

College Algebra

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LSSU Math 111

1 Functions, Graphs, and Models; Linear Functions

- Some Calculator Tips
- Functions and Models
- Graphs of Functions
- Linear Functions
- Equations of Lines

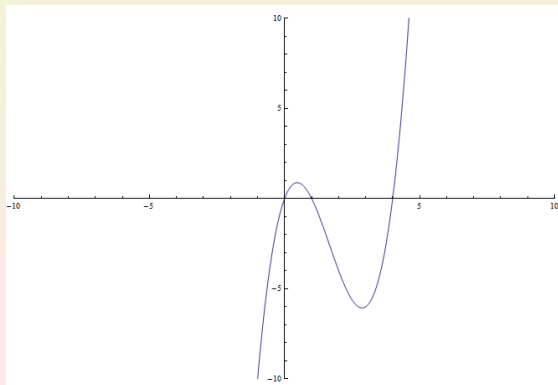
Subsection 1

Some Calculator Tips

Graphing a Function

- We obtain the graph of $y = x^3 - 5x^2 + 4x$:

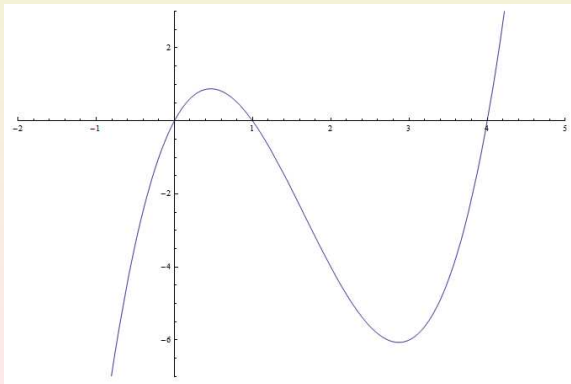
1. Press **Y=**
2. Insert $Y_1 = X^3 - 5X^2 + 4X$
3. Press **GRAPH**.



Adjusting the Window

- We want to focus more on the “interesting interval” $[-2, 5]$.

1. Press **WINDOW**
2. Adjust $X_{\min} = -2$, $X_{\max} = 5$, $Y_{\min} = -7$ and $Y_{\max} = 3$.
3. Press **GRAPH**



Setting and Looking at Table of Values

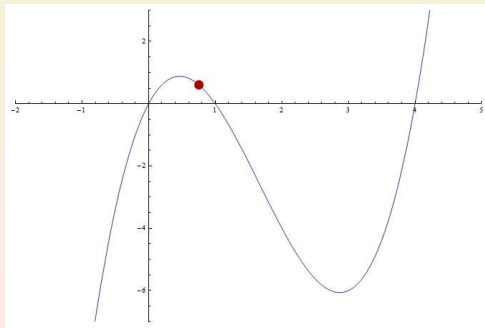
- We want to look at (x, y) pairs of values of points on $y = x^3 - 5x^2 + 4x$, for every x between -2 and 5 with 0.5 resolution.
 1. Press **TBLSET**
 2. Set TblStart = -2 and $\Delta\text{Tbl} = 0.5$
 3. Press **TABLE**

X	Y_1
-2	-36
-1.5	-20.63
-1	-10
-0.5	-3.375
0	0
0.5	0.875
\vdots	\vdots

Computing a Specific Value

- We find the y value for input $x = 0.75$.

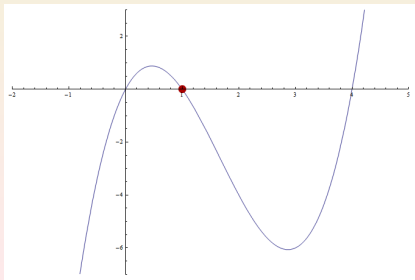
1. Press **GRAPH**
2. Press **TRACE**
3. Press **↑**
4. Insert $X = 0.75$
5. Press **ENTER**



$$Y = 0.609375.$$

Finding Zeros

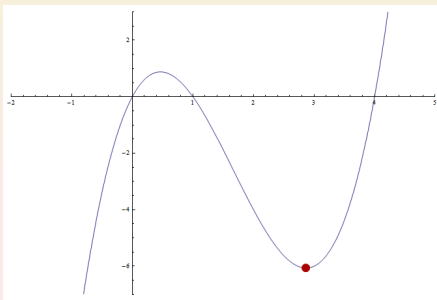
- We find the second point where the graph of $y = x^3 - 5x^2 + 4x$ crosses the x -axis.
 1. Press **CALC**
 2. Go **CALCULATE** \rightarrow 2:zero
 3. Press **ENTER**
 4. Choose LeftBound? to the left of the point and press **ENTER**.
 5. Choose RightBound? to the right of the point and press **ENTER**.
 6. Guess? Press **ENTER**



$$X = 1, Y = 0.$$

Finding Maxima and Minima

- We find the local minimum of the graph of $y = x^3 - 5x^2 + 4x$.
 1. Press **CALC**
 2. Go **CALCULATE** → 3:minimum
 3. Press **ENTER**
 4. Choose LeftBound? to the left of the point and press **ENTER**.
 5. Choose RightBound? to the right of the point and press **ENTER**.
 6. Guess? Press **ENTER**



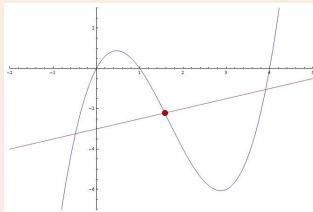
$$X = 2.8685$$
$$Y = -6.0646.$$

Finding Intersection Points

- We find the second intersection point of $y = x^3 - 5x^2 + 4x$ and $y = \frac{1}{2}x - 3$.
 1. Press **Y=**
 2. Insert $Y_2 = (1/2)X - 3$
 3. Press **GRAPH**
 4. Press **CALC**
 5. Go CALCULATE \rightarrow 5:intersect
 6. Press **ENTER**
 7. Choose Firstcurve? close to intersection and press **ENTER**.
 8. Choose Secondcurve? close to intersection and press **ENTER**.
 9. Guess? Press **ENTER**

$$X = 1.5784$$

$$Y = -2.2108.$$



Inserting Data Points in a List

- Consider the following data points:

x	-2	-1	0	1	2	3	4
y	-4	-3.4	-3.1	-2.51	-1.75	-1.6	-0.8

We insert the data points into a list.

- Press **STAT**
- Choose **EDIT** → 1:Edit
- Press **ENTER**
- Fill in column L_1 with the values $-2, -1, 0, 1, 2, 3, 4$
 - Fill in column L_2 with the values $-4, -3.4, -3.1, -2.51, -1.75, -1.6, -0.8$

We should now have

L_1	L_2
-2	-4
-1	-3.4
0	-3.1
1	-2.51
2	-1.75
3	-1.6
4	-0.8
-----	-----

Plotting Data and Fitting Graphs to Data Points

- We fit a line $y = ax + b$ between the data.

1. Press **STAT**
2. Choose **CALC** → 4:LinReg($ax + b$)
3. Press **ENTER**
4. Go to Store RegEQ: and choose **VARS** → Y-VARS → 1:Function
→ 1 : Y_1 and press **ENTER**
4. Choose Calculate and press **ENTER**

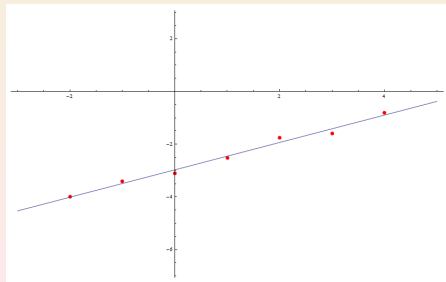
We now have

$$y = ax + b$$

$$a = 0.5196$$

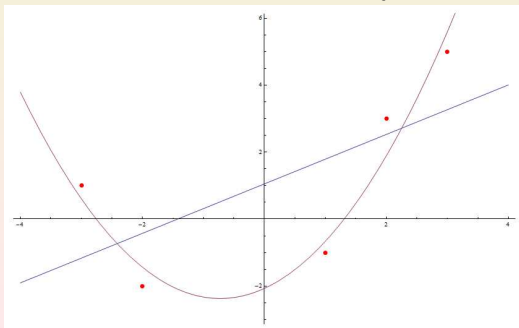
$$b = -2.9711$$

Also $Y_1 = 0.5196X - 2.9711$,
so we are ready to graph.



Another Data Drill

- Consider the data
- | | | | | | |
|-----|----|----|----|---|---|
| x | -3 | -2 | 1 | 2 | 3 |
| y | 1 | -2 | -1 | 3 | 5 |
- (a) Insert the data points in the list editor.
 - (b) Graph them by setting a window $[-4, 4]$ by $[-3, 6]$.
 - (c) Fit a line $y = ax + b$ assigning it also to Y_1 .
 - (d) Fit a quadratic $y = ax^2 + bx + c$ assigning it to Y_2 .
 - (e) Graph data and both models on the same system of axes.



Subsection 2

Functions and Models

Presenting Functional Relations

- Ways to show how Fahrenheit degree measurements are related to Celsius degree measurements:

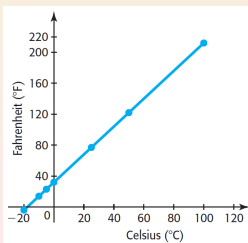
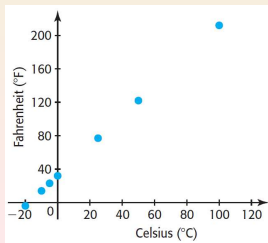
- By listing some Celsius measurements and the corresponding Fahrenheit measurements in the form of **data**:

Celsius Degrees ($^{\circ}\text{C}$)	-20	-10	-5	0	25	50	100
Fahrenheit Degrees ($^{\circ}\text{F}$)	-4	14	23	32	77	122	212

- By giving the set of **ordered pairs**:

$$\{(-20, -4), (-10, 14), (-5, 23), (0, 32), (25, 77), (50, 122), (100, 212)\}$$

- Using a **graph**: We can show a **scatter plot** of the data or a **graph of a function** relating the two degree measurements.



Functions

- All these methods define a **function** with a set of Celsius temperature **inputs** (called the **domain** of the function) and a set of corresponding Fahrenheit **outputs** (called the **range** of the function).
- A function giving the Fahrenheit temperature measurement F that corresponds to any Celsius temperature measurement C between -20°C and 100°C is
$$F = \frac{9}{5}C + 32.$$
- This equation defines F as a function of C because **each input** C results in **exactly one output** F .
- If we consider only Celsius temperatures from -20 to 100 , then the domain is $-20 \leq C \leq 100$ and the resulting range is $-4 \leq F \leq 212$.

A **function** is a rule or correspondence that assigns to **each** element of one set (called the **domain**) **exactly one** element of a second set (called the **range**).

The function may be defined by a set of ordered pairs, a table, a graph, an equation, or a verbal description.

Domains and Ranges

- Consider the function defined by

Celsius Degrees ($^{\circ}\text{C}$)	−20	−10	−5	0	25	50	100
Fahrenheit Degrees ($^{\circ}\text{F}$)	−4	14	23	32	77	122	212

- Its domain is the finite set $\{-20, -10, -5, 0, 25, 50, 100\}$ with all values measured in degrees Celsius;
- Its range is the set $\{-4, 14, 23, 32, 77, 122, 212\}$ with all values measured in degrees Fahrenheit.
- The function defined by $F = \frac{9}{5}C + 32$ has the inputs (domain) and outputs (range) restricted to $-20 \leq C \leq 100$ and $-4 \leq F \leq 212$, respectively.
- If x represents any element in the domain, then x is called the **independent variable**.
- If y represents an output of the function from an input x , then y is called the **dependent variable**. This output is symbolized by $f(x)$.

Specifying Unspecified Domains

- If the domain of a function is not specified or restricted by the application, it is assumed that it consists of **all real number inputs that result in real number outputs in the range**.
- Two important special cases where the domain of a function may be limited:
 1. Functions with **variables in the denominator** may have input values that give 0 in the denominator.

To find values **not in the domain**, set the denominator equal to 0 and solve for the variable.
 2. Functions with **variables inside an even root** may have one or more input values that give negative values inside the even root.

To find values **in the domain**, set the expression inside the even root greater than or equal to 0 and solve for the variable.

Domains and Ranges

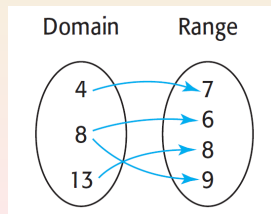
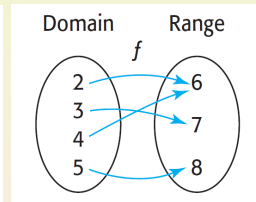
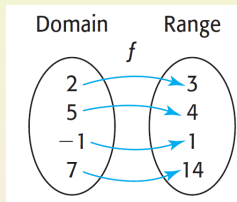
- Determine the domain in (a)-(c) and the range in parts (a) and (c):

$$(a) y = 4x^2 \quad (b) y = 1 + \frac{1}{x-2} \quad (c) y = \sqrt{4-x}$$

- (a) Since no denominators or even-index roots appear, the domain is the set of all real numbers.
Because $y = 4x^2$ cannot be negative for any x , the range is the set of all nonnegative real numbers ($y \geq 0$).
- (b) The denominator will be 0 when $x = 2$. So the domain contains all real numbers **except** 2.
- (c) Because for negative $4 - x$ the root is not defined, x must satisfy $4 - x \geq 0$, or $4 \geq x$, so the domain is $x \leq 4$.
Because $\sqrt{4-x}$ can never be negative, the range is $y \geq 0$.

Arrow Diagrams of Functions

- Functions may also be represented by **arrow diagrams** showing the unique output associated with each individual input.
 - Each arrow in these two figures goes from an input to **exactly one output**: So each of these diagrams defines a function.
 - On the other hand, the last arrow diagram does not define a function because one input, 8, goes to two different outputs, 6 and 9.



Recognizing Functions I

- Determine whether or not the indicated relationship represents a function and, if yes, give the domain and range.

The amount N of U.S. sales of personal computers, in millions of dollars, determined by the year x , as defined in the table.

Year x	1997	2000	2004	2008	2009	2010
U.S. Sales N in \$ M	15,950	16,400	18,233	23,412	21,174	26,060

Is N a function of x ?

For each year (input), only one value is given for the amount of sales (output). So the table represents N as a function of x .

The domain is the set $\{1997, 2000, 2004, 2008, 2009, 2010\}$.

The range is the set

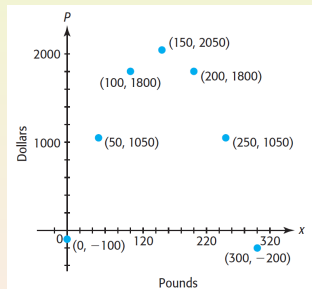
$\{15,950, 16,400, 18,233, 23,412, 21,174, 26,060\}$ million dollars.

Recognizing Functions II

- Determine whether or not the indicated relationship represents a function and, if yes, give the domain and range.

The daily profit P (in dollars) from the sale of x pounds of candy, as shown in the figure.

Is P a function of x ?



Each input x corresponds to only one daily profit P . So this scatter plot represents P as a function of x .

The domain is $\{0, 50, 100, 150, 200, 250, 300\}$ pounds.

The range is $\{-200, -100, 1050, 1800, 2050\}$ dollars.

Recognizing Functions III

- Determine whether or not the indicated relationship represents a function and, if yes, give the domain and range.

The number of tons x of coal sold determined by the profit P that is made from the sale of the product

x tons	0	500	1000	1500	2000	2500	3000
P \$	-100,000	109,000	480,000	505,000	480,000	109,000	-100,000

Is x a function of P ?

The number of tons x of coal sold is not a function of the profit P that is made, because some values of P result in two values of x .

For example, a profit of \$ 480,000 corresponds to both 1000 tons of coal and 2000 tons of coal.

Recognizing Functions IV

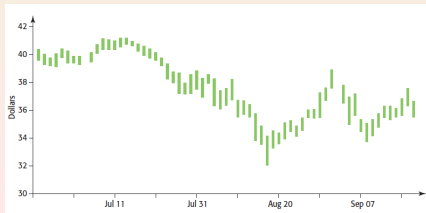
- (a) Does the equation $y^2 = 3x - 3$ define y as a function of x ?

The relation is not a function because there can be more than one output for each input: E.g., both $y = 3$ and $y = -3$ are related to the input $x = 4$.

- (b) Does the equation $y = -x^2 + 4x$ define y as a function of x ?

Each value of x results in exactly one value of y . So this equation defines y as a function of x .

- (c) Does the graph give the price of Home Depot, Inc., stock as a function of the day for three months in 2007?



The vertical bar above each day shows that the stock has many prices between its daily high and low. Thus, the price of a share of stock is not a function of the day.

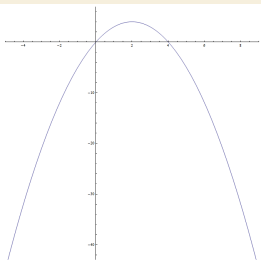
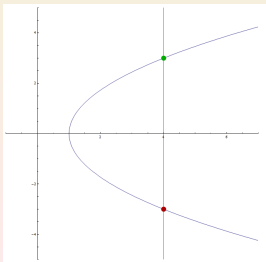
The Vertical Line Test

- To determine whether an equation defines a function we can use the

Vertical Line Test

A set of points in a coordinate plane is the graph of a function if and only if no vertical line intersects the graph in more than one point.

Example: Since $(4, 3)$ and $(4, -3)$ are on the graph of $y^2 = 3x - 3$, the equation does not represent y as a function of x .



For the graph of $y = -x^2 + 4x$, any vertical line will intersect the graph in at most one point. So the equation $y = -x^2 + 4x$ represents y as a function of x .

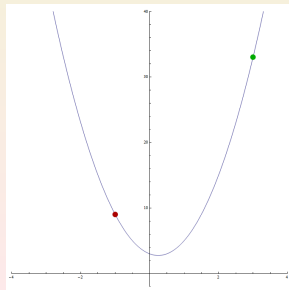
Function Notation

- The **function notation** $y = f(x)$, read “ y equals f of x ” to indicate that the variable y is a function of the variable x .

For specific values of x , $f(x)$ represents the resulting outputs, or y -values:

- The point $(a, f(a))$ lies on the graph of $y = f(x)$, for any number a in the domain of the function.
- We also say that $f(a)$ is $f(x)$ evaluated at $x = a$.

Example: If $f(x) = 4x^2 - 2x + 3$, then
 $f(3) = 4(3)^2 - 2(3) + 3 = 33$ and
 $f(-1) = 4(-1)^2 - 2(-1) + 3 = 9$.
This means that $(3, 33)$ and $(-1, 9)$ are points on the graph of $f(x) = 4x^2 - 2x + 3$.



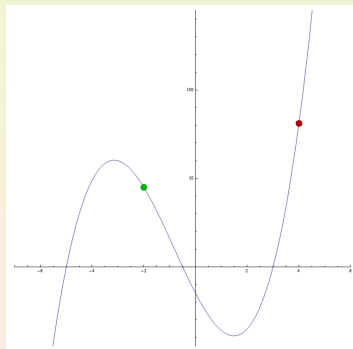
Function Notation: Another Example

- The figure shows the graph of $f(x) = 2x^3 + 5x^2 - 28x - 15$.

- (a) Use the points shown on the graph to find $f(-2)$ and $f(4)$.

$$f(-2) = 45 \text{ and } f(4) = 81.$$

- (b) Use the equation to find $f(-2)$ and $f(4)$.



$$f(-2) = 2(-2)^3 + 5(-2)^2 - 28(-2) - 15 = 45,$$

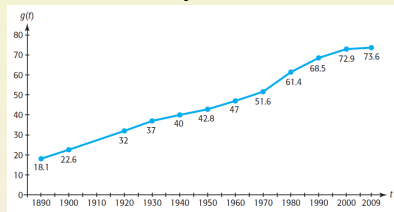
$$f(4) = 2(4)^3 + 5(4)^2 - 28(4) - 15 = 81.$$

Men in the Workforce

- The points on the graph give the number of men in the workforce (in millions) as a function g of the year t for selected years from 1890 to 2009.

(a) Find and interpret $g(1940)$.

$g(1940) = 40$. There were 40 million men in the workforce in 1940.



- (b) What is the input t if the output is $g(t) = 51.6$ million men?
Since $g(1970) = 51.6$, $t = 1970$ is the input.
- (c) What can be said about the number of men in the workforce during 1890-2009?
The number increased during the period 1890-2009.
- (d) What is the maximum number of men in the workforce during the period shown on the graph?
The maximum was 73.6 million, in 2009.

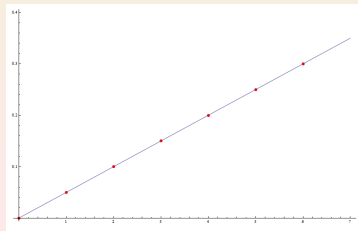
Mathematical Models

- The process of translating real-world information into a mathematical form is called **modeling**.
- A **mathematical model** is a functional relationship (usually in the form of an equation) that includes (a) the **function rule**, (b) **descriptions of all variables** involved and (c) their **units** of measure.
- A mathematical model can sometimes provide an **exact description** of a real situation, but a model frequently provides only an **approximate description** of a real-world situation.

Example: The number of drinks and resulting blood alcohol percent for a 90-pound woman is shown:

Drinks	0	1	2	3	4	5	6
Alc. %	0	0.05	0.10	0.15	0.20	0.25	0.30

The graph of $f(x) = 0.05x$ lies on every data point, i.e., this model fits the data points exactly.



Public Health Care Expenditures

- Public health care expenditures for the period 1990-2012 can be modeled by the function

$$E(t) = 738.1(1.065)^t,$$

where $E(t)$ is in billions of dollars and t is the number of years after 1990.

- (a) What value of t represents 2010?
 $t = 20$.
- (b) Approximate the public health care expenditure for 2010.

$$E(20) = 738.1(1.065)^{20} \approx 2600.8 \text{ billion.}$$

- (c) Use the model to estimate the public health care expenditure for 2015.

$$E(25) = 738.1(1.065)^{25} \approx 3563.3 \text{ billion.}$$

Subsection 3

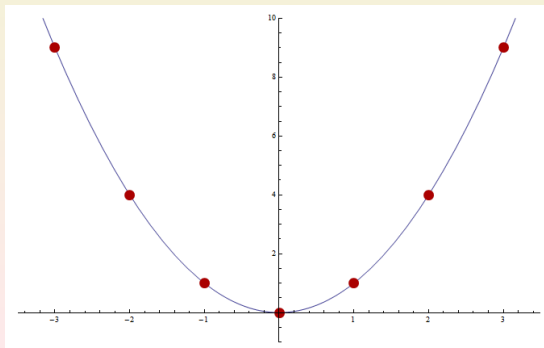
Graphs of Functions

The Point-Plotting Method

- If an equation defines y as a function of x , we can sketch the graph of the function by plotting enough points to determine the shape of the graph and then drawing a line or curve through the points.

Example: Graph the equation $y = x^2$ by drawing a smooth curve through points determined by integer values of x between -3 and 3 .

x	$y = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

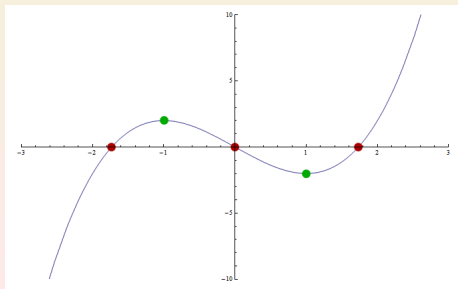


Complete Graph

Complete Graph

A graph is a **complete graph** if it shows the **basic shape** of the graph and **important points** on the graph (including points where the graph **crosses the axes** and points where the graph **turns**) and **suggests what the unseen portions of the graph will be**.

Example: Sketch the complete graph of the equation $f(x) = x^3 - 3x$, using the fact that the graph has at most two turning points.



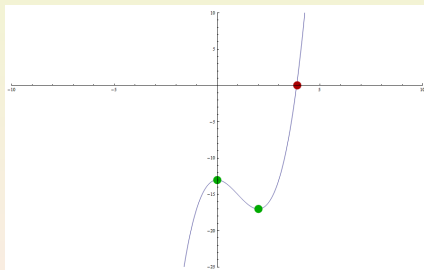
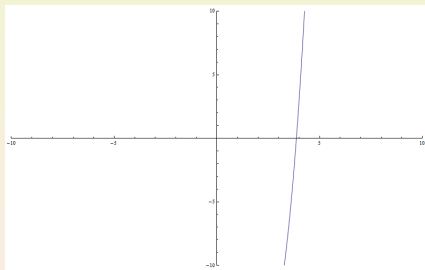
Using a Graphing Calculator to Draw a Graph

Using a Graphing Calculator to Draw a Graph

1. Write the function in the form $y = f(x)$.
 2. Enter the function in the equation editor of the graphing utility.
 3. Activate the graph by pressing the ZOOM or GRAPH key.
 4. To see parts of the graph of a function other than those shown in a standard window, press WINDOW to set the x - and y -boundaries of the viewing window before pressing GRAPH.
- Knowing more about specific classes of functions, helps in determining viewing windows that give complete graphs (including all important points).

Graphing a Complete Graph

- Sketch the graph of $y = x^3 - 3x^2 - 13$ using the standard window.



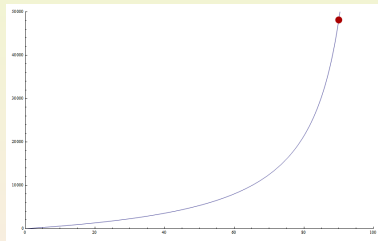
Use instead the window $x_{\min} = -10$, $x_{\max} = 10$, $y_{\min} = -25$, $y_{\max} = 10$.

Cost-Benefit

- Suppose that the cost C of removing $p\%$ of the pollution from drinking water is given by the model $C = \frac{5350p}{100-p}$ dollars.

- (a) Use the restriction on p to determine the limitations on the horizontal-axis values (which are the x -values on a calculator).

Since p is a percentage, we must have $0 \leq p \leq 100$.



- (b) Graph the function on the viewing window $[0, 100]$ by $[0, 50,000]$ and explain why it is reasonable to graph the model on a viewing window with the limitation $C \geq 0$.
- (c) Find the point on the graph that corresponds to $p = 90$ and interpret its coordinates.

Since $C(90) = \frac{5350 \cdot 90}{100 - 90} = \frac{481500}{10} = 48150$, the point is $(90, 48150)$. To remove 90% of the pollution the cost is \$48,150.

Aligning Data

- When finding a model to fit a set of data, it is often easier to use aligned inputs rather than the actual data values.

Aligned inputs are input values that have been converted to smaller numbers by subtracting the same number from each input.

Example: Between 1950 and 2008, the percent of the voting population who voted in presidential elections (during election years) is given by $f(x) = 0.0155x^2 - 1.213x + 75.26$, where x is the number of years after 1950.

- (a) What are the values of x that correspond to the years 1960 and 2008?
They range from 10 to 58.
- (b) Find $f(10)$ and explain its meaning.
We have $f(10) = 64.7$. This is the percent of the voting population in 1960.
- (c) If this model is accurate for 2008, find the percent of the voting population who voted in the 2008 presidential election.
This would be $f(58) = 57.0$.

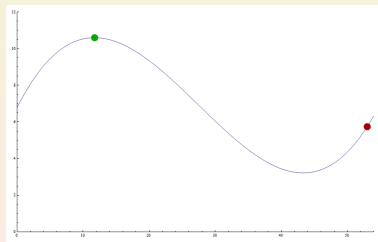
Personal Savings

- Using data from 1960 to 2009, the personal savings rate (as a percent of disposable income) of Americans can be modeled by the function $y = 0.000469x^3 - 0.0387x^2 + 0.714x + 6.787$, where x is the number of years after 1960.

(a) Choose an appropriate window and graph the function with a graphing calculator.

(b) Use the model to estimate the personal savings rate in 2013.

$$y(53) = 5.7\%.$$



(c) Use the graph to estimate the year in which the personal savings rate is a maximum.

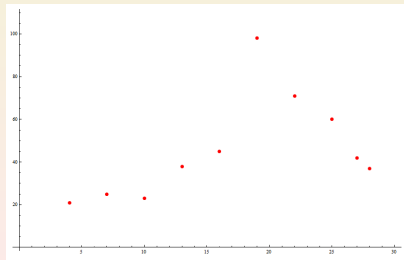
$x \approx 11.72$ and $y \approx 10.6$. Thus, the max percentage is 10.6% and occurs in 1972.

Executions in The United States

- The table gives the number of executions in the United States for selected years from 1984 to 2008.

Year	1984	1987	1990	1993	1996	1999	2002	2005	2007	2008
Number	21	25	23	38	45	98	71	60	42	37

- (a) Align the data so that $x =$ the number of years after 1980, and enter these x -values in list L1 of your graphing calculator. Enter the number of executions in L2.
- (b) Use a graphing command to create the scatter plot of these data points.

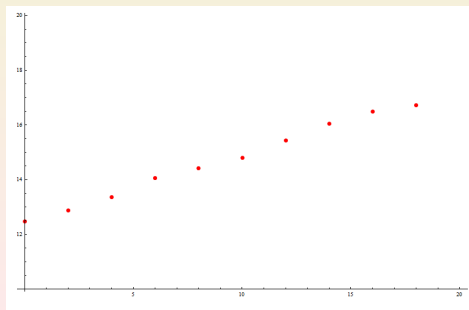


High School Enrollment

- The table shows the enrollment (in thousands) in grades 9-12 at U.S. public and private schools for the years 1990-2008.

Year	1990	1992	1994	1996	1998	2000	2002	2004	2006	2008
Enrol.	12,488	12,882	13,376	14,060	14,428	14,802	15,426	16,048	16,498	16,715

Align the data with x representing the number of years after 1990 and y representing the enrollment in millions and plot.



Medical School

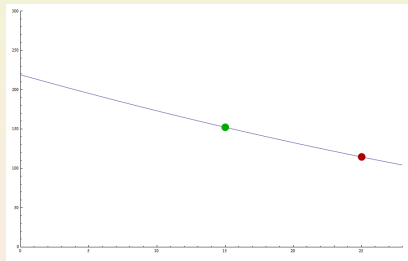
- The number of students (in thousands) of osteopathic medicine in the United States can be described by $S = 0.027t^2 - 4.85t + 218.93$, where t is the number of years after 1980.

- (a) Graph this function on the viewing window $[0, 28]$ by $[0, 300]$.
- (b) Use technology to find S when t is 15.

$$S(15) = 152.255.$$

- (c) Use the model to estimate the number of osteopathic students in 2005.

Compute $S(25) = 0.027 \cdot 25^2 - 4.85 \cdot 25 + 218.93 = 114.555$. Thus, the number of students in 2005 was 114,555.



State Lotteries

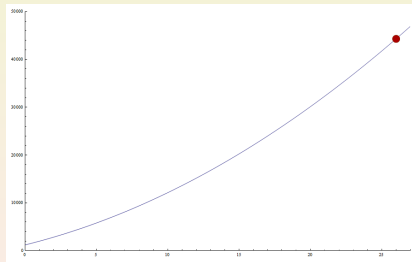
- The cost (in millions of dollars) of prizes and expenses for state lotteries can be described by $L = 35.3t^2 + 740.2t + 1207.2$, where t is the number of years after 1980.

- (a) Graph this function on the viewing window $[0, 27]$ by $[1200, 45,000]$.
- (b) Use technology to find L when t is 26.

$$L(26) = 44,315.$$

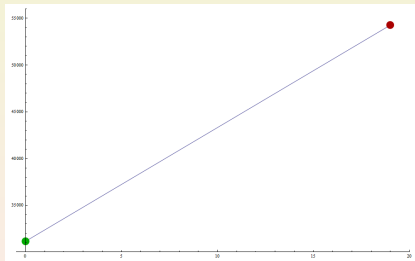
- (c) What was the cost of prizes and expenses for state lotteries in 2006?

Compute $L(26) = 44,315$. Thus, the cost was 44,315,000,000.



Teacher Salaries

- The average salary of a U.S. classroom teacher is given by $f(t) = 1216.879t + 31,148.869$, where t is the number of years from 1990.
 - (a) What inputs correspond to the years 1990-2009?
Inputs 0-19.
 - (b) What outputs correspond to the inputs determined for 1990 and 2009?
31,148.869 to 54,269.57.
 - (c) Use the answers to Parts (a) and (b) and the fact that the function increases to find an appropriate viewing window and graph this function.



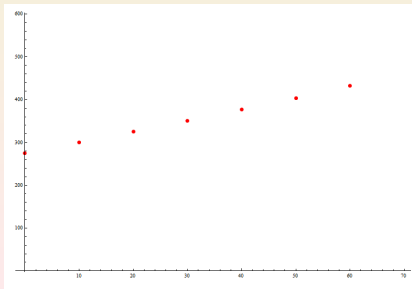
U.S. Population

- The projected population of the United States for selected years from 2000 to 2060 is shown in the table below, with the population given in millions.

Year	2000	2010	2020	2030	2040	2050	2060
Population	275.3	299.9	324.9	351.1	377.4	403.7	432.0

Align the data so that x corresponds to years after 2000 and y is in millions and graph the aligned data as a scatter plot.

Year	Population
0	275.3
10	299.9
20	324.9
30	351.1
40	377.4
50	403.7
60	432.0



Unemployment Rate

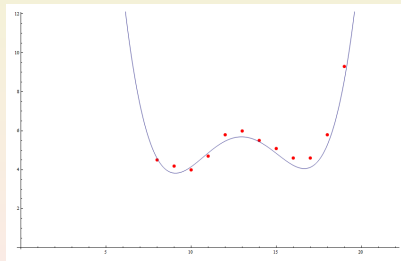
- The U.S. civilian unemployment rate (as a percent) is given by the table.

Year	98	99	00	01	02	03	04	05	06	07	08	09
Unempl.	4.5	4.2	4.0	4.7	5.8	6.0	5.5	5.1	4.6	4.6	5.8	9.3

- (a) According to this table, what was the unemployment rate in 2003?

It was 6%.

- (b) Graph the data from this table as a scatter plot, using the number of years after 1990 as x .



- (c) Graph the equation $y = 0.0085x^4 - 0.4385x^3 + 8.2396x^2 - 66.627x + 199.962$ on the same axes as the scatter plot.

Dropout Rates

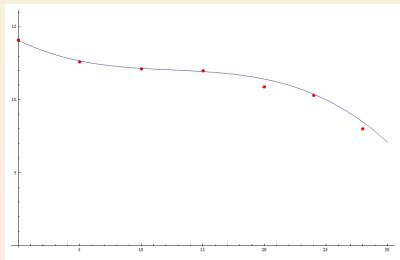
- The table gives the dropout rates (as percents) for students ages 16 to 24 during given years.

Year	1980	1985	1990	1995	2000	2004	2008
Dropout	14.1	12.6	12.1	12.0	10.9	10.3	8.0

- (a) What is the dropout rate in 2004, according to the data?

The dropout rate in 2004 was 10.3%.

- (b) Graph the data from this table as a scatter plot, using the number of years after 1980 as x .



- (c) On the same axes as the scatter plot, graph the equation $y = -0.0008x^3 + 0.030x^2 - 0.413x + 14.081$.

Subsection 4

Linear Functions

Linear Functions

Linear Function

A **linear function** is a function that can be written in the form

$$f(x) = ax + b,$$

where a and b are constants.

Example: Determine whether each equation represents a linear function:

(a) $0 = 2t - s + 1$
 $s = 2t + 1$ is linear.

(b) $y = 5$
This is also linear.

(c) $xy = 2$
Since $y = \frac{2}{x}$, this is not linear.

The Intercepts

- The points where a graph crosses or touches the x -axis and the y -axis are called the **x -intercepts** and **y -intercepts**, respectively, of the graph.

Finding Intercepts Algebraically

- To find the y -intercept of a graph of $y = f(x)$, set $x = 0$ in the equation and solve for y . If the solution is b , we say the y -intercept is b and the graph intersects the y -axis at the point $(0, b)$.
- To find the x -intercept(s) of the graph of $y = f(x)$, set $y = 0$ in the equation and solve for x . If the solution is a , we say the x -intercept is a and the graph intersects the x -axis at the point $(a, 0)$.

Example: Find the x -intercept and the y -intercept of the graph of $2x - 3y = 12$ algebraically.

x -intercept: Set $y = 0$: $2x = 12$, whence $x = 6$.

y -intercept: Set $x = 0$: $-3y = 12$, whence $y = -4$.

Finding Intercepts Graphically

Finding Intercepts Graphically

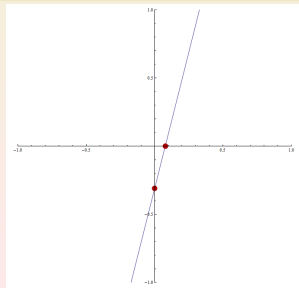
To find the intercept(s) of a graph of $y = f(x)$, first graph the function in a window that shows all intercepts.

- To find the y -intercept, TRACE to $x = 0$ and the y -intercept will be displayed.
- To find the x -intercept(s) of the graph of $y = f(x)$, use the ZERO command under the CALC menu.

Example: Find graphically the intercepts of $52x - 13y = 4$.

We solve for y : $y = \frac{52}{13}x - \frac{4}{13}$ and graph.

Then, follow the directions to obtain: $x \approx 0.077$ and $y \approx -0.308$.



Loan Balance

- A business property is purchased with a promise to pay off a \$60,000 loan plus the \$16,500 interest on this loan by making 60 monthly payments of \$1275. The amount of money, y , remaining to be paid on \$76,500 (the loan plus interest) is reduced by \$1275 each month. Although the amount of money remaining to be paid changes every month, it can be modeled by the linear function

$$y = 76,500 - 1275x,$$

where x is the number of monthly payments made. We recognize that only integer values of x from 0 to 60 apply to this application.

- (a) Find the x -intercept and the y -intercept of the graph of this linear equation.

For the y -intercept, set $x = 0$: $y = 76,500$.

For the x -intercept, set $y = 0$: $x = \frac{76500}{1275} = 60$.

- (b) Interpret the intercepts in the context of this problem situation.

The y -intercept represents the total debt. The x -intercept represents the time when the total debt will be paid off.

Loan Balance (Cont'd)

- The amount of money remaining to be paid can be modeled by the linear function

$$y = 76,500 - 1275x,$$

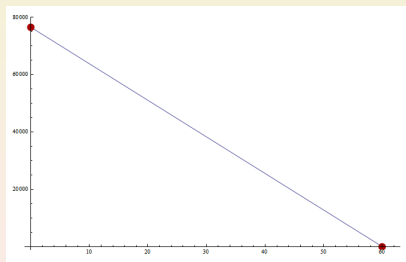
where x is the number of monthly payments made.

- (c) How should x and y be limited in this model so that they make sense in the application?

They make sense if both represent positive quantities.

Thus, $0 \leq x \leq 60$ and
 $0 \leq y \leq 76500$.

- (d) Use the intercepts and the results of Part (c) to sketch the graph of the given equation.



Slope of a Line

Slope of a Line

The **slope** of a line is defined as

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}.$$

If a nonvertical line passes through the two points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, its slope, denoted by m , is found by using

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example: Find the slope of the line ℓ that passes through $(-3, 12)$ and $(2, -13)$.

$$m = \frac{-13-12}{2-(-3)} = \frac{-25}{5} = -5.$$

Example: Find the slope of the line with x -intercept -14 and y -intercept 52 .

The two points are $(-14, 0)$ and $(0, 52)$. Thus, we get

$$m = \frac{52-0}{0-(-14)} = \frac{52}{14} = \frac{26}{7}.$$

Slope and y -Intercept of a Line

Slope and y -Intercept of a Line

The slope of the graph of the equation

$$y = mx + b$$

is m and the y -intercept of the graph is b , so the graph crosses the y -axis at $(0, b)$.

Example: The amount of money y remaining to be paid on the loan of \$60,000 with \$16,500 interest is $y = 76,500 - 1275x$, where x is the number of monthly payments that have been made.

(a) What are the slope and y -intercept of the graph of this function?

The slope is -1275 and the y -intercept is 76500 .

(b) How does the amount owed on the loan change as the number of months increases?

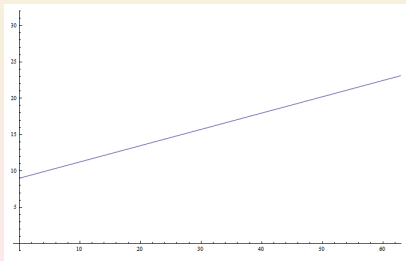
It is reduced by 1275 per month, i.e., its rate of change is equal to the slope of the line.

Constant Rate of Change

Constant Rate of Change

The **rate of change** of the linear function $y = mx + b$ is the constant m , the slope of the graph of the function.

Example: Using data and projections from 1990 through 2050, the percent of Hispanics in the U.S. population can be modeled by $H(x) = 0.224x + 9.01$, with x number of years after 1990.



- (a) What is the slope of the graph of this function?

$$m = 0.224.$$

- (b) What does this slope tell us about the annual rate of change in the percent of Hispanics in the United States?

It is increasing by 0.224 per year.

Revenue, Cost and Profit

- The profit that a company makes on its product is the difference between the revenue and the cost.

If x units are produced and sold, we can write

$$P(x) = R(x) - C(x),$$

- $P(x)$ = profit from sale of x units;
 - $R(x)$ = total revenue from sale of x units;
 - $C(x)$ = total cost of production and sale of x units.
- Revenue is found by using the equation

$$\text{revenue} = (\text{price per unit})(\text{number of units}).$$

- The total cost is the sum of fixed costs and variable costs
$$\text{cost} = \text{variable costs} + \text{fixed costs}.$$
 - Fixed costs (FC), such as depreciation, rent and utilities, remain constant regardless of the number of units produced.
 - Variable costs (VC) are those directly related to the number of units produced.

Cost, Revenue and Profit: An Example

- Suppose that a company manufactures 50-inch 3D plasma TVs and sells them for \$1800 each. The costs incurred in the production and sale of the TVs are \$400,000 plus \$1000 for each TV produced and sold. Write the profit function for the production and sale of x TVs. For the revenue:

$$R(x) = (\text{price per unit})(\text{number of units}) = 1800x.$$

For the cost

$$C(x) = \text{variable costs} + \text{fixed costs} = 1000x + 400000.$$

Thus, for the profit, we have

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1800x - (1000x + 400000) \\ &= 800x - 400000. \end{aligned}$$

Marginal Cost, Revenue, and Profit

- For total cost, total revenue and profit functions that are linear, the rates of change are called **marginal cost**, **marginal revenue**, and **marginal profit**, respectively.

Example: Suppose that the cost to produce and sell a product is

$$C(x) = 54.36x + 6790 \text{ dollars,}$$

where x is the number of units produced and sold.

The rate of change of this cost function, called the **marginal cost**, is \$54.36 per unit produced and sold.

This means that the production and sale of each additional unit will cost an additional \$54.36.

Marginal Revenue and Marginal Profit

- A company produces and sells a BlackBerry smartphone with revenue given by

$$R(x) = 89.50x \text{ dollars}$$

and cost given by

$$C(x) = 54.36x + 6790 \text{ dollars,}$$

where x is the number of BlackBerries produced and sold.

- (a) What is the marginal revenue for this BlackBerry, and what does it mean?

It is \$89.50 and it means that each additional unit increases the revenue by that amount.

- (b) Find the profit function.

$$P(x) = R(x) - C(x) = 35.14x - 6790.$$

- (c) What is the marginal profit for this BlackBerry, and what does it mean?

It is \$35.14 and means that each additional phone sold increases the profit by that amount.

Cigarette Use

- For the years 1965-2009, the percent p of adults who have tried cigarettes can be modeled by

$$p = 43.3 - 0.504t,$$

where t is the number of years after 1960.

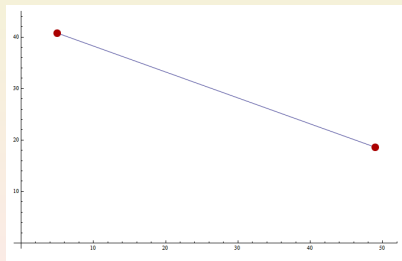
- (a) Is the rate of change of the percent positive or negative?

The rate of change is -0.504 , so it is negative.

- (b) How fast was the percent of adults who tried cigarettes during this period changing?

It is declining by 0.504 percentage points per year.

- (c) Sketch the graph of the function for the window for which it makes sense.



Marijuana Use

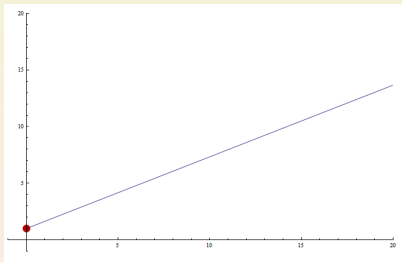
- The percent p of high school seniors using marijuana daily can be modeled by

$$30p - 19x = 30,$$

where x is the number of years after 1990.

- (a) Use this model to determine the slope of the graph of this function if x is the independent variable.

Solve for p : $p(x) = \frac{19}{30}x + 1$.
Thus, the slope is $m = \frac{19}{30}$.



- (b) What is the rate of change of the percent of high school seniors using marijuana per year?
It is increasing by $\frac{19}{30}$ per year.
- (c) Sketch the graph of $p(x)$ for appropriate values of x .

Wireless Service Spending

- The total amount S (in billions of dollars) spent in the United States for wireless communication services can be modeled by the function

$$S = 6.205 + 11.23t,$$

where t is the number of years after 1995.

- (a) Find the slope and the y -intercept of the graph of this equation.
The slope is 11.23 and the y -intercept is 6.205.
- (b) What interpretation could be given to the y -intercept?
The y -intercept is the amount spent in billions in 1995 ($t = 0$).
- (c) What interpretation could be given to the slope?
The slope represents the rate of increase of the amount spent (in billions) per year.

Profit

- A company charting its profits notices that the relationship between the number of units sold x and the profit P is linear. Suppose 300 units sold results in \$4650 profit and 375 units sold results in \$9000 profit.

- (a) Find the marginal profit.

The marginal profit is the slope of the $y = P(x)$ line:

$$m = \frac{9000 - 4650}{375 - 300} = \frac{4350}{75} = 58.$$

- (b) What is the rate of change of the profit?

It is \$58 per unit sold.

Marginal Cost

- Suppose the monthly total cost for the manufacture of 19-inch television sets is

$$C(x) = 2546 + 98x,$$

where x is the number of TVs produced each month.

- (a) What is the slope of the graph of the total cost function?
The slope is 98.
- (b) What is the marginal cost for the product?
It is \$98 per TV.
- (c) Interpret the marginal cost for this product.
It is the increase in cost per additional TV produced and sold.

Subsection 5

Equations of Lines

Slope-Intercept Form of the Equation of a Line

Slope-Intercept Form of the Equation of a Line

The **slope-intercept form** of the equation of a line with slope m and y -intercept b is

$$y = mx + b.$$

In an applied context, m is the rate of change and b is the initial value (when $x = 0$).

Example: Find the equation of the line with slope $\frac{1}{3}$ and y -intercept 3.
 $y = \frac{1}{3}x + 3.$

Example: For interstate calls, AT&T charges 10 cents per minute plus a base charge of \$2.99 each month. Write an equation for the monthly charge y as a function of the number of minutes of use.
 $y = 0.10x + 2.99.$

Appliance Repair

- An appliance repairman charges \$60 for a service call plus \$25 per hour for each hour spent on the repair. Assuming his service call charges can be modeled by a linear function of the number of hours spent on the repair, write the equation of the function.

Let y be the total charges in dollars and x be the number of hours sent on the repair. Then,

$$y = 25x + 60.$$

Magazine Advertising

- The amount spent on magazine advertising was \$15.5 billion in 2009 and is expected to decrease at a rate of \$0.65 billion per year for the next five years.
 - (a) Write an equation for the amount of magazine advertising spending as a function of the number of years after 2009.

$$y = -0.65x + 15.5.$$

- (b) Use the function to estimate the amount that will be spent on magazine advertising in 2015.

$$y(6) = -0.65 \cdot 6 + 15.5 = \$11.6 \text{ billion.}$$

Point-Slope Form of the Equation of a Line

Point-Slope Form of the Equation of a Line

The equation of the line with slope m that passes through a known point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

Example: Write an equation for the line with slope $-\frac{3}{4}$ passing through $(4, -6)$.

$$\begin{aligned}y - (-6) &= -\frac{3}{4}(x - 4) \\ \Rightarrow y + 6 &= -\frac{3}{4}x + 3 \\ \Rightarrow y &= -\frac{3}{4}x - 3.\end{aligned}$$

Blood Alcohol Percent

- The table gives the number of drinks and the resulting blood alcohol percent for a 180 pound man.

(One drink is equal to

- 1.25 oz of 80-proof liquor,
- 12 oz of regular beer, or
- 5 oz of table wine;

many states have set 0.08% as the legal limit for DUI.)

Number of Drinks x	3	4	5	6	7	8	9	10
Blood Alcohol Percent y	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21

- Is the rate of change of the blood alcohol percent for a 180 pound man a constant? What is it?
It is 0.02 percent per drink.
- Write the equation of the function that models the blood alcohol percent as a function of the number of drinks.

$$y - 0.07 = 0.02(x - 3)$$

$$\Rightarrow y = 0.02x + 0.01.$$

Using Two Points to Write an Equation of a Line

- Write the equation of the line that passes through the points $(-1, 5)$ and $(2, 4)$.

First compute the slope

$$m = \frac{4 - 5}{2 - (-1)} = \frac{-1}{3}.$$

Now use the point-slope form: $y - 5 = -\frac{1}{3}(x + 1)$ or $y = -\frac{1}{3}x + \frac{14}{3}$.

- Find the equation of the line passing through $(-1, 3)$ and $(7, 35)$.

First compute the slope

$$m = \frac{35 - 3}{7 - (-1)} = \frac{32}{8} = 4.$$

Now use the point-slope form: $y - 3 = 4(x + 1)$ or $y = 4x + 7$.

Profit

- A company charting its profits notices that the relationship between the number of units sold, x , and the profit, P , is linear.
 - 300 units sold results in \$4650 profit.
 - 375 units sold results in \$9000 profit.

Write the equation that models the company's profit.

The slope is

$$m = \frac{9000 - 4650}{375 - 300} = \frac{4350}{75} = 58.$$

Thus, we get

$$\begin{aligned}y - 4650 &= 58(x - 300) \\ \Rightarrow y &= 58x - 12750.\end{aligned}$$

Earnings and Race

- Data from 2003 for various age groups show that for each \$100 increase in median weekly income for whites, the median weekly income for blacks increases by \$61.90. For these workers, the median weekly income for whites was \$676 and for blacks was \$527. Write the equation that gives the median weekly income for blacks as a function of the median weekly income for whites.

Let y be the median weekly income for blacks and x that of whites.

The slope of y as a function of x , is

$$m = \frac{61.90}{100} = 0.619.$$

Thus, the model is

$$\begin{aligned} y - 527 &= 0.619(x - 676) \\ \Rightarrow y &= 0.619x + 108.556. \end{aligned}$$

Vertical and Horizontal Lines

Vertical and Horizontal Lines

- A vertical line has the form $x = a$, where a is a constant and a is the x -coordinate of any point on the line.
- A horizontal line has the form $y = b$, where b is a constant and b is the y -coordinate of any point on the line.

Example: Find an equation for the line that passes through the points $(5, 2)$ and $(-3, 2)$.

$$y = 2.$$

Example: Find an equation for the line that passes through the points $(9, 2)$ and $(9, 7)$.

$$x = 9.$$

Example: Find an equation for the horizontal and for the vertical line that passes through the points $(-5, 13)$.

$$y = 13 \text{ and } x = -5.$$

Parallel and Perpendicular Lines

Parallel and Perpendicular Lines

- Parallel lines have identical slopes.
- If a line has slope $m \neq 0$, any line perpendicular to it will have slope $-\frac{1}{m}$. That is, the slopes of perpendicular lines are negative reciprocals of each other if neither line is horizontal.

Example: Find an equation for the line passing through $(5, -3)$ that is parallel to the line $2x + y = -3$.

The given line has equation $y = -2x - 3$. Thus, its slope is -2 .

Thus, the parallel line has slope -2 . Therefore, its equation is $y + 3 = -2(x - 5)$ or $y = -2x + 7$.

Example: Find an equation for the line passing through $(-4, 5)$ that is perpendicular to the line $3x + 2y = -8$.

The given line has equation $y = -\frac{3}{2}x - 4$. Thus, its slope is $-\frac{3}{2}$. So, the perpendicular line has slope $\frac{2}{3}$. Therefore, its equation is $y - 5 = \frac{2}{3}(x + 4)$ or $y = \frac{2}{3}x + \frac{23}{3}$.

Average Rate of Change

Average Rate of Change

The average rate of change of $f(x)$ with respect to x over the interval from $x = a$ to $x = b$ (where $a < b$) is calculated as

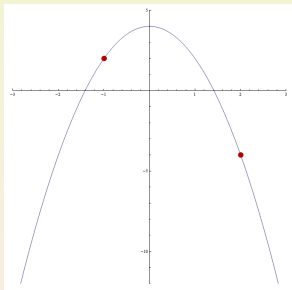
$$\text{average rate of change} = \frac{\text{change in } f(x) \text{ values}}{\text{corresponding change in } x \text{ values}} = \frac{f(b) - f(a)}{b - a}.$$

Example: For $f(x) = x^3$ calculate the average rate of change between $x = -1$ and $x = 2$.

$$\begin{aligned}\text{average rate of change} &= \frac{f(2) - f(-1)}{2 - (-1)} \\ &= \frac{2^3 - (-1)^3}{3} \\ &= \frac{9}{3} = 3.\end{aligned}$$

Calculating an Average Rate of Change

- For the function in the figure, find the average rate of change between $x = -1$ and $x = 2$.



$$\begin{aligned}\text{average rate of change} &= \frac{f(2) - f(-1)}{2 - (-1)} \\ &= \frac{-4 - (-4)}{3} \\ &= \frac{0}{3} \\ &= 0.\end{aligned}$$

High School Enrollment

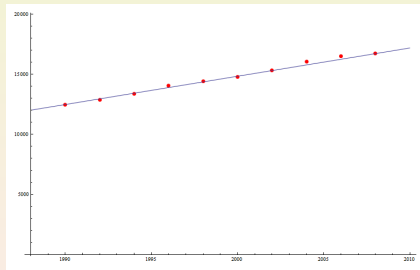
- The table shows the enrollment (in thousands) in grades 9-12 at U.S. public and private schools for the even years 1990-2008.

Year x	1990	1992	1994	1996	1998	2000	2002	2004	2006	2008
Enrollment	12,488	12,882	13,376	14,060	14,428	14,802	15,332	16,048	16,498	16,715

- (a) Create a scatter plot of the data. Does a line fit the data points exactly?

Not exactly, but close.

- (b) Find the average rate of change of the high school enrollment between 1990 and 2008.

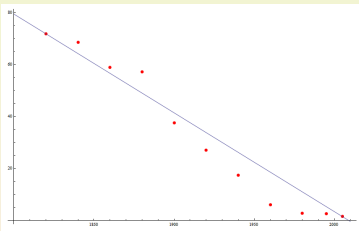
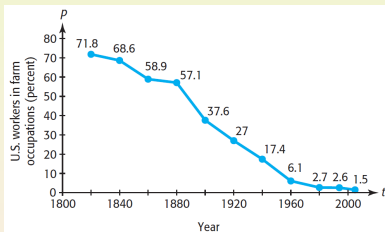


$$\text{average rate of change} = \frac{16715 - 12488}{2008 - 1990} = \frac{4227}{18} \approx 235.$$

- (c) Write the equation of the line determined by this rate of change and one of the two points.
- $$y - 12488 = 235(x - 1990) \text{ or } y = 235x - 455,162.$$

Farm Workers

- The figure below shows the percent p of U.S. workers in farm occupations for selected years t .



- (a) Write the equation of a line connecting the points (1820, 71.8) and (2005, 1.5), with values rounded to two decimal places.

The slope is $m = \frac{1.5 - 71.8}{2005 - 1820} = -0.38$. Thus, the equation is $p - 1.5 = -0.38(t - 2005)$ or $p = -0.38t + 763.4$.

- (b) Does this line appear to be a reasonable fit to the data points?
- (c) Interpret the slope of this line as a rate of change of the percent of farm workers.

The percentage decreases by 0.38 point per year.

Women in the Workforce

- The number of women in the workforce, based on data and projections from 1950 to 2050, can be modeled by a linear function.
 - The number was 18.4 million in 1950.
 - The number is projected to be 81.6 million in 2030.

Let x represent the number of years after 1950.

- (a) What is the slope of the line through $(0, 18.4)$ and $(80, 81.6)$?

$$m = \frac{81.6 - 18.4}{80 - 0} = 0.79.$$

- (b) What is the average rate of change in the number of women in the workforce during this time period?
0.79 million per year.
- (c) Use the slope from Part (a) and the number of millions of women in the workforce in 1950 to write the equation of the line.

$$\begin{aligned}y - 18.4 &= 0.79(x - 0) \\ \Rightarrow y &= 0.79x + 18.4.\end{aligned}$$

Social Agency

- A social agency provides emergency food and shelter to two groups of clients. The first group has x clients who need an average of \$300 for emergencies. The second group has y clients who need an average of \$200 for emergencies. The agency has \$100,000 to spend for these two groups.
 - (a) Write the equation that gives the number of clients who can be served in each group.
$$300x + 200y = 100000.$$
 - (b) Find the y -intercept and the slope of the graph of this equation. Interpret each value.
Since $y = -\frac{3}{2}x + 500$, the y -intercept is 500 and the slope is $-\frac{3}{2}$.
 - The y intercept is the number of clients from the second group that can be served when no clients from the first group are served.
 - The slope is the average decline in clients served from the second group per 1 more client served from the first group.
 - (c) If 10 clients are added from the first group, what happens to the number served in the second group?
Then 15 less clients from the second group can be served.