a to b is given by

$$\begin{aligned} L &= \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt. \end{aligned}$$

- Recall that $\boldsymbol{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$. Therefore, $\|\boldsymbol{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$. This shows that

$$L = \int_a^b \|\boldsymbol{r}'(t)\| dt.$$

Computing Arc Length

- Compute the length of the arc of the circular helix with vector equation $\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + t\boldsymbol{k}$ from $(1,0,0)$ to $(1,0,2\pi)$.

  Note that:
  - $(1,0,0)$ corresponds to $t = 0$;
  - $(1,0,2\pi)$ corresponds to $t = 2\pi$.

  Moreover, $x'(t) = -\sin t,\ y'(t) = \cos t$ and $z'(t) = 1$. Therefore,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(-\sin t)^2 + \cos^2 t + 1}dt \\ &= \int_0^{2\pi} \sqrt{2}dt \\ &= \sqrt{2}t\ \Big|_0^{2\pi} \\ &= 2\sqrt{2}\pi. \end{aligned}$$

Arc Length Function and Speed

- We define the **arc length function** as the distance traveled during the interval $[a, t]$:
$$s(t) = \int_a^t \|\boldsymbol{r}'(u)\| du.$$
- By definition, **speed** is the rate of change of distance traveled with respect to time t:
$$\text{Speed at time } t = \frac{ds}{dt} = \frac{d}{dt} \int_a^t \|\boldsymbol{r}'(u)\| du \stackrel{\text{FTC}}{=} \|\boldsymbol{r}'(t)\|.$$

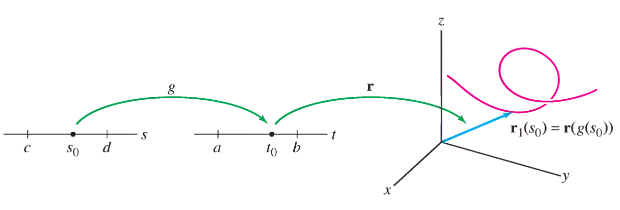
Calculating Speed

- Find the speed at time $t = 2$ s of a particle whose position vector is
$$\boldsymbol{r}(t) = t^3\boldsymbol{i} - e^t\boldsymbol{j} + 4t\boldsymbol{k}.$$
The velocity vector is
$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = 3t^2\boldsymbol{i} - e^t\boldsymbol{j} + 4\boldsymbol{k}.$$
At $t = 2$, $\boldsymbol{v}(2) = 12\boldsymbol{i} - e^2\boldsymbol{j} + 4\boldsymbol{k}$.
Therefore, the particle's speed is
$$v(2) = \|\boldsymbol{v}(2)\| = \sqrt{12^2 + (-e^2)^2 + 4^2} = \sqrt{160 + e^4}.$$

Switching Between Parametrizations

- Parametrizations are not unique.

  Example: By elimination of parameters, it is easy to see that both $\boldsymbol{r}_1(t) = \langle t, t^2 \rangle$ and $\boldsymbol{r}_2(s) = \langle s^3, s^6 \rangle$ parametrize the parabola $y = x^2$. In this case $\boldsymbol{r}_2(s)$ is obtained by substituting $t = s^3$ in $\boldsymbol{r}_1(t)$.
- In general, we obtain a new parametrization by making a substitution $t = g(s)$,



  i.e., by replacing $\boldsymbol{r}(t)$ with $\boldsymbol{r}_1(s) = \boldsymbol{r}(g(s))$. If $t = g(s)$ increases from a to b as s varies from c to d, then the path $\boldsymbol{r}(t)$ for $a \leq t \leq b$ is also parametrized by $\boldsymbol{r}_1(s)$ for $c \leq s \leq d$.

Example

- Parametrize the path $\boldsymbol{r}(t) = \langle t^2, \sin t, t \rangle$, for $3 \leq t \leq 9$, using the parameter s, where $t = g(s) = e^s$.

  Substituting $t = e^s$ in $\boldsymbol{r}(t)$, we obtain the parametrization

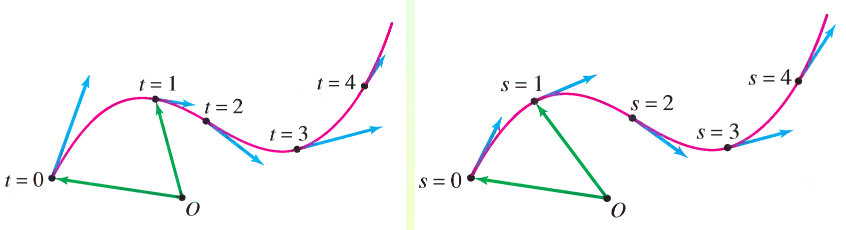
  $$\boldsymbol{r}_1(s) = \boldsymbol{r}(g(s)) = \langle e^{2s}, \sin e^s, e^s \rangle.$$

  Because $s = \ln t$, the parameter t varies from 3 to 9 as s varies from $\ln 3$ to $\ln 9$. Therefore, the path is parametrized by

  $$\boldsymbol{r}_1(s), \text{ for } \ln 3 \leq s \leq \ln 9.$$

Arc Length Parametrization

- One way of parametrizing a path is to choose a starting point and "walk along the path" at unit speed.



Such a parametrization is called an **arc length parametrization** and is defined by the property that the speed has constant value 1:

$$\|\boldsymbol{r}'(t)\| = 1, \text{ for all } t.$$

Process for Arc Length Parametrization

- To find an arc length parametrization:
  - Start with any parametrization $\boldsymbol{r}(t)$ such that $\boldsymbol{r}'(t) \neq \mathbf{0}$, for all t;
  - Form the arc length integral $s(t) = \int_0^t \|\boldsymbol{r}'(u)\| du$;
  - Notice that $\boldsymbol{r}'(t) \neq \mathbf{0}$ implies that $s(t)$ is an increasing function and therefore has an inverse $t = g(s)$.
  - The parametrization
    $$\boldsymbol{r}_1(s) = \boldsymbol{r}(g(s))$$
    is an arc length parametrization.
- We show why:
  - By the formula for the derivative of an inverse, we get
    $$g'(s) = \frac{1}{s'(g(s))} = \frac{1}{\|\boldsymbol{r}'(g(s))\|}.$$
  - Now we get, using the Chain Rule,
    $$\|\boldsymbol{r}_1'(s)\| \overset{\text{Chain}}{=} \|g'(s)\boldsymbol{r}'(g(s))\| = \frac{1}{\|\boldsymbol{r}'(g(s))\|}\|\boldsymbol{r}'(g(s))\| = 1.$$

Finding an Arc Parametrization

- Find the arc length parametrization of the helix

$$\boldsymbol{r}(t) = \langle \cos 4t, \sin 4t, 3t \rangle.$$

First, we evaluate the arc length function

$$\begin{aligned} \|\boldsymbol{r}'(t)\| &= \|\langle -4\sin 4t, 4\cos 4t, 3 \rangle\| \\ &= \sqrt{16\sin^2 4t + 16\cos^2 4t + 3^2} = 5; \\ s(t) &= \int_0^t \|\boldsymbol{r}'(t)\| dt = \int_0^t 5dt = 5t. \end{aligned}$$

Then we observe that the inverse of $s(t) = 5t$ is $t = \frac{s}{5}$, i.e., $g(s) = \frac{s}{5}$. Thus, an arc length parametrization is

$$\boldsymbol{r}_1(s) = \boldsymbol{r}(g(s)) = \boldsymbol{r}\left(\frac{s}{5}\right) = \left\langle \cos\frac{4s}{5}, \sin\frac{4s}{5}, \frac{3s}{5} \right\rangle.$$

Subsection 4

Curvature

Unit Tangent Vector

- Consider a path with parametrization $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, such that $\boldsymbol{r}'(t) \neq \mathbf{0}$, for all t in the domain of $\boldsymbol{r}(t)$.

  A parametrization with this property is called **regular**.
- At every point P along the path there is a **unit tangent vector** $\mathbf{T} = \mathbf{T}_P$ that points in the direction of motion of the parametrization

$$\text{Unit Tangent Vector} = \mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}.$$

Computing a Unit Tangent Vector

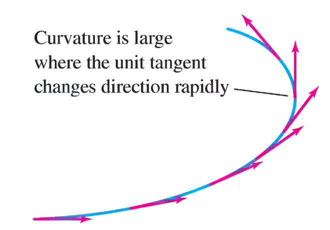
- If $\boldsymbol{r}(t) = \langle t, t^2, t^3 \rangle$, compute the unit tangent vector at $P = (1, 1, 1)$.
  We have $\boldsymbol{r}'(t) = \langle 1, 2t, 3t^2 \rangle$.
  Note that P is the terminal point of $\boldsymbol{r}(1)$.
  Thus, the unit tangent vector at $P = (1, 1, 1)$ is

$$\begin{aligned} \mathbf{T}_P &= \frac{\boldsymbol{r}'(1)}{\|\boldsymbol{r}'(1)\|} = \frac{\langle 1,2,3 \rangle}{\|\langle 1,2,3 \rangle\|} \\ &= \frac{\langle 1,2,3 \rangle}{\sqrt{1^2+2^2+3^2}} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle. \end{aligned}$$

Definition of Curvature

- Imagine walking along a path and observing how the unit tangent vector **T** changes direction.

  A change in **T** indicates that the path is bending, and the more rapidly **T** changes, the more the path bends. Thus, $\left\|\frac{d\mathbf{T}}{dt}\right\|$ would seem to be a good measure of curvature. However, this depends on how fast you walk.

  To counter this, we assume an arc length parametrization.
- Let $\boldsymbol{r}(s)$ be an arc length parametrization and **T** the unit tangent vector. The curvature at $\boldsymbol{r}(s)$ is the quantity

$$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\|.$$

A Line Has Zero Curvature

- Compute the curvature at each point on the line $\boldsymbol{r}(t) = \langle x_0, y_0, z_0 \rangle + t\boldsymbol{u}$, where $\|\boldsymbol{u}\| = 1$.

  Since $\boldsymbol{u}$ is a unit vector, $\boldsymbol{r}(t)$ is an arc length parametrization: $\boldsymbol{r}'(t) = \boldsymbol{u}$ and, thus, $\|\boldsymbol{r}'(t)\| = \|\boldsymbol{u}\| = 1$.

  Thus, we have $\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \boldsymbol{r}'(t)$. Hence, $\mathbf{T}'(t) = \boldsymbol{r}''(t) = \mathbf{0}$ (because $\boldsymbol{r}'(t) = \boldsymbol{u}$ is constant). As expected, the curvature is zero at all points on a line:

$$\kappa(t) = \left\| \frac{d\mathbf{T}}{dt} \right\| = \|\boldsymbol{r}''(t)\| = 0.$$

The Curvature of a Circle of Radius R is $1/R$

- Compute the curvature of a circle of radius R.
  Assume the circle is centered at the origin $\boldsymbol{r}(\theta) = \langle R\cos\theta, R\sin\theta\rangle$.
  We find an arc length parametrization:
  $$s(\theta) = \int_0^\theta \|\boldsymbol{r}'(u)\| du = \int_0^\theta R du = R\theta.$$
  Thus, $s = R\theta$, and the inverse function is $\theta = g(s) = \frac{s}{R}$. Thus, an arc length parametrization is
  $$\boldsymbol{r}_1(s) = \boldsymbol{r}(g(s)) = r\left(\frac{s}{R}\right) = \left\langle R\cos\frac{s}{R}, R\sin\frac{s}{R}\right\rangle.$$
  The unit tangent vector and its derivative are
  $\mathbf{T}(s) = \frac{d\boldsymbol{r}_1}{ds} = \frac{d}{ds}\langle R\cos\frac{s}{R}, R\sin\frac{s}{R}\rangle = \langle -\sin\frac{s}{R}, \cos\frac{s}{R}\rangle$. Therefore, $\frac{d\mathbf{T}}{ds} = -\frac{1}{R}\langle\cos\frac{s}{R}, \sin\frac{s}{R}\rangle$. By definition of curvature,
  $$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{1}{R}\left\|\left\langle\cos\frac{s}{R}, \sin\frac{s}{R}\right\rangle\right\| = \frac{1}{R}.$$

Derivative of the Unit Tangent Vector and Curvature

- Suppose that $\mathbf{T}(s) = \mathbf{T}(s(t))$.

  So the derivatives of $\mathbf{T}$ with respect to t and s are related by the Chain Rule:

$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{ds}{dt}\frac{d\mathbf{T}}{ds}.$$

- Now note that
  - $\frac{ds}{dt} = \|\boldsymbol{r}'(t)\| = v(t)$;
  - $\|\frac{d\mathbf{T}}{ds}\| = \kappa(t)$.
- So we get:

$$\|\mathbf{T}'(t)\| = v(t)\kappa(t).$$

Formula for Curvature

- If $\boldsymbol{r}(t)$ is a regular parametrization, then the curvature at $\boldsymbol{r}(t)$ is
$$\kappa(t) = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}.$$
Since $v(t) = \|\boldsymbol{r}'(t)\|$, we have $\boldsymbol{r}'(t) = v(t)\mathbf{T}(t)$. By the Product Rule,
$$\boldsymbol{r}''(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t).$$
Now using the fact that $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$, we get:
$$\begin{aligned} \boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= v(t)\mathbf{T}(t) \times (v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)) \\ &= v(t)^2\mathbf{T}(t) \times \mathbf{T}'(t). \end{aligned}$$
Now we get
$$\|\mathbf{r}'(t) \times \boldsymbol{r}''(t)\| = v(t)^2\|\mathbf{T}(t)\|\|\mathbf{T}'(t)\| \sin\frac{\pi}{2} = v(t)^2\|\mathbf{T}'(t)\|.$$
Finally, we obtain
$$\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| = v(t)^2\|\mathbf{T}'(t)\| = v(t)^3\kappa(t) = \|\boldsymbol{r}'(t)\|^3\kappa(t).$$

Twisted Cubic Curve

- Calculate the curvature $\kappa(t)$ of the twisted cubic $\boldsymbol{r}(t) = \langle t, t^2, t^3 \rangle$.

$$\begin{array}{rcl} \boldsymbol{r}'(t) & = & \langle 1, 2t, 3t^2 \rangle \\ \boldsymbol{r}''(t) & = & \langle 0, 2, 6t \rangle \\ \boldsymbol{r}'(t) \times \boldsymbol{r}''(t) & = & \left| \begin{array}{ccc} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{array} \right| = \langle 6t^2, -6t, 2 \rangle. \end{array}$$

Therefore, we get

$$\begin{array}{rcl} \kappa(t) & = & \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3} \\ & = & \frac{\sqrt{(6t^2)^2 + (-6t)^2 + 2^2}}{\sqrt{(1^2 + (2t)^2 + (3t^2)^2)^3}} \\ & = & \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{(1 + 4t^2 9t^4)^3}}. \end{array}$$

Curvature of a Graph in the Plane

- The curvature at the point $(x, f(x))$ on the graph of $y = f(x)$ is equal to
$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$
The curve $y = f(x)$ has parametrization $\boldsymbol{r}(x) = \langle x, f(x) \rangle$. Therefore, $\boldsymbol{r}'(x) = \langle 1, f'(x) \rangle$ and $\boldsymbol{r}''(x) = \langle 0, f''(x) \rangle$. To apply the formulas for $\kappa(x)$, we treat $\boldsymbol{r}'(x)$ and $\boldsymbol{r}''(x)$ as vectors in $\mathbb{R}^3$ with z-component equal to zero. Then
$$\boldsymbol{r}'(x) \times \boldsymbol{r}''(x) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\boldsymbol{k}.$$
Now we get
$$\kappa(x) = \frac{\|\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)\|}{\|\boldsymbol{r}'(x)\|^3} = \frac{\sqrt{f''(x)^2}}{\sqrt{(1 + f'(x)^2)^3}} = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

Computing the Curvature of a Graph in the Plane

- Compute the curvature of $f(x) = x^3 - 3x^2 + 4$ at $x = 0, 1, 2$.

  We have
  $$\begin{aligned} f'(x) &= 3x^2 - 6x = 3x(x-2); \\ f''(x) &= 6x - 6. \end{aligned}$$

  So we get
  $$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} = \frac{|6x - 6|}{(1 + (3x(x-2))^2)^{3/2}}.$$

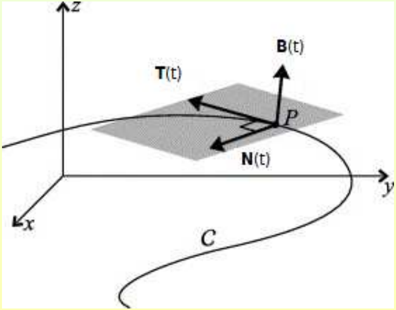
  We obtain the following values:
  $$\kappa(0) = \frac{6}{1} = 6, \quad \kappa(1) = \frac{0}{10^{3/2}} = 0, \quad \kappa(2) = \frac{6}{1} = 6.$$

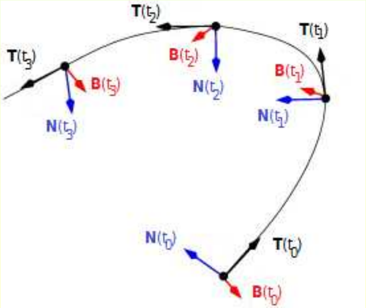
Unit Normal and Binormal Vectors

- Given a curve $\boldsymbol{r}(t)$, the **unit normal** $\mathbf{N}(t)$ is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|};$$

  Note that $\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t) = v(t)\kappa(t)\mathbf{N}(t)$.
- The **binormal** is defined by $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$;





Finding the Unit Normal and Binormal Vectors

- Curve: $\boldsymbol{r}(t)$; Unit Normal $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$;

  Binormal: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

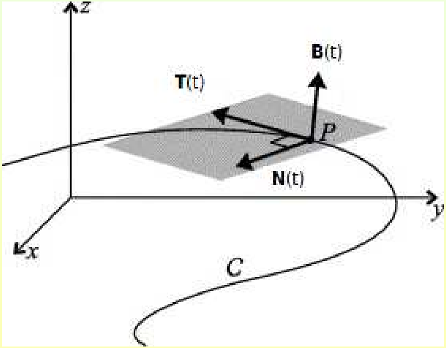
  Example: Find the unit normal and the binormal to the curve $\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + t\boldsymbol{k}$.

  - $\boldsymbol{r}'(t) = \langle -\sin t, \cos t, 1\rangle$ and $\|\boldsymbol{r}'(t)\| = \sqrt{2}$;
  - $\mathbf{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \langle -\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\rangle$;
  - $\mathbf{T}'(t) = \langle -\frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{2}}\sin t, 0\rangle$ and $\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{2}}$;
  - $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle -\cos t, -\sin t, 0\rangle$;
  - $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\frac{1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} =$

    $\langle \frac{1}{\sqrt{2}}\sin t, -\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\rangle$.

Normal Plane and Osculating Plane

- Let $\boldsymbol{r}(t)$ be a vector function determining a space curve $\mathcal{C}$.
- The **normal plane** of $\mathcal{C}$ at a point P is the plane determined by the normal $\mathbf{N}$ and the binormal $\mathbf{B}$ vectors of $\mathcal{C}$ at P.
- The **osculating** (kissing) **plane** of C at P is the plane determined by the tangent $\mathbf{T}$ and normal $\mathbf{N}$ vectors of $\mathcal{C}$ at P.



Remarks on Normal Plane and Osculating Plane

- Curve $\mathcal{C}$ with vector $\boldsymbol{r}(t)$;
  - Normal plane at a point P determined by the normal $\mathbf{N}$ and the binormal $\mathbf{B}$;
  - Osculating plane at P determined by the tangent $\mathbf{T}$ and normal $\mathbf{N}$.
- Since the normal plane at t is determined by the normal $\mathbf{N}(t)$ and the binormal $\mathbf{B}(t)$, the tangent vector $\boldsymbol{r}'(t)$ is a normal vector to the normal plane;
- Similarly, since the osculating plane at t is determined by the tangent $\mathbf{T}(t)$ and normal $\mathbf{N}(t)$, the binormal vector $\mathbf{B}(t)$ is a normal vector to the osculating plane.

Example

- Determine the normal and the osculating plane of $x = 2\sin 3t, y = t, z = 2\cos 3t$ at the point $(0, \pi, -2)$.
  We are focusing at the point with $t = \pi$;
  - We have $\boldsymbol{r}(t) = \langle 2\sin 3t, t, 2\cos 3t\rangle$;
  - $\boldsymbol{r}'(t) = \langle 6\cos 3t, 1, -6\sin 3t\rangle$; So $\boldsymbol{r}'(\pi) = \langle -6, 1, 0\rangle$;
  - $\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \langle \frac{6}{\sqrt{37}}\cos 3t, \frac{1}{\sqrt{37}}, -\frac{6}{\sqrt{37}}\sin 3t\rangle$;
    So $\mathbf{T}(\pi) = \langle -\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0\rangle$;
  - Now we get $\mathbf{T}'(t) = \langle -\frac{18}{\sqrt{37}}\sin 3t, 0, -\frac{18}{\sqrt{37}}\cos 3t\rangle$;
    So $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle -\sin 3t, 0, -\cos 3t\rangle$;
    Hence $\mathbf{N}(\pi) = \langle 0, 0, 1\rangle$;
  - Finally, $\mathbf{B}(\pi) = \mathbf{T}(\pi) \times \mathbf{N}(\pi) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\frac{6}{\sqrt{37}} & \frac{1}{\sqrt{37}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \frac{1}{\sqrt{37}}, \frac{6}{\sqrt{37}}, 0\rangle$;

  The normal plane is $-6x + (y - \pi) = 0$;
  The osculating plane is $\frac{1}{\sqrt{37}}x + \frac{6}{\sqrt{37}}(y - \pi) = 0$, or $x + 6(y - \pi) = 0$.

Subsection 5

Motion in Three-Space

Velocity, Speed and Acceleration

- Assume that $\boldsymbol{r}(t)$ is the position vector at time t of a particle moving through space.
- Then, the **velocity vector** $\boldsymbol{v}(t)$ at time t is

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t).$$

- The **speed** of the particle is the magnitude or length of the velocity vector, i.e., $v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|$.
- Finally, its **acceleration** is the derivative of the velocity

$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t).$$

Example: If the position vector of a particle is $\boldsymbol{r}(t) = t^3\boldsymbol{i} + t^2\boldsymbol{j}$, what are its velocity, speed and acceleration at time $t = 1$?
We compute the following:

$$\begin{array}{lll} \boldsymbol{v}(t) = 3t^2\boldsymbol{i} + 2t\boldsymbol{j}, & \|\boldsymbol{v}(t)\| = \sqrt{9t^4 + 4t^2}, & \boldsymbol{a}(t) = \boldsymbol{v}'(t) = 6t\boldsymbol{i} + 2\boldsymbol{j}, \\ \boldsymbol{v}(1) = 3\boldsymbol{i} + 2\boldsymbol{j}, & \|\boldsymbol{v}(1)\| = \sqrt{13}, & \boldsymbol{a}(1) = 6\boldsymbol{i} + 2\boldsymbol{j}. \end{array}$$

Examples on Velocity, Speed and Acceleration

- Example: Find the velocity, acceleration and speed of a particle whose position vector is $\boldsymbol{r}(t) = \langle t^2, e^t, te^t \rangle$.
  $\boldsymbol{v}(t) = \boldsymbol{r}'(t) = \langle 2t, e^t, e^t + te^t \rangle$;
  $\|\boldsymbol{v}(t)\| = \sqrt{4t^2 + e^{2t} + (e^t + te^t)^2} = \sqrt{4t^2 + (t^2 + 2t + 2)e^{2t}}$;
  $\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \langle 2, e^t, 2e^t + te^t \rangle$.

- Example: Find the velocity and the position vector at time t of a particle, whose position vector at time 0 is $\boldsymbol{r}(0) = \langle 1, 0, 0 \rangle$, whose velocity at time 0 is $\boldsymbol{v}(0) = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and whose acceleration is $\boldsymbol{a}(t) = 4t\boldsymbol{i} + 6t\boldsymbol{j} + \boldsymbol{k}$.
  $\boldsymbol{v}(t) = \int \boldsymbol{a}(t)dt = 2t^2\boldsymbol{i} + 3t^2\boldsymbol{j} + t\boldsymbol{k} + \boldsymbol{c}$. But $\boldsymbol{v}(0) = \boldsymbol{c} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, whence $\boldsymbol{v}(t) = (2t^2 + 1)\boldsymbol{i} + (3t^2 - 1)\boldsymbol{j} + (t + 1)\boldsymbol{k}$.
  $\boldsymbol{r}(t) = \int \boldsymbol{v}(t)dt = (\frac{2}{3}t^3 + t)\boldsymbol{i} + (t^3 - t)\boldsymbol{j} + (\frac{1}{2}t^2 + t)\boldsymbol{k} + \boldsymbol{c}$; As before, $\boldsymbol{r}(0) = \boldsymbol{c} = \boldsymbol{i}$, whence $\boldsymbol{r}(t) = (\frac{2}{3}t^3 + t + 1)\boldsymbol{i} + (t^3 - t)\boldsymbol{j} + (\frac{1}{2}t^2 + t)\boldsymbol{k}$.

Newton's Second Law of Motion

- If at time t a force $\boldsymbol{F}(t)$ acts on an object of mass m producing an acceleration $\boldsymbol{a}(t)$, then
$$\boldsymbol{F}(t) = m\boldsymbol{a}(t).$$
Example: The position vector of an object with mass m moving in a circular path with constant angular speed ω is
$$\boldsymbol{r}(t) = a\cos\omega t\boldsymbol{i} + a\sin\omega t\boldsymbol{j}.$$
What is the force acting on the object and what is its direction?
We have
$$\begin{aligned} \boldsymbol{v}(t) = \boldsymbol{r}'(t) &= -a\omega\sin\omega t\boldsymbol{i} + a\omega\cos\omega t\boldsymbol{j} \\ \boldsymbol{a}(t) = \boldsymbol{v}'(t) &= -a\omega^2\cos\omega t\boldsymbol{i} - a\omega^2\sin\omega t\boldsymbol{j} \\ \boldsymbol{F}(t) = m\boldsymbol{a}(t) &= -ma\omega^2\cos\omega t\boldsymbol{i} - ma\omega^2\sin\omega t\boldsymbol{j} \end{aligned}$$
Therefore $\|\boldsymbol{F}(t)\| = ma\omega^2$ and $\boldsymbol{F}(t) = -m\omega^2\boldsymbol{r}(t)$, i.e., $\boldsymbol{F}(t)$ is opposite to the position (radius) vector.

Position Vector of a Projectile

- A projectile is fired from initial position $\boldsymbol{r}_0 = \mathbf{0}$, with angle of elevation α and initial velocity $\boldsymbol{v}_0$. If the only external force is due to gravity g, what is the position function $\boldsymbol{r}(t)$ of the projectile?

  We have

$$\begin{aligned} \boldsymbol{a}(t) &= -g\boldsymbol{j}; \\ \boldsymbol{v}(t) &= \int \boldsymbol{a}(t)dt = -gt\boldsymbol{j} + \boldsymbol{v}_0; \\ \mathbf{r}(t) &= \int \boldsymbol{v}(t)dt = -\frac{1}{2}gt^2\boldsymbol{j} + t\boldsymbol{v}_0 + \boldsymbol{r}_0 \\ &= -\frac{1}{2}gt^2\boldsymbol{j} + t\boldsymbol{v}_0. \end{aligned}$$

  Since $\boldsymbol{v}_0 = v_0 \cos\alpha\boldsymbol{i} + v_0 \sin\alpha\boldsymbol{j}$, the above vector equation can be rewritten as

$$\boldsymbol{r}(t) = v_0 t \cos\alpha\boldsymbol{i} + (v_0 t \sin\alpha - \frac{1}{2}gt^2)\boldsymbol{j}.$$

Tangential and Normal Components of Acceleration

- Suppose a traveling particle has position vector $\boldsymbol{r}(t)$, velocity vector $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$ and speed $v = \|\boldsymbol{v}(t)\|$.
- Then the unit tangent to its position is $\mathbf{T}(t) = \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \dfrac{\boldsymbol{v}(t)}{v(t)}$, showing that $\mathbf{v}(t) = v(t)\mathbf{T}(t)$.
- Recall the formula for the curvature $\kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\boldsymbol{r}'(t)\|} = \dfrac{\|\mathbf{T}'(t)\|}{v(t)}$, which gives $\|\mathbf{T}'(t)\| = \kappa(t)v(t)$.
- Recall, also, the formula for the unit normal $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, which gives $\mathbf{T}'(t) = \|\mathbf{T}'(t)\|\mathbf{N}(t) = \kappa(t)v(t)\mathbf{N}(t)$.
- Differentiating the velocity vector and putting these formulas together gives the resolution of the acceleration into a tangential and a normal component to the motion

$$\boldsymbol{a}(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) = v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t).$$

Obtain Expressions for the Components in Terms of $\boldsymbol{r}(t)$

- From the previous slide $\boldsymbol{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t)$, where
  - $a_T(t) = v'(t)$;
  - $a_N(t) = \kappa(t)v^2(t)$.
- Now note that

$$\begin{aligned} a_T(t) &= v'(t) = \frac{v(t)v'(t)}{v(t)} \\ &= \frac{v(t)v'(t)\mathbf{T}(t)\cdot\mathbf{T}(t) + \kappa(t)v^3(t)\mathbf{T}(t)\cdot\mathbf{N}(t)}{v(t)} \\ &= \frac{v(t)\mathbf{T}(t)\cdot(v'(t)\mathbf{T}(t) + \kappa(t)v^2(t)\mathbf{N}(t))}{v(t)} \\ &= \frac{\boldsymbol{v}(t)\cdot\boldsymbol{a}(t)}{v(t)} = \frac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{\|\boldsymbol{r}'(t)\|}. \end{aligned}$$

- Also, we get

$$a_N(t) = \kappa(t)v^2(t) = \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}\|\boldsymbol{r}'(t)\|^2 = \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|}.$$

Computing the Acceleration Components

- Suppose $\boldsymbol{r}(t) = \langle t^2, t^2, t^3 \rangle$.

  Then we have

$$\begin{aligned}
\boldsymbol{r}'(t) &= \langle 2t, 2t, 3t^2 \rangle; \\
\|\boldsymbol{r}'(t)\| &= \sqrt{8t^2 + 9t^4}; \\
\boldsymbol{r}''(t) &= \langle 2, 2, 6t \rangle; \\
\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t) &= \langle 2t, 2t, 3t^2 \rangle \cdot \langle 2, 2, 6t \rangle = 8t + 18t^3; \\
\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t^2, 0 \rangle; \\
\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| &= 6\sqrt{2}t^2.
\end{aligned}$$

  Therefore,

$$a_T(t) = \frac{\boldsymbol{r}'(t) \cdot \boldsymbol{r}''(t)}{\|\boldsymbol{r}'(t)\|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}};$$

$$a_N(t) = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|} = \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}.$$