## College Algebra

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LSSU Math 111

## (1) Linear Models, Equations and Inequalities

- Solution of Linear Equations
- Modeling Linear Functions
- Systems of Linear Equations in Two Variables
- Solutions of Linear Inequalities


## Subsection 1

## Solution of Linear Equations

## Solving a Linear Equation in a Single Variable

## Solving a Linear Equation in a Single Variable

Multiply both sides by a number to clear denominators.
2. Perform multiplications to remove parentheses.
3. Isolate all terms containing the variable on one side.
4. Divide both sides by the coefficient of the variable.

Example: Solve the linear equation $5(x-6)=18-2 x$

$$
\begin{aligned}
& 5(x-6)=18-2 x \quad \Rightarrow \quad 5 x-30=18-2 x \\
& \Rightarrow \quad 7 x=48 \quad \Rightarrow \quad x=\frac{48}{7} \text {. }
\end{aligned}
$$

## Solving Linear Equations

- Solve the linear equation $\frac{5(x-3)}{6}-x=1-\frac{x}{9}$.

$$
\begin{aligned}
& \frac{5(x-3)}{6}-x=1-\frac{x}{9} \\
& \Rightarrow \quad 18\left(\frac{5(x-3)}{6}-x\right)=18\left(1-\frac{x}{9}\right) \\
& \Rightarrow \quad 15(x-3)-18 x=18-2 x \\
& \Rightarrow \quad 15 x-45-18 x=18-2 x \\
& \Rightarrow \quad-3 x-45=18-2 x \\
& \Rightarrow \quad x=-63 .
\end{aligned}
$$

- Solve the linear equation $\frac{3}{4}+\frac{1}{5} x-\frac{1}{3}=\frac{4}{5} x$.

$$
\begin{aligned}
& \frac{3}{4}+\frac{1}{5} x-\frac{1}{3}=\frac{4}{5} x \quad \Rightarrow \quad 60\left(\frac{3}{4}+\frac{1}{5} x-\frac{1}{3}\right)=60 \cdot \frac{4}{5} x \\
& \Rightarrow \quad 45+12 x-20=48 x \quad \Rightarrow \quad 12 x+25=48 x \\
& \Rightarrow \quad 36 x=25 \quad \Rightarrow \quad x=\frac{25}{36} .
\end{aligned}
$$

## Credit Card Debt

- The interest paid on a $\$ 10,000$ debt over 3 years is approximated by

$$
y=175.393 x-116.287 \text { dollars, }
$$

when the interest rate is $x \%$. What is the interest rate if the interest is $\$ 1637.60$ ?

$$
\begin{aligned}
& 1637.60=175.393 x-116.287 \\
& \Rightarrow \quad 175.393 x=1753.887 \\
& \Rightarrow \quad x=\frac{1753.887}{175.393} \approx 10
\end{aligned}
$$

Thus, the interest rate is approximately $10 \%$.

## Stock Market

- For a period of time, a man is very successful speculating on an Internet stock, with its value growing to $\$ 100,000$. However, the stock drops rapidly until its value has been reduced by $40 \%$. What percent increase will have to occur before the latest value returns to $\$ 100,000$ ?
Suppose the increase needed must be $x$ written in decimal.

$$
\begin{aligned}
& (1+x)(1-0.4) 10000=10000 \\
& \Rightarrow \quad 0.6(1+x)=1 \\
& \Rightarrow \quad 0.6 x+0.6=1 \\
& \Rightarrow \quad 0.6 x=0.4 \\
& \Rightarrow \quad x=0 . \overline{6}
\end{aligned}
$$

Thus, a $60.66 \%$ increase will be needed.

## Solutions, Zeros and x-Intercepts

## Zero of a Function

Any number $a$, such that $f(a)=0$ is called a zero of the function $f(x)$. If $a$ is real, then $a$ is an $x$-intercept of the graph of the function.

- The following three concepts are numerically the same:
- The $x$-intercepts of the graph of $y=f(x)$;
- The real zeros of the function $f(x)$;
- The real solutions of the equation $f(x)=0$.


## Solving an Equation Using the $x$-Intercept Method

## Solving an Equation Using the $x$-Intercept Method

Rewrite the equation with one side 0 , i.e., in the form $y=0$.
2. Graph the function $y$ in a window showing all $x$-intercepts.
3. Find the $x$-intercept using the ZERO of your calculator.

Example: Solve the equation

$$
\frac{2 x-3}{4}=\frac{x}{3}+1
$$

We have $\frac{2 x-3}{4}-\frac{x}{3}-1=0$.
We graph and find the $x$-intercept.

Thus, $x=10.5$.


## Solving an Equation Using the $x$-Intercept Method

Solve the equation

$$
\frac{x}{3}-\frac{1}{2}=\frac{x+4}{9}
$$

We make one side 0 :

$$
\frac{x}{3}-\frac{1}{2}-\frac{x+4}{9}=0
$$

We graph $y=\frac{x}{3}-\frac{1}{2}-\frac{x+4}{9}$ and find the $x$-intercept.
Thus, $x=\frac{17}{4}$.

## Solving an Equation Using the Intersection Method

## Solving an Equation Using the Intersection Method

Graph the left-hand side $y_{1}$ of the equation and the right-hand side $y_{2}$ of the equation in a window showing all points of intersection.
3. Find the $x$-coordinates of the points of intersection using the INTERSECT of your calculator.

Example: Solve the equation

$$
4-\frac{x}{6}=\frac{3(x-2)}{4}
$$

We graph $y_{1}=4-\frac{x}{6}$ and $y_{2}=\frac{3(x-2)}{4}$ and find the $x$-coordinate of the intersection point.

Thus, $x=6$.


## Solving an Equation Using the Intersection Method

Solve the equation

$$
3(x-8)=5(x-4)+6
$$

We graph $y_{1}=3(x-8)$ and $y_{2}=5(x-4)+6$ and find the $x$-coordinate of the point of intersection.

Thus, $x=-5$.

## Cell Phone Bills

- The average monthly bill for wireless telephone subscribers from 1985 to 2008 can be modeled by

$$
B(x)=-1.871 x+95.793
$$

where $x$ is the number of years after 1980. If this model remains valid, in what year will the average monthly bill be $\$ 20.95$ ?

$$
\begin{aligned}
& B(x)=20.95 \\
& \Rightarrow \quad-1.871 x+95.793=20.95 \\
& \Rightarrow \quad-1.871 x=-74.843 \\
& \Rightarrow \quad x=\frac{-74.843}{-1.871} \\
& \Rightarrow \quad x \approx 40 .
\end{aligned}
$$



## Profit

- The profit from the production and sale of specialty golf hats is given by the function

$$
P(x)=20 x-4000
$$

where $x$ is the number of hats produced and sold.
(a) Producing and selling how many units will give a profit of $\$ 8000$ ?

$$
\begin{aligned}
& P(x)=8000 \\
& \Rightarrow \quad 20 x-4000=8000 \\
& \Rightarrow \quad 20 x=12000 \\
& \Rightarrow \quad x=600 .
\end{aligned}
$$

(b) How many units must be produced and sold to avoid a loss?
$P \geq 0$, i.e., $20 x-4000 \geq 0$.
Thus, $x \geq 200$.


## Inmates

- The total number of inmates in custody between 1990 and 2005 in state and federal prisons is given approximately by

$$
y=76 x+115 \text { thousand prisoners, }
$$

where $x$ is the number of years after 1990. If the model remains accurate, in what year should the number of inmates be $1,787,000$ ?

$$
\begin{aligned}
& y=1,787 \\
& \Rightarrow \quad 76 x+115=1,787 \\
& \Rightarrow \quad 76 x=1672 \\
& \Rightarrow \quad x=\frac{1672}{76}=22 .
\end{aligned}
$$

Thus, this will happen in 2012.


## Cell Phone Subscribers

- The number of cell phone subscribers (in millions) between 2001 and 2009 can be modeled by

$$
S(x)=21 x+101.7
$$

where $x$ is the number of years after 1995. In what year does this model indicate that there were $311,700,000$ subscribers?

$$
\begin{aligned}
& S(x)=311.7 \\
& \Rightarrow \quad 21 x+101.7=311.7 \\
& \Rightarrow \quad 21 x=210 \\
& \Rightarrow \quad x=10 .
\end{aligned}
$$

Thus, this will happen in 2005.


## Solving for a Variable: Simple Interest

- The formula for the future value of an investment of $P$ dollars at simple interest rate $r$ for $t$ years is

$$
A=P(1+r t)
$$

Solve the formula for $r$, the interest rate.

$$
\begin{aligned}
& A=P(1+r t) \\
& \Rightarrow \quad A=P+P r t \\
& \Rightarrow \quad P r t=A-P \\
& \Rightarrow \quad r=\frac{A-P}{P t} .
\end{aligned}
$$

## Solving an Equation for a Specified Variable

- Solve the equation

$$
2(2 x-b)=\frac{5 c x}{3}
$$

for $x$.

$$
\begin{aligned}
& 2(2 x-b)=\frac{5 c x}{3} \\
& \Rightarrow \quad 6(2 x-b)=5 c x \\
& \Rightarrow \quad 12 x-6 b=5 c x \\
& \Rightarrow \quad 12 x-5 c x=6 b \\
& \Rightarrow \quad x(12-5 c)=6 b \\
& \Rightarrow \quad x=\frac{6 b}{12-5 c} .
\end{aligned}
$$

## Solving for a Variable and Graphing

- Express $y$ as a function of $x$ and graph the resulting equation:

$$
4 x^{2}+2 y=8
$$

$$
\begin{aligned}
& 4 x^{2}+2 y=8 \\
& \Rightarrow \quad 2 y=-4 x^{2}+8 \\
& \Rightarrow \quad y=-2 x^{2}+4 .
\end{aligned}
$$



## Marijuana Use

- The percent $p$ of high school seniors using marijuana daily can be related to $x$, the number of years after 1990, by the equation

$$
30 p-19 x=1
$$

Assuming the model remains accurate, during what year should the percent using marijuana daily equal $12.7 \%$ ?

$$
\begin{aligned}
& 30 p-19 x=1 \\
& \Rightarrow \quad 19 x=30 p-1 \\
& \Rightarrow \quad x=\frac{30 p-1}{19} \\
& \Rightarrow \quad x=\frac{30 \cdot 12.7-1}{19} \\
& \Rightarrow \quad x=20 .
\end{aligned}
$$

Thus, this will happen in 2010.


## Tobacco Judgment

- As part of a damage award in a jury trial, the July 2000 penalty handed down against the tobacco industry included a $\$ 36$ billion judgment against R.J. Reynolds. If this amount was $479 \%$ of this company's 1999 revenue, how much was R.J. Reynolds 1999 revenue?
Suppose R.J. Reynolds 1999 revenue was $\$ x$ billion. Then

$$
\begin{aligned}
& 4.79 \cdot x=36 \\
& x=\frac{36}{4.79} \\
& x \approx 7.516 \text { billions of dollars. }
\end{aligned}
$$

## Direct Variation: Blood Alcohol Percent

## Direct Variation

Let $x$ and $y$ be two variables. We say $y$ is directly proportional to $x$ or that $y$ varies directly with $x$ if $y$ and $x$ are related by

$$
y=k x
$$

where $k$ is called the constant of proportionality or the constant of variation.

Example: The blood alcohol percent of a 130 -pound man is directly proportional to the number of drinks consumed, and 3 drinks give a blood alcohol percent of 0.087 .
(a) Find the constant of proportionality.

Let $P$ be the blood alcohol percent and $N$ the number of drinks. With $k$ the constant of proportionality, we have:

$$
P=k N \quad \Rightarrow \quad 0.087=k \cdot 3 \quad \Rightarrow \quad k=0.029
$$

Thus, the proportionality relation is $P=0.029 \mathrm{~N}$.
(b) Find the blood alcohol percent resulting from 5 drinks.

$$
P=0.029 \cdot 5=0.145 .
$$

## Calories

- The amount of heat produced in the human body by burning protein is directly proportional to the amount of protein burned. If burning 1 gram of protein produces 32 calories of heat, how much protein should be burned to produce 180 calories?
Let $H$ be the amount of heat and $P$ the amount of protein burned. With $k$ the constant of proportionality,

$$
\begin{aligned}
& H=k P \\
& \Rightarrow \quad 32=k \cdot 1 \\
& \Rightarrow \quad k=32 .
\end{aligned}
$$

Thus, the proportionality relation is

$$
H=32 P
$$

The amount of protein to be burned to produce 180 calories is

$$
P=\frac{H}{32}=\frac{180}{32}=5.625 \mathrm{~g} .
$$

## Land Cost

- The cost of land in Savannah is directly proportional to the size of the land. If a 2500 -square foot piece of land costs $\$ 172,800$, what is the cost of a piece of land that is 5500 square feet?
Let $C$ be the cost of land and $A$ the land area. With $k$ the constant of proportionality,

$$
\begin{aligned}
& C=k A \\
& \Rightarrow \quad 172800=k \cdot 2500 \\
& \Rightarrow \quad k=69.12 .
\end{aligned}
$$

Thus, the proportionality relation is

$$
C=69.12 A
$$

The cost of a piece of land that is 5500 square feet is

$$
C=69.12 \cdot 5500=\$ 380,160
$$

## Subsection 2

## Modeling Linear Functions

## Exact and Approximate Linear Models

## Exact and Approximate Linear Models

- If the first differences of outputs are constant for uniform inputs, the rate of change is constant and a linear function fits the data exactly.
- If the first differences are "nearly constant", a linear function can be found that is an approximate fit for the data.
Example: Consider the data

| $x$ | 1 | 3 | 5 | 7 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | 4 | 7 | 10 | 13 | 16 |

(a) Does a linear function fit exactly the data?

| $y$ | 4 |  | 7 |  | 10 |  | 13 |  | 16 |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\Delta y$ |  | 3 |  | 3 |  | 3 |  | 3 |  |

Yes, because the first differences are constant.
(b) Find a linear model that fits the data.

Notice that the slope is $m=\frac{\Delta y}{\Delta x}=\frac{3}{2}$ and the line passes through $(1,4)$. Thus, it has equation $y-4=\frac{3}{2}(x-1)$ or $y=\frac{3}{2} x+\frac{5}{2}$.

## Retirement

- The table gives the annual retirement payment to a 62 -year-old retiree with 21 or more years of service at Clarion State University as a function of the number of years of service.

| Year | 21 | 22 | 23 | 24 | 25 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Retirement Payment | 40,950 | 42,900 | 44,850 | 46,800 | 48,750 |

(a) Does a linear function fit exactly the data?

| 40,950 | 42,900 | 44,850 | 46,800 | 48,750 |
| :---: | :---: | :---: | :---: | :---: |
| 1950 | 1950 | 1950 | 1950 |  |

Thus, a linear model fits the data exactly.
(b) Find a linear model that fits the data.

The slope is $\frac{\Delta y}{\Delta x}=\frac{1950}{1}=1950$. Since the point $(21,40950)$ is on the line, we get

$$
y-40950=1950(x-21) \text { or } y=1950 x
$$

## Health Service Employment

- The table gives the number of full- and part-time employees in thousands in offices and clinics of dentists for selected years between 1990 and 2005.

| Year | 1990 | 1995 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Employees | 513 | 592 | 688 | 705 | 725 | 744 | 760 | 771 |

(a) Draw a scatter plot of the data with the $x$-value of each point representing the number of years after 1990 and the $y$-value representing the number of dental employees (in thousands) corresponding to that year.

(b) Graph the equation $y=16 x+510$ on the same graph as the scatter plot and determine if the line appears to be a good fit. Not the best possible fit.
(c) The line $y=17.830 x+508.732$ is the best fit.

- The table shows some sample incomes and the income tax due for each taxable income.

| Taxable Income | 30,000 | 30,050 | 30,100 | 30,150 | 30,200 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Income Tax Due | 3665.00 | 3672.50 | 3680.00 | 3687.50 | 3695.00 |

(a) Can a linear function model exactly the points from the table?

| 3665.00 | 3672.50 | 3680.00 | 3687.50 |  | 3695.00 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 7.5 | 7.5 | 7.5 | 7.5 |  |  |

(b) If so, find a linear function $T=f(x)$ that models the points. $T-3665=\frac{7.5}{50}(x-30000)$ or $T=0.15 x-835$.
(c) If the model can be interpreted continuously, use it to find the tax due on taxable income of $\$ 30,125$.
$T(30125)=0.15 \cdot 30125-835=3683.75$.

## Steps for Modeling Data

## Modeling Data

Step 1: Enter the data into lists of a graphing utility.
Step 2: Create a scatter plot of the data to see if a linear model is reasonable.

Step 3: Use the graphing utility to obtain the linear equation that is the best fit for the data.
Step 4: Graph the linear function (unrounded) and the data points on the same graph to see how well the function fits the data.
Step 5: Report the function and/or numerical results in a way that makes sense in the context of the problem, with the appropriate units and with the variables identified.

- We usually report functions with coefficients rounded to three decimal places.


## An Example

- Consider the data

$$
\begin{array}{c|ccccccc}
x & 1 & 2 & 3 & 5 & 7 & 9 & 12 \\
\hline y & 1 & 3 & 6 & 1 & 9 & 2 & 6
\end{array}
$$

(a) Create a scatter plot for the data
(b) Does a linear model fit the data exactly?

No line passes through all points.
(c) Find a linear function that best fits the data.
 $y=0.282 x+2.428$.

## Earnings and Gender

- The table shows the earnings in thousands of year-round full-time workers by gender and educational attainment.

| Education | 9th | Some High | High | Some College | Associate | Bachelor | Master | Ph.D. | Professional |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Male | 21.659 | 26.277 | 35.725 | 41.875 | 44.404 | 57.220 | 71.530 | 82.401 | 100.000 |
| Female | 17.659 | 19.162 | 26.029 | 30.816 | 33.481 | 41.681 | 51.316 | 68.875 | 75.036 |

(a) Let $x$ represent earnings for males, let $y$ represent earnings for females, and create a scatter plot of the data.
(b) Create a linear model that expresses female annual earnings as a function of male annual earnings.

$y=0.776 x-1.050$.
(c) Graph the linear function and the data points on the same graph.

## U.S. Population

- The total U.S. population in millions for selected years beginning in 1960 and projected to 2050 is shown

| Year | 1960 | 1965 | 1970 | 1975 | 1980 | 1985 | 1990 | 1995 | 2000 | 2025 | 2050 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Population | 180.671 | 194.303 | 205.052 | 215.973 | 227.726 | 238.466 | 249.948 | 263.044 | 281.422 | 358.030 | 408.695 |

(a) Align the data to represent the number of years after 1960, and draw a scatter plot of the data.
(b) Create the linear equation that is the best fit for these data, where $y$ is in millions and $x$ is the number of years after 1960 .

$y=2.60441 x+177.35$ million.
(c) Graph the equation of the linear model on the same graph with the scatter plot and discuss how well the model fits the data.

## Diabetes

- Projections indicate that the percent of U.S. adults with diabetes could dramatically increase:

| Year | 2010 | 2015 | 2020 | 2025 | 2030 | 2035 | 2040 | 2045 | 2050 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Percent | 15.7 | 18.9 | 21.1 | 24.2 | 27.2 | 29.0 | 31.4 | 32.1 | 34.3 |

(a) Find a linear model that fits the data in the table, with $x=0$ for 2000 .
$y=0.465 x+12.049$.
(b) Use the model to predict the percent of U.S. adults with diabetes in 2018.
$y(18)=$ $0.465 \cdot 18+12.049=20.4 \%$

(c) In what year does this model predict the percent to be $25.05 \%$ ? $0.465 x+12.049=25.05$ implies $0.465 x=13.001$, whence $x \approx 28$, i.e., in 2028.

## U.S. Domestic Trave

- The graph gives the number of millions of persons who took trips of 50 miles or more for the years 2000 through 2008.

(a) Find the equation of the line which is the best fit for these data, with $x$ equal to the number of years after 2000 and $y$ in millions of people.
$y=25.568 x+1328.171$
(b) Use the model to estimate how many took trips in 2010. $y(10)=25.568 \cdot 10+1328.171=1583.9$ million.
(c) In what year does the model estimate the number of people taking trips as 1737.3 million?
$25.568 x+1328.171=1737.3 \Rightarrow 25.568 x=409.129 \Rightarrow x \approx 16$, i.e., in 2016.


## Gross Domestic Product

- The table gives the gross domestic product in billions of dollars of the United States for selected years from 1970 to 2009.

| Year | 1970 | 1980 | 1990 | 2000 | 2005 | 2007 | 2008 | 2009 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| GDP | 1038.3 | 2788.1 | 5800.5 | 9951.5 | $12,638.4$ | $14,061.8$ | $14,369.1$ | $14,119.0$ |

(a) Create a scatter plot of the data, with $y$ representing the GDP in billions of dollars and $x$ representing the number of years after 1970 .
(b) Find the linear function that best fits the data, with $x$ equal to the number of years
 after 1970.

$$
y=365.910 x-213.561
$$

(c) Graph the model with the scatter plot to see if the line is a good fit.

## Smoking

- The table gives the percent of adults aged 18 and over in the United States who reported smoking for selected years.

| Year | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Smoking | 23.5 | 23.2 | 22.7 | 22.4 | 21.6 | 20.9 | 20.9 | 20.8 | 19.7 | 20.5 | 20.6 |

(a) Write the equation that is the best fit for the data, with $x$ equal to the number of years after 1990 .
$y=-0.347 x+26.389$.
(b) What does the model
estimate the percent to be in 2012?


$$
y(22)=-0.347 \cdot 22+26.389=18.75 \%
$$

(c) When will the percent be 16 , according to the model?
$-0.347 x+26.389=16 \Rightarrow-0.347 x=-10.389 \Rightarrow x \approx 30$, i.e., in 2020.

- Using a model to estimate the value of $y$ between two given values of $x$ is called interpolation.
- When a model is evaluated for a prediction using input(s) outside the given data points, the process is called extrapolation.
- The goodness of fit of a line to a set of data points can be observed from the graph of the line and the data on the same set of axes. It can be measured using the correlation coefficient, the number $r$, with $-1 \leq r \leq 1$, that measures the strength of the linear relationship that exists between the two variables:
- The closer $|r|$ is to 1 , the more closely the data points fit the linear regression line.
There is no linear relationship between the two variables if $r=0$.
- Positive values of $r$ indicate that the output variable increases as the input variable increases
Negative values of $r$ indicate that the output variable decreases as the input variable increases.


## Drug Doses

- The table below shows the usual dosage for a certain prescription drug that combats bacterial infections for a person's weight.

| Weight (lbs) | 88 | 99 | 110 | 121 | 132 | 143 | 154 | 165 | 176 | 187 | 198 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Dosage (mg) | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 |

(a) Find a linear function $D=f(W)$ that models the dosage given in the table as a function of the patient's weight.
$D=\frac{5}{11} W$.
(b) How well does the model fit the data?


It is a perfect fit!
(c) What does the model give as the dosage for a 150 pound person? $D(150)=\frac{5}{11} \cdot 150 \approx 68.18 \mathrm{mg}$.

## U.S. Households with Internet Access

- The following table gives the percentage of U.S. households with Internet access in various years.

| Year | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2003 | 2007 | 2008 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Percent | 8.5 | 14.3 | 26.2 | 28.6 | 41.5 | 50.5 | 52.4 | 61.7 | 78.0 |

(a) Create a scatter plot of the data, with $x$ equal to the number of years from 1995.
(b) Create a linear equation that models the data.
$y=5.167 x+9.189$.

(c) Graph the function and the data on the same graph, to see how well the function models (fits) the data.

## Marriage Rate

- The marriage rate per 1000 population for selected years from 1991 to 2009 is shown

| Year | 1991 | 1993 | 1995 | 1997 | 1999 | 2001 | 2003 | 2005 | 2007 | 2009 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Rate | 9.4 | 9.0 | 8.9 | 8.9 | 8.6 | 8.2 | 7.9 | 7.5 | 7.3 | 6.8 |

(a) Create a scatter plot of the data, where $x$ is the number of years after 1990 .
(b) Create a linear function that models the data, with $x$ equal to the number of years after 1990.
$y=-0.138 x+9.635$.

(c) Graph the function and the data on the same axes.
(d) In what year is the marriage rate expected to be 6.5?
$-0.138 x+9.635=6.5 \Rightarrow-0.138 x=-3.135 \Rightarrow x \approx 22.72$, i.e., in 2013.

## Subsection 3

## Systems of Linear Equations in Two Variables

## Solving a System of Linear Equations Graphically

- Solve the system $\left\{\begin{array}{l}2 x-4 y=6 \\ 3 x+5 y=20\end{array}\right\}$ graphically.

First, solve both equations for $y$ :

$$
\left\{\begin{array}{l}
y=\frac{1}{2} x-\frac{3}{2} \\
y=-\frac{3}{5} x+4
\end{array}\right.
$$

Graph both equations and find the coordinates of the point of intersection. Thus, $(x, y)=(5,1)$.


## Solving a System of Linear Equations Graphically

- Solve the system $\left\{\begin{array}{l}5 x-6 y=22 \\ 4 x-4 y=16\end{array}\right\}$ graphically.

First, solve both equations for $y$ :

$$
\left\{\begin{array}{l}
y=\frac{5}{6} x-\frac{11}{3} \\
y=x-4
\end{array}\right.
$$

Graph both equations and find the coordinates of the point of intersection. Thus, $(x, y)=(2,-2)$.


## Break-Even

- A manufacturer of automobile air conditioners has total revenue given by $R=136.50 x$ and total cost given by $C=9661.60+43.60 x$, where $x$ is the number of units produced and sold. Use the graphical method to find the number of units that gives break-even for this product.

Graph both equations and find the coordinates of the point of intersection. Thus, the breakeven point occurs at

$$
x=104 \text { units. }
$$



## Military

- The number of active-duty U.S. Navy personnel (in thousands) is given by

$$
y=-5.686 x+676.173
$$

and the number of active-duty U.S. Air Force personnel is given by

$$
y=-11.997 x+847.529
$$

where $x$ is the number of years after 1960 .
(a) Use graphical methods to find the year in which the number of Navy personnel reached the number of Air Force personnel. At around $x \approx 27$, i.e., in 1987.
(b) How many were in each service when the numbers of personnel were equal?
$\approx 521,787$.


## The Substitution Method

## Solving Systems by Substitution

Solve one of the equations for one of the variables.
2. Substitute the expression into the other equation.
3. Solve the second equation for the remaining variable.
4. Substitute the value into the first equation to find the value of the other variable.

Example: Solve $\left\{\begin{array}{l}3 x+4 y=10 \\ 4 x-2 y=6\end{array}\right\}$.
Solve the second equation for $y$ : $y=2 x-3$.
Substitute into the first equation: $3 x+4(2 x-3)=10$, whence $3 x+8 x-12=10$, giving $11 x=22$ and, therefore, $x=2$.
Substitute into the first equation: $y=2 \cdot 2-3=1$.
Thus $(x, y)=(2,1)$.

## Applying the Substitution Method

- Solve $\left\{\begin{aligned} 2 x-3 y & =2 \\ 5 x-y & =18\end{aligned}\right\}$.

Solve the second equation for $y$ : $y=5 x-18$.
Substitute into the first equation: $2 x-3(5 x-18)=2$, whence $2 x-15 x+54=2$, giving $-13 x=-52$ and, therefore, $x=4$.
Substitute into the first equation: $y=5 \cdot 4-18=2$.
Thus $(x, y)=(4,2)$.

- Solve $\left\{\begin{array}{ll}4 x-5 y & =-17 \\ 3 x+2 y & =-7\end{array}\right\}$.

Solve the second equation for $y$ : $y=-\frac{3}{2} x-\frac{7}{2}$.
Substitute into the first equation: $4 x-5\left(-\frac{3}{2} x-\frac{7}{2}\right)=-17$, whence $4 x+\frac{15}{2} x+\frac{35}{2}=-17$, giving $\frac{23}{2} x=-\frac{69}{2}$ and, therefore, $x=-3$.
Substitute into the first equation: $y=-\frac{3}{2} \cdot(-3)-\frac{7}{2}=1$.
Thus $(x, y)=(-3,1)$.

## Supply and Demand

- A certain product has supply and demand functions given by

$$
p=5 q+20 \quad \text { and } \quad p=128-4 q
$$

respectively.
(a) If the price $p$ is $\$ 60$, how many units $q$ are supplied and how many are demanded?
We substitute $p=60$ and solve each equation for $q$.

$$
\begin{array}{ll}
\text { Supply: } & 60=5 q+20 \Rightarrow 5 q=40 \Rightarrow q=8 \\
\text { Demand: } & 60=128-4 q \Rightarrow 4 q=68 \Rightarrow q=17 .
\end{array}
$$

(b) What price gives market equilibrium, and how many units are demanded and supplied at this price?
We need to solve

$$
5 q+20=128-4 q \quad \Rightarrow \quad 9 q=108 \quad \Rightarrow \quad q=12
$$

Thus the market is at equilibrium when $p=80$ and $q=12$.

## Market Equilibrium

- Wholesalers' willingness to sell laser printers is given by the supply function $p=50.50+0.80 q$ and retailers' willingness to buy the printers is given by $p=400-0.70 q$, where $p$ is the price per printer in dollars and $q$ is the number of printers. What price will give market equilibrium for the printers?
In equilibrium

$$
\begin{aligned}
& 50.50+0.80 q=400-0.70 q \\
& \Rightarrow \quad 1.50 q=349.50 \\
& \Rightarrow \quad q=233 .
\end{aligned}
$$

Thus, the equilibrium price is

$$
p=400-0.70 \cdot 233=\$ 236.90
$$

## Solution by Elimination

## Solving Systems by Elimination

Multiply one or both equations by a nonzero number to make the coefficients of one of the variables of equal value, but of opposite sign.
2. Add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute into one of the original euqations to get the value of the other variable.
Example: Use elimination to solve $\left\{\begin{array}{ll}3 x+4 y & =10 \\ 4 x-2 y & =6\end{array}\right\}$.
Multiply the second equation by $2:\left\{\begin{array}{ll}3 x+4 y & =10 \\ 8 x-4 y & =12\end{array}\right\}$.
Add side by side $11 x=22$, whence $x=2$.
Substituting into $3 x+4 y=10$, we get $6+4 y=10$ or $y=1$. Thus $(x, y)=(2,1)$.

## Solving Systems by Elimination

- Use elimination to solve $\left\{\begin{array}{l}4 x-3 y=-13 \\ 5 x+6 y=13\end{array}\right\}$.

Multiply the first equation by $2:\left\{\begin{array}{ll}8 x-6 y & =-26 \\ 5 x+6 y & =\end{array}\right\}$.
Add side by side $13 x=-13$, whence $x=-1$.
Substituting into $4 x-3 y=-13$, we get $-4-3 y=-13$ or $y=3$.
Thus $(x, y)=(-1,3)$.

- Use elimination to solve $\left\{\begin{array}{l}3 x+3 y=5 \\ 2 x+4 y=8\end{array}\right\}$.

Multiply the first equation by 2 and the second by -3 :
$\left\{\begin{aligned} & 6 x+6 y=10 \\ &-6 x-12 y= \\ &-24\end{aligned}\right\}$.
Add side by side $-6 y=-14$, whence $y=\frac{7}{3}$.
Substituting into $3 x+3 y=5$, we get $3 x+7=5$ or $x=-\frac{2}{3}$.
Thus $(x, y)=\left(-\frac{2}{3}, \frac{7}{3}\right)$.

- The sum of the 2011 revenue and twice the 2008 revenue for Mama Joan's International, Inc., is $\$ 2144.9$ million. The difference between the 2011 and 2008 revenues is $\$ 135.5$ million. If Mama Joan's revenue between 2008 and 2011 is an increasing linear function, find the 2008 and 2011 revenues.
Suppose the 2008 revenue is $x$ and the 2011 revenue is $y$. Then, we have $\left\{\begin{aligned} 2 x+y & =2144.9 \\ -x+y & =135.5\end{aligned}\right\}$.
Multiply the second equation by $2:\left\{\begin{aligned} 2 x+y & =2144.9 \\ -2 x+2 y & =271\end{aligned}\right\}$.
Add side-by-side to eliminate $x: 3 y=2415.9$, whence $y=805.3$. Substituting into $-x+y=135.5$, we get $-x+805.3=135.5$ or $x=669.8$.


## Stock Prices

- The sum of the high and low prices of a share of stock in Johns, Inc., in 2012 is $\$ 83.50$, and the difference between these two prices in 2012 is $\$ 21.88$. Find the high and low prices.
Suppose the high is $h$ and the low is $\ell$.
Then $\left\{\begin{aligned} h+\ell & =83.5 \\ h-\ell & =21.88\end{aligned}\right\}$.
Add side-by-side to get $2 h=105.38$ or $h=52.69$.
Substituting back into $h+\ell=83.5$, we get $52.69+\ell=83.5$ or $\ell=30.81$.
- A concert promoter needs to make $\$ 84,000$ from the sale of 2400 tickets. The promoter charges $\$ 30$ for some tickets and $\$ 45$ for the others.
(a) If there are $x$ of the $\$ 30$ tickets sold and $y$ of the $\$ 45$ tickets sold, write an equation that states that the total number of tickets sold is 2400. $x+y=2400$
(b) Write an equation that states that the total amount received from the sale of $x \$ 30$-tickets and $y \$ 45$-tickets is $\$ 84,000$.
$30 x+45 y=84000$
(c) Solve the equations simultaneously to find how many tickets of each type must be sold to yield the $\$ 84,000$.

$$
\begin{aligned}
& \left\{\begin{array}{l}
x+y=2400 \\
30 x+45 y=84000
\end{array}\right\} \Rightarrow\left\{\begin{array}{l}
y=2400-x \\
30 x+45(2400-x)=84000
\end{array}\right\} \Rightarrow \\
& \left\{\begin{array}{l}
y=2400-x \\
30 x+108000-45 x=84000
\end{array}\right\} \Rightarrow\left\{\begin{array}{l}
y=2400-x \\
15 x=24000
\end{array}\right\} \Rightarrow \\
& \left\{\begin{array}{l}
y=800 \\
x=1600
\end{array}\right\}
\end{aligned}
$$

## Investment

- A woman invests $\$ 52,000$ in two different mutual funds, one that averages $10 \%$ per year and another that averages $14 \%$ per year. If her average annual return on the two mutual funds is $\$ 5720$, how much did she invest in each fund?

Suppose she invested $x$ in the first kind and $y$ in the second. Then

$$
\begin{aligned}
& \left\{\begin{array}{l}
x+y=52000 \\
0.1 x+0.14 y=5720
\end{array}\right\} \Rightarrow \\
& \left\{\begin{array}{l}
x=52000-y \\
0.1(52000-y)+0.14 y=5720
\end{array}\right\} \Rightarrow \\
& \left\{\begin{array}{l}
x=52000-y \\
5200-0.1 y+0.14 y=5720
\end{array}\right\} \Rightarrow\left\{\begin{array}{l}
x=52000-y \\
0.04 y=520
\end{array}\right\} \Rightarrow \\
& \left\{\begin{array}{l}
x=39000 \\
y=13000
\end{array}\right\}
\end{aligned}
$$

## Medication

- A pharmacist wants to mix two solutions to obtain 200 cc of a solution that has a $12 \%$ concentration of a certain medicine. If one solution has a $16 \%$ concentration of the medicine and the second has a $6 \%$ concentration, how much of each solution should she mix?
Suppose she should mix $x$ cc of the first and $y \mathrm{cc}$ of the second solution. Then, $\left\{\begin{array}{l}x+y=200 \\ 0.16 x+0.06 y=0.12 \cdot 200\end{array}\right\} \Rightarrow$
$\left\{\begin{array}{l}y=200-x \\ 0.16 x+0.06(200-x)=24\end{array}\right\} \Rightarrow$
$\left\{\begin{array}{l}y=200-x \\ 0.16 x+12-0.06 x=24\end{array}\right\} \Rightarrow\left\{\begin{array}{l}y=200-x \\ 0.1 x=12\end{array}\right\} \Rightarrow$
$\left\{\begin{array}{l}y=80 \\ x=120\end{array}\right\}$


## Social Agency

- A social agency provides emergency food and shelter to two groups of clients. The first group has $x$ clients who need an average of $\$ 300$ for emergencies, and the second group has $y$ clients who need an average of $\$ 200$ for emergencies. The agency has $\$ 100,000$ to spend for these two groups.
(a) Write an equation that describes the maximum number of clients who can be served with the $\$ 100,000$.

$$
300 x+200 y=100000
$$

(b) If the first group has twice as many clients as the second group, how many clients are in each group if all the money is spent?
We have

$$
\begin{aligned}
& \left\{\begin{array}{l}
x=2 y \\
300 x+200 y=100000
\end{array}\right\} \Rightarrow\left\{\begin{array}{l}
x=2 y \\
300 \cdot 2 y+200 y=100000
\end{array}\right\} \Rightarrow \\
& \left\{\begin{array}{l}
x=2 y \\
800 y=100000
\end{array}\right\} \Rightarrow\left\{\begin{array}{l}
x=250 \\
y=125
\end{array}\right\}
\end{aligned}
$$

## Dependent and Inconsistent Systems

- Solve the system $\left\{\begin{array}{l}3 x+6 y=12 \\ 4 y-8=-2 x\end{array}\right\}$.

Solve the second equation for $x$ : $x=-2 y+4$. Substitute into the first equation: $3(-2 y+4)+6 y=12$, giving $-6 y+12+6 y=12$ or $0=0$. Thus, the system has infinitely many solutions, given by

$$
(x, y)=(-2 y+4, y), y \text { any real number. }
$$

- Solve the system $\left\{\begin{array}{l}4 x-8 y=5 \\ 6 x-12 y=10\end{array}\right\}$.

Solve the first equation for $x: x=2 y+\frac{5}{4}$. Substitute into the second equation: $6\left(2 y+\frac{5}{4}\right)-12 y=10$, giving $12 y+\frac{15}{2}-12 y=10$ or $\frac{15}{2}=10$. Thus, the system is inconsistent, i.e., has no solutions.

## More on Dependent and Inconsistent Systems

- Solve the system $\left\{\begin{array}{l}6 y-12=4 x \\ 10 x-15 y=-30\end{array}\right\}$.

Solve the first equation for $x: x=\frac{3}{2} y-3$. Substitute into the second equation: $10\left(\frac{3}{2} y-3\right)-15 y=-30$, giving $15 y-30-15 y=-30$ or $0=0$. Thus, the system has infinitely many solutions, given by

$$
(x, y)=\left(\frac{3}{2} y-3, y\right), y \text { any real number. }
$$

- Solve the system $\left\{\begin{array}{l}5 x-4 y=8 \\ -15 x+12 y=-12\end{array}\right\}$.

Multiply the first equation by 3 : $\left\{\begin{array}{l}15 x-12 y=24 \\ -15 x+12 y=-12\end{array}\right\}$.
Add side by side to eliminate $x$ : $0=12$.
Thus, the system is inconsistent, i.e., has no solutions.

- Members of an investment club have set a goal of earning $15 \%$ on the money they invest in stocks. They are considering buying two stocks, for which the cost per share and the projected growth per share (both in dollars) are summarized in

|  | Utility | Technology |
| :--- | :---: | :---: |
| Cost/share | $\$ 30$ | $\$ 45$ |
| Growth/share | $\$ 4.50$ | $\$ 6.75$ |

(a) If they have $\$ 180,000$ to invest, how many shares of each stock should they buy to meet their goal?
$\left\{\begin{array}{l}30 x+45 y=180000 \\ 4.5 x+6.75 y=27000\end{array}\right\} \Rightarrow\left\{\begin{array}{l}x=-\frac{3}{2} y+6000 \\ 4.5\left(-\frac{3}{2} y+6000\right)+6.75 y=27000\end{array}\right\}$
$\Rightarrow\left\{\begin{array}{c}x=-\frac{3}{2} y+6000 \\ -6.75 y+27000+6.75 y=27000\end{array}\right\} \Rightarrow\left\{\begin{array}{l}x=-\frac{3}{2} y+6000 \\ 0=0\end{array}\right\}$
Thus, $(x, y)=\left(-\frac{3}{2} y+6000, y\right)$, $y$ any real number will help them achieve the goal.
(b) If they buy 1800 shares of the utility stock, how many shares of the technology stock should they buy to meet their goal?
$-\frac{3}{2} y+6000=1800$ implies $y=2800$.

## Subsection 4

## Solutions of Linear Inequalities

## Algebraic Solution of Linear Inequalities

## Linear Inequalities

A linear inequality in the variable $x$ is an inequality that can be written in the form $a x+b>0$, where $a \neq 0$. The inequality symbol can be $>, \geq,<$ or $\leq$.

## Solving a Linear Inequality

Multiply both sides by a positive number to clear denominators.
2. Multiply to remove parentheses.
3. Isolate on one side all terms containing the variable.
4. Divide by the coefficient of the variable. Be careful to reverse the inequality if the number is negative!

Example: Solve the linear inequality $3 x-\frac{1}{3} \leq-4+x$.

$$
\begin{aligned}
& 3 x-\frac{1}{3} \leq-4+x \quad \Rightarrow \quad 9 x-1 \leq-12+3 x \\
& \Rightarrow \quad 6 x \leq-11 \quad \Rightarrow \quad x \leq-\frac{11}{6}
\end{aligned}
$$

## Solving Linear Inequalities Algebraically

- Solve the following inequalities:
(a) $2(x+3)>4 x+5$

$$
\begin{aligned}
& 2(x+3)>4 x+5 \quad \Rightarrow \quad 2 x+6>4 x+5 \\
& \Rightarrow \quad 1>2 x \quad \Rightarrow \quad x<\frac{1}{2} .
\end{aligned}
$$

Thus, the solution set is $\left(-\infty, \frac{1}{2}\right)$.
(b) $4 x-\frac{1}{2} \leq-2+\frac{x}{3}$

$$
\begin{aligned}
& 4 x-\frac{1}{2} \leq-2+\frac{x}{3} \quad \Rightarrow \quad 6\left(4 x-\frac{1}{2}\right) \leq 6\left(-2+\frac{x}{3}\right) \\
& \Rightarrow \quad 24 x-3 \leq-12+2 x \quad \Rightarrow \quad 22 x \leq-9 \quad \Rightarrow \quad x \leq-\frac{9}{22} .
\end{aligned}
$$

Thus, the solution set is $\left(-\infty,-\frac{9}{22}\right]$.
(c) $\frac{2(x-4)}{3} \geq \frac{3 x}{5}-8$

$$
\begin{aligned}
& \frac{2(x-4)}{3} \geq \frac{3 x}{5}-8 \quad \Rightarrow \quad 15 \frac{2(x-4)}{3} \geq 15\left(\frac{3 x}{5}-8\right) \\
& \Rightarrow \quad 10(x-4) \geq 9 x-120^{3} \Rightarrow 10 x-40 \geq 9 x-120 \\
& \Rightarrow \quad x \geq-80
\end{aligned}
$$

Thus, the solution set is $[-80, \infty)$.

## Job Selection

- Beata Schmidt is given the choice of two positions, one paying €3100 per month and the other paying $€ 2000$ per month plus a $5 \%$ commission on all sales made during the month. What amount must she sell in a month for the second position to be more profitable?
Suppose that the monthly sales are $x$ euros. Then, the second position would offer $2000+0.05 x$ euros per month. Thus, for this position to be more profitable, we must have:

$$
\begin{aligned}
& 2000+0.05 x>3100 \Rightarrow 0.05 x>1100 \\
& \Rightarrow \quad x>\frac{1100}{0.05} \text { or } x>22000 .
\end{aligned}
$$

Thus, she must sell more than $€ 22000$ per month for the second position to be more profitable.

## Stock Market

- Yoko Bono purchased 1000 shares of stock for $\$ 22$ per share, and 3 months later the value had dropped by $20 \%$. What is the minimum percent increase required for her to make a profit?
Assume that the increase percent will be $x$ written in decimal. After the drop her amount was $(1-0.2) \cdot 22000$. Thus, after the increase her amount will be $(1+x) \cdot[(1-0.2) \cdot 22000]$. For her to be able to make a profit, this amount must be more than the original:

$$
\begin{aligned}
& (1+x) \cdot[(1-0.2) \cdot 22000]>22000 \quad \Rightarrow \quad(1+x) \cdot 0.8>1 \\
& \Rightarrow \quad 1+x>\frac{5}{4} \quad \Rightarrow \quad x>0.25 .
\end{aligned}
$$

Thus, more than $25 \%$ increase is required for her to make a profit.

## Body Temperature

- A child's health is at risk when his or her body temperature is $103^{\circ} \mathrm{F}$ or higher. The Fahrenheit temperature is given as a function of the Celsius temperature by $F=\frac{9}{5} C+32$. What Celsius temperature reading would indicate that a child's health was at risk?
The child is at risk when $F \geq 103$. This gives

$$
\begin{aligned}
& \frac{9}{5} C+32 \geq 103 \quad \\
& \Rightarrow \quad C \geq \frac{9}{9} \cdot 71 \quad \Rightarrow \quad C \geq 41
\end{aligned}
$$

The child's health is at risk when the Celsius temperature reading is at least 40.

## Graphical Solution: The Intersection Method

## Solving Linear Inequalities Using Intersection

Set the left side equal to $y_{1}$ and the right to $y_{2}$ and graph $y_{1}$ and $y_{2}$ using a graphing utility.
Adjust the window to contain the point of intersection and find its $x$-coordinate $a$. At $x=a, y_{1}=y_{2}$.
To solve the inequality choose the values of $x$, such that:

- $y_{1}$ is below $y_{2}$, if the inequality is $y_{1}<y_{2}$;
- $y_{1}$ is above $y_{2}$, if the inequality is $y_{1}>y_{2}$.

Example: Solve graphically

$$
7 x+3 \geq 2 x-7
$$

We graph $y=7 x+3$ and $y=2 x-7$.
The point of intersection has $x=-2$.
Since we want the first graph above the second, the solution set is $[-2, \infty)$.


## The Intersection Method

- Solve the linear inequality graphically

$$
2.2 x-2.6 \geq 6-0.8 x
$$

Graph $y_{1}=2.2 x-2.6$ and $y_{2}=6-0.8 x$.


The point of intersection has $x=2.8 \overline{6}$. Since we want $y_{1} \geq y_{2}$, we choose the interval over which $y_{1}$ lies above $y_{2}:[2.8 \overline{6}, \infty)$.

## SAT Scores

- The College Board began reporting SAT scores with a new scale in 1996, with the new scale score $y$ defined as a function of the old scale score $x$ by the equation $y=0.97 x+128.3829$. Suppose a college requires a new scale score greater than or equal to 1000 to admit a student. To determine what old score values would be equivalent to the new scores that would result in admission to this college, do the following:
(a) Write an inequality to represent the problem, and solve it algebraically.
$y \geq 1000 \Rightarrow 0.97 x+128.3829 \geq 1000 \Rightarrow$ $0.97 x \geq 871.6171 \Rightarrow x \geq \frac{871.6171}{0.97} \approx$ 898.5743 .
(b) Solve the inequality from Part (a) graphically to verify your result.



## Graphical Solution: The $x$-Intercept Method

## Solving Linear Inequalities Using $x$-Intercept

Rewrite the inequality with one side equal to 0 , i.e., in one of the forms $f(x)>0, f(x)<0, f(x) \geq 0$ or $f(x) \leq 0$.
Graph $y=f(x)$ and pick a window showing the $x$-intercept $a$. At $x=a, f(x)=0$.
To solve the inequality choose the values of $x$, such that:

- $f(x)$ is below the $x$-axis, if the inequality is $f(x)<0$;
- $f(x)$ is above the $x$-axis, if the inequality is $f(x)>0$.

Example: Solve graphically

$$
-3(x-4)<2(3 x-1)
$$

We rewrite $9 x-14>0$. We graph $y=9 x-14$. The $x$-intercept is $x=\frac{14}{9}$. Since we want $y>0$, the solution set is $\left(\frac{14}{9}, \infty\right)$.


## Apparent Temperature

- When the temperature is $100^{\circ} \mathrm{F}$, the apparent temperature $A$ (or heat index) depends on the humidity $h$ (expressed as a decimal) according to

$$
A=90.2+41.3 h
$$

For what humidity levels is the apparent temperature at least $110^{\circ} \mathrm{F}$ ?
We solve $A \geq 110$ using the $x$-intercept method.
Rewrite $90.2+41.3 h \geq 110$ in the form $41.3 h-19.8 \geq 0$. Graph

$$
y=41.3 h-19.8
$$

The $h$-intercept is $h=0.479$. Since we want $y \geq 0$, we must have humidity at least 47.9\%.


## Car Sales Profit

- A car dealer purchases 12 new cars for $\$ 32,500$ each and sells 11 of them at a profit of $5.5 \%$. For how much must he sell the remaining car to average a profit of at least $6 \%$ on the 12 cars?
Suppose that the remaining car sells for $\$ x$. Then, the total profit of the dealership is

$$
0.055 \cdot 32500 \cdot 11+(x-32500)
$$

On the other hand, an average $6 \%$ profit would give

$$
0.06 \cdot 32500 \cdot 12
$$

Therefore, to attain at least $6 \%$ profit on average, we must have $0.055 \cdot 32500 \cdot 11+(x-32500) \geq 0.06 \cdot 32500 \cdot 12$ or, equivalently, $19662.5+x-32500 \geq 23400$. This yields $x \geq 36,237.50$.

- The yearly profit from the production and sale of Plumber's Helpers is

$$
P(x)=-40,255+9.80 x \text { dollars }
$$

where $x$ is the number of Plumber's Helpers produced and sold. What level of production and sales gives a profit of more than $\$ 84,355$ ? We must solve $P(x) \geq 84,355$.

We solve using the $x$-intercept method. Rewrite $-40,255+9.80 x \geq 84,355$ in the form $9.8 x-124610 \geq 0$. Graph

$$
y=9.8 x-124610
$$

The $x$-intercept is $x=12,715$. Since we want $y \geq 0$, we must sell at least 12,715 units.


## Solving Double Inequalities: The AND Case

- Solve the double inequality

$$
\begin{array}{rlrl}
16 x-8 & >12 & \text { and } & \\
& 16 x-8<32 \\
16 x-8 & >12 & \text { and } & \\
16 x-8<32 \\
16 x>20 & \text { and } & 16 x<40 \\
x>\frac{5}{4} & \text { and } & x<\frac{5}{2} .
\end{array}
$$

Thus, the solution set is $\left(\frac{5}{4}, \frac{5}{2}\right)$.

- Solve the double inequality

$$
\begin{gathered}
\frac{3}{4} x-2 \leq 6-2 x \quad \text { and } \quad \frac{2}{3} x-1 \leq 2 x-2 \\
\frac{3}{4} x-2 \leq 6-2 x \quad \text { and } \quad \frac{2}{3} x-1 \leq 2 x-2 \\
3 x-8 \leq 24-8 x \quad \text { and } 2 x-3 \leq 6 x-6 \\
11 x \leq 32 \text { and } 3 \leq 4 x \\
x \leq \frac{32}{11} \text { and } \frac{3}{4} \leq x
\end{gathered}
$$

Thus, the solution set is $\left[\frac{3}{4}, \frac{32}{11}\right]$.

## Solving Double Inequalities: The OR Case

- Solve the double inequality

$$
\begin{aligned}
& 6 x-2 \leq-5 \\
& \text { or } 3 x+4>9 . \\
& 6 x-2 \leq-5
\end{aligned} \text { or } 3 x+4>9 .
$$

Thus, the solution set is $\left(-\infty,-\frac{1}{2}\right] \cup\left(\frac{5}{3}, \infty\right)$.

- Solve the double inequality

$$
\begin{aligned}
& \frac{1}{2} x-3<5 x \quad \text { or } \quad \frac{2}{5} x-5>6 x \\
& \frac{1}{2} x-3<5 x \text { or } \frac{2}{5} x-5>6 x \\
& x-6<10 x \text { or } 2 x-25>30 x \\
& -6<9 x \text { or }-25>28 x \\
& -\frac{2}{3}<x \text { or }-\frac{25}{28}>x .
\end{aligned}
$$

Thus, the solution set is $\left(-\infty,-\frac{25}{28}\right) \cup\left(-\frac{2}{3}, \infty\right)$.

## Course Grades

- A student has taken four tests and has earned grades of $90 \%, 88 \%$, $93 \%$ and $85 \%$. If all five tests count the same, what grade must the student earn on the final test so that his course average is a B, i.e., at least $80 \%$ and less than $90 \%$ ?

Suppose $x \%$ is the score on his final. Then, the average score is

$$
y=\frac{90+88+93+85+x}{5}=\frac{356+x}{5}
$$

So, to get a $B$ average in the course, we must have

$$
\begin{gathered}
80 \leq \frac{356+x}{5}<90 \\
400 \leq 356+x<450 \\
44 \leq x<94
\end{gathered}
$$

## Expected Prison Sentences

- The mean (expected) time $y$ served in prison for a serious crime can be approximated by a function of the mean sentence length $x$, with

$$
y=0.55 x-2.886
$$

where $x$ and $y$ are measured in months. According to this model, to how many months should a judge sentence a convicted criminal so that the criminal will serve between 37 and 78 months?
We solve $37 \leq y \leq 78$ graphically. Graph $y_{1}=0.55 x-2.886$ and the lines $y_{2}=37$ and $y_{3}=78$. The intersection points have $x$ coordinates $x=72.52$ and $x=$ 147.07. Since we want $y_{2} \leq y_{1} \leq$ $y_{3}$, the sentence must be between
 73 months and 147 months.

## Grades

- If André has a course average score between 70 and 79 , he will earn a grade of $C$ in his algebra course. Suppose that he has three exam scores of 78,62 , and 82 and that his teacher said the final exam score has twice the weight of the other exams. What range of scores on the final exam will result in André earning a grade of C?
Suppose his final exam score is $x / 100$. Since the final exam has twice the weight of the other three, his weighted average will be

$$
\frac{78+62+82+2 x}{5}=\frac{222+2 x}{5}
$$

For him to get a C, this average must fall between 70 and 79:

$$
\begin{aligned}
& 70 \leq \frac{222+2 x}{5} \leq 79 \quad \Rightarrow \quad 350 \leq 222+2 x \leq 395 \\
& \Rightarrow \quad 128 \leq 2 x \leq 173 \quad \Rightarrow \quad 64 \leq x \leq \frac{173}{2} .
\end{aligned}
$$

Thus, André must get a score between 64 and 86.5.

## Marriage Rate

- The marriage rate (marriages per 1000) can be described by

$$
y=-0.146 x+11.074
$$

where $x$ is the number of years after 1980. For what years does this model indicate that the marriage rate was below 8 marriages per 1000 or above 9 marriages per 1000?
We must solve the double inequality

$$
\begin{aligned}
& y<8 \text { or } \\
&-0.146 x+11.074<8 \text { or } \\
&-0.146 x<-3.074 \text { or } \\
& x>21.055 \text { or } \\
& x<146 x+11.074 \gg-205
\end{aligned}
$$

Thus, the marriage rate was below 8 marriages per 1000 or above 9 marriages per 1000 before 1994 or after 2001.

## Home Appraisal

- A home purchased in 1996 for $\$ 190,000$ was appraised at $\$ 270,000$ in 2000. Assuming the rate of increase in the value of the home is constant, do the following:
(a) Write an equation for the value of the home as a function of the number of years, $x$, after 1996.
The rate is the slope of the line:

$$
m=\frac{270000-190000}{4}=20000 .
$$

Since the $y$-intercept $(x=0)$ is 190000 , we get

$$
y=20000 x+190000
$$

(b) Assuming that the equation in Part (a) remained accurate, write an inequality that gives the range of years (until the end of 2010) when the value of the home was greater than $\$ 400,000$.

$$
\begin{aligned}
& y \geq 400000 \quad \Rightarrow \quad 20000 x+190000 \geq 400000 \\
& \Rightarrow \quad 20000 x \geq 210000 \quad \Rightarrow \quad x \geq 10.5
\end{aligned}
$$

Thus, $11 \leq x \leq 14$.

