Calculus III

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LSSU Math 251

Subsection 1

Functions of Several Variables

Functions of Several Variables

- A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set $\mathcal{D}$ a unique real number $f(x, y)$.
- The set $\mathcal{D}$ is the **domain** of f and its **range** is the set of values that f takes on, i.e., the set $\{f(x, y) : (x, y) \in \mathcal{D}\}$.
- The variables x, y are called **independent variables** and $z = f(x, y)$ is the **dependent variable**.
- If $f(x, y)$ is specified by a formula, then the domain is understood to be the set of all pairs (x, y) for which the given formula yields a well defined real number.

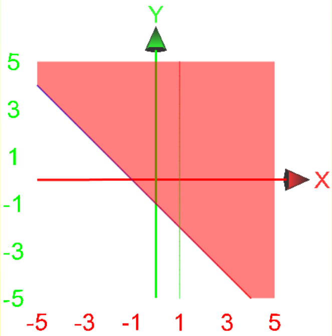
Finding and Graphing the Domain

- Find and graph the domain of $f(x,y)=\dfrac{\sqrt{x+y+1}}{x-1}$.

  The domain of $f(x,y)=\dfrac{\sqrt{x+y+1}}{x-1}$ is specified by enforcing the following conditions:

  - $x+y+1\geq 0$, giving $y\geq -x-1$;
  - $x-1\neq 0$, giving $x\neq 1$.

  Thus, the domain is $\mathcal{D}=\{(x,y):y\geq -x-1 \text{ and } x\neq 1\}$.



Another Example of a Domain

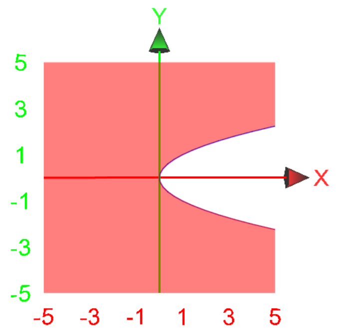
- Find and graph the domain of $f(x,y) = x \ln(y^2 - x)$.

  The domain of $f(x,y) = x \ln(y^2 - x)$ is specified by enforcing the following condition:

  - $y^2 - x > 0$, giving $y^2 > x$.

  Thus, the domain is

$$\mathcal{D} = \{(x,y) : y^2 > x\}.$$



A Third Example of a Domain

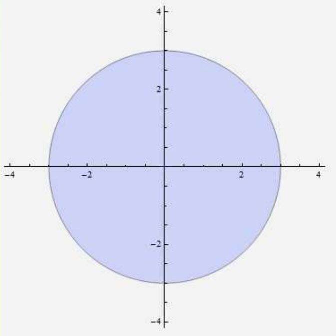
- Find and graph the domain of $f(x,y) = \sqrt{9 - x^2 - y^2}$.

  The domain of $f(x,y) = \sqrt{9 - x^2 - y^2}$ is specified by enforcing the following condition:

  - $9 - x^2 - y^2 \geq 0$, giving $x^2 + y^2 \leq 9$.

  Thus, the domain is

$$\mathcal{D} = \{(x,y) : x^2 + y^2 \leq 9\}.$$



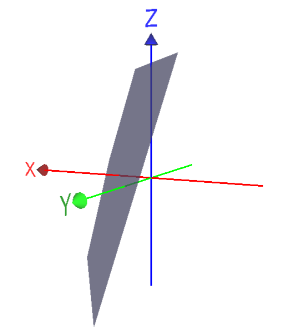
Graphs of Functions of Two Variables

- If $f(x,y)$ is a function of two variables, with domain $\mathcal{D}$, the **graph** of f is the set of points

$$\{(x,y,z) \in \mathbb{R}^3 : z = f(x,y), (x,y) \in \mathcal{D}\}.$$

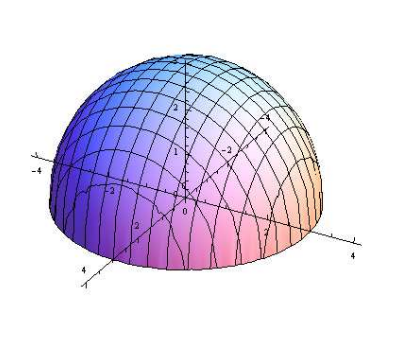
- The graphs of functions of two variables are 3-dimensional surfaces.

  Example: Sketch the graph of the function $f(x,y) = 6 - 3x - 2y$.
  $3x + 2y + z = 6$ is the equation of a plane in space.
  It intersects the coordinate axes at the points $(2,0,0)$, $(0,3,0)$, $(0,0,6)$.



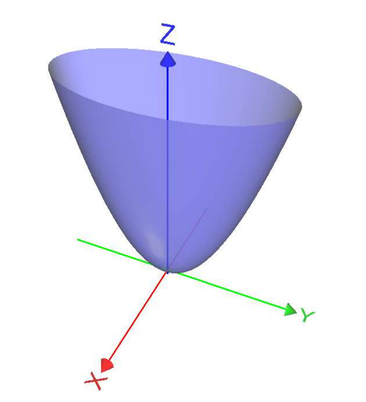
A Second Graph

- Sketch the graph of the function $f(x, y) = \sqrt{9 - x^2 - y^2}$.
  Rewriting $z = \sqrt{9 - x^2 - y^2}$ as $x^2 + y^2 + z^2 = 9$, we get the equation of a sphere with center at the origin and radius 3. But the positive square root allows only the upper hemisphere.



A Third Graph

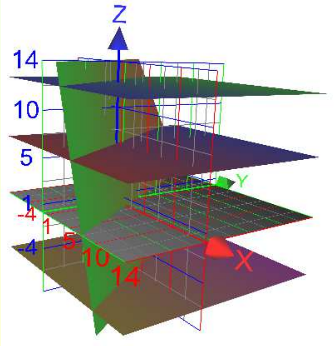
- Sketch the graph of the function $f(x, y) = 4x^2 + y^2$.
  Calculating traces, we see that $z = 4x^2 + y^2$ is the equation of an elliptic paraboloid.

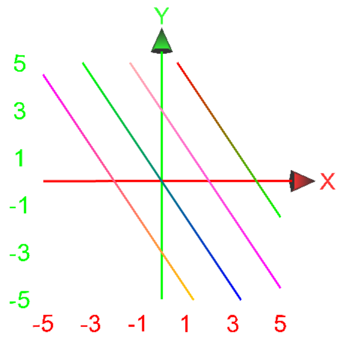


Level Curves

- The **level curves** of a function $f(x, y)$ of two variables are the curves with equations $f(x, y) = c$, where c is a constant in the range of f.

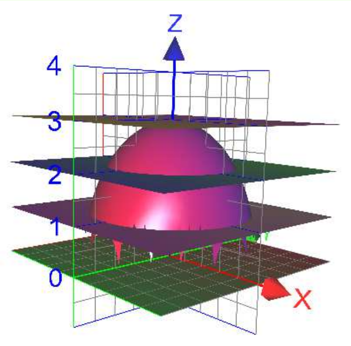
  Example: Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for $c = -6, 0, 6, 12$.

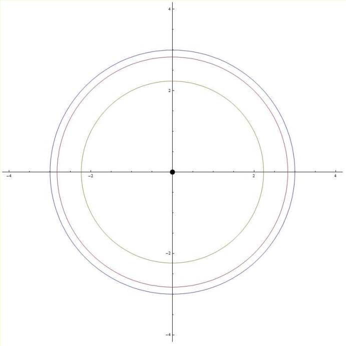




Level Curves: Second Example

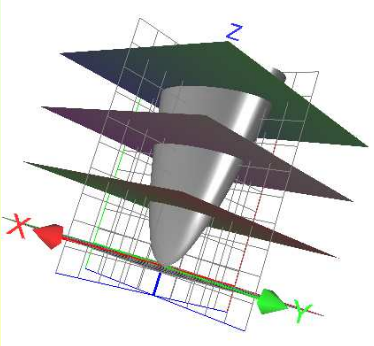
- Sketch the level curves of the function $f(x,y) = \sqrt{9 - x^2 - y^2}$ for $c = 0, 1, 2, 3$.

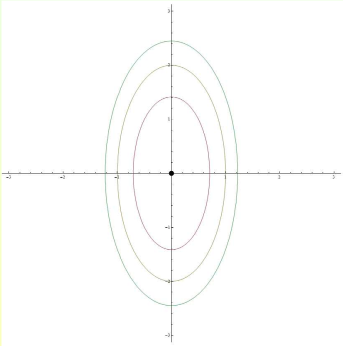




Level Curves: Third Example

- Sketch the level curves of the function $f(x, y) = 4x^2 + y^2$ for $c = 0, 2, 4, 6$.





Functions of Three Variables

- A **function of three variables** $f(x, y, z)$ is a rule that assigns to each ordered triple (x, y, z) in a domain $\mathcal{D}$ a unique real number $f(x, y, z)$.

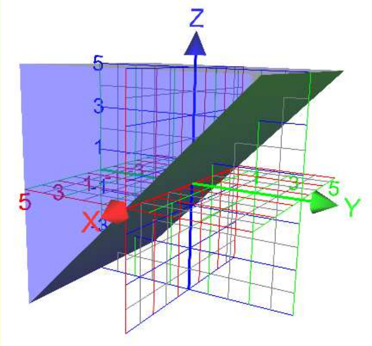
  Example: What is the domain $\mathcal{D}$ of the function

$$f(x, y, z) = \ln(z - y) + xy \sin z?$$

We must have $z - y > 0$, i.e., $z > y$. Thus, the domain of f is the following half-space

$$\mathcal{D} = \{(x, y, z) \in \mathbb{R}^3 : z > y\}$$

of $\mathbb{R}^3$:



Subsection 2

Limits and Continuity in Several Variables

Limits

- Suppose f is a function of two variables whose domain $\mathcal{D}$ includes points arbitrarily close to the point (a,b).

  We say that the **limit of** $f(x,y)$ **as** (x,y) **approaches** (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

  if the values of $f(x,y)$ approach the number L as the point (x,y) approaches the point (a,b) along any path that stays within $\mathcal{D}$.

- The definition implies that, if
  - $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_1$ in $\mathcal{D}$,
  - $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_2$ in $\mathcal{D}$,
  - $L_1 \neq L_2$,

  then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example of Non-Existence

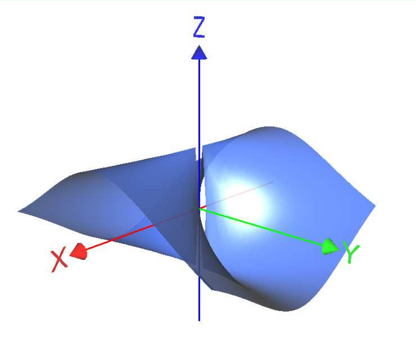
- Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

  If $(x,y) \to (0,0)$ along the x-axis, then $y=0$, whence

  $$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2}{x^2} \to 1.$$

  If $(x,y) \to (0,0)$ along the y-axis, then $x=0$, whence

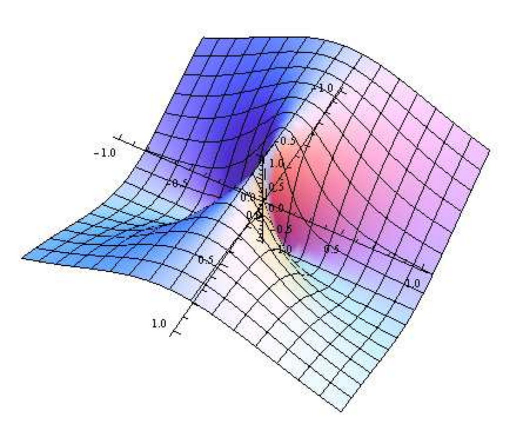
  $$\frac{x^2-y^2}{x^2+y^2} = \frac{-y^2}{y^2} \to -1.$$



  Since f approaches two different values along two different paths, the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Example of Non-Existence (Another Point of View)

$$f(x) = \frac{x^2 - y^2}{x^2 + y^2}$$



Another Example of Non-Existence

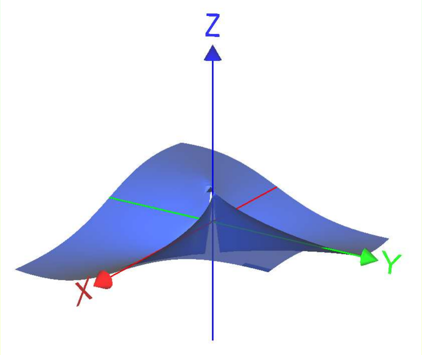
- Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

If $(x, y) \to (0, 0)$ along the x-axis, then $y = 0$, whence

$$\frac{xy}{x^2+y^2} = \frac{x \cdot 0}{x^2+0} \to 0.$$

If $(x, y) \to (0, 0)$ along the line $y = x$, then

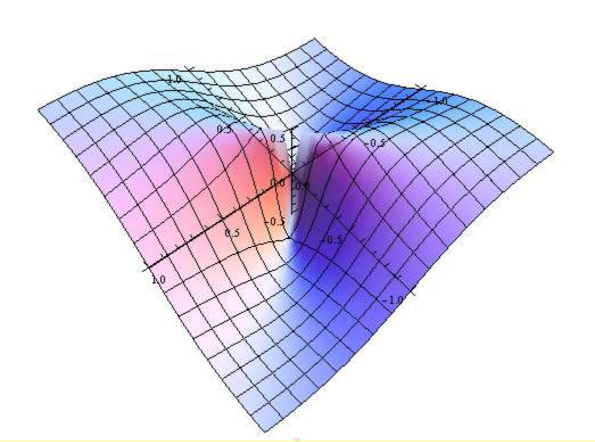
$$\frac{xy}{x^2+y^2} = \frac{x^2}{x^2+x^2} \to \frac{1}{2}.$$



Since f approaches two different values along two different paths, the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist;

Another Example of Non-Existence (Second Point of View)

$$f(x) = \frac{xy}{x^2 + y^2}.$$



A More Difficult Example of Non-Existence

- Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

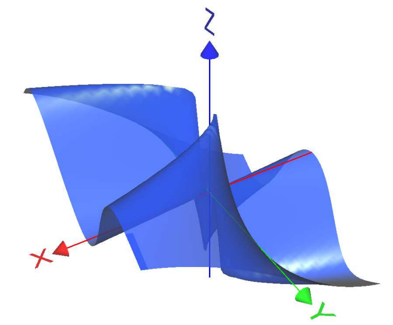
  If $(x,y)\to(0,0)$ along any line $y=mx$ through the origin,

$$\frac{xy^2}{x^2+y^4} = \frac{xm^2x^2}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2} \to 0.$$

  If $(x,y)\to(0,0)$ along the parabola $x=y^2$, then

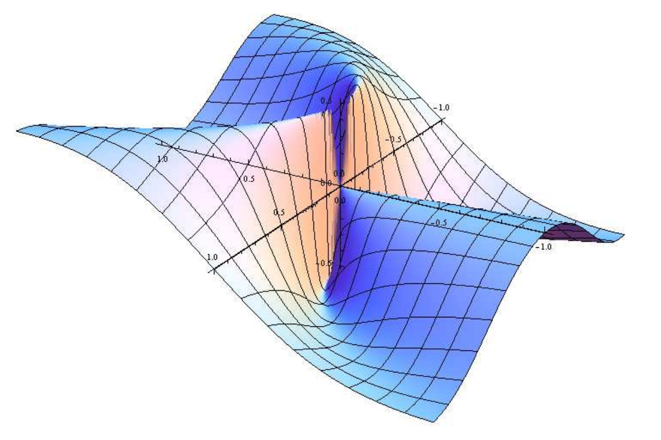
$$\frac{xy^2}{x^2+y^4} = \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} \to \frac{1}{2}.$$

  Since f approaches two different values along two different paths, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.



More Difficult Example (Second Point of View)

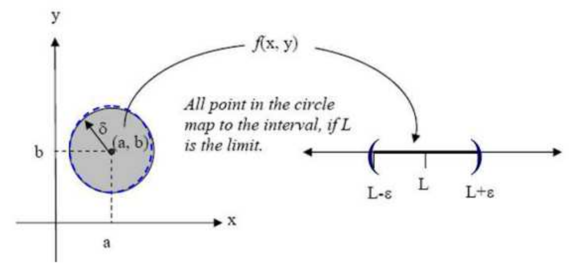
$$f(x) = \frac{xy^2}{x^2 + y^4}$$



Formal Definition of Limit

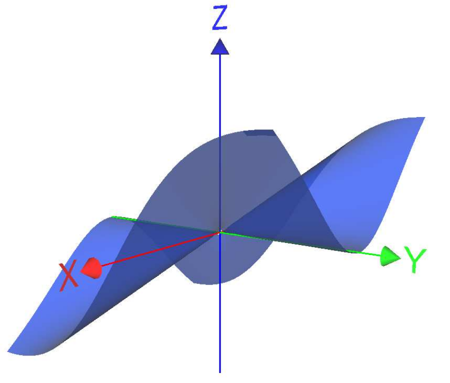
- Let f be a function of two variables whose domain $\mathcal{D}$ includes points arbitrarily close to (a, b).
  The **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L, written $\lim_{(x,y)\to(a,b)} f(x, y) = L$, if for every number $\epsilon > 0$, there exists a number $\delta > 0$, such that

  if $(x, y) \in \mathcal{D}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.



Showing Existence of Limits

- Because there are many paths a point may follow to approach a fixed point, showing that a limit exists is rather difficult.
- We show formally that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$;



The Limit of the Function $f(x, y) = \frac{3x^2y}{x^2+y^2}$

- Assume that the distance from $(x, y) \neq (0, 0)$ to $(0, 0)$ is less than δ, i.e., $0 < \sqrt{x^2 + y^2} < \delta$. Since $\frac{x^2}{x^2 + y^2} \leq \frac{x^2}{x^2} = 1$, we obtain
$$\left|\frac{3x^2y}{x^2 + y^2} - 0\right| = \frac{3x^2|y|}{x^2 + y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2 + y^2}.$$
Thus, we have that the distance of $f(x, y)$ from 0 is
$$\left|\frac{3x^2y}{x^2 + y^2} - 0\right| \leq 3\sqrt{x^2 + y^2} < 3\delta.$$
This shows that we can make $|f(x, y) - 0| < \epsilon$ (i.e., arbitrarily small) by taking $0 < \sqrt{x^2 + y^2} < \delta = \frac{\epsilon}{3}$ (i.e., (x, y) sufficiently close to $(0, 0)$) and verifies that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0$.

Limit Laws

- Assume that $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ exist. Then:
  (i) **Sum Law:**
  $$\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y)) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y).$$
  (ii) **Constant Multiple Law:** For any number k,
  $$\lim_{(x,y)\to(a,b)} kf(x,y) = k \lim_{(x,y)\to(a,b)} f(x,y).$$
  (iii) **Product Law:**
  $$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = \left(\lim_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim_{(x,y)\to(a,b)} g(x,y)\right).$$
  (iv) **Quotient Law:** If $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$, then
  $$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}.$$

Continuity

- A function f of two variables is called **continuous at** (a,b) if
$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$
- A function f is **continuous on** $\mathcal{D}$ if it is continuous at all (a,b) in $\mathcal{D}$.
Examples:
  - $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is continuous on $\mathbb{R}^2$ because it is a polynomial.
  - $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ is continuous at all $(a,b) \neq (0,0)$ as a rational function defined, for all $(a,b) \neq (0,0)$. It is discontinuous at $(0,0)$, since it is not defined at $(0,0)$.
  - $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at all $(a,b) \neq (0,0)$ as a rational function defined there. It is also continuous at $(a,b) = (0,0)$, since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

Evaluating Limits by Substitution

- Show that $f(x,y)=\frac{3x+y}{x^2+y^2+1}$ is continuous.

  Then evaluate $\lim_{(x,y)\to(1,2)} f(x,y)$.

  The function $f(x,y)$ is continuous at all points (a,b) because it is a rational function whose denominator $Q(x,y)=x^2+y^2+1$ is never zero.

  Therefore, we can evaluate the limit by substitution:

$$\lim_{(x,y)\to(1,2)} \frac{3x+y}{x^2+y^2+1}=f(1,2)=\frac{3\cdot 1+2}{1^2+2^2+1}=\frac{5}{6}.$$

Product Functions

- Evaluate $\lim_{(x,y)\to(3,0)} x^3 \frac{\sin y}{y}$.

  The limit is equal to a product of limits:

$$\begin{aligned}\lim_{(x,y)\to(3,0)} x^3 \frac{\sin y}{y} &= \left(\lim_{(x,y)\to(3,0)} x^3\right)\left(\lim_{(x,y)\to(3,0)} \frac{\sin y}{y}\right) \\ &= 3^3 \cdot 1 = 27.\end{aligned}$$

A Composite of Continuous Functions Is Continuous

- If
  - $f(x, y)$ is continuous at (a, b),
  - $G(u)$ is continuous at $c = f(a, b)$,

  then the composite function $G(f(x, y))$ is continuous at (a, b).

  Example: Write $H(x, y) = e^{-x^2+2y}$ as a composite function and evaluate $\lim_{(x,y)\to(1,2)} H(x, y)$.

  We have $H(x, y) = G \circ f$, where
  - $G(u) = e^u$;
  - $f(x, y) = -x^2 + 2y$.

  Both f and G are continuous. So H is also continuous. This allows computing the limit as follows:

$$\lim_{(x,y)\to(1,2)} H(x, y) = \lim_{(x,y)\to(1,2)} e^{-x^2+2y} = e^{-(1)^2+2\cdot 2} = e^3.$$

Subsection 3

Partial Derivatives

Partial Derivative With Respect to x

- If f is a function of x and y, by keeping y constant, say $y = b$, we can consider a function of a single variable x:

$$g(x) = f(x, b).$$

- If g has a derivative at $x = a$, we call it the **partial derivative of** f **with respect to** x at (a, b) and denote it by $f_x(a, b)$.
- Thus, $f_x(a, b) = g'(a)$, where $g(x) = f(x, b)$.
- More formally, the **partial derivative** f_x of $f(x, y)$ is the function

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

- Sometimes we write $f_x(x, y) = \frac{\partial f}{\partial x} = D_1 f = D_x f$.

Partial Derivative With Respect to y

- If f is a function of x and y, by keeping x constant, say $x = a$, we can consider a function of a single variable y:

$$h(y) = f(a, y).$$

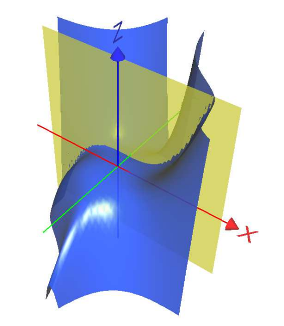
- If h has a derivative at $y = b$, we call it the **partial derivative of** f **with respect to** y at (a, b) and denote it by $f_y(a, b)$.
- Thus, $f_y(a, b) = h'(b)$, where $h(y) = f(a, y)$.
- More formally, the **partial derivative** f_y of $f(x, y)$ is the function

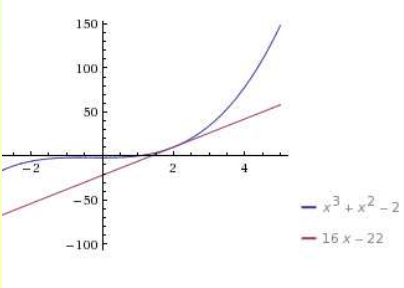
$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

- Sometimes we write $f_y(x, y) = \frac{\partial f}{\partial y} = D_2 f = D_y f$.

Computing the Partials

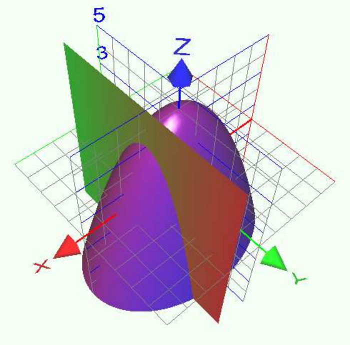
- To find f_x regard y as a constant and differentiate with respect to x.
Example: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, then $f_x(x,y) = 3x^2 + 2xy^3$ and $f_x(2,1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$.
- To find f_y regard x as a constant and differentiate with respect to y.
Example: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, then $f_y(x,y) = 3x^2y^2 - 4y$ and $f_y(2,1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$.

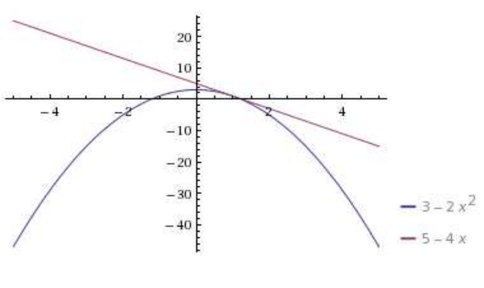




Another Example of Partials

- Let $f(x,y) = 4 - x^2 - 2y^2$.
  Then $f_x(x,y) = -2x$ and $f_x(1,1) = -2$.
  Moreover, $f_y(x,y) = -4y$ and $f_y(1,1) = -4$.



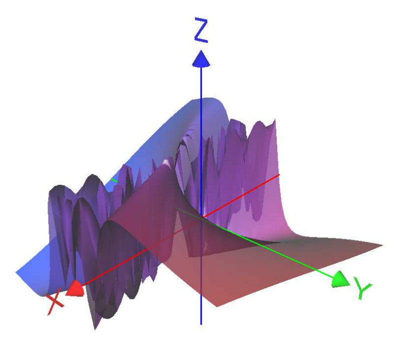


A Third Example of Partials

- Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$.

  Then $\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$ and

  $$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}.$$



Implicit Partial Differentiation

- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x, y by

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Take partials with respect to x: $\dfrac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz) = \dfrac{\partial(1)}{\partial x}$.
Thus, we get $3x^2 + 3z^2\dfrac{\partial z}{\partial x} + 6y(z + x\dfrac{\partial z}{\partial x}) = 0$. To solve for $\frac{\partial z}{\partial x}$, we separate $(3z^2 + 6xy)\dfrac{\partial z}{\partial x} = -3x^2 - 6yz$ and, therefore,

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

- Do similar work for $\frac{\partial z}{\partial y}$.

Answer: $\dfrac{\partial z}{\partial y} = -\dfrac{y^2 + 2xz}{z^2 + 2xy}$.

Second Order Partial Derivatives

- For a function f of two variables x, y it is possible to consider four **second-order partial derivatives**:
  - $(f_x)_x = f_{xx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$
  - $(f_x)_y = f_{xy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$
  - $(f_y)_x = f_{yx} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$
  - $(f_y)_y = f_{yy} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}$

  Example: Calculate all four second order derivatives of $f(x, y) = x^3 + x^2 y^3 - 2y^2$.
  - $f_x = \frac{\partial f}{\partial x} = 3x^2 + 2xy^3$ and $f_y = \frac{\partial f}{\partial y} = 3x^2 y^2 - 4y$.
  - $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x + 2y^3$ and $f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$.
  - $f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$ and $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6x^2 y - 4$.

  Note that $f_{xy} = f_{yx}$.

Clairaut's Theorem

Clairaut's Theorem

If f is defined on a disk $\mathcal{D}$ containing the point (a, b) and the partial derivatives f_{xy} and f_{yx} are both continuous on $\mathcal{D}$, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example: Show that, if $f(x, y) = x \sin(x + 2y)$, then $f_{xy} = f_{yx}$.

For the first-order partials, we have

$$f_x = \sin(x + 2y) + x\cos(x + 2y), \quad f_y = 2x\cos(x + 2y).$$

Therefore, we obtain

$$f_{xy} = 2\cos(x + 2y) - 2x\sin(x + 2y),$$

and

$$f_{yx} = 2\cos(x + 2y) - 2x\sin(x + 2y).$$

Verifying Clairaut's Theorem

- If $W(T,U) = e^{U/T}$, verify that $\frac{\partial^2 W}{\partial U \partial T} = \frac{\partial^2 W}{\partial T \partial U}$.

$$\begin{aligned}
\frac{\partial W}{\partial T} &= e^{U/T}\frac{\partial}{\partial T}\left(\frac{U}{T}\right) = -\frac{U}{T^2}e^{U/T};\\
\frac{\partial W}{\partial U} &= e^{U/T}\frac{\partial}{\partial U}\left(\frac{U}{T}\right) = \frac{1}{T}e^{U/T};\\
\frac{\partial^2 W}{\partial U \partial T} &= \frac{\partial}{\partial U}\left(-\frac{U}{T^2}\right)e^{U/T} + \left(-\frac{U}{T^2}\right)\frac{\partial}{\partial U}(e^{U/T})\\
&= -\frac{1}{T^2}e^{U/T} - \frac{U}{T^3}e^{U/T};\\
\frac{\partial^2 W}{\partial T \partial U} &= \frac{\partial}{\partial T}\left(\frac{1}{T}\right)e^{U/T} + \frac{1}{T}\frac{\partial}{\partial T}(e^{U/T})\\
&= -\frac{1}{T^2}e^{U/T} - \frac{U}{T^3}e^{U/T}.
\end{aligned}$$

Using Clairaut's Theorem

- Although Clairaut's Theorem is stated for f_{xy} and f_{yx}, it implies more generally that partial differentiation may be carried out in any order, provided that the derivatives in question are continuous.

  Example: Calculate the partial derivative f_{zzwx}, where $f(x,y,z,w) = x^3w^2z^2 + \sin(\frac{xy}{z^2})$.

  We differentiate with respect to w first:

  $$\frac{\partial}{\partial w}(x^3w^2z^2 + \sin(\frac{xy}{z^2})) = 2x^3wz^2.$$

  Next, differentiate twice with respect to z and once with respect to x:

  $$\begin{aligned} f_{wz} &= \tfrac{\partial}{\partial z}(2x^3wz^2) = 4x^3wz; \\ f_{wzz} &= \tfrac{\partial}{\partial z}(4x^3wz) = 4x^3w; \\ f_{wzzx} &= \tfrac{\partial}{\partial x}(4x^3w) = 12x^2w. \end{aligned}$$

  We conclude that $f_{zzwx} = f_{wzzx} = 12x^2w$.

Partial Differential Equations (PDEs)

- Verify that $f(x,y) = e^x \sin y$ is a solution of **Laplace's partial differential equation** $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

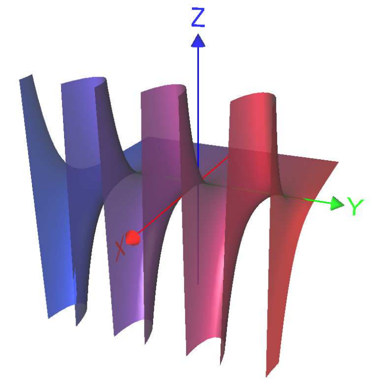
  We have

  $$f_x = e^x \sin y, \quad f_y = e^x \cos y,$$

  $$f_{xx} = e^x \sin y, \quad f_{yy} = -e^x \sin y.$$

  Thus,

  $$f_{xx} + f_{yy} = 0.$$

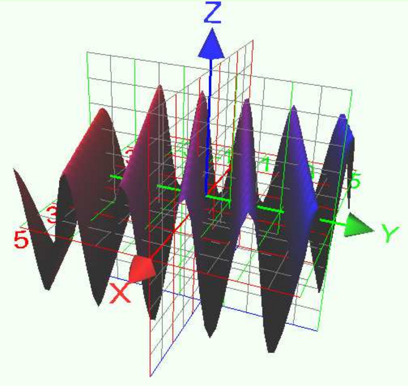


Partial Differential Equations (PDEs)

- Verify that $f(x,t) = \sin(x - at)$ is a solution of the **wave partial differential equation** $\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$.

$$\begin{aligned} \frac{\partial f}{\partial t} &= -a\cos(x - at), \\ \frac{\partial f}{\partial x} &= \cos(x - at), \\ \frac{\partial^2 f}{\partial t^2} &= -a^2 \sin(x - at), \\ \frac{\partial^2 f}{\partial x^2} &= -\sin(x - at). \end{aligned}$$

Thus, $\frac{\partial^2 f}{\partial t^2} = a^2 \frac{\partial^2 f}{\partial x^2}$.

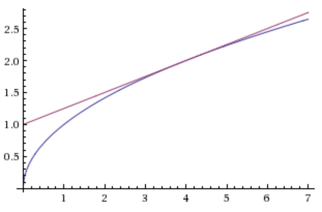


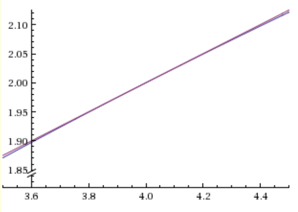
Subsection 4

Differentiability and Tangent Planes

Tangent Lines and Linear Approximations

- Consider the function $f(x) = \sqrt{x}$.
  Calculate $f'(x) = \frac{1}{2\sqrt{x}}$ and $f'(4) = \frac{1}{4}$. Thus, the equation of the tangent line to f at $x = 4$ is
  $$y - 2 = \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{1}{4}x + 1.$$
- Very close to $x = 4$, $y = \sqrt{x}$ can be very accurately approximated by $y = \frac{1}{4}x + 1$.
  Therefore, e.g., $1.994993734 = \sqrt{3.98} \approx \frac{1}{4} \cdot 3.98 + 1 = 1.995$.





Tangent Planes and Linear Approximations

- Consider $f(x, y)$ with continuous partial derivatives.
- An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P = (a, b, c)$, where $c = f(a, b)$, is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

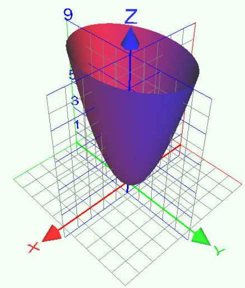
Example: Consider the elliptic paraboloid $f(x, y) = 2x^2 + y^2$.

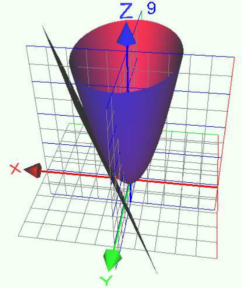
Since $f_x(x, y) = 4x$ and $f_y(x, y) = 2y$,

we have $f_x(1, 1) = 4$ and $f_y(1, 1) = 2$. Therefore, the plane

$$z - 3 = 4(x - 1) + 2(y - 1)$$

is the tangent plane to the paraboloid at $(1, 1, 3)$.





Linearization of f at (a, b)

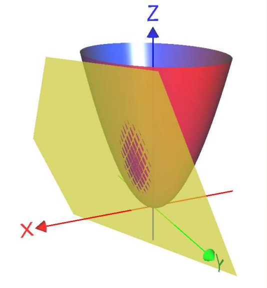
- Given a function $f(x, y)$ with continuous partial derivatives f_x, f_y, an equation of the tangent plane to $f(x, y)$ at $(a, b, f(a, b))$ is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

- The linear function whose graph is this tangent plane

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

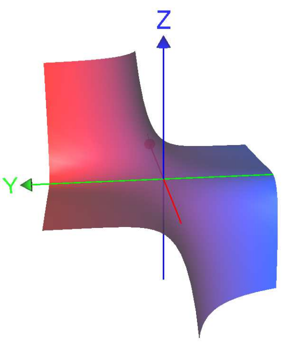
is called the **linearization** of f at (a, b). The approximation $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is called the **linear approximation** of f at (a, b).
Example: We saw for $f(x, y) = 2x^2 + y^2$, that $f(x, y) \approx 3 + 4(x - 1) + 2(y - 1)$ near $(1, 1, 3)$.

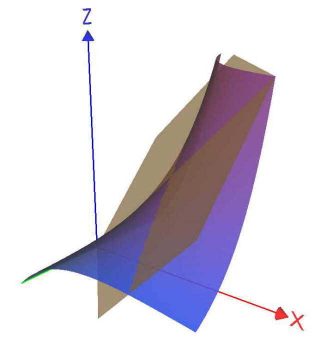


Another Example of a Linearization

- Consider the function $f(x,y) = xe^{xy}$.
  We have $f_x(x,y) = e^{xy} + xye^{xy}$ and $f_y(x,y) = x^2e^{xy}$.
  Thus, $f_x(1,0) = 1$ and $f_y(1,0) = 1$.
  So the linearization of $f(x,y)$ at $(1,0,1)$ is

$$f(x,y) \approx 1 + (x-1) + (y-0) = x + y.$$





Differentiability

- Assume that $f(x,y)$ is defined in a disk $\mathcal{D}$ containing (a,b) and that $f_x(a,b)$ and $f_y(a,b)$ exist.

  $f(x,y)$ is **differentiable at** (a,b) if it is **locally linear**, i.e.,

  $$f(x,y) = L(x,y) + e(x,y),$$

  where $e(x,y)$ satisfies $\lim_{(x,y)\to(a,b)} \frac{e(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$.

  In this case, the **tangent plane** to the graph at $(a,b,f(a,b))$ is the plane with equation

  $$z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

- If $f(x,y)$ is differentiable at all points in a domain $\mathcal{D}$, we say that $f(x,y)$ is **differentiable on** $\mathcal{D}$.

Criterion for Differentiability

- The following theorem provides a criterion for differentiability and shows that all familiar functions are differentiable on their domains.

Criterion for Differentiability

If $f_x(x, y)$ and $f_y(x, y)$ exist and are continuous on an open disk $\mathcal{D}$, then $f(x, y)$ is differentiable on $\mathcal{D}$.

Example: Show that $f(x, y) = 5x + 4y^2$ is differentiable and find the equation of the tangent plane at $(a, b) = (2, 1)$.

The partial derivatives exist and are continuous functions: $f_x(x, y) = 5$, $f_y(x, y) = 8y$. Therefore, $f(x, y)$ is differentiable for all (x, y), by the criterion.

To find the tangent plane, we evaluate the partial derivatives at $(2, 1)$: $f(2, 1) = 14$, $f_x(2, 1) = 5$, and $f_y(2, 1) = 8$. The linearization at $(2, 1)$ is $L(x, y) = 14 + 5(x - 2) + 8(y - 1) = -4 + 5x + 8y$. Thus, the tangent plane through $P = (2, 1, 14)$ has equation $z = -4 + 5x + 8y$.

Tangent Plane

- Find a tangent plane of the graph of $f(x,y) = xy^3 + x^2$ at $(2,-2)$.

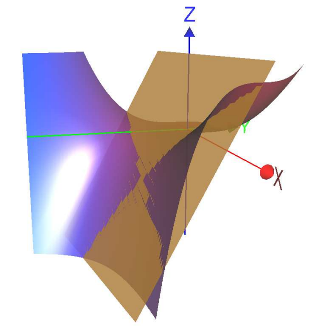
  The partial derivatives are continuous, so $f(x,y)$ is differentiable:

  $$f_x(x,y) = y^3 + 2x, \quad f_x(2,-2) = -4,$$
  $$f_y(x,y) = 3xy^2, \quad f_y(2,-2) = 24.$$

  Since $f(2,-2) = -12$, the tangent plane through $(2,-2,-12)$ has equation

  $$z = -12 - 4(x-2) + 24(y+2).$$

  This can be rewritten as $z = 44 - 4x + 24y$.



Differentials

- For $z = f(x, y)$ a differentiable function of two variables, the **differentials** dx, dy are independent variables, i.e., can be assigned any values.
- The **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

- If we set $dx = x - a$ and $dy = y - b$ in the formula for the linear approximation of f, we have

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) = f(a, b) + dz.$$

Example: Consider $f(x, y) = x^2 + 3xy - y^2$. Then $dz = f_x(x, y)dx + f_y(x, y)dy = (2x + 3y)dx + (3x - 2y)dy$. If x changes from 2 to 2.05 and y changes from 3 to 2.96, then $dx = 0.05, dy = -0.04$ and $(a, b) = (2, 3)$, whence $dz = f_x(2, 3) \cdot 0.05 + f_y(2, 3) \cdot (-0.04) = 0.65$ and $f(2.05, 2.96) \approx f(2, 3) + dz = 13 + 0.65 = 13.65$.

Using Differentials for Error Estimation

- If the base radius and the height of a right circular cone are measured as 10 cm and 25 cm, respectively, with possible maximum error 0.1 cm in each, estimate the max possible error in calculating the volume of the cone, given that the volume formula is $V(r,h) = \frac{1}{3}\pi r^2 h$.

  We have $dV = V_r dr + V_h dh = \frac{2}{3}\pi rhdr + \frac{1}{3}\pi r^2 dh$.

  Therefore

$$\begin{aligned} dV &= \tfrac{2}{3}\pi \cdot 10 \cdot 25 \cdot (\pm 0.1) + \tfrac{1}{3}\pi \cdot 10^2 \cdot (\pm 0.1) \\ &= (\tfrac{500}{3}\pi + \tfrac{100}{3}\pi) \cdot (\pm 0.1) \\ &= \pm\, 20\pi \text{ cm}^3. \end{aligned}$$

Application: Change in Body Mass Index (BMI)

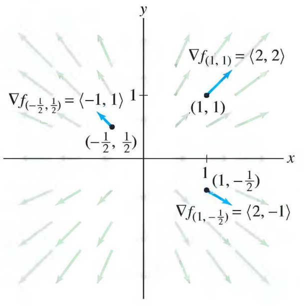
- A person's BMI is $I = \frac{W}{H^2}$, where W is the body weight (in kilograms) and H is the body height (in meters). Estimate the change in a child's BMI if (W, H) changes from $(40, 1.45)$ to $(41.5, 1.47)$.
  We have
  $$\frac{\partial I}{\partial W} = \frac{1}{H^2}, \quad \frac{\partial I}{\partial H} = -\frac{2W}{H^3}.$$
  At $(W, H) = (40, 1.45)$, we get
  $$\left.\frac{\partial I}{\partial W}\right|_{(40,1.45)} = \frac{1}{1.45^2}, \quad \left.\frac{\partial I}{\partial H}\right|_{(40,1.45)} = -\frac{2 \cdot 40}{1.45^3}.$$
  The differential $dI \approx \frac{1}{1.45^2} dW - \frac{80}{1.45^3} dH$.
  If (W, H) changes from $(40, 1.45)$ to $(41.5, 1.47)$, then $dW = 1.5$ and $dH = 0.02$. Therefore,
  $$\Delta I \approx dI = \frac{1}{1.45^2} dW - \frac{2 \cdot 40}{1.45^3} dH = \frac{1}{1.45^2} \cdot 1.5 - \frac{80}{1.45^3} \cdot 0.02.$$

Subsection 5

The Gradient and Directional Derivatives

The Gradient Vector

- The **gradient of a function** $f(x, y)$ **at a point** $P = (a, b)$ is the vector
$$\nabla f_P = \langle f_x(a, b), f_y(a, b) \rangle.$$
In three variables, if $P = (a, b, c)$,
$$\nabla f_P = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle.$$
- We also write $\nabla f_{(a,b)}$ or $\nabla f(a, b)$ for the gradient. Sometimes, we omit reference to the point P and write
$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle.$$
The gradient ∇f assigns a vector ∇f_P to each point in the domain of f.



Examples

- Let $f(x,y) = x^2 + y^2$. Calculate the gradient ∇f and compute ∇f_P at $P = (1,1)$.

  The partial derivatives are $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$. So $\nabla f = \langle 2x, 2y \rangle$. At $(1,1)$, $\nabla f_P = \nabla f(1,1) = \langle 2, 2 \rangle$.

- If $f(x,y) = \sin x + e^{xy}$, compute ∇f.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle.$$

- Calculate $\nabla f_{(3,-2,4)}$, where $f(x,y,z) = ze^{2x+3y}$.

  The partial derivatives and the gradient are $\frac{\partial f}{\partial x} = 2ze^{2x+3y}$, $\frac{\partial f}{\partial y} = 3ze^{2x+3y}$, $\frac{\partial f}{\partial z} = e^{2x+3y}$. So $\nabla f = \langle 2ze^{2x+3y}, 3ze^{2x+3y}, e^{2x+3y} \rangle$. Finally, $\nabla f_{(3,-2,4)} = \langle 8, 12, 1 \rangle$.

Properties of the Gradient Vector

- If $f(x, y, z)$ and $g(x, y, z)$ are differentiable and c is a constant, then:
  - (i) $\nabla(f+g) = \nabla f + \nabla g$ (**Sum Rule**)
  - (ii) $\nabla(cf) = c\nabla f$ (**Constant Multiple Rule**)
  - (iii) $\nabla(fg) = f\nabla g + g\nabla f$ (**Product Rule**)
  - (iv) If $F(t)$ is a differentiable function of one variable, then

$$\nabla(F(f(x, y, z))) = F'(f(x, y, z))\nabla f \quad (\textbf{Chain Rule}).$$

Using the Chain Rule

- Find the gradient of

$$g(x, y, z) = (x^2 + y^2 + z^2)^8.$$

The function g is a composite $g(x, y, z) = F(f(x, y, z))$, with:

  - $F(t) = t^8$;
  - $f(x, y, z) = x^2 + y^2 + z^2$.

Now we have

$$\begin{array}{rcl} \nabla g & = & \nabla((x^2 + y^2 + z^2)^8) \\ & = & 8(x^2 + y^2 + z^2)^7 \nabla(x^2 + y^2 + z^2) \\ & = & 8(x^2 + y^2 + z^2)^7 \langle 2x, 2y, 2z \rangle \\ & = & 16(x^2 + y^2 + z^2)^7 \langle x, y, z \rangle. \end{array}$$

Chain Rule for Paths

- If $z = f(x, y)$ is a differentiable function of x and y, where $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot \langle x'(t), y'(t)\rangle.$$

- Alternatve formulation: If $f(x, y)$ is a differentiable function of x and y and $\boldsymbol{c}(t) = \langle x(t), y(t)\rangle$ a differentiable function of t, then

$$\frac{d}{dt}f(\boldsymbol{c}(t)) = \nabla f_{\boldsymbol{c}(t)} \cdot \boldsymbol{c}'(t)$$

  also written

$$\frac{d}{dt}f(\boldsymbol{c}(t)) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \langle x'(t), y'(t)\rangle.$$

Applying The Chain Rule for Paths

- Suppose that $f(x,y) = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$. Compute $\frac{dz}{dt}$ at $t = 0$.
  We have

$$\frac{\partial f}{\partial x} = 2xy + 3y^4,\ \frac{\partial f}{\partial y} = x^2 + 12xy^3,\ \frac{dx}{dt} = 2\cos 2t,\ \frac{dy}{dt} = -\sin t.$$

At $t = 0$, $x = \sin 0 = 0$, $y = \cos 0 = 1$, whence

$$\frac{\partial f}{\partial x} = 3, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 0.$$

Since $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$, we get, $\left.\frac{dz}{dt}\right|_{t=0} = 3 \cdot 2 + 0 \cdot 0 = 6$.

Application

- The pressure P in kilopascals, the volume V in liters and the temperature T in kelvins of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/sec and the volume is 100 L and increasing at a rate of 0.2 L/sec.

  Note, first, that $P = \frac{8.31T}{V}$.

  Thus, we have

$$\frac{\partial P}{\partial T} = \frac{8.31}{V}, \ \frac{\partial P}{\partial V} = -\frac{8.31T}{V^2}, \ \frac{dT}{dt} = 0.1, \ \frac{dV}{dt} = 0.2.$$

  Moreover, since $T = 300$ and $V = 100$,

$$\frac{\partial P}{\partial T} = \frac{8.31}{100}, \quad \frac{\partial P}{\partial V} = -\frac{8.31 \cdot 300}{100^2}.$$

  Therefore, $\frac{dP}{dt} = \frac{8.31}{100} \cdot 0.1 + (-\frac{8.31 \cdot 300}{100^2}) \cdot 0.2$ kPa/sec.

The Chain Rule for Paths in Three Variables

- In general, if $f(x_1,\ldots,x_n)$ is a differentiable function of n variables and $\boldsymbol{c}(t)=\langle x_1(t),\ldots,x_n(t)\rangle$ is a differentiable path, then
$$\frac{d}{dt}f(\boldsymbol{c}(t))=\nabla f\cdot\boldsymbol{c}'(t)=\frac{\partial f}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial f}{\partial x_2}\frac{dx_2}{dt}+\cdots+\frac{\partial f}{\partial x_n}\frac{dx_n}{dt}.$$
Example: Calculate $\frac{d}{dt}f(\boldsymbol{c}(t))\big|_{t=\pi/2}$, where $f(x,y,z)=xy+z^2$ and $\boldsymbol{c}(t)=\langle\cos t,\sin t,t\rangle$.
We have $\boldsymbol{c}(\frac{\pi}{2})=\langle\cos\frac{\pi}{2},\sin\frac{\pi}{2},\frac{\pi}{2}\rangle=\langle 0,1,\frac{\pi}{2}\rangle$.
Compute the gradient: $\nabla f=\langle y,x,2z\rangle$ and $\nabla f_{\boldsymbol{c}(0,1,\frac{\pi}{2})}=\langle 1,0,\pi\rangle$.
Then compute the tangent vector:
$$\boldsymbol{c}'(t)=\langle-\sin t,\cos t,1\rangle,\quad \boldsymbol{c}'(\frac{\pi}{2})=\langle-1,0,1\rangle.$$
By the Chain Rule,
$$\frac{d}{dt}(f(\boldsymbol{c}(t))\bigg|_{t=\pi/2}=\nabla f_{\boldsymbol{c}(\frac{\pi}{2})}\cdot\boldsymbol{c}'(\frac{\pi}{2})=\langle 1,0,\pi\rangle\cdot\langle-1,0,1\rangle=\pi-1.$$

Application

- The temperature at (x,y) is $T(x,y) = 20 + 10e^{-0.3(x^2+y^2)}$ °C. A bug carries a tiny thermometer along the path $\mathbf{c}(t) = \langle \cos(t-2), \sin 2t \rangle$ (t in seconds). How fast is the temperature changing at time t?

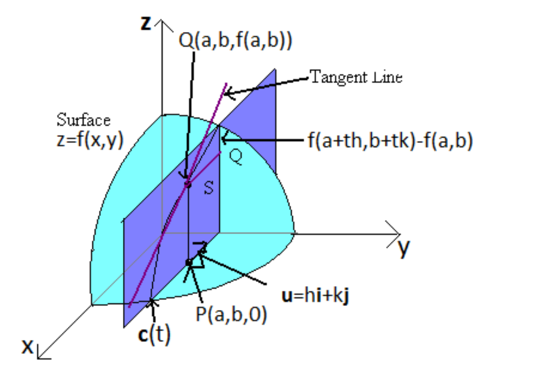
$$\begin{aligned}
\frac{dT}{dt} &= \nabla T_{\mathbf{c}(t)} \cdot \mathbf{c}'(t); \\
\nabla T_{\mathbf{c}(t)} &= \langle -6xe^{-0.3(x^2+y^2)}, -6ye^{-0.3(x^2+y^2)} \rangle_{\mathbf{c}(t)} \\
&= \langle -6\cos(t-2)e^{-0.3(\cos^2(t-2)+\sin^2(2t))}, \\
&\qquad -6\sin(2t)e^{-0.3(\cos^2(t-2)+\sin^2(2t))} \rangle; \\
\mathbf{c}'(t) &= \langle -\sin(t-2), 2\cos(2t) \rangle.
\end{aligned}$$

So, we get

$$\begin{aligned}
\frac{dT}{dt} &= 6\sin(t-2)\cos(t-2)e^{-0.3(\cos^2(t-2)+\sin^2(2t))} \\
&\qquad - 12\sin(2t)\cos(2t)e^{-0.3(\cos^2(t-2)+\sin^2(2t))}.
\end{aligned}$$

Directional Derivatives

- The **directional derivative of** f at $P = (a, b)$ **in the direction** of a unit vector $\mathbf{u} = \langle h, k \rangle$ is
$$D_{\mathbf{u}} f(a, b) = \lim_{t \to 0} \frac{f(a + th, b + tk) - f(a, b)}{t}.$$



Computing Directional Derivatives Using Partials

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle h, k \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)h + f_y(x, y)k = \nabla f \cdot \boldsymbol{u}.$$

Example: What is the directional derivative $D_{\mathbf{u}} f(x, y)$ of $f(x, y) = x^3 - 3xy + 4y^2$ in the direction of the unit vector with angle $\theta = \frac{\pi}{6}$? What is $D_{\mathbf{u}} f(1, 2)$?

The unit vector $\mathbf{u}$ with direction $\theta = \frac{\pi}{6}$ is $\mathbf{u} = \langle h, k \rangle = \langle 1 \cos \frac{\pi}{6}, 1 \sin \frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$. Moreover, we have $\frac{\partial f}{\partial x} = 3x^2 - 3y$ and $\frac{\partial f}{\partial y} = -3x + 8y$. Therefore,

$$D_{\mathbf{u}} f(x, y) = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y).$$

In particular, for $(x, y) = (1, 2)$, $D_{\mathbf{u}}(1, 2) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$.

Graphical Illustration

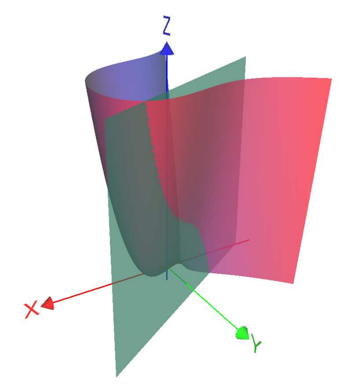
- The graph of the function $f(x,y) = x^3 - 3xy + 4y^2$.

  The plane passing through $(1,2,11)$, with direction $\mathbf{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$.

  The directional derivative

$$D_{\mathbf{u}}(1,2) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$$

  is the slope of the tangent to the curve of intersection of the surface $z = f(x,y)$ with the plane at $(1,2,11)$.



Directional Derivatives Generalized

- To evaluate directional derivatives, it is convenient to define $D_{\boldsymbol{v}} f(a,b)$ even when $\boldsymbol{v} = \langle h, k \rangle$ is not a unit vector:

$$D_{\boldsymbol{v}} f(a,b) = \lim_{t \to 0} \frac{f(a+th, b+tk) - f(a,b)}{t}.$$

  We call $D_{\boldsymbol{v}} f$ the **derivative with respect to** $\boldsymbol{v}$.

- We have

$$D_{\boldsymbol{v}} f(a,b) = \nabla f(a,b) \cdot \boldsymbol{v}.$$

- It $\boldsymbol{v} \neq \boldsymbol{0}$, then $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ is the unit vector in the direction of $\boldsymbol{v}$, and the directional derivative is given by

$$D_{\boldsymbol{u}} f(P) = \frac{1}{\|\boldsymbol{v}\|} \nabla f_P \cdot \boldsymbol{v}.$$

Example

- Let $f(x,y) = xe^y$, $P = (2,-1)$ and $\boldsymbol{v} = \langle 2,3\rangle$.
  (a) Calculate $D_{\boldsymbol{v}} f(P)$.
  (b) Then calculate the directional derivative in the direction of $\boldsymbol{v}$.

(a) First compute the gradient at $P = (2,-1)$:

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle e^y, xe^y\rangle \quad \Rightarrow \quad \nabla f_P = \nabla f_{(2,-1)} = \langle \frac{1}{e}, \frac{2}{e}\rangle.$$

Now we get

$$D_{\boldsymbol{v}} f_P = \nabla f_P \cdot \boldsymbol{v} = \langle \frac{1}{e}, \frac{2}{e}\rangle \cdot \langle 2,3\rangle = \frac{8}{e}.$$

(b) The directional derivative is $D_{\boldsymbol{u}} f(P)$, where $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$.
We get

$$D_{\boldsymbol{u}} f(P) = \frac{1}{\|\boldsymbol{v}\|} D_{\boldsymbol{v}} f(P) = \frac{8/e}{\sqrt{2^2+3^2}} = \frac{8}{\sqrt{13}e}.$$

Applying $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$ Directly

- Find the directional derivative of $f(x, y) = x^2 y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\boldsymbol{v} = 2\boldsymbol{i} + 5\boldsymbol{j}$.

  For the gradient vector, we have $\nabla f(x, y) = \langle 2xy^3, 3x^2y^2 - 4\rangle$ and, hence, $\nabla f(2, -1) = \langle -4, 8\rangle$.

  The unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = \langle 2, 5\rangle$ is $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle$.

  Therefore, the directional derivative $D_{\boldsymbol{u}} f(2, -1)$ of f in the direction of $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f(2, -1) = \nabla f(2, -1) \cdot \boldsymbol{u} = \langle -4, 8\rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle = \frac{32}{\sqrt{29}}.$$

Applying $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$ in Three Variables

- If $f(x,y,z) = x \sin yz$, find ∇f and the directional derivative of f at $(1,3,0)$ in the direction of $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

  For the gradient vector, we have
  $\nabla f(x,y,z) = \langle \sin yz, xz \cos yz, xy \cos yz \rangle$ and, hence,
  $\nabla f(1,3,0) = \langle 0,0,3 \rangle$.

  The unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = \langle 1,2,-1 \rangle$ is
  $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \rangle$.

  Therefore, the directional derivative $D_{\boldsymbol{u}} f(1,3,0)$ of f in the direction of $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f(1,3,0) = \nabla f(1,3,0) \cdot \boldsymbol{u} = \langle 0,0,3 \rangle \cdot \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \rangle = -\frac{3}{\sqrt{6}}.$$

Maximum Directional Derivative

Theorem

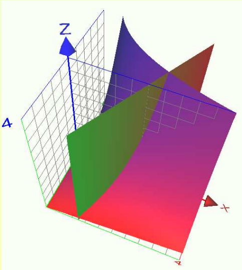
If f is a differentiable function of two or three variables, the maximum value of $D_{\boldsymbol{u}} f(\mathbf{x})$ is $\|\nabla f(x,y)\|$ and it occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $\nabla f(x,y)$.

Example: Suppose that $f(x,y) = xe^y$. Find the rate of change of f at $P = (2,0)$ in the direction from P to $Q = (\frac{1}{2}, 2)$.

We have $\nabla f(x,y) = \langle e^y, xe^y \rangle$, whence $\nabla f(2,0) = \langle 1,2 \rangle$. Moreover, $\overrightarrow{PQ} = \langle -\frac{3}{2}, 2 \rangle$, whence the unit vector in the direction of $\overrightarrow{PQ}$ is

$\boldsymbol{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$. Therefore, we get $D_{\boldsymbol{u}} f(2,0) = \langle 1,2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = 1$.

According to the Theorem, the max change occurs in the direction of $\nabla f(2,0) = \langle 1,2 \rangle$ and equals $\|\nabla f(2,0)\| = \sqrt{5}$.



Example

- Let $f(x,y) = \frac{x^4}{y^2}$ and $P = (2,1)$. Find the unit vector that points in the direction of maximum rate of increase at P.

  The gradient at P points in the direction of maximum rate of increase:

$$\nabla f = \langle \frac{4x^3}{y^2}, -\frac{2x^4}{y^3} \rangle \quad \Rightarrow \quad \nabla f_{(2,1)} = \langle 32, -32 \rangle.$$

  The unit vector in this direction is

$$\boldsymbol{u} = \frac{\langle 32, -32 \rangle}{\| \langle 32, -32 \rangle \|} = \frac{\langle 32, -32 \rangle}{32\sqrt{2}} = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle.$$

Application

- If the temperature at a point (x, y, z) is given by $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$ in degrees Celsius, where x, y, z are in meters, in which direction does the temperature increase the fastest at $(1, 1, -2)$ and what is the maximum rate of increase?

  We have that $\nabla T(x, y, z) = \langle -\frac{160x}{(1+x^2+2y^2+3z^2)^2}, -\frac{320y}{(1+x^2+2y^2+3z^2)^2}, -\frac{480z}{(1+x^2+2y^2+3z^2)^2}\rangle$.

  Thus, $\nabla T(1, 1, -2) = \langle -\frac{5}{8}, -\frac{5}{4}, \frac{15}{4}\rangle$.

  Therefore, the temperature increases the fastest in the direction of the vector $\nabla T(1, 1, -2) = \langle -\frac{5}{8}, -\frac{5}{4}, \frac{15}{4}\rangle$ and the fastest rate of increase is

$$\|\nabla T(1, 1, -2)\| = \sqrt{\frac{25}{64} + \frac{25}{16} + \frac{225}{16}} = \frac{\sqrt{25 + 100 + 900}}{4} = \frac{5\sqrt{41}}{8}.$$

Gradient Vectors and Level Surfaces

- Consider a surface $\mathcal{S}$, with equation $F(x, y, z) = k$.

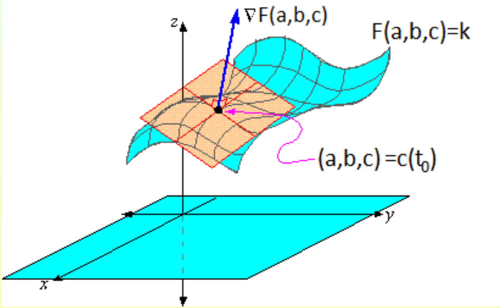
  Let $\mathcal{C}$ be a curve $\mathbf{c}(t) = \langle x(t), y(t), z(t) \rangle$ on the surface $\mathcal{S}$, passing through a point $\mathbf{c}(t_0) = \langle a, b, c \rangle$ on $\mathcal{C}$.

  Recall that

  $$\left. \frac{dF}{dt} \right|_{t=t_0} = \nabla F_{\mathbf{c}(t_0)} \cdot \mathbf{c}'(t_0).$$

  Hence, we get

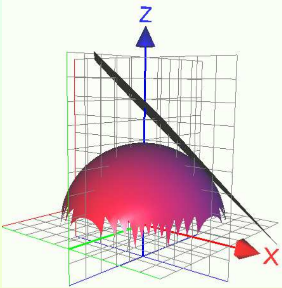
  $$\nabla F_{\mathbf{c}(t_0)} \cdot \mathbf{c}'(t_0) = 0.$$



  Therefore, $\nabla F_{\mathbf{c}(t_0)}$ is perpendicular to the tangent vector $\mathbf{c}'(t_0)$ to any curve $\mathcal{C}$ on $\mathcal{S}$ passing through $\mathbf{c}(t_0)$.

Tangent Plane to a Level Surface

- We define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P = (a, b, c)$ as the plane passing through P, with normal vector $\nabla F(a, b, c)$.



  This plane has equation

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

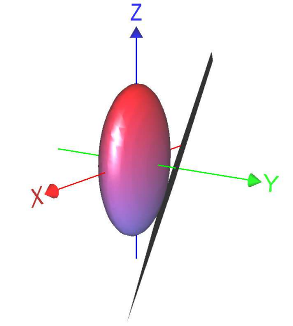
- Moreover, the **normal line** to $\mathcal{S}$ at P that passes through P and is perpendicular to the tangent plane has parametric equations

$$x = a + tF_x(a,b,c),\ \ y = b + tF_y(a,b,c),\ \ z = c + tF_z(a,b,c).$$

Finding a Tangent Plane and a Normal Line

- Let us find the equations of the tangent plane and of the normal line at $P = (-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$;
  We consider $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$.
  We have $F_x(x, y, z) = \frac{1}{2}x$, $F_y(x, y, z) = 2y$, $F_z(x, y, z) = \frac{2}{9}z$.
  So, $F_x(-2, 1, -3) = -1$, $F_y(-2, 1, -3) = 2$ and $F_z(-2, 1, -3) = -\frac{2}{3}$.
  Therefore, the equation of the tangent plane is $-(x+2)+2(y-1)-\frac{2}{3}(z+3) = 0$, i.e., $3x - 6y + 2z + 18 = 0$,
  and the parametric equations of the normal line are

$$\left\{ \begin{array}{lcl} x & = & -2 - t \\ y & = & 1 + 2t \\ z & = & -3 - \frac{2}{3}t \end{array} \right\}.$$



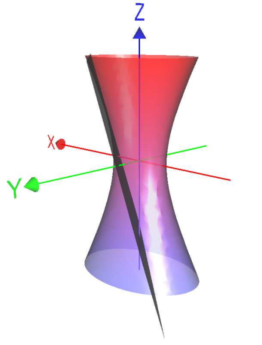
Finding a Normal Vector and a Tangent Plane

- Find an equation of the tangent plane to the surface $4x^2 + 9y^2 - z^2 = 16$ at $P = (2, 1, 3)$.

  Let $F(x, y, z) = 4x^2 + 9y^2 - z^2$. Then $\nabla F = \langle 8x, 18y, -2z \rangle$ and

  $$\nabla F_P = \nabla F(2, 1, 3) = \langle 16, 18, -6 \rangle.$$

  The vector $\langle 16, 18, -6 \rangle$ is normal to the surface $F(x, y, z) = 16$.



  So the tangent plane at P has equation

  $$16(x - 2) + 18(y - 1) - 6(z - 3) = 0 \quad \text{or} \quad 16x + 18y - 6z = 32.$$

Subsection 6

The Chain Rule

The Chain Rule

- If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}, \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

Example: If $f(x, y) = e^x \sin y$, $x = st^2$, $y = s^2 t$, what are $\frac{\partial f}{\partial s}$, $\frac{\partial f}{\partial t}$?
We have

$$\frac{\partial f}{\partial x} = e^x \sin y, \quad \frac{\partial f}{\partial y} = e^x \cos y.$$

We also have

$$\frac{\partial x}{\partial s} = t^2, \quad \frac{\partial x}{\partial t} = 2st, \quad \frac{\partial y}{\partial s} = 2st, \quad \frac{\partial y}{\partial t} = s^2.$$

Therefore,

$$\frac{\partial f}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st, \ \frac{\partial f}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2.$$

The Chain Rule: General Version

- If f is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then f is a differentiable function of $t_1, \ldots, t_m$ and, for all $i = 1, \ldots, m$,

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}.$$

  This may be expressed using the dot product:

$$\frac{\partial f}{\partial t_i} = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \cdot \left\langle \frac{\partial x_1}{\partial t_i}, \frac{\partial x_2}{\partial t_i}, \ldots, \frac{\partial x_n}{\partial t_i} \right\rangle.$$

Using the Chain Rule

- Let $f(x,y,z) = xy + z$. Calculate $\frac{\partial f}{\partial s}$, where $x = s^2$, $y = st$, $z = t^2$. Compute the primary derivatives.

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 1.$$

Next, we get

$$\frac{\partial x}{\partial s} = 2s, \quad \frac{\partial y}{\partial s} = t, \quad \frac{\partial z}{\partial s} = 0.$$

Now apply the Chain Rule:

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \\ &= y \cdot 2s + x \cdot t + 1 \cdot 0 \\ &= (st) \cdot 2s + s^2 \cdot t = 3s^2 t. \end{aligned}$$

Evaluating the Derivative

- If $f = x^4y + y^2z^3$, $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2s\sin t$, find $\frac{\partial f}{\partial s}$ when $r = 2, s = 1$ and $t = 0$.
  Note, first, that for $(r, s, t) = (2, 1, 0)$, we have $(x, y, z) = (2, 2, 0)$. Moreover,

$$\frac{\partial f}{\partial x} = 4x^3y, \quad \frac{\partial f}{\partial y} = x^4 + 2yz^3, \quad \frac{\partial f}{\partial z} = 3y^2z^2.$$

  Thus, for $(r, s, t) = (2, 1, 0)$, we get $\frac{\partial f}{\partial x} = 64, \quad \frac{\partial f}{\partial y} = 16, \quad \frac{\partial f}{\partial z} = 0$. Furthermore,

$$\frac{\partial x}{\partial s} = re^t, \quad \frac{\partial y}{\partial s} = 2rse^{-t}, \quad \frac{\partial z}{\partial s} = r^2\sin t.$$

  Thus, for $(r, s, t) = (2, 1, 0)$, we get $\frac{\partial x}{\partial s} = 2, \quad \frac{\partial y}{\partial s} = 4, \quad \frac{\partial z}{\partial s} = 0$.
  Therefore, $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} = 64 \cdot 2 + 16 \cdot 4 + 0 \cdot 0 = 192$.

Polar Coordinates

- Let $f(x,y)$ be a function of two variables, and let (r,θ) be polar coordinates.
  (a) Express $\frac{\partial f}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
  (b) Evaluate $\frac{\partial f}{\partial \theta}$ at $(x,y)=(1,1)$ for $f(x,y)=x^2y$.

(a) Since $x=r\cos\theta$ and $y=r\sin\theta$, $\frac{\partial x}{\partial \theta}=-r\sin\theta$, $\frac{\partial y}{\partial \theta}=r\cos\theta$.
By the Chain Rule,

$$\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}=-r\sin\theta\frac{\partial f}{\partial x}+r\cos\theta\frac{\partial f}{\partial y}.$$

Since $x=r\cos\theta$ and $y=r\sin\theta$, we can write $\frac{\partial f}{\partial \theta}$ in terms of x and y alone: $\frac{\partial f}{\partial \theta}=-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}$.

(b) Apply the preceding equation to $f(x,y)=x^2y$:

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= -y\frac{\partial}{\partial x}(x^2y)+x\frac{\partial}{\partial y}(x^2y)=-2xy^2+x^3;\\ \frac{\partial f}{\partial \theta}\Big|_{(x,y)=(1,1)} &= -2\cdot 1\cdot 1^2+1^3=-1.\end{aligned}$$

An Abstract Example on the Chain Rule

- If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the PDE $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.
  Notice that $g(s,t) = f(x,y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$.
  Thus, by the chain rule, we get

$$\begin{aligned} \frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= 2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y}; \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= -2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}. \end{aligned}$$

  Therefore,

$$\begin{aligned} t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} &= t(2s\frac{\partial f}{\partial x} - 2s\frac{\partial f}{\partial y}) + s(-2t\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}) \\ &= 2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y} - 2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y} \\ &= 0. \end{aligned}$$

Implicit Differentiation: $y = y(x)$

- Suppose that the equation $F(x, y) = 0$ defines y implicitly as a function of x.

  By the chain rule $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$, whence

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

  Example: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

  We have $F(x, y) = x^3 + y^3 - 6xy = 0$, whence

$$\frac{\partial F}{\partial x} = 3x^2 - 6y, \quad \frac{\partial F}{\partial y} = 3y^2 - 6x.$$

  Therefore, $\displaystyle\frac{dy}{dx} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$.

Implicit Differentiation $z = z(x, y)$

- Suppose that the equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y.
  By the chain rule $\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$.
  But, we also have $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, whence $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$, giving
  $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}. \quad \text{Similarly} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$
  Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.
  We have $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$, whence
  $$\frac{\partial F}{\partial x} = 3x^2 + 6yz, \quad \frac{\partial F}{\partial y} = 3y^2 + 6xz, \quad \frac{\partial F}{\partial z} = 3z^2 + 6xy.$$
  Therefore, $\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$;
  $\frac{\partial z}{\partial y} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}$.

Subsection 7

Optimization in Several Variables

Maxima and Minima

- A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$, when (x, y) is near (a, b). The z-value $f(a, b)$ is called the **local maximum value**.
- A function of two variables has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$, when (x, y) is near (a, b). The z-value $f(a, b)$ is called the **local minimum value**.

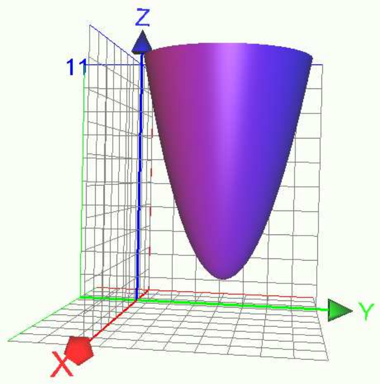
Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

- A point (a, b) is called a **critical point** of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.
- As was the case with functions of a single variable the critical points are candidates for local extrema. At a critical point the function may have a local maximum, a local minimum or neither.

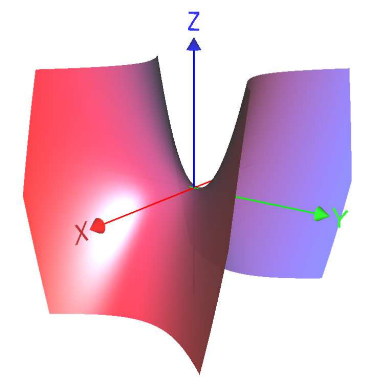
Finding Critical Points

- Suppose $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Then, we have $f_x(x,y) = 2x - 2$ and $f_y = 2y - 6$. Therefore, f has a critical point $(x,y) = (1,3)$. By rewriting $f(x,y) = 4 + (x-1)^2 + (y-3)^2$, we see that $f(x,y) \geq 4 = f(1,3)$. Therefore, f has an absolute minimum at $(1,3)$ equal to 4.



Another Example of Finding Critical Points

- Suppose $f(x,y) = y^2 - x^2$. Then, we have $f_x(x,y) = -2x$ and $f_y = 2y$. Therefore, f has a critical point $(x,y) = (0,0)$. Note, however, that for points on x-axis $f(x,0) = -x^2 \leq f(0,0)$ and for points on the y-axis $f(0,y) = y^2 \geq f(0,0)$. Thus, $f(0,0)$ can be neither a local max nor a local min.



The kind of point that occurs at $(0,0)$ is this case is called a **saddle point** because of its shape.

Second Derivative Test

- Suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ and that f has continuous second partial derivatives on a disk with center (a,b).

  Define
  $$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

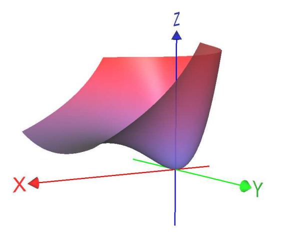
  Then, the following possibilities may occur:
  - If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum;
  - If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum;
  - If $D < 0$, then $f(a,b)$ is neither a local max nor a local min;
    In this case f has a **saddle point** at (a,b) and the graph of f crosses the tangent plane at (a,b);
  - If $D = 0$, the test is inconclusive;
    In this case, f could have a local min, a local max, a saddle point or none of the above.

Example I

- Find the local extrema and the saddle points of $f(x,y) = (x^2+y^2)e^{-x}$.

  We have $f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x} = (2x-x^2-y^2)e^{-x}$.
  Moreover, $f_{xx}(x,y) = (2-4x+x^2+y^2)e^{-x}$ and $f_{xy}(x,y) = -2ye^{-x}$.
  Also $f_y(x,y) = 2ye^{-x}$ and $f_{yy}(x,y) = 2e^{-x}$.
  We now obtain $2ye^{-x} = 0$ implies $y = 0$ and, thus, $2x - x^2 = x(2-x) = 0$. This implies $x = 0$ or $x = 2$.
  Therefore, we get critical points $(0,0),(2,0)$.
  We compute

  $$D(0,0) = 2 \cdot 2 - 0^2 = 4 > 0$$
  $$f_{xx}(0,0) = 2 > 0$$
  $$D(2,0) = \frac{-2}{e^2}\frac{2}{e^2} - 0^2 = -\frac{4}{e^4} < 0$$



Example II

- Find the local extrema and the saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

  We have $f_x(x,y) = 4x^3 - 4y = 4(x^3 - y)$. Moreover, $f_{xx}(x,y) = 12x^2$ and $f_{xy}(x,y) = -4$.

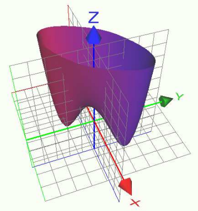
  Also $f_y(x,y) = 4y^3 - 4x = 4(y^3 - x)$. Also, $f_{yy}(x,y) = 12y^2$.

  The system $\left\{ \begin{array}{rcl} x^3 - y & = & 0 \\ y^3 - x & = & 0 \end{array} \right\}$ gives $x^9 - x = 0$, and, thus, $x(x^8 - 1) = 0$. This implies $x = 0$ or $x^8 = 1$, whence $x = 0, x = \pm 1$.

  Therefore, we get critical points $(0,0), (-1,-1)$ and $(1,1)$.

  We compute

  $$D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0$$
  $$D(-1,-1) = 12 \cdot 12 - (-4)^2 = 128 > 0$$
  $$f_{xx}(-1,-1) = 12 > 0$$
  $$D(1,1) = 12 \cdot 12 - (-4)^2 = 128 > 0$$
  $$f_{xx}(1,1) = 12 > 0$$



Example III

- Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$.
  The distance of $(1, 0, -2)$ from a point (x, y, z) is given by
  $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$.
  If the point (x, y, z) is on the plane $x + 2y + z = 4$, then
  $z = 4 - x - 2y$, whence the distance formula becomes a function of two variables only

  $$d(x, y) = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}.$$

  We want to minimize this function. We look instead at minimizing the square function
  $f(x, y) = d^2(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$. We compute partial derivatives and set them equal to zero to find critical points:

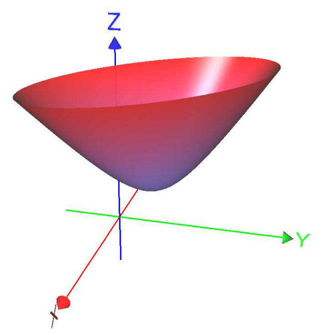
  $$f_x(x, y) = 2(x-1) - 2(6 - x - 2y) = 2(2x + 2y - 7) = 0$$
  $$f_y(x, y) = 2y - 4(6 - x - 2y) = 2(2x + 5y - 12) = 0;$$

Example III (Cont'd)

- We have

$$\left\{\begin{array}{lcl} 2x+2y & = & 7 \\ 2x+5y & = & 12 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y = \frac{5}{3} \\ x = -\frac{5}{3}+\frac{7}{2}=\frac{11}{6}. \end{array}\right.$$

We can verify using the second derivative test that at $(\frac{11}{6}, \frac{5}{3})$ we have a minimum, but this is clear from the interpretation of $d(x, y)$.



Moreover, we can compute

$$z = 4 - x - 2y = 4 - \frac{11}{6} - \frac{10}{3} = -\frac{7}{6}.$$

Thus the point is $(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6})$.

Example IV

- What is the max possible volume of a rectangular box without a lid that can be made of 12 square meters of cardboard?

  The volume equation is $V = \ell wh$ and the equation for the amount of cardboard gives $\ell w + 2\ell h + 2wh = 12$.

  The latter equation solved for h gives $h = \frac{12-\ell w}{2(\ell+w)}$.

  Therefore, the equation for the volume becomes $V = \frac{12\ell w-\ell^2 w^2}{2(\ell+w)}$.

  We compute V_ℓ using the quotient rule:

$$\begin{aligned} V_\ell &= \frac{(12w-2\ell w^2)2(\ell+w)-2(12\ell w-\ell^2 w^2)}{4(\ell+w)^2} \\ &= \frac{(12w-2\ell w^2)(\ell+w)-(12\ell w-\ell^2 w^2)}{2(\ell+w)^2} \\ &= \frac{12w\ell+12w^2-2\ell^2 w^2-2\ell w^3-12\ell w+\ell^2 w^2}{2(\ell+w)^2} \\ &= \frac{12w^2-\ell^2 w^2-2\ell w^3}{2(\ell+w)^2} = \frac{w^2(12-\ell^2-2\ell w)}{2(\ell+w)^2}. \end{aligned}$$

Example IV (Cont'd)

- By symmetry, we get

$$V_\ell = \frac{w^2(12 - \ell^2 - 2\ell w)}{2(\ell + w)^2}, \quad V_w = \frac{\ell^2(12 - w^2 - 2\ell w)}{2(\ell + w)^2}.$$

The system $\left\{ \begin{array}{rcl} 12 - 2\ell w - \ell^2 & = & 0 \\ 12 - 2\ell w - w^2 & = & 0 \end{array} \right\}$ gives $\ell^2 - w^2 = 0$ or $(\ell + w)(\ell - w) = 0$, yielding (since $\ell, w > 0$) $\ell = w$.
So $12 - 3\ell^2 = 0 \Rightarrow \ell^2 = 4 \Rightarrow \ell = 2$. Thus, since $h = \frac{12 - \ell w}{2(\ell + w)}$, we obtain that

$$\ell = 2, \quad w = 2 \quad \text{and} \quad h = 1.$$

The maximum volume is, therefore, 4 cubic meters.

Extreme Value Theorem

Extreme Value Theorem: Functions of Two Variables

If f is continuous on a closed and bounded set $\mathcal{D}$ in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in $\mathcal{D}$.

- To find those absolute extrema in a closed and bounded set $\mathcal{D}$, we use

The Closed and Bounded Region Method

1. Find the values of f at the critical points of f in $\mathcal{D}$;
2. Find the extreme values of f on the boundary of $\mathcal{D}$;
3. The largest of the values from the previous steps is the absolute maximum value and the smallest of these values is the absolute minimum value.

Finding Absolute Extrema in Closed Bounded Set

- Find the absolute extrema of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $\mathcal{D} = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$.

  Compute the partial derivatives: $f_x(x,y) = 2x - 2y$, $f_y(x,y) = -2x + 2$.

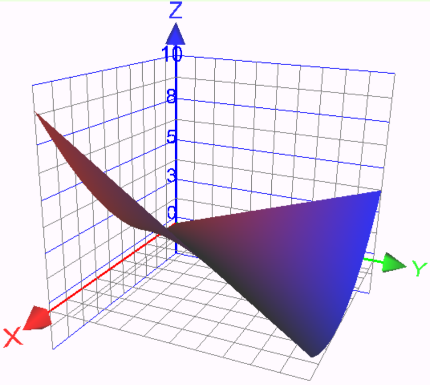
  Therefore, the only critical point is $(1,1)$ and $f(1,1) = 1$.

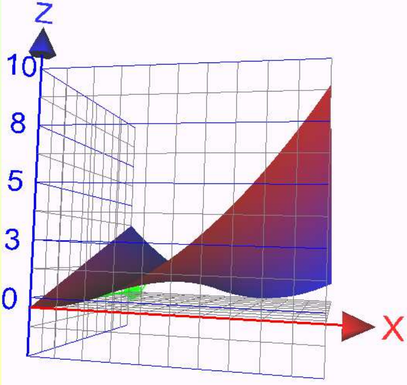
  On the boundary, we have
  - If $0 \le x \le 3, y = 0$, then $f(x,0) = x^2$ has min $f(0,0) = 0$ and max $f(3,0) = 9$.
  - If $x = 3, 0 \le y \le 2$, then $f(3,y) = 9 - 4y$ has min $f(3,2) = 1$ and max $f(3,0) = 9$.
  - If $0 \le x \le 3, y = 2$, then $f(x,2) = (x-2)^2$ has min $f(2,2) = 0$ and max $f(0,2) = 4$.
  - If $x = 0, 0 \le y \le 2$, then $f(0,y) = 2y$ has min $f(0,0) = 0$ and max $f(0,2) = 4$.

Illustration of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $\mathcal{D}$

- Thus, on the boundary, the min value is $f(0,0) = f(2,2) = 0$ and the max value is $f(3,0) = 9$.

  Since $f(1,1) = 1$ these are also the absolute extrema on $\mathcal{D}$.



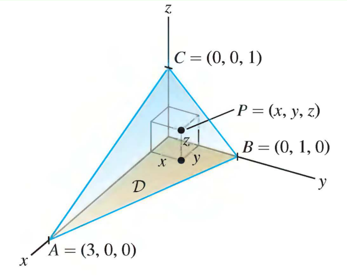


Application

- What is the max possible volume of a box inscribed in the tetrahedron bounded by the coordinate planes and the plane $\frac{1}{3}x + y + z = 1$?

The volume equation is $V = xyz$. Since the (x, y, z) is a point on $\frac{1}{3}x + y + z = 1$, we must have $z = 1 - \frac{1}{3}x - y$. Therefore, $V = xy(1 - \frac{1}{3}x - y) = xy - \frac{1}{3}x^2y - xy^2$. We get:

$\frac{\partial V}{\partial x} = y - \frac{2}{3}xy - y^2 = y(1 - \frac{2}{3}x - y)$,
$\frac{\partial V}{\partial y} = x - \frac{1}{3}x^2 - 2xy = x(1 - \frac{1}{3}x - 2y)$.



Therefore,

$$\left\{ \begin{array}{rcl} \frac{2}{3}x + y & = & 1 \\ \frac{1}{3}x + 2y & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} \frac{4}{3}x + 2y & = & 2 \\ \frac{1}{3}x + 2y & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} x & = & 1 \\ y & = & \frac{1}{3} \end{array} \right\}.$$

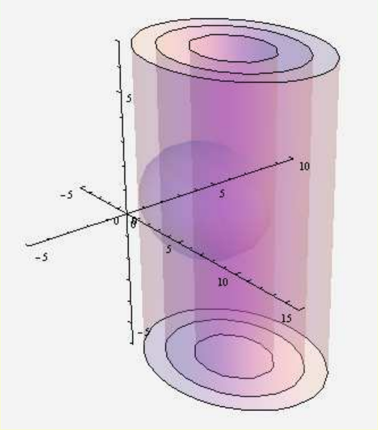
Since the maximum cannot occur on the boundary, we get that the maximum volume is $1 \cdot \frac{1}{3} - \frac{1}{3} \cdot 1^2 \cdot \frac{1}{3} - 1 \cdot (\frac{1}{3})^2 = \frac{1}{9}$ cubic meters.

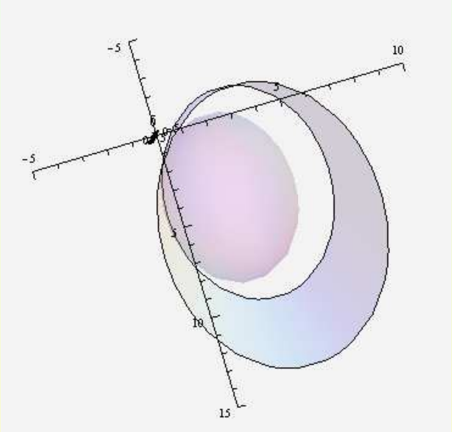
Subsection 8

Lagrange Multipliers

Illustration of General Idea of Lagrange Multipliers

Problem: Maximize or minimize an objective function $f(x,y,z) = (x-5)^2 + 3(y-3)^2$ subject to a constraint $g(x,y,z) = (x-4)^2 + 3(y-2)^2 + 4(z-1)^2 = 20 = k$.





Lagrange Multipliers

- **Problem:** Maximize or minimize an objective function $f(x,y,z)$ subject to a constraint $g(x,y,z)=k$.
  Example: Maximize the volume $V(\ell,w,h)=\ell wh$ subject to $S(\ell,w,h)=\ell w+2\ell h+2wh=12$.

The Method of Lagrange Multipliers

(a) Find all values of (x,y,z) and λ (a parameter called a **Lagrange multiplier**), such that

$$\left\{\begin{array}{rcl}\nabla f(x,y,z) & = & \lambda\nabla g(x,y,z)\\ g(x,y,z) & = & k\end{array}\right\} \qquad (1)$$

(b) Evaluate f at all (x,y,z) found in (a): The largest value is the max of f and the smallest value is the min of f.

- Recall that $\nabla f=\langle f_x,f_y,f_z\rangle$ and $\nabla g=\langle g_x,g_y,g_z\rangle$.
  So, the System (1) may be rewritten in the form:

$$f_x=\lambda g_x,\quad f_y=\lambda g_y,\quad f_z=\lambda g_z,\quad g=k.$$

Example I: Lagrange Multiplier Method

- Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

  Set $g(x,y) = x^2 + y^2$ and we want $g(x,y) = 1$.

  We get the system

$$\left\{ \begin{array}{rcl} f_x(x,y) & = & \lambda g_x(x,y) \\ f_y(x,y) & = & \lambda g_y(x,y) \\ g(x,y) & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 2x & = & \lambda 2x \\ 4y & = & \lambda 2y \\ x^2 + y^2 & = & 1 \end{array} \right\} \Rightarrow$$

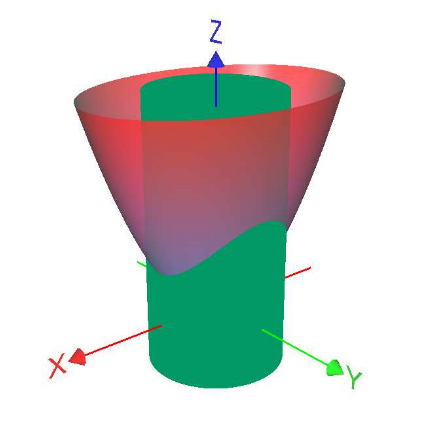
$$\left\{ \begin{array}{lcl} x = 0 & \text{or} & \lambda = 1 \\ y = 0 & \text{or} & \lambda = 2 \end{array} \right.$$

  Therefore, we get for (x,y) the values $(0,\pm 1)$ and $(\pm 1, 0)$.

  Since $f(0,\pm 1) = 2$ and $f(\pm 1, 0) = 1$, f has max 2 and min 1, subject to $x^2 + y^2 = 1$.

Example I Illustrated

- The extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.
  Max: $f(0,\pm 1) = 2$ and Min: $f(\pm 1, 0) = 1$.



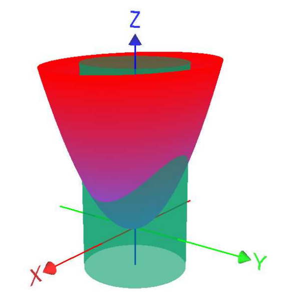
Example I Modified

- Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

  Recall the method for finding extreme values on a closed and bounded region!

  First, we find critical points of f: We have $f_x = 2x$ and $f_y = 4y$; Thus, the only critical point is $(x,y) = (0,0)$ and $f(0,0) = 0$.

  Then we compute min and max on the boundary: We did this using Lagrange multipliers and found min $f(\pm 1, 0) = 1$ and max $f(0, \pm 1) = 2$.
  Therefore, on the disk $x^2 + y^2 \leq 1$, f has absolute min $f(0,0) = 0$ and absolute max $f(0, \pm 1) = 2$.



Example II: Lagrange Multiplier Method

- Find the extreme values of $f(x,y) = 2x + 5y$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

  Set $g(x,y) = \frac{x^2}{16} + \frac{y^2}{9}$ and we want $g(x,y) = 1$.

  We get the system

$$\left\{ \begin{array}{rcl} f_x(x,y) & = & \lambda g_x(x,y) \\ f_y(x,y) & = & \lambda g_y(x,y) \\ g(x,y) & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} 2 & = & \lambda \frac{x}{8} \\ 5 & = & \lambda \frac{2y}{9} \\ \frac{x^2}{16} + \frac{y^2}{9} & = & 1 \end{array} \right\} \Rightarrow$$

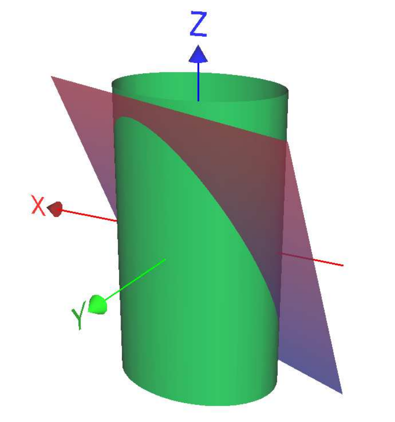
$$\left\{ \begin{array}{l} x = \frac{16}{\lambda} \\ y = \frac{45}{2\lambda} \\ \frac{16^2}{16\lambda^2} + \frac{45^2}{36\lambda^2} = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \frac{16}{\lambda} \\ y = \frac{45}{2\lambda} \\ \frac{64}{4\lambda^2} + \frac{225}{4\lambda^2} = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \pm\frac{32}{17} \\ y = \pm\frac{45}{17} \\ \lambda = \pm\frac{17}{2} \end{array} \right.$$

  Therefore, we get for (x,y) the values $(\frac{32}{17}, \frac{45}{17})$ and $(-\frac{32}{17}, -\frac{45}{17})$.

  We compute $f(\frac{32}{17}, \frac{45}{17}) = 17$ and $f(-\frac{32}{17}, -\frac{45}{17}) = -17$.

Example II Illustrated

- The extreme values of $f(x, y) = 2x + 5y$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Max: $f(\frac{32}{17}, \frac{45}{17}) = 17$ and Min: $f(-\frac{32}{17}, -\frac{45}{17}) = -17$.



Example III: Lagrange Multiplier Method

- Find the points on the sphere $x^2+y^2+z^2=4$ with smallest and largest square distance from the point $(3,1,-1)$.
  Set $f(x,y,z)=(x-3)^2+(y-1)^2+(z+1)^2$ be the square distance from (x,y,z) to $(3,1,-1)$ and $g(x,y,z)=x^2+y^2+z^2$ so that $g(x,y,z)=4$.
  We get the system

$$\left\{\begin{array}{rcl} f_x(x,y,z) & = & \lambda g_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z) \\ g(x,y,z) & = & 4 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 2(x-3) & = & \lambda 2x \\ 2(y-1) & = & \lambda 2y \\ 2(z+1) & = & \lambda 2z \\ x^2+y^2+z^2 & = & 4 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{rcl} \frac{1}{\lambda-1} & = & -\frac{1}{3}x \\ \frac{1}{\lambda-1} & = & -y \\ \frac{1}{\lambda-1} & = & z \\ x^2+y^2+z^2 & = & 4 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & -3z \\ y & = & -z \\ x^2+y^2+z^2 & = & 4 \end{array}\right\}$$

Example III: Lagrange Multiplier Method (Cont'd)

- The system gives

$$\left\{\begin{array}{rcl} x & = & -3z \\ y & = & -z \\ 9z^2+z^2+z^2 & = & 4 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = \mp\frac{6}{\sqrt{11}} \\ y = \mp\frac{2}{\sqrt{11}} \\ z = \pm\frac{2}{\sqrt{11}} \end{array}\right\}$$

Therefore, we get

$$\begin{array}{l} (x,y,z) = (\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}) \text{ or} \\ (x,y,z) = (-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}) \end{array}.$$

f has smallest value at one of those points and the largest at the other.

$f(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}) = \frac{165-44\sqrt{11}}{11} = 15 - 11\sqrt{11}$,

$f(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}) = \frac{165+44\sqrt{11}}{11} = 15 + 11\sqrt{11}$.

Lagrange Multipliers with Two Constraints

- **Problem:** Maximize or minimize an objective function $f(x,y,z)$ subject to the constraints $g(x,y,z)=k$ and $h(x,y,z)=c$.

The Method of Lagrange Multipliers Revisited

(a) Find all values of (x,y,z) and λ, μ (two parameters called **Lagrange multipliers**), such that

$$\left\{\begin{array}{rcl} \nabla f(x,y,z) & = & \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) & = & k \\ h(x,y,z) & = & c \end{array}\right\} \qquad (2)$$

(b) Evaluate f at all (x,y,z) resulting from (a): The largest value is the max of f and the smallest value is the min of f.

- Since $\nabla f = \langle f_x, f_y, f_z\rangle$, $\nabla g = \langle g_x, g_y, g_z\rangle$ and $\nabla h = \langle h_x, h_y, h_z\rangle$ the System (2) may be rewritten in the form:

$$f_x = \lambda g_x + \mu h_x, \quad f_y = \lambda g_y + \mu h_y, \quad f_z = \lambda g_z + \mu h_z, \quad g = k, \quad h = c.$$

Example IV: Lagrange Multiplier Method

- Find the extreme values of $f(x,y,z) = x + 2y + 3z$ on the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.
  Set $g(x,y,z) = x - y + z$ and $h(x,y,z) = x^2 + y^2$ so that $g(x,y,z) = 1$ and $h(x,y,z) = 1$.
  We get the system

$$\left\{\begin{array}{rcl} f_x(x,y,z) & = & \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) & = & 1 \\ h(x,y,z) & = & 1 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} 1 & = & \lambda + \mu 2x \\ 2 & = & -\lambda + \mu 2y \\ 3 & = & \lambda \\ x - y + z & = & 1 \\ x^2 + y^2 & = & 1 \end{array}\right\} \Rightarrow$$

Example IV: Lagrange Multiplier Method (Cont'd)

$$\left\{\begin{array}{l} 1=\lambda+\mu 2x \\ 2=-\lambda+\mu 2y \\ 3=\lambda \\ x-y+z=1 \\ x^2+y^2=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \lambda=3 \\ x=-\frac{1}{\mu} \\ y=\frac{5}{2\mu} \\ x-y+z=1 \\ \frac{1}{\mu^2}+\frac{25}{4\mu^2}=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \lambda=3 \\ \mu=\pm\frac{\sqrt{29}}{2} \\ x=\mp\frac{2}{\sqrt{29}} \\ y=\pm\frac{5}{\sqrt{29}} \\ z=1\pm\frac{7}{\sqrt{29}} \end{array}\right.$$

Therefore, we get for (x, y, z) the values $(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}})$ and $(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}})$.

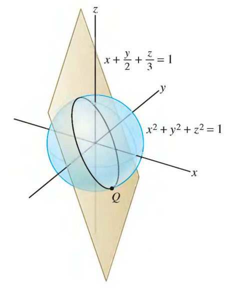
The max of f occurs at the first point and is $3+\sqrt{29}$.

Example V: Lagrange Multiplier Method

- The intersection of the plane $x + \frac{1}{2}y + \frac{1}{3}z = 0$ with the unit sphere $x^2 + y^2 + z^2 = 1$ is a great circle. Find the point on this great circle with the largest x coordinate.

  Set $f(x,y,z) = x$, $g(x,y,z) = x + \frac{1}{2}y + \frac{1}{3}z$ and $h(x,y,z) = x^2 + y^2 + z^2$ so that $g(x,y,z) = 0$ and $h(x,y,z) = 1$. We get the system

$$\left\{ \begin{array}{lcl} f_x(x,y,z) & = & \lambda g_x(x,y,z) + \mu h_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z) + \mu h_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z) + \mu h_z(x,y,z) \\ g(x,y,z) & = & 0 \\ h(x,y,z) & = & 1 \end{array} \right\}.$$



Example V: Lagrange Multiplier Method (Cont'd)

- Since $f(x,y,z)=x$, $g(x,y,z)=x+\frac{1}{2}y+\frac{1}{3}z$ and $h(x,y,z)=x^2+y^2+z^2$, we get
$$\left\{\begin{array}{lcl} f_x(x,y,z) & = & \lambda g_x(x,y,z)+\mu h_x(x,y,z) \\ f_y(x,y,z) & = & \lambda g_y(x,y,z)+\mu h_y(x,y,z) \\ f_z(x,y,z) & = & \lambda g_z(x,y,z)+\mu h_z(x,y,z) \\ g(x,y,z) & = & 0 \\ h(x,y,z) & = & 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 1=\lambda+2\mu x \\ 0=\frac{1}{2}\lambda+2\mu y \\ 0=\frac{1}{3}\lambda+2\mu z \\ x+\frac{1}{2}y+\frac{1}{3}z=0 \\ x^2+y^2+z^2=1 \end{array}\right\}.$$
Note that μ cannot be zero. The second and third equations yield $\lambda=-4\mu y$ and $\lambda=-6\mu z$. Thus, $-4\mu y=-6\mu z$, i.e., since $\mu\neq 0$, $y=\frac{3}{2}z$. Applying $x+\frac{1}{2}y+\frac{1}{3}z=0$, we get $x=-\frac{13}{12}z$. Finally, we substitute into $x^2+y^2+z^2=1$ to get $(-\frac{13}{12}z)^2+(\frac{3}{2}z)^2+z^2=1$, whence $\frac{637}{144}z^2=1$, yielding $z=\pm\frac{12}{7\sqrt{13}}$.
Therefore, we obtain the critical points $(-\frac{\sqrt{13}}{7},\frac{18}{7\sqrt{13}},\frac{12}{7\sqrt{13}})$ $(\frac{\sqrt{13}}{7},-\frac{18}{7\sqrt{13}},-\frac{12}{7\sqrt{13}})$. We conclude that the max x occurs at the second point and is equal to $\frac{\sqrt{13}}{7}$.