Calculus III

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LSSU Math 251

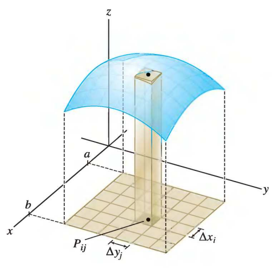
1 Functions
- Integration in Two Variables
- Double Integrals Over More General Regions
- Double Integrals in Polar Coordinates
- Triple Integrals
- Triple Integrals in Cylindric Coordinates
- Triple Integrals in Spherical Coordinates

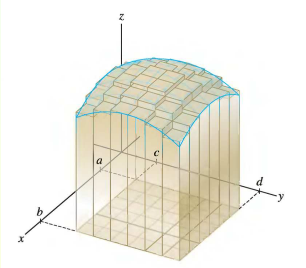
Subsection 1

Integration in Two Variables

Approximating Volumes by Sums of Volumes of Boxes

- Consider the function of two variables $f(x, y)$.
- The elementary volume under the graph of $z = f(x, y)$ over an elementary area ΔA_{ij} that contains the point $P_{ij} = (x_{ij}^*, y_{ij}^*)$ is approximated by the volume ΔV_{ij} of a box $\Delta V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$.





- To obtain an approximation of the entire volume we sum:

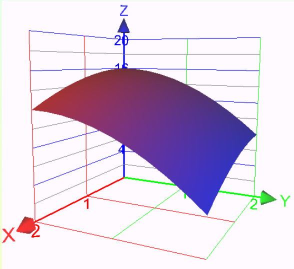
$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}.$$

Double Integral

- The limit of the sum as the numbers of x- and y-subintervals become infinite, or, equivalently, as the lengths Δx_i of each x- and Δy_j of each y-subinterval approach 0 is the actual volume under the curve
$$V = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}.$$
- The **double integral** of f over a rectangle R is
$$\iint\limits_R f(x,y)dA = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij},$$
if the limit exists. If it does exist, we call f **integrable**.
- Thus, we have $V = \iint\limits_R f(x,y)dA$, where V is the volume under f over the rectangle R.

Approximating a Volume via Rectangles

- Approximate roughly the volume of the solid lying above $R = [0,2] \times [0,2]$ and below $f(x,y) = 16 - x^2 - 2y^2$ using two subintervals and right endpoints.



Each subinterval has length 1, so each of the four rectangles formed has area $\Delta A = 1 \cdot 1 = 1$.

Thus, we get

$$\begin{array}{rcl} V & \approx & f(1,1) \cdot 1 + f(1,2) \cdot 1 + f(2,1) \cdot 1 + f(2,2) \cdot 1 \\ & = & 13 + 7 + 10 + 4 = 34. \end{array}$$

A Double Integral via a Volume

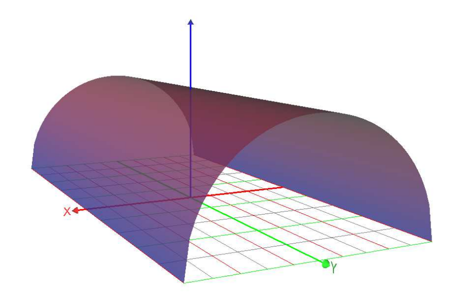
- Suppose $R = [-1,1] \times [-2,2]$. Evaluate $\iint\limits_R \sqrt{1-x^2}dA$.

  The face is a semi-disk with radius 1, so it has area

$$A = \frac{1}{2}\pi \cdot 1^2 = \frac{\pi}{2}.$$

  The length is equal to 4. Thus, the volume is

$$\begin{aligned} V &= \iint\limits_R \sqrt{1-x^2}dA \\ &= \tfrac{\pi}{2} \cdot 4 = 2\pi \text{ units}^3. \end{aligned}$$



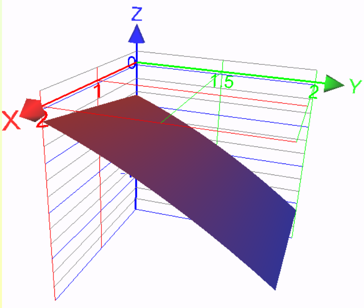
The Midpoint Rule

Midpoint Rule for Double Integrals

$$\iint_R f(x,y)dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta A$$ where $\bar{x}_i$ is the midpoint of $[x_{i-1},x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1},y_j]$.

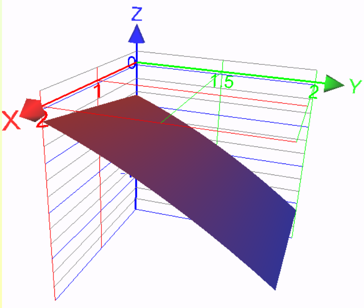
Example: Use the Midpoint Rule with $m=n=2$ to estimate the value of the integral $\iint_R (x-3y^2)dA$, where

$$R=\{(x,y): 0\le x\le 2, 1\le y\le 2\}.$$



Approximating $\iint_R (x - 3y^2)dA$

$$\begin{aligned}\iint\limits_R (x-3y^2)dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\bar{x}_i,\bar{y}_j)\Delta A \\ &= f(\tfrac{1}{2},\tfrac{5}{4})\tfrac{1}{2} + f(\tfrac{1}{2},\tfrac{7}{4})\tfrac{1}{2} + f(\tfrac{3}{2},\tfrac{5}{4})\tfrac{1}{2} + f(\tfrac{3}{2},\tfrac{7}{4})\tfrac{1}{2} \\ &= (-\tfrac{67}{16})\tfrac{1}{2} + (-\tfrac{139}{16})\tfrac{1}{2} + (-\tfrac{51}{16})\tfrac{1}{2} + (-\tfrac{123}{16})\tfrac{1}{2} \\ &= -\tfrac{95}{8}.\end{aligned}$$



Iterated Integrals

- Let f be a function of two variables x, y that is continuous on a rectangle $R = [a, b] \times [c, d]$.
- The notation $\int_c^d f(x, y) dy$ means that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This process is called **partial integration with respect to** y.
- The partial integral $\int_c^d f(x, y) dy$ depends on the value of x, i.e., it is a function of x: $A(x) = \int_c^d f(x, y) dy$.
- If we integrate $A(x)$ with respect to x from $x = a$ to $x = b$, we get the **iterated integral**
$$\int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$
- In the notation, the brackets are omitted and we write
$$\int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

Example of Iterated Integral

- Compute $\int_2^4 \int_1^9 ye^x dy\,dx$.

$$\begin{aligned} \int_2^4 \int_1^9 ye^x dy\,dx &= \int_2^4 e^x \int_1^9 ydy\,dx \\ &= \int_2^4 e^x \left(\left.\frac{y^2}{2}\right|_1^9 \right) dx \\ &= \int_2^4 40e^x dx \\ &= 40 \left. e^x \right|_2^4 \\ &= 40(e^4 - e^2). \end{aligned}$$

Changing the Order of Iterated Integration

- Compute $\int_0^3 \int_1^2 x^2 y dy dx$.

$$\begin{array}{rcl} \int_0^3 \int_1^2 x^2 y dy dx & = & \int_0^3 x^2 \int_1^2 y dy dx \\ & = & \int_0^3 x^2 (\frac{1}{2} y^2 \mid_1^2) dx \\ & = & \int_0^3 \frac{3}{2} x^2 dx \\ & = & \frac{1}{2} x^3 \mid_0^3 = \frac{27}{2}. \end{array}$$

- Compute $\int_1^2 \int_0^3 x^2 y dx dy$.

$$\begin{array}{rcl} \int_1^2 \int_0^3 x^2 y dx dy & = & \int_1^2 y \int_0^3 x^2 dx dy \\ & = & \int_1^2 y (\frac{1}{3} x^3 \mid_0^3) dy \\ & = & \int_1^2 9 y dy \\ & = & \frac{9}{2} y^2 \mid_1^2 \\ & = & 18 - \frac{9}{2} = \frac{27}{2}. \end{array}$$

Fubini's Theorem

Fubini's Theorem

If f is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

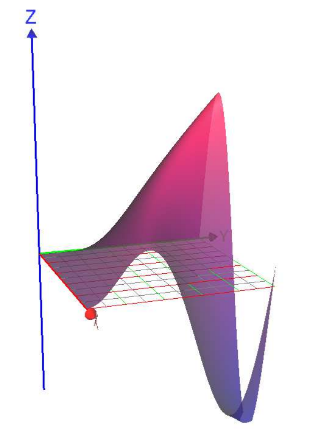
This is also true under the weaker conditions that f is bounded on R, discontinuous only on a finite number of smooth curves and the iterated integrals exist.

Example: Evaluate $\iint_R (x - 3y^2)dA$, where $R = [0,2] \times [1,2]$.

$$\begin{aligned}
\iint_R (x - 3y^2)dA &= \int_0^2 \int_1^2 (x - 3y^2)dydx \\
&= \int_0^2 (xy - y^3) \big|_1^2 \, dx \\
&= \int_0^2 (2x - 8 - (x - 1))dx \\
&= \int_0^2 (x - 7)dx \\
&= (\tfrac{1}{2}x^2 - 7x) \big|_0^2 \; = \; -12.
\end{aligned}$$

Double Integration through Iterated Integrals I

- Compute $\iint_R y\sin(xy)dA$, where $R = [1,2] \times [0,\pi]$.

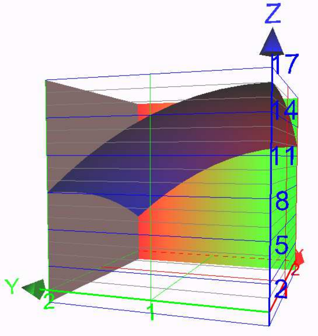


$$\begin{aligned}
&\iint_R y\sin(xy)dA \\
&= \int_0^\pi \int_1^2 y\sin(xy)dxdy \\
&= \int_0^\pi -\cos(xy)\ |_1^2\ dy \\
&= \int_0^\pi (-\cos 2y + \cos y)dy \\
&= (-\tfrac{1}{2}\sin 2y + \sin y)\ |_0^\pi \\
&= 0.
\end{aligned}$$

Double Integration through Iterated Integrals II

- Compute the volume of the solid S bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$ and the three coordinate planes.

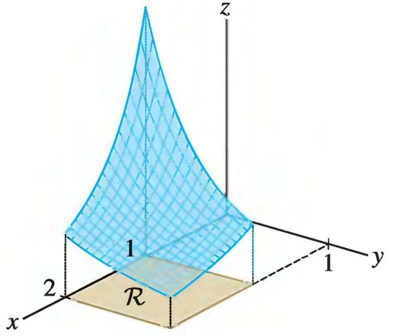
$$
\begin{aligned}
&\iint_R (16 - x^2 - 2y^2)\,dA \\
&= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\,dx\,dy \\
&= \int_0^2 (16x - \tfrac{1}{3}x^3 - 2y^2x) \Big|_0^2 \,dy \\
&= \int_0^2 (\tfrac{88}{3} - 4y^2)\,dy \\
&= (\tfrac{88}{3}y - \tfrac{4}{3}y^3) \Big|_0^2 \\
&= 48 \text{ units}^3.
\end{aligned}
$$



Double Integration through Iterated Integrals III

- Calculate $\iint_R \frac{dA}{(x+y)^2}$, where $R = [1,2] \times [0,1]$.

$$\begin{aligned}
&\iint_R \frac{dA}{(x+y)^2} \\
&= \int_1^2 \int_0^1 \frac{dy}{(x+y)^2} dx \\
&= \int_1^2 (-\frac{1}{x+y} \mid_0^1) dx \\
&= \int_1^2 (-\frac{1}{x+1} + \frac{1}{x}) dx \\
&= (\ln x - \ln(x+1)) \mid_1^2 \\
&= (\ln 2 - \ln 3) - (\ln 1 - \ln 2) \\
&= 2\ln 2 - \ln 3 = \ln \frac{4}{3}.
\end{aligned}$$



Properties of Double Integrals

- **Sum Rule:**
$$\iint\limits_R [f(x,y)+g(x,y)]dA = \iint\limits_R f(x,y)dA + \iint\limits_R g(x,y)dA;$$
- **Constant Factor Rule:**
$$\iint\limits_R cf(x,y)dA = c\iint\limits_R f(x,y)dA;$$
- **Comparison Property:** If $f(x,y) \geq g(x,y)$, for all (x,y) in R, then
$$\iint\limits_R f(x,y)dA \geq \iint\limits_R g(x,y)dA.$$

Subsection 2

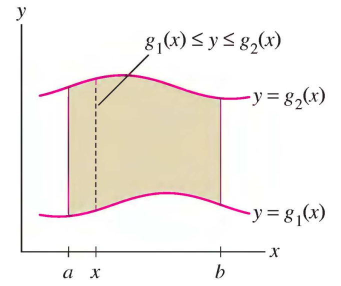
Double Integrals Over More General Regions

Double Integrals Over Type I Regions

- A plane region $\mathcal{D}$ is of **type I** or **vertically simple** if it lies between the graphs of two continuous functions of x, that is

$$\mathcal{D} = \{(x,y) : a \leq x \leq b, \\ g_1(x) \leq y \leq g_2(x)\},$$

where g_1, g_2 are continuous on $[a, b]$.



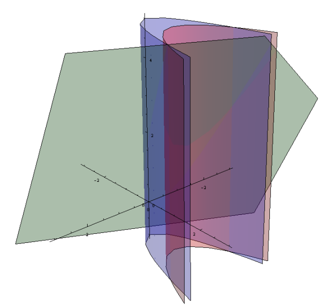
- If $f(x,y)$ is continuous on a type I region $\mathcal{D}$, as above, then

$$\iint_{\mathcal{D}} f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

Example of Double Integral Over a Type I Region

- Evaluate $\iint\limits_{\mathcal{D}}(x+2y)dA$, where $\mathcal{D}$ is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$.
  Note that $\mathcal{D}$ is of type I:

$$\mathcal{D}=\{(x,y): -1\leq x\leq 1, \\ 2x^2\leq y\leq 1+x^2\}.$$

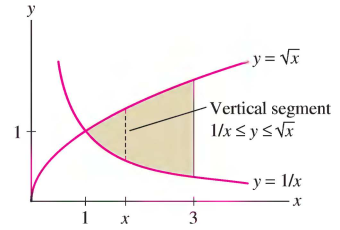


$$\begin{aligned}\iint\limits_{\mathcal{D}}(x+2y)dA &= \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx \\ &= \int_{-1}^{1}(xy+y^2)\mid_{2x^2}^{1+x^2} dx \\ &= \int_{-1}^{1}(x(1+x^2)+(1+x^2)^2-x(2x^2)-(2x^2)^2)dx \\ &= \int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)dx \\ &= (-\tfrac{3}{5}x^5-\tfrac{1}{4}x^4+\tfrac{2}{3}x^3+\tfrac{1}{2}x^2+x)\mid_{-1}^{1} = \tfrac{32}{15}.\end{aligned}$$

Example II of Double Integral Over a Type I Region

- Evaluate $\iint\limits_{\mathcal{D}} x^2 y dA$, where $\mathcal{D}$ is the region shown in the figure.
  Note that $\mathcal{D}$ is of type I:

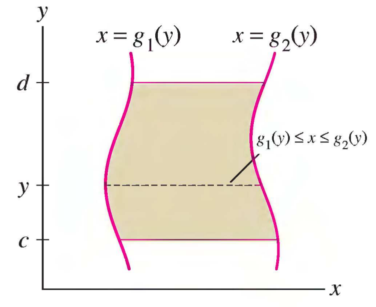
$$\mathcal{D} = \{(x,y) : 1 \leq x \leq 3, \\ \frac{1}{x} \leq y \leq \sqrt{x}\}.$$



$$\begin{aligned}
\iint\limits_{\mathcal{D}} x^2 y dA &= \int_1^3 \int_{1/x}^{\sqrt{x}} x^2 y dy dx = \int_1^3 \frac{1}{2} x^2 (y^2) \Big|_{1/x}^{\sqrt{x}} dx \\
&= \int_1^3 \frac{1}{2} x^2 (x - \frac{1}{x^2}) dx = \int_1^3 (\frac{1}{2} x^3 - \frac{1}{2}) dx \\
&= (\frac{1}{8} x^4 - \frac{1}{2} x) \Big|_1^3 = (\frac{81}{8} - \frac{3}{2}) - (\frac{1}{8} - \frac{1}{2}) = \frac{72}{8} = 9.
\end{aligned}$$

Double Integrals Over Type II Regions

- A plane region $\mathcal{D}$ is of **type II** or **horizontally simple** if it lies between the graphs of two continuous functions of y, that is
$$\mathcal{D} = \{(x,y) : c \le y \le d,\\ g_1(y) \le x \le g_2(y)\},$$
with g_1, g_2 are continuous on $[c,d]$.



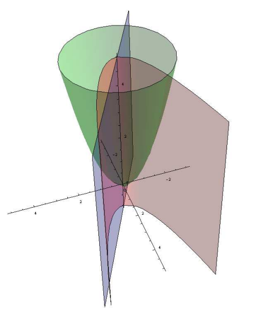
- If $f(x,y)$ is continuous on a type II region $\mathcal{D}$, as above, then
$$\iint\limits_{\mathcal{D}} f(x,y)dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy.$$

Example of a Double Integral Over a Type II Region

- Evaluate $\iint\limits_{\mathcal{D}}(x^2+y^2)dA$, where $\mathcal{D}$ is the region bounded by the line $y=2x$ and the parabola $y=x^2$.

  $\mathcal{D}$ is both of type I and of type II:

$$\begin{array}{rcl}\mathcal{D} & = & \{(x,y): 0 \le x \le 2, x^2 \le y \le 2x\} \\ & = & \{(x,y): 0 \le y \le 4, \frac{1}{2}y \le x \le \sqrt{y}\}.\end{array}$$



  We evaluate the integral using the type II expression:

$$\begin{array}{rcl}\iint\limits_{\mathcal{D}}(x^2+y^2)dA & = & \int_0^4\int_{y/2}^{\sqrt{y}}(x^2+y^2)dxdy = \int_0^4(\frac{1}{3}x^3+y^2x)\mid_{y/2}^{\sqrt{y}} dy \\ & = & \int_0^4(\frac{1}{3}y^{3/2}+y^{5/2}-\frac{1}{24}y^3-\frac{1}{2}y^3)dy \\ & = & (\frac{2}{15}y^{5/2}+\frac{2}{7}y^{7/2}-\frac{13}{96}y^4)\mid_0^4 = \frac{216}{35}.\end{array}$$

Example II of a Double Integral Over a Type II Region

- Evaluate $\iint\limits_{\mathcal{D}} xydA$, where $\mathcal{D}$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

  $\mathcal{D}$ can be written as type II:

  $$\mathcal{D} = \{(x, y) : -2 \leq y \leq 4, \frac{1}{2}y^2 - 3 \leq x \leq y + 1\}.$$

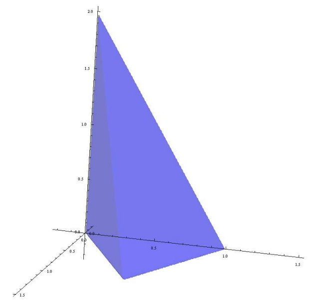
  We evaluate the integral using type II integration:

  $$\begin{aligned} \iint\limits_{\mathcal{D}} xydA &= \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xydxdy \\ &= \int_{-2}^{4} (\tfrac{1}{2}x^2y) \Big|_{\frac{1}{2}y^2-3}^{y+1} dy \\ &= \tfrac{1}{2} \int_{-2}^{4} y((y+1)^2 - (\tfrac{1}{2}y^2 - 3)^2)dy \\ &= \tfrac{1}{2} \int_{-2}^{4} (-\tfrac{1}{4}y^5 + 4y^3 + 2y^2 - 8y)dy \\ &= \tfrac{1}{2}(-\tfrac{1}{24}y^6 + y^4 + \tfrac{2}{3}y^3 - 4y^2) \Big|_{-2}^{4} = 36. \end{aligned}$$

A Double Integral Over a Type I Region

- Evaluate the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ and $z = 0$.
  This can be expressed as the volume under $z = 2 - x - 2y$ and above the type I region

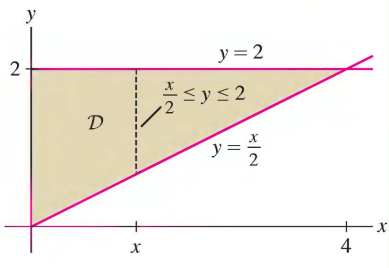
$$\mathcal{D} = \{(x, y) : 0 \leq x \leq 1, \\ \tfrac{1}{2}x \leq y \leq 1 - \tfrac{1}{2}x\}.$$



$$\begin{aligned}
\iint_{\mathcal{D}} (2 - x - 2y) dA &= \int_0^1 \int_{\frac{1}{2}x}^{1-\frac{1}{2}x} (2 - x - 2y) dy dx \\
&= \int_0^1 (2y - xy - y^2) \Big|_{\frac{1}{2}x}^{1-\frac{1}{2}x} dx \\
&= \int_0^1 (2 - x - x(1 - \tfrac{1}{2}x) - (1 - \tfrac{1}{2}x)^2 - x + \tfrac{1}{2}x^2 + \tfrac{1}{4}x^2) dx \\
&= \int_0^1 (x^2 - 2x + 1) dx = (\tfrac{1}{3}x^3 - x^2 + x) \Big|_0^1 = \tfrac{1}{3}.
\end{aligned}$$

Choosing the Order Carefully

- Evaluate $\iint_{\mathcal{D}} e^{y^2} dA$, where $\mathcal{D}$ is the region shown in the figure.

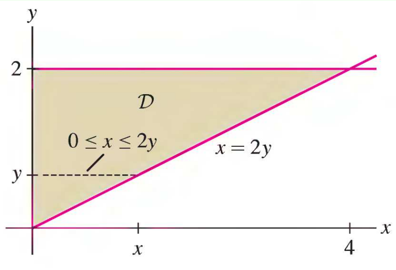


If we attempt to integrate over a type I region $D = \{(x, y) : 0 \leq x \leq 4, \frac{1}{2}x \leq y \leq 2\}$, we will fail.

$$\iint_D e^{y^2} dA = \int_0^4 \int_{x/2}^2 e^{y^2} dy\, dx = ?$$

Choosing the Order Carefully (Cont'd)

- So we switch and evaluate over a type II region



$$\mathcal{D} = \{(x,y) : 0 \leq y \leq 2, 0 \leq x \leq 2y\}.$$

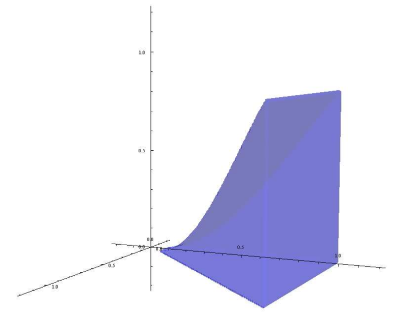
$$\begin{aligned} \iint_{\mathcal{D}} e^{y^2} dA &= \int_0^2 \int_0^{2y} e^{y^2} dx dy = \int_0^2 (xe^{y^2} \mid_0^{2y}) dy \\ &= \int_0^2 2ye^{y^2} dy = e^{y^2} \mid_0^2 = e^4 - 1. \end{aligned}$$

Reversing the Order

- To compute $\int_0^1 \int_x^1 \sin(y^2)dydx$, we must first reverse the order of integration.

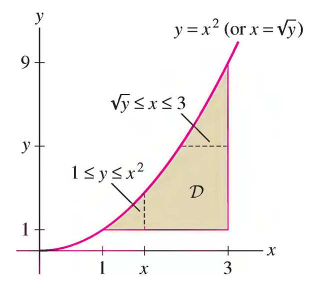
  But this needs care as far as limits are concerned!! Note that $\mathcal{D} = \{(x,y) : 0 \leq x \leq 1, x \leq y \leq 1\} = \{(x,y) : 0 \leq y \leq 1, 0 \leq x \leq y\}$.

$$\begin{aligned}
&\int_0^1 \int_x^1 \sin(y^2)dydx \\
&= \int_0^1 \int_0^y \sin(y^2)dxdy \\
&= \int_0^1 x\sin(y^2)\,|_0^y\, dy \\
&= \int_0^1 y\sin(y^2)dy \\
&= -\tfrac{1}{2}\cos(y^2)\,|_0^1 \\
&= \tfrac{1}{2}(1-\cos 1).
\end{aligned}$$



Reversing the Order II

- Sketch the domain $\mathcal{D}$ of integration of
$$\int_1^9 \int_{\sqrt{y}}^3 xe^y \, dx dy.$$
Then change the order of integration and evaluate.
The domain as given is $\mathcal{D} = \{(x,y) : 1 \leq y \leq 9, \sqrt{y} \leq x \leq 3\}$.

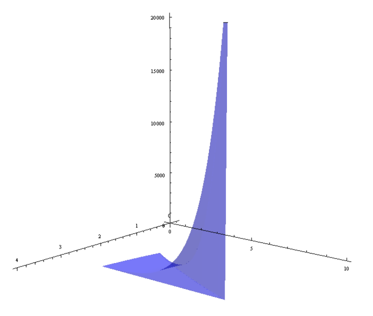


This can be rewritten as $\mathcal{D} = \{(x,y) : 1 \leq x \leq 3, 1 \leq y \leq x^2\}$.

Reversing the Order Again (Cont'd)

- We got $\mathcal{D} = \{(x, y) : 1 \leq x \leq 3, 1 \leq y \leq x^2\}$.

$$\begin{aligned}
&\int_1^9 \int_{\sqrt{y}}^3 xe^y \, dx \, dy \\
&= \int_1^3 \int_1^{x^2} xe^y \, dy \, dx \\
&= \int_1^3 (xe^y) \Big|_1^{x^2} dx \\
&= \int_1^3 (xe^{x^2} - ex) dx \\
&= \tfrac{1}{2}(e^{x^2} - ex^2) \Big|_1^3 \\
&= \tfrac{1}{2}(e^9 - 9e) - 0 \\
&= \tfrac{1}{2}(e^9 - 9e).
\end{aligned}$$



Properties of Double Integrals over Regions

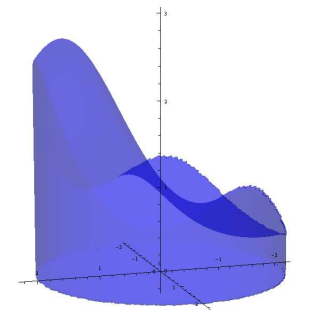
- $\displaystyle\iint_{\mathcal{D}}[f(x,y)+g(x,y)]dA=\iint_{\mathcal{D}}f(x,y)dA+\iint_{\mathcal{D}}g(x,y)dA;$
- $\displaystyle\iint_{\mathcal{D}}cf(x,y)dA=c\iint_{\mathcal{D}}f(x,y)dA;$
- If $f(x,y)\geq g(x,y)$, for all (x,y) in $\mathcal{D}$, then
$$\iint_{\mathcal{D}}f(x,y)dA\geq\iint_{\mathcal{D}}g(x,y)dA;$$
- $\displaystyle\iint_{\mathcal{D}}1dA=A(\mathcal{D});$
- If $m\leq f(x,y)\leq M$, for all (x,y) in $\mathcal{D}$, then
$$mA(\mathcal{D})\leq\iint_{\mathcal{D}}f(x,y)dA\leq MA(\mathcal{D}).$$

Estimating Double Integrals

- Estimate the double integral $\iint_{\mathcal{D}} e^{\sin x \cos y} dA$, where $\mathcal{D}$ is disk with center at the origin and radius 2.

  We have

$$-1 \leq \sin x \leq 1,$$
$$-1 \leq \cos y \leq 1.$$



  Since e^x is an increasing function, we get

$$e^{-1} \leq e^{\sin x \cos y} \leq e^1.$$

  Note, also, that $A(\mathcal{D}) = \pi 2^2 = 4\pi$.

  Therefore, by the inequality above,

$$\frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y} dA \leq 4\pi e.$$

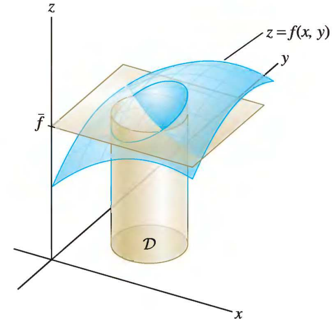
Average Value

- The average value or mean value of a function $f(x,y)$ on a domain $\mathcal{D}$, denoted $\overline{f}$, is the quantity:

$$\begin{aligned} \overline{f} &= \frac{1}{A(\mathcal{D})}\iint_{\mathcal{D}} f(x,y)dA \\ &= \frac{\iint_{\mathcal{D}} f(x,y)dA}{\iint_{\mathcal{D}} 1dA} \end{aligned}$$

Equivalently, $\overline{f}$ is the value satisfying

$$\iint_{\mathcal{D}} f(x,y)dA = \overline{f}\cdot A(\mathcal{D}).$$

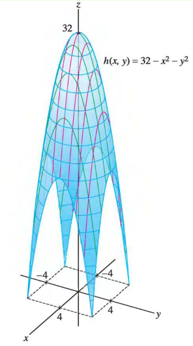


Computing Average Value

- An architect needs to know the average height $\overline{H}$ of the ceiling of a pagoda whose base $\mathcal{D}$ is the square $[-4,4] \times [-4,4]$ and roof is the graph of $H(x,y) = 32 - x^2 - y^2$, where distances are in feet. Calculate $\overline{H}$.

  Compute the integral of $H(x,y)$ over $\mathcal{D}$:

$$\begin{aligned}
&\iint_{\mathcal{D}} (32 - x^2 - y^2) dA \\
&= \int_{-4}^{4} \int_{-4}^{4} (32 - x^2 - y^2) dy dx \\
&= \int_{-4}^{4} (32y - x^2 y - \tfrac{1}{3} y^3) \big|_{-4}^{4} dx \\
&= \int_{-4}^{4} (\tfrac{640}{3} - 8x^2) dx \\
&= (\tfrac{640}{3} x - \tfrac{8}{3} x^3) \big|_{-4}^{4} \\
&= \tfrac{4096}{3}.
\end{aligned}$$



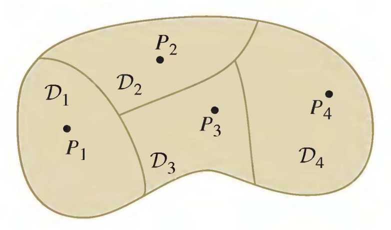
  The area of $\mathcal{D}$ is $8 \times 8 = 64$. So the average height of the pagoda's ceiling is $\overline{H} = \frac{1}{64} \cdot \frac{4096}{3} = \frac{64}{3}$ feet.

Decomposing the Domain Into Smaller Domains

- If $\mathcal{D}$ is the union of domains $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_N$, that do not overlap except possibly on boundary curves, then
$$\iint_{\mathcal{D}} f(x,y)dA = \iint_{\mathcal{D}_1} f(x,y)dA + \cdots + \iint_{\mathcal{D}_N} f(x,y)dA.$$
- If $f(x,y)$ is a continuous function on a small domain $\mathcal{D}$, then
$$\iint_{\mathcal{D}} f(x,y)dA \approx \underbrace{f(P)}_{\text{Function Value}} \cdot \underbrace{A(\mathcal{D})}_{\text{Area}},$$
where P is any sample point in $\mathcal{D}$.
- If the domain $\mathcal{D}$ is not small, we may partition it into N smaller subdomains $\mathcal{D}_1, \ldots, \mathcal{D}_N$ and choose sample points P_j in $\mathcal{D}_j$. Using both preceding properties, we get
$$\iint_{\mathcal{D}} f(x,y)dA \approx \sum_{j=1}^{N} f(P_j)A(\mathcal{D}_j).$$

Example of Decomposing the Domain and Approximating

- Estimate $\iint_{\mathcal{D}} f(x,y)dA$ for the domain $\mathcal{D}$ of the figure, using the areas and function values given.



j	1	2	3	4
$A(\mathcal{D}_j)$	1	1	0.9	1.2
$f(P_j)$	1.8	2.2	2.1	2.4

$$\begin{aligned} \iint_{\mathcal{D}} f(x,y)dA &\approx \sum_{j=1}^{4} f(P_j)A(\mathcal{D}_j) \\ &= 1.8 \cdot 1 + 2.2 \cdot 1 + 2.1 \cdot 0.9 + 2.4 \cdot 1.2 \\ &= 8.8. \end{aligned}$$

Subsection 3

Double Integrals in Polar Coordinates

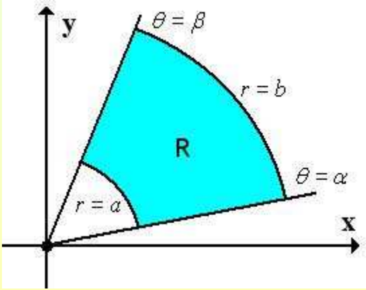
Polar Rectangles

- Recall the formulas relating Cartesian coordinate pairs (x, y) with polar coordinate pairs (r, θ) of the same point on the plane:

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

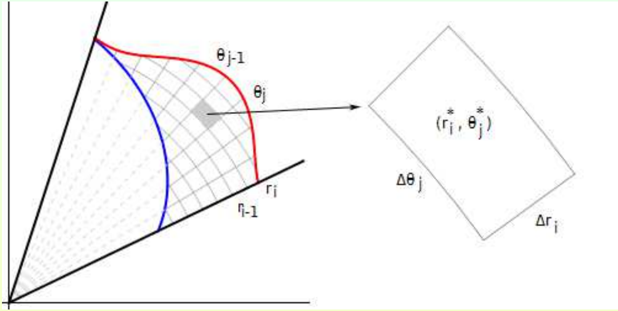
- A **polar rectangle** is the set of points

$$\mathcal{R} = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}.$$



Area of Elementary Polar Sub-rectangle

- The polar subrectangle $\mathcal{R}_{ij} = \{(r,\theta) : r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j\}$.



- Its center has polar coordinates $r_i^* = \frac{1}{2}(r_{i-1} + r_i), \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$.
- Since area of a sector of circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$, we get for the elementary polar rectangular area:
$$\Delta A_{ij} = \frac{1}{2}r_i^2\Delta\theta_j - \frac{1}{2}r_{i-1}^2\Delta\theta_j = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta_j = r_i^*\Delta r_i\Delta\theta_j.$$

Approximating Volumes by Sums in Polar Coordinates

- Given a function $f(x,y)$ defined over the polar rectangle $\mathcal{R}$, we can approximate the volume under f over $\mathcal{R}$ by a sum of volumes over elementary polar rectangles:

$$\begin{aligned}\iint_{\mathcal{R}} f(x,y)dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*,y_j^*)\Delta A_{ij} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)r_i^*\Delta r_i\Delta\theta_j \\ &\approx \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)rdrd\theta.\end{aligned}$$

Changing Double Integrals to Polar Coordinates

If f is continuous on polar rectangle $\mathcal{R}$, with $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$,

$$\iint_{\mathcal{R}} f(x,y)dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Example I

- Evaluate $\iint_{\mathcal{R}}(3x+4y^2)dA$, where $\mathcal{R}$ is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

  The region of integration is

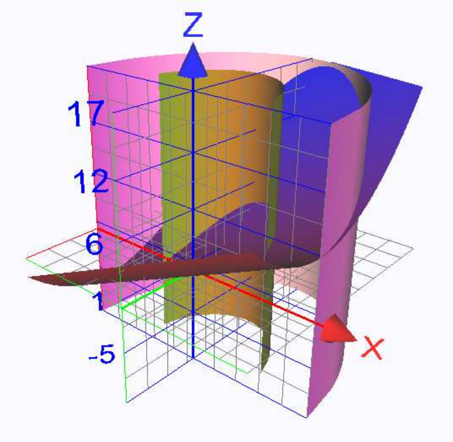
$$\begin{array}{rcl}\mathcal{R} & = & \{(x,y): y\geq 0, 1\leq x^2+y^2\leq 4\}\\ & = & \{(r,\theta): 1\leq r\leq 2, 0\leq\theta\leq\pi\}.\end{array}$$

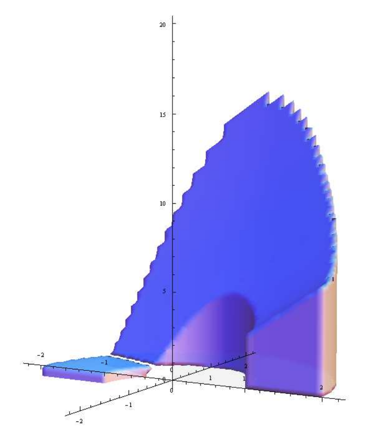
  Thus, we have

$$\begin{array}{rcl}\iint_{\mathcal{R}}(3x+4y^2)dA & = & \int_0^\pi\int_1^2(3r\cos\theta+4r^2\sin^2\theta)rdrd\theta\\ & = & \int_0^\pi\int_1^2(3r^2\cos\theta+4r^3\sin^2\theta)drd\theta\\ & = & \int_0^\pi(r^3\cos\theta+r^4\sin^2\theta)\mid_1^2 d\theta\\ & = & \int_0^\pi(7\cos\theta+15\sin^2\theta)d\theta\\ & = & \int_0^\pi(7\cos\theta+\frac{15}{2}(1-\cos 2\theta))d\theta\\ & = & (7\sin\theta+\frac{15}{2}\theta-\frac{15}{4}\sin 2\theta)\mid_0^\pi \quad = \quad \frac{15}{2}\pi.\end{array}$$

Example I Illustrated

- The volume $\int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)r\,dr\,d\theta = \frac{15}{2}\pi$ units^3.



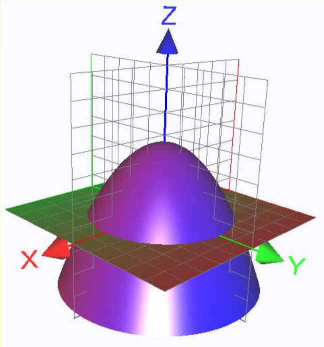


Example II

- Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

  The region of integration is

$$\begin{array}{rcl} \mathcal{R} & = & \{(x,y) : x^2 + y^2 \le 1\} \\ & = & \{(r,\theta) : 0 \le r \le 1, \\ & & \qquad 0 \le \theta \le 2\pi\}. \end{array}$$



  Thus, we have

$$\begin{array}{l} \iint_{\mathcal{R}} (1 - x^2 - y^2) dA \\ = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\ = \int_0^{2\pi} (\frac{1}{2} r^2 - \frac{1}{4} r^4) \mid_0^1 d\theta \\ = \int_0^{2\pi} \frac{1}{4} d\theta \\ = \frac{1}{4}\theta \mid_0^{2\pi} = \frac{\pi}{2}. \end{array}$$

Example III

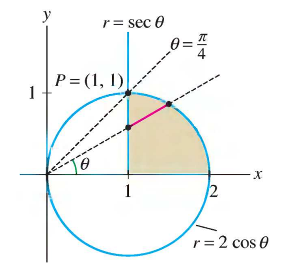
- Calculate $\iint_{\mathcal{D}} \frac{1}{(x^2+y^2)^2} dA$, for the domain $\mathcal{D}$ shaded in the figure.

The region of integration is

$$\mathcal{D} = \{(r,\theta) : 0 \leq \theta \leq \frac{\pi}{4}, \sec\theta \leq r \leq 2\cos\theta\}.$$

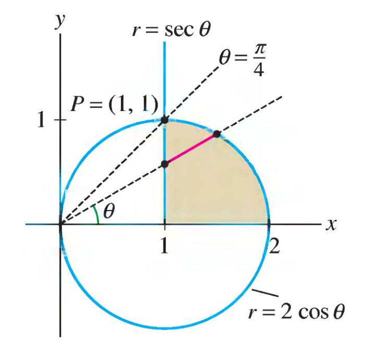
Thus, we have

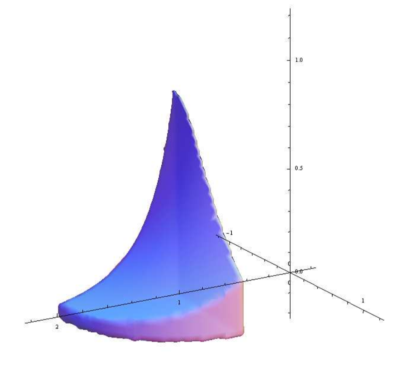
$$\begin{aligned}\iint_{\mathcal{D}} \frac{1}{(x^2+y^2)^2} dA &= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} \frac{1}{r^4} r dr d\theta \\ &= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} \frac{1}{r^3} dr d\theta \\ &= \int_0^{\frac{\pi}{4}} (-\frac{1}{2r^2}) \Big|_{\sec\theta}^{2\cos\theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} (-\frac{1}{8}\sec^2\theta + \frac{1}{2}\cos^2\theta) d\theta \\ &= [-\frac{1}{8}\tan\theta + \frac{1}{4}(\theta + \frac{1}{2}\sin 2\theta)] \Big|_0^{\pi/4} \\ &= -\frac{1}{8} + \frac{1}{4}(\frac{\pi}{4} + \frac{1}{2}) = \frac{\pi}{16}.\end{aligned}$$



Example III Illustrated

- The volume $\iint_{\mathcal{D}} \frac{1}{(x^2+y^2)^2} dA$, for the domain $\mathcal{D}$ shaded in the figure on the left.





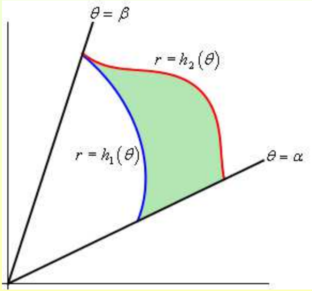
Double Integrals over Polar Regions Between Two Curves

Polar Integration Between Two Curves

If f is continuous on a polar region

$$\mathcal{D} = \{(r,\theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\},$$

then $$\iint\limits_{\mathcal{D}} f(x,y)dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$$



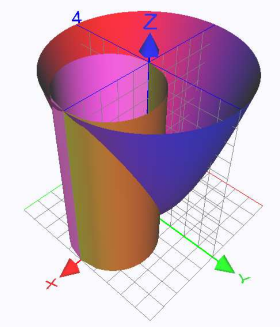
Double Integration Between Two Curves

- Find the volume of the solid under the paraboloid $z = x^2 + y^2$ above the xy-plane inside the cylinder $x^2 + y^2 = 2x$.

  The region of integration is

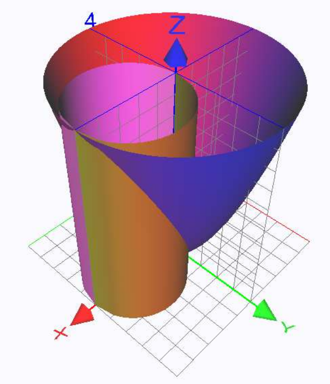
$$\begin{aligned} \mathcal{D} &= \{(x,y) : (x-1)^2 + y^2 \leq 1\} \\ &= \{(r,\theta) : -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}, 0 \leq r \leq 2\cos\theta\}. \end{aligned}$$

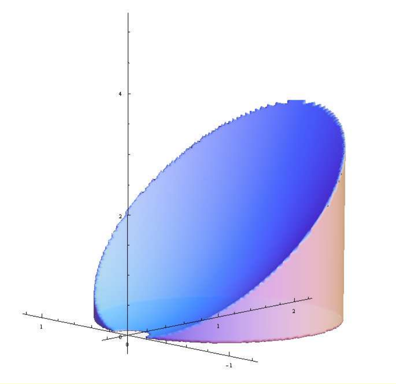
$$\begin{aligned} &\iint_{\mathcal{D}} (x^2 + y^2) dA \\ &= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \tfrac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta \\ &= 8 \int_0^{\pi/2} (\tfrac{1+\cos 2\theta}{2})^2 d\theta \\ &= 2 \int_0^{\pi/2} (1 + 2\cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)) d\theta \\ &= 2[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta]_0^{\pi/2} \\ &= \tfrac{3}{2}\pi. \end{aligned}$$



Double Integration Between Two Curves Ilustrated

- The volume of the solid under the paraboloid $z = x^2 + y^2$ above the xy-plane inside the cylinder $x^2 + y^2 = 2x$.





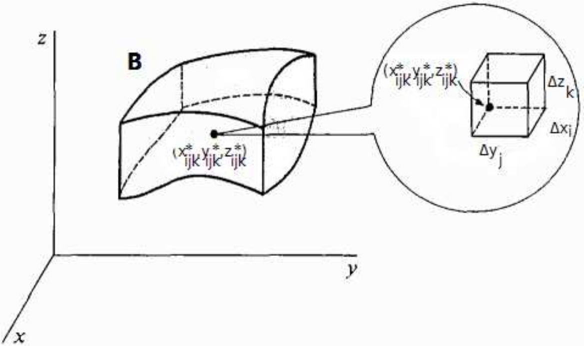
Subsection 4

Triple Integrals

Triple Integrals

- The **triple integral** of $f(x,y,z)$ over a box $\mathcal{B}$ is defined by

$$\iiint\limits_{\mathcal{B}} f(x,y,z)dV = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0 \\ \Delta z_k \to 0}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}.$$



Fubini's Theorem for Triple Integrals

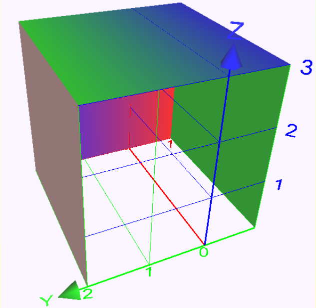
Fubini's Theorem

If f is continuous on the rectangular box $\mathcal{B} = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint\limits_{\mathcal{B}} f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz.$$

Example: Evaluate the integral $\iiint_{\mathcal{B}} xyz^2 dV$, where $\mathcal{B}$ is the rectangular box given by

$$\begin{aligned} \mathcal{B} &= \{(x,y,z) : 0 \leq x \leq 1, \\ &\quad -1 \leq y \leq 2, 0 \leq z \leq 3\}. \end{aligned}$$



Computing the Triple Integral

$$\begin{aligned}
\iiint_{\mathcal{B}} xyz^2 dV &= \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx dy dz \\
&= \int_0^3 \int_{-1}^2 (\tfrac{1}{2}x^2yz^2) \big|_0^1 \, dy dz \\
&= \int_0^3 \int_{-1}^2 \tfrac{1}{2}yz^2 dy dz \\
&= \int_0^3 \tfrac{1}{4}y^2z^2 \big|_{-1}^2 \, dz \\
&= \int_0^3 \tfrac{3}{4}z^2 dz \\
&= \tfrac{1}{4}z^3 \big|_0^3 \\
&= \tfrac{27}{4}.
\end{aligned}$$

Computing Another Triple Integral

- Compute the integral $\iiint_{\mathcal{B}} x^2 e^{y+3z} dV$, where $\mathcal{B} = [1,4] \times [0,3] \times [2,6]$.

$$\begin{aligned}
\int\int\int\int_{\mathcal{B}} x^2 e^{y+3z} dV &= \int_1^4 \int_0^3 \int_2^6 x^2 e^{y+3z} dz dy dx \\
&= \int_1^4 \int_0^3 \int_2^6 x^2 e^y e^{3z} dz dy dx \\
&= \int_1^4 \int_0^3 \tfrac{1}{3} x^2 e^y (e^{3z}) \mid_2^6 dy dx \\
&= \int_1^4 \int_0^3 \tfrac{1}{3} x^2 e^y (e^{18} - e^6) dy dx \\
&= \int_1^4 \tfrac{1}{3} x^2 (e^{18} - e^6)(e^y) \mid_0^3 dx \\
&= \int_1^4 \tfrac{1}{3} x^2 (e^{18} - e^6)(e^3 - 1) dx \\
&= \tfrac{1}{9}(e^{18} - e^6)(e^3 - 1)(x^3) \mid_1^4 \\
&= \tfrac{1}{9}(e^{18} - e^6)(e^3 - 1) \cdot 63 \\
&= 7(e^{18} - e^6)(e^3 - 1).
\end{aligned}$$

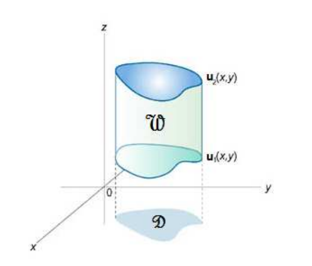
Triple Integrals Over Type I Solid Regions

- A solid region $\mathcal{W}$ is said to be of **type I** if it lies between the graphs of two continuous functions of x and y, i.e., if it is of the form

$$\mathcal{W} = \{(x,y,z) : (x,y) \in \mathcal{D}, u_1(x,y) \leq z \leq u_2(x,y)\}.$$

- For a type I region $\mathcal{W}$,

$$\iiint_{\mathcal{W}} f(x,y,z)dV = \iint_{\mathcal{D}} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdA.$$



Two Special Cases of Type I Solid Regions

- If the projection $\mathcal{D}$ of $\mathcal{W}$ on the xy-plane is a type I plane region $\mathcal{D} = \{(x,y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\mathcal{W} = \{(x,y,z) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x,y) \leq z \leq u_2(x,y)\}$$

and

$$\iiint_{\mathcal{W}} f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx.$$

- If the projection $\mathcal{D}$ of $\mathcal{W}$ on the xy-plane is a type II plane region $\mathcal{D} = \{(x,y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then

$$\mathcal{W} = \{(x,y,z) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x,y) \leq z \leq u_2(x,y)\}$$

and

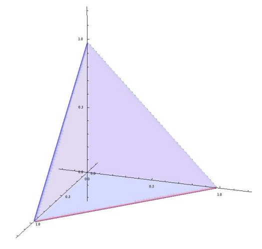
$$\iiint_{\mathcal{W}} f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy.$$

Calculating a Type I Triple Integral

- Evaluate $\iiint_{\mathcal{E}} z dV$, where $\mathcal{E}$ is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.
  The tetrahedral region may be expressed as

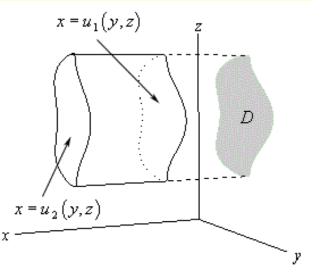
$$\mathcal{E}=\{(x,y,z): 0\leq x\leq 1, 0\leq y\leq 1-x, 0\leq z\leq 1-x-y\}.$$

$$\begin{aligned}
\iiint_{\mathcal{E}} z dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z dz dy dx \\
&= \int_0^1 \int_0^{1-x} \tfrac{1}{2} z^2 \Big|_0^{1-x-y} dy dx \\
&= \tfrac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 dy dx \\
&= \tfrac{1}{2} \int_0^1 (-\tfrac{1}{3}(1-x-y)^3) \Big|_0^{1-x} dx \\
&= \tfrac{1}{6} \int_0^1 (1-x)^3 dx \\
&= \tfrac{1}{6}(-\tfrac{1}{4}(1-x)^4) \Big|_0^1 = \tfrac{1}{24}.
\end{aligned}$$



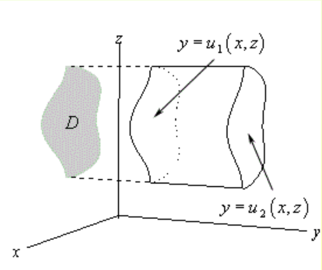
Triple Integrals Over Type II Solid Regions

- A solid region $\mathcal{W}$ is said to be of **type II** if it lies between the graphs of two continuous functions of y and z, i.e., if it is of the form
$$\mathcal{W} = \{(x,y,z) : (y,z) \in \mathcal{D}, u_1(y,z) \leq x \leq u_2(y,z)\}.$$
- For a type II region $\mathcal{W}$,
$$\iiint\limits_{\mathcal{W}} f(x,y,z)dV = \iint\limits_{\mathcal{D}} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)dxdA.$$



Triple Integrals Over Type III Solid Regions

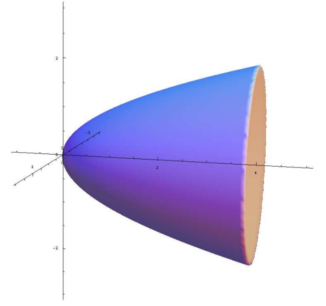
- A solid region $\mathcal{W}$ is said to be of **type III** if it lies between the graphs of two continuous functions of x and z, i.e., if it is of the form
$$\mathcal{W} = \{(x,y,z) : (x,z) \in \mathcal{D}, u_1(x,z) \leq y \leq u_2(x,z)\}.$$
- For a type III region $\mathcal{W}$,
$$\iiint\limits_{\mathcal{W}} f(x,y,z)dV = \iint\limits_{\mathcal{D}} \int\limits_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)dydA.$$



Calculating a Type III Triple Integral

- Evaluate $\iiint_{\mathcal{W}} \sqrt{x^2+z^2}dV$, where $\mathcal{W}$ is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$.
  Let $\mathcal{D} = \{(x,z) : x^2+z^2 \leq 4\}$.
  The paraboloid region may be expressed as
  $$\mathcal{W} = \{(x,y,z) : (x,z) \in \mathcal{D}, x^2+z^2 \leq y \leq 4\}.$$

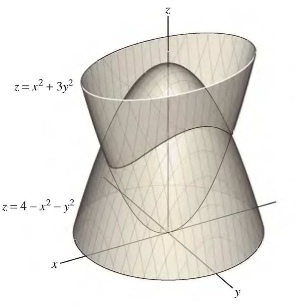
$$\begin{aligned}
&\iiint_{\mathcal{W}} \sqrt{x^2+z^2}dV \\
&= \iint_{\mathcal{D}} \int_{x^2+z^2}^4 \sqrt{x^2+z^2}dydA \\
&= \iint_{\mathcal{D}} (4-x^2-z^2)\sqrt{x^2+z^2}dA \\
&= \int_0^{2\pi} \int_0^2 (4-r^2)rrdrd\theta \\
&= \int_0^{2\pi} d\theta \int_0^2 (4r^2-r^4)dr \\
&= 2\pi(\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5)\mid_0^2 \\
&= \frac{128\pi}{15}.
\end{aligned}$$

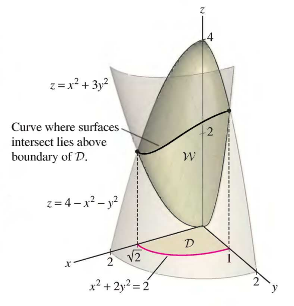


Region Between Intersecting Surfaces

- Integrate $f(x,y,z)=x$ over the region $\mathcal{W}$ bounded above by $z=4-x^2-y^2$ and below by $z=x^2+3y^2$ in the octant $x\geq 0$, $y\geq 0$, $z\geq 0$.

  We have $\iiint_{\mathcal{W}} x dV = \iint_{\mathcal{D}} \int_{x^2+3y^2}^{4-x^2-y^2} x dz dA$.





For the boundary of $\mathcal{D}$ set $x^2+3y^2=4-x^2-y^2 \Rightarrow x^2+2y^2=2$.
We conclude that $\mathcal{D}=\{(x,y):0\leq y\leq 1, 0\leq x\leq \sqrt{2-2y^2}\}$.

Region Between Intersecting Surfaces (Cont'd)

- Now we have:

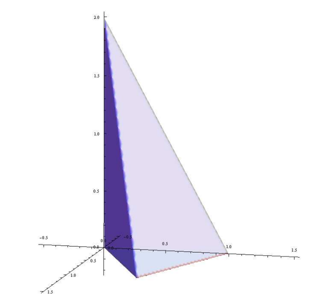
$$
\begin{aligned}
\iiint_{\mathcal{W}} x dV &= \int_0^1 \int_0^{\sqrt{2-2y^2}} \int_{x^2+3y^2}^{4-x^2-y^2} x dz dx dy \\
&= \int_0^1 \int_0^{\sqrt{2-2y^2}} (xz) \Big|_{x^2+3y^2}^{4-x^2-y^2} dx dy \\
&= \int_0^1 \int_0^{\sqrt{2-2y^2}} (4x - 2x^3 - 4y^2 x) dx dy \\
&= \int_0^1 (2x^2 - \tfrac{1}{2}x^4 - 2x^2y^2) \Big|_0^{\sqrt{2-2y^2}} dy \\
&= \int_0^1 (2(2-2y^2) - \tfrac{1}{2}(2-2y^2)^2 - 2(2-2y^2)y^2) dy \\
&= \int_0^1 (4 - 4y^4 - 2 + 4y^2 - 2y^4 - 4y^2 + 4y^4) dy \\
&= \int_0^1 (2 - 4y^2 + 2y^4) dy \\
&= (2y - \tfrac{4}{3}y^3 + \tfrac{2}{5}y^5)_0^1 = 2 - \tfrac{4}{3} + \tfrac{2}{5} = \tfrac{16}{15}.
\end{aligned}
$$

Volumes

- If $f(x,y,z)=1$ throughout a solid region $\mathcal{W}$, then the triple integral of f over $\mathcal{W}$ is equal to the volume of $\mathcal{W}$: $V(\mathcal{W})=\iiint\limits_{\mathcal{W}} 1dV$.

  Example: Compute the volume of the tetrahedron $\mathcal{T}$ bounded by the planes $x+2y+z=2, x=2y, x=0$ and $z=0$.

$$
\begin{aligned}
V(\mathcal{T}) &= \iiint_{\mathcal{T}} dV \\
&= \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dzdydx \\
&= \int_0^1 \int_{x/2}^{1-x/2} (2-x-2y)dydx \\
&= \int_0^1 ((2-x)y-y^2)\,\Big|_{x/2}^{1-x/2} dx \\
&= \int_0^1 (x^2-2x+1)dx \\
&= \tfrac{1}{3}.
\end{aligned}
$$

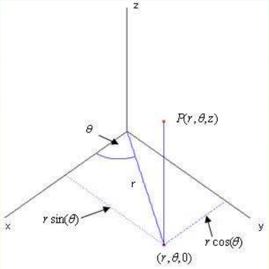


Subsection 5

Triple Integrals in Cylindric Coordinates

Cylindrical Coordinate System

- In **cylindrical Coordinates** a point P is represented by a triple (r, θ, z), where
  - r and θ are polar coordinates of the projection of P onto the xy-plane;
  - z is the directed distance from the xy-plane to P.



- Conversion from Cylindrical to Rectangular:

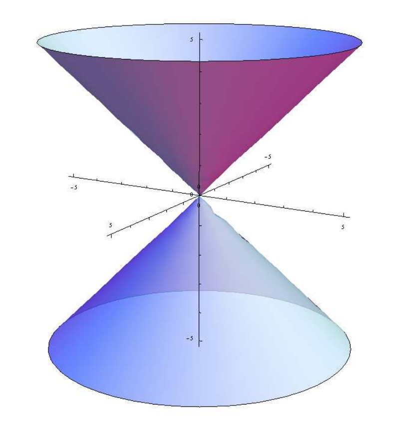
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

- Conversion from Rectangular to Cylindrical:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z.$$

Surface with Cylindrical Coordinates $z = r$

- In rectangular $z = r$ translates to $z^2 = x^2 + y^2$, which represents a cone with axis the z-axis.



Triple Integrals in Cylindrical Coordinates

- Assume f is continuous on

$$\mathcal{W} = \{(x, y, z) : (x, y) \in \mathcal{D}, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Assume also that

$$\mathcal{D} = \{(r, \theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}.$$

Then, the triple integral of f over $\mathcal{W}$ is given by

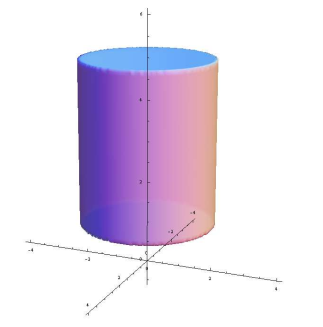
$$\iiint\limits_{\mathcal{W}} f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta.$$

Example I

- Integrate $f(x, y, z) = z\sqrt{x^2 + y^2}$ over the cylinder $x^2 + y^2 \leq 4$, for $1 \leq z \leq 5$.

We have

$$\mathcal{W} = \{(r, \theta, z) : 0 \leq \theta \leq 2\pi, \\ 0 \leq r \leq 2, 1 \leq z \leq 5\}.$$



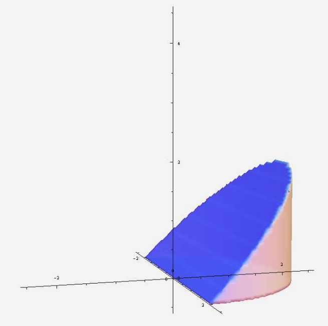
Therefore, we obtain

$$\begin{aligned}
&\iiint_{\mathcal{W}} z\sqrt{x^2 + y^2} dV \\
&= \int_0^{2\pi} \int_0^2 \int_1^5 (zr) r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^2 \tfrac{1}{2} r^2 (z^2 \mid_1^5) dr d\theta \\
&= \int_0^{2\pi} \int_0^2 12 r^2 dr d\theta = \int_0^{2\pi} 4(r^3) \mid_0^2 d\theta \\
&= \int_0^{2\pi} 32 d\theta = 32(\theta) \mid_0^{2\pi} = 64\pi.
\end{aligned}$$

Example II

- Compute the integral of $f(x,y,z) = z$ over the region $\mathcal{W}$ within the cylinder $x^2 + y^2 \le 4$ where $0 \le z \le y$.

  We have
  $$\mathcal{W} = \{(r,\theta,z) : 0 \le \theta \le \pi, \\ 0 \le r \le 2, 0 \le z \le r\sin\theta\}.$$
  Therefore, we obtain



$$\begin{aligned}
&\iiint_{\mathcal{W}} z dV \\
&= \int_0^\pi \int_0^2 \int_0^{r\sin\theta} zrdzdrd\theta \\
&= \int_0^\pi \int_0^2 \frac{1}{2} r(z^2 \mid_0^{r\sin\theta})drd\theta
\end{aligned}$$
$$= \int_0^\pi \int_0^2 \frac{1}{2} r^3 \sin^2\theta drd\theta = \int_0^\pi \frac{1}{8}\sin^2\theta (r^4) \mid_0^2 d\theta = \int_0^\pi 2\sin^2\theta d\theta$$
$$= \int_0^\pi (1 - \cos 2\theta)d\theta = (\theta - \frac{1}{2}\sin 2\theta) \mid_0^\pi = \pi.$$

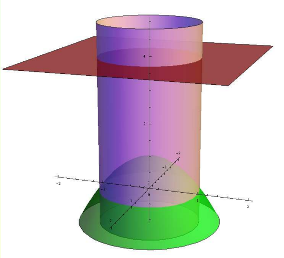
Computing a Mass

- Compute the mass of a solid $\mathcal{W}$ that lies within the cylinder $x^2+y^2=1$, below $z=4$ and above $z=1-x^2-y^2$, with density proportional to the distance from the axis of the cylinder.

The region $\mathcal{W}$ can be expressed as

$$\mathcal{W}=\{(r,\theta,z): 0\le\theta\le 2\pi,\ 0\le r\le 1, 1-r^2\le z\le 4\}$$

Density is $\rho(x,y,z)=K\sqrt{x^2+y^2}=Kr$.



$$m = \iiint_E K\sqrt{x^2+y^2}dV$$

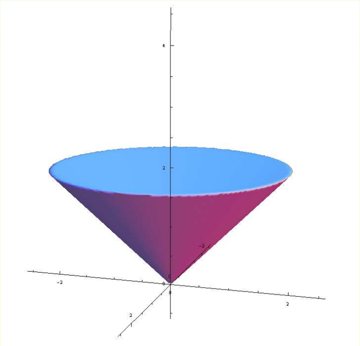
$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (Kr)rdzdrd\theta = \int_0^{2\pi}\int_0^1 Kr^2(4-(1-r^2))drd\theta$$

$$= K\int_0^{2\pi}d\theta\int_0^1(3r^2+r^4)dr = 2\pi K(r^3+\frac{1}{5}r^5)\mid_0^1 = \frac{12\pi K}{5}.$$

Another Example

- Evaluate
$$I=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)dzdydx;$$

The region $\mathcal{W}$ can be expressed as
$$\mathcal{W}=\{(r,\theta,z):0\leq\theta\leq 2\pi,\\ 0\leq r\leq 2, r\leq z\leq 2\}.$$



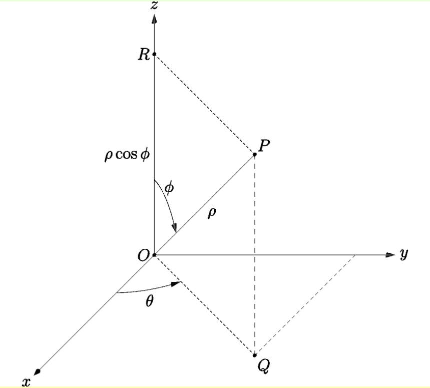
$$\begin{aligned}
I &= \iiint_{\mathcal{W}}(x^2+y^2)dV \\
&= \int_0^{2\pi}\int_0^2\int_r^2 r^2 rdzdrd\theta \\
&= \int_0^{2\pi}d\theta\int_0^2 r^3 z\mid_r^2 dr \\
&= \int_0^{2\pi}d\theta\int_0^2 r^3(2-r)dr \\
&= 2\pi(\tfrac{1}{2}r^4-\tfrac{1}{5}r^5)\mid_0^2 \\
&= \frac{16\pi}{5}.
\end{aligned}$$

Subsection 6

Triple Integrals in Spherical Coordinates

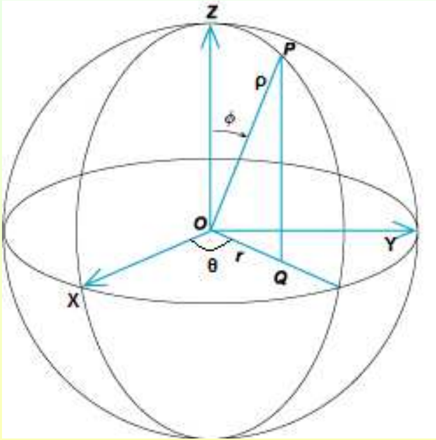
Spherical Coordinate System

- The **spherical coordinates** (ρ, θ, ϕ) of a point P consist of
  - the distance $\rho = OP$ of P from the origin O;
  - the same angle θ as in cylindrical coordinates;
  - the angle ϕ between the positive z-axis and the line segment OP.



Why "Spherical"?

- The sphere centered at origin with radius c has equation $\rho = c$.

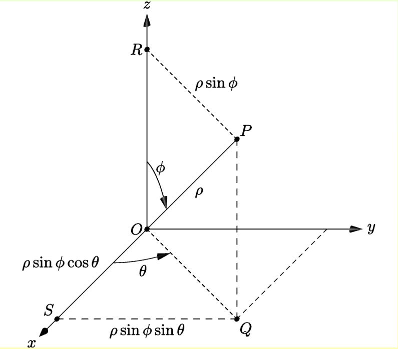


From Spherical to Rectangular

- Recall again that $z = \rho\cos\phi$ and $r = \rho\sin\phi$.
  Thus, the equations to convert from Spherical to Rectangular are:

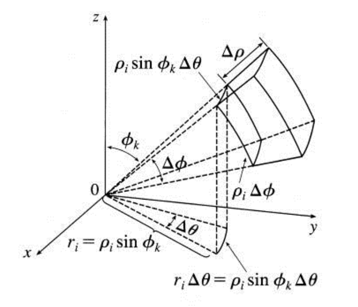
$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

- Recall, also, that $\rho^2 = x^2 + y^2 + z^2$.



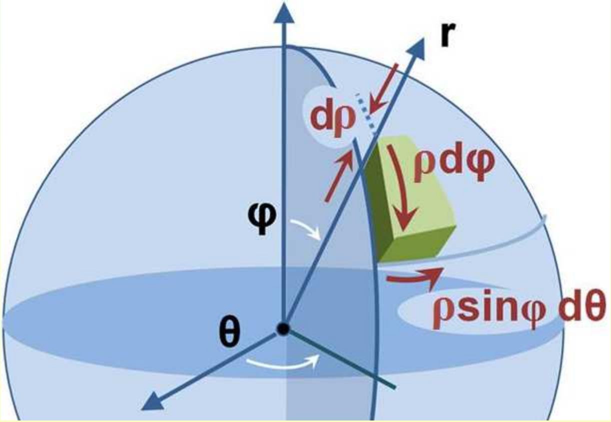
Triple Integrals Using Spherical Coordinates

- A **spherical wedge** is a set of the form
$$\mathcal{W} = \{(\rho, \theta, \phi) : a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \delta\}.$$
- The elementary volume ΔV_{ijk} of a small wedge, whose center radius is ρ_i and whose spherical "dimensions" are $\Delta\rho_i, \Delta\theta_j$ and $\Delta\phi_k$ is given by
$$\begin{aligned} \Delta V_{ijk} &\approx (\Delta\rho_i)(\rho_i \Delta\phi_k)(\rho_i \sin\phi_k \Delta\theta_j) \\ &= \rho_i^2 \sin\phi_k \Delta\rho_i \Delta\theta_j \Delta\phi_k. \end{aligned}$$



Illustrating an Elementary Spherical Volume

- Recall $\Delta V_{ijk} \approx \rho_i^2 \sin \phi_k \Delta\rho_i \Delta\theta_j \Delta\phi_k$.
- Volume differential: $dV = d\rho(\rho d\phi)(\rho \sin \phi d\theta) = \rho^2 \sin \phi d\rho d\theta d\phi$.



Triple Integrals in Spherical Coordinates

- Recall $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$.
- Recall, also, $\Delta V_{ijk} = \rho_i^2 \sin\phi_k \Delta\rho_i \Delta\theta_j \Delta\phi_k$.
- So, we get that

$$\iiint_{\mathcal{W}} f(x,y,z)dV \approx \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}$$

$$= \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho_i \Delta\theta_j \Delta\phi_k.$$

- We, therefore get the formula $\iiint_{\mathcal{W}} f(x,y,z)dV =$

$$= \lim_{\substack{\Delta\rho_i \to 0 \\ \Delta\theta_j \to 0 \\ \Delta\phi_k \to 0}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho_i \Delta\theta_j \Delta\phi_k$$

$$= \int_{\gamma}^{\delta}\int_{\alpha}^{\beta}\int_{a}^{b} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

Example I

- Evaluate $\iiint_{\mathcal{W}} 16zdV$, where $\mathcal{W}$ is the upper half of the sphere $\mathcal{B} = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.
  In spherical coordinates
  $$\mathcal{W} = \{(\rho, \theta, \phi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{2}\}.$$
  Taking into account that $z = \rho \cos \phi$, we get
  $$\begin{array}{rcl} \iiint_{\mathcal{W}} 16zdV & = & \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (16\rho \cos \phi)(\rho^2 \sin \phi) d\rho d\theta d\phi \\ & = & \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 8\rho^3 \sin 2\phi d\rho d\theta d\phi \\ & = & \int_0^{\pi/2} \int_0^{2\pi} 2\rho^4 \sin 2\phi \big|_0^1 d\theta d\phi \\ & = & \int_0^{\pi/2} \int_0^{2\pi} 2 \sin 2\phi d\theta d\phi \\ & = & \int_0^{\pi/2} 2\theta \sin 2\phi |_0^{2\pi} d\phi \\ & = & \int_0^{\pi/2} 4\pi \sin 2\phi d\phi \\ & = & -2\pi \cos 2\phi |_0^{\pi/2} = 4\pi. \end{array}$$

Example II

- Evaluate $\iiint_{\mathcal{B}} e^{(x^2+y^2+z^2)^{3/2}} dV$, where $\mathcal{B}$ is the unit ball $\mathcal{B} = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$.

  In spherical coordinates

  $$\mathcal{B} = \{(\rho,\theta,\phi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

  Taking into account that $x^2+y^2+z^2 = \rho^2$, we get

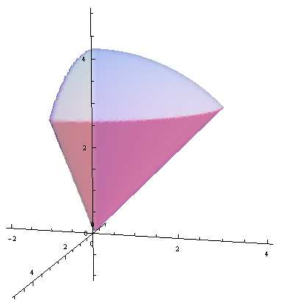
  $$\begin{aligned} \iiint_{\mathcal{B}} e^{(x^2+y^2+z^2)^{3/2}} dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho d\theta d\phi \\ &= \int_0^\pi \sin\phi d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{\rho^3} d\rho \\ &= -\cos\phi \,\big|_0^\pi \cdot 2\pi \cdot \left(\tfrac{1}{3} e^{\rho^3}\right) \big|_0^1 \\ &= \tfrac{4}{3}\pi(e-1). \end{aligned}$$

Example III

- Compute the integral $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dzdxdy$.

  The equation of the sphere in spherical coordinates is $\rho^2 = 18$ or $\rho = 3\sqrt{2}$.

  The equation of the cone is $z = \rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi$. So $\phi = \frac{\pi}{4}$.

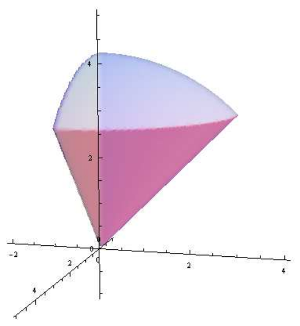


  Finally, the solid $\mathcal{W}$ in spherical coordinates is given by

$$\mathcal{W} = \{(\rho,\theta,\phi) : 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{4}, 0 \le \rho \le 3\sqrt{2}\}.$$

Example III (Cont'd)

$$\begin{aligned}
&\iiint_{\mathcal{W}} \rho^2 dV \\
&= \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{3\sqrt{2}} \rho^2 (\rho^2 \sin\phi) d\rho d\phi d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/4} \frac{1}{5}\rho^5 \sin\phi \Big|_0^{3\sqrt{2}} d\phi d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/4} \frac{243\cdot 4\sqrt{2}}{5} \sin\phi d\phi d\theta \\
&= \int_0^{\pi/2} -\frac{972\sqrt{2}}{5} \cos\phi \Big|_0^{\pi/4} d\theta \\
&= \int_0^{\pi/2} (-\frac{972\sqrt{2}}{5}(\frac{\sqrt{2}}{2} - 1)) d\theta \\
&= \int_0^{\pi/2} \frac{972(\sqrt{2}-1)}{5} d\theta \\
&= \frac{972(\sqrt{2}-1)\pi}{10}.
\end{aligned}$$

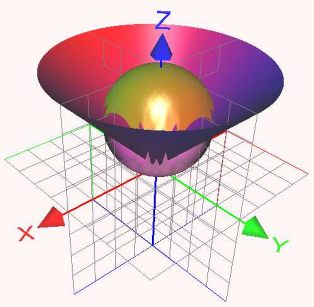


Example IV

- Compute the volume of the solid lying above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

  The equation of the sphere in spherical coordinates is $\rho^2 = \rho \cos \phi$ or $\rho = \cos \phi$.

  The equation of the cone is

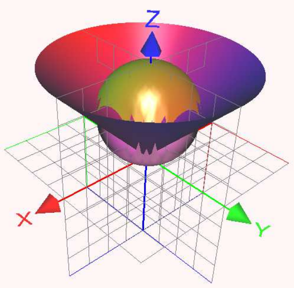


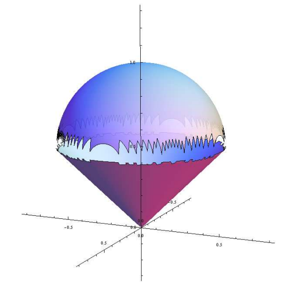
$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta} = \rho \sin \phi. \text{ So } \phi = \frac{\pi}{4}.$$

Finally, the solid $\mathcal{W}$ in spherical coordinates is given by

$$\mathcal{W} = \{(\rho, \theta, \phi) : 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \rho \leq \cos \phi\}.$$

Example IV (Cont'd)





$$\begin{aligned}
V(\mathcal{W}) &= \iiint_{\mathcal{W}} dV \\
&= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi (\tfrac{1}{3}\rho^3) \Big|_0^{\cos\phi} d\phi \\
&= \tfrac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi \\
&= \tfrac{2\pi}{3} (-\tfrac{1}{4}\cos^4\phi) \Big|_0^{\pi/4} = \tfrac{\pi}{8}.
\end{aligned}$$