## College Algebra

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LSSU Math 111

## (1) Quadratic, Piece-Wise Defined and Power Functions

- Quadratic Functions and Parabolas
- Solving Quadratic Equations
- Piece-Wise Defined and Power Functions
- Quadratic and Power Models


## Subsection 1

## Quadratic Functions and Parabolas

## Quadratic Functions and Graphs

- A second-degree function or quadratic function is given by

$$
f(x)=a x^{2}+b x+c, \quad a \neq 0
$$

- The graph is a parabola:

- The vertex is the point $V=\left(-\frac{b}{2 a}, f\left(-\frac{b}{2 a}\right)\right)$.
- The parabola opens up, if $a>0$, and the parabola opens down, if $a<0$.
- The $y$-intercept is $Y=(0, c)$.
- The $x$-intercepts are the solutions of $a x^{2}+b x+c=0$.

They are given by $x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$.

## Graph of a Quadratic Function I

- Consider the function $f(x)=2 x^{2}-8 x+6$. Do a complete study and sketch the graph of $y=f(x)$.

- The vertex is at $x=-\frac{b}{2 a}=-\frac{-8}{2 \cdot 2}=2$. The $y$-coordinate is $f(2)=2 \cdot 2^{2}-8 \cdot 2+6=-2$.
- The parabola opens up, since $a=2>0$.
- The $y$-intercept is $Y=(0,6)$.
- The $x$-intercepts are the solutions of $2 x^{2}-8 x+6=0$.

We solve the equation: $x^{2}-4 x+3=0$, whence $(x-3)(x-1)=0$, and, therefore, $x=1$ or $x=3$.

## Graph of a Quadratic Function II

- Consider the function $f(x)=-x^{2}+6 x-4$. Do a complete study and sketch the graph of $y=f(x)$.

- The vertex is at $x=-\frac{b}{2 a}=-\frac{6}{2 \cdot(-1)}=3$. The $y$-coordinate is

$$
f(2)=-3^{2}+6 \cdot 3-4=5 .
$$

- The parabola opens down, since $a=-1<0$.
- The $y$-intercept is $Y=(0,-4)$.
- The $x$-intercepts are the solutions of $-x^{2}+6 x-4=0$.

We solve the equation: $-x^{2}+6 x-4=0$ :

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}=\frac{-6 \pm \sqrt{36-4 \cdot(-1) \cdot(-4)}}{2 \cdot(-1)}=\frac{-6 \pm \sqrt{20}}{-2}=3 \pm \sqrt{5}
$$

## Maximizing Revenue

- Suppose the monthly revenue from the sale of Carlson 42-inch 3D TVs is given by $R(x)=-0.1 x^{2}+600 x$ dollars, where $x$ is the number of televisions sold.

Find the vertex and the axis of symmetry of the graph of this function.
$x=-\frac{b}{2 a}=-\frac{600}{2 \cdot(-0.1)}=3000$ and $R(3000)=-0.1 \cdot 3000^{2}+$
$600 \cdot 3000=900000$. The axis of symmetry is $x=3000$.


Determine if the vertex represents a maximum or minimum point.

It is a maximum, since $a<0$.
(c) Interpret the vertex in the context of the application.

It gives the maximum revenue.
(d) Graph the function.
(e) For what $x$-values is the function increasing? decreasing? It is increasing for $x<3000$ and decreasing for $x>3000$.

## Foreign-Born Population

- Using data from 1900 through 2008, the percent of the U.S. population that was foreign born can be modeled by the equation

$$
y=0.0034 x^{2}-0.439 x+20.185
$$

where $x$ is the number of years after 1900 .
During what year does the model indicate that the percent of foreign-born population was a minimum?


The minimum is at $x=-$
$\frac{b}{2 a}=-\frac{-0.439}{2 \cdot 0.0034} \approx 64.56$ and $y=0.0034 \cdot 64.56^{2}-$ $0.439 \cdot 64.56+20.185 \approx 6$. Thus, it was $6 \%$ at around 1965.

## Profit

- The daily profit for a product is given by

$$
P=420 x-0.1 x^{2}-4100 \text { dollars, }
$$

where $x$ is the number of units produced and sold.
(a) Graph this function for $x$ between 0 and 4200 .

(b) Is the graph of the function concave up or down? Concave down ( $a=-0.1<0$ ).

## World Population

- A low-projection scenario of world population for the years 1995-2150 by the United Nations is given by the function

$$
y=-0.36 x^{2}+38.52 x+5822.86
$$

where $x$ is the number of years after 1990 and the world population is measured in millions of people.
(a) Graph this function for $x=0$ to $x=120$.

(b) What would the world population have been in 2010 if the projections made using this model had been accurate?
$y(20)=-0.36 \cdot 20^{2}+38.52 \cdot 20+5822.86=6449.26$ million.

## Flight of a Ball

- If a ball is thrown upward at 96 feet per second from the top of a building that is 100 feet high, the height of the ball can be modeled by

$$
S(t)=100+96 t-16 t^{2} \text { feet }
$$

where $t$ is the number of seconds after the ball is thrown.
(a) Describe the graph of the model.

The graph is a parabola that opens down.
(b) Find the $t$-coordinate and S-coordinate of the vertex of the graph of this quadratic function.
The $t$-coordinate is

$$
t=-\frac{b}{2 a}=-\frac{96}{2 \cdot(-16)}=-\frac{96}{-32}=3
$$

The $S$-coordinate is

$$
S(3)=100+96 \cdot 3-16 \cdot 3^{2}=244
$$

(c) Explain the meaning of the coordinates of the vertex for this model. The ball will be at the maximum height $S=244$ feet after 3 seconds.

## The Vertex Form of a Quadratic Function

- The vertex form of a quadratic function is

$$
f(x)=a(x-h)^{2}+k
$$

and it represents a parabola with vertex at $(h, k)$.

- It opens up if $a>0$ and down if $a<0$.
- It has $y$-intercept $\left(0, a h^{2}+k\right)$.
- It has $x$-intercepts $x=h \pm \sqrt{-\frac{k}{a}}$. Example: Find the coordinates of the vertex and graph the function

$$
f(x)=(x+10)^{2}-6
$$

The vertex is at $(-10,-6)$.


## Graphing Using the Vertex Form

- Find the coordinates of the vertex and graph the function

$$
f(x)=-(x-12)^{2}+50
$$

The vertex is at $(12,50)$.


## Obtaining the Vertex Form from the Graph I

- What is the quadratic function whose graph is shown below?


Since the vertex is at $(2,-4)$, we must have the form $f(x)=a(x-2)^{2}-4$.
Since the point $(4,0)$ is on the graph, we must have

$$
\begin{aligned}
& f(4)=0 \Rightarrow a(4-2)^{2}-4=0 \\
& \Rightarrow 4 a-4=0 \Rightarrow a=1 .
\end{aligned}
$$

Thus, the equation is

$$
f(x)=(x-2)^{2}-4
$$

## Obtaining the Vertex Form from the Graph II

- What is the quadratic function whose graph is shown below?


Since the vertex is at $(3,5)$, we must have the form $f(x)=a(x-3)^{2}+5$.
Since the point $(1,1)$ is on the graph, we must have

$$
\begin{aligned}
& f(1)=1 \Rightarrow a(1-3)^{2}+5=1 \\
& \Rightarrow 4 a+5=1 \Rightarrow a=-1 .
\end{aligned}
$$

Thus, the equation is

$$
f(x)=-(x-3)^{2}+5
$$

## Minimizing Cost

- The cost for producing $x$ Champions golf hats is given by the function

$$
C(x)=0.2(x-40)^{2}+200 \text { dollars }
$$

(a) Find the vertex of this function.

The vertex is $V=(40,200)$.
(b) Is the vertex a maximum or minimum? Interpret the vertex in the context of the application.
It is a minimum since $a>1$. Thus, the minimum cost is $\$ 200$ when 40 hats are produced.
(c) Graph the function using a window that includes the vertex.


## Profit

- Right Sports Management had its monthly maximum profit, $\$ 450,000$, when it produced and sold 5500 Waist Trimmers. Its fixed cost is $\$ 155,000$. If the profit can be modeled by a quadratic function of $x$, the number of Waist Trimmers produced and sold each month, find this quadratic function $P(x)$.
Since the max is $P=450000$ when $x=5500$, we must have an equation of the form

$$
P(x)=a(x-5500)^{2}+450000
$$

Since the fixed cost is 155000 , we get

$$
\begin{aligned}
& P(0)=-155000 \quad \Rightarrow \quad a \cdot 5500^{2}+450000=-155000 \\
& \Rightarrow \quad 30250000 a=605000 \quad \Rightarrow \quad a=-0.02 .
\end{aligned}
$$

Hence, we obtain

$$
P(x)=-0.02(x-5500)^{2}+450000
$$

## Drug Sensitivity

- The sensitivity $S$ to a drug is related to the dosage size $x$ by $S=1000 x-x^{2}$.
(a) Sketch the graph of this model using a domain and range with nonnegative $x$ and $S$.
(b) Is the function increasing or decreasing for x between 0 and 500 ?

It is increasing.

(c) What is the positive $x$-intercept of the graph?

It is $x=1000$.
(d) Why is this $x$-intercept important in the context of this application? It is the dosage value for which sensitivity is 0 .

## Apartment Rental

- The owner of an apartment building can rent all 100 apartments if he charges $\$ 1200$ per apartment per month. The number of apartments rented is reduced by 2 for every $\$ 40$ increase in the monthly rent.
(a) Construct a table that gives the revenue if the rent charged is $\$ 1240$, $\$ 1280$, and $\$ 1320$.

| Rent | 1240 | 1280 | 1320 |
| ---: | :---: | :---: | :---: |
| Revenue | $98 \cdot 1240=121,520$ | $96 \cdot 1280=122,880$ | $94 \cdot 1320=124,080$ |

(b) Does $R(x)=(1200+40 x)(100-2 x)$ model the revenue from these apartments if $x$ represents the number of $\$ 40$ increases?
Yes!
(c) What monthly rent gives the maximum revenue for the apartments? Since

$$
R(x)=-80 x^{2}+1600 x+120000
$$

the maximum occurs at $x=-\frac{b}{2 a}=-\frac{1600}{2 \cdot(-80)}=10$, i.e., when the rent is $1200+40 \cdot 10=\$ 1,600$.

## Rink Rental

- The owner of a skating rink rents the rink for parties at $\$ 720$ if 60 or fewer skaters attend, so the cost is $\$ 12$ per person if 60 attend. For each 6 skaters above 60 , she reduces the price per skater by $\$ 0.50$.
(a) Construct a table that gives the revenue if the number attending is 66 , 72 , and 78.

| Attendees | 66 | 72 | 78 |
| ---: | :---: | :---: | :---: |
| Revenue | $66 \cdot 11.5=759$ | $72 \cdot 11=792$ | $78 \cdot 10.5=819$ |

(b) Does the function

$$
R(x)=(60+6 x)(12-0.5 x)
$$

model the revenue from the party if $x$ represents the number of increases of 6 people each?
Yes!
(c) How many people should attend for the rink's revenue to be a maximum?
Since $\quad R(x)=-3 x^{2}+42 x+720$,
the maximum occurs at $x=-\frac{b}{2 a}=-\frac{42}{2 \cdot(-3)}=7$, i.e., when the party should consist of $60+6 \cdot 7=102$ people.

## Subsection 2

## Solving Quadratic Equations

## Quadratic Equations and Zero Factor Property

- An equation that can be written in the form

$$
a x^{2}+b x+c=0, a \neq 0
$$

is called a quadratic equation.

## Zero Product Property

If the product of two real numbers is 0 , then at least one of them must be 0 . I.e., if for real numbers $a, b$, the product $a b=0$, then $a=0$ or $b=0$.

Example: Solve the equation $x^{2}+3 x-10=0$.

$$
\begin{aligned}
& x^{2}+3 x-10=0 \quad \Rightarrow \quad(x+5)(x-2)=0 \\
& \Rightarrow \quad x+5=0 \text { or } x-2=0 \quad \Rightarrow \quad x=-5 \text { or } x=2
\end{aligned}
$$

Example: Solve the equation $6 x^{2}-13 x+6=0$.

$$
\begin{aligned}
& 6 x^{2}-13 x+6=0 \quad \Rightarrow \quad(2 x-3)(3 x-2)=0 \\
& \Rightarrow \quad 2 x-3=0 \text { or } 3 x-2=0 \quad \Rightarrow \quad x=\frac{3}{2} \text { or } x=\frac{2}{3} .
\end{aligned}
$$

## Falling Object

- A tennis ball is thrown into a swimming pool from the top of a tall hotel. The height of the ball above the pool is modeled by

$$
D(t)=-16 t^{2}-4 t+200 \text { feet }
$$

where $t$ is the time, in seconds, after the ball is thrown. How long after the ball is thrown is it 44 feet above the pool?
We must find $t$ so that $D(t)=44$.

$$
\begin{aligned}
& D(t)=44 \quad \Rightarrow \quad-16 t^{2}-4 t+200=44 \\
& \Rightarrow \quad 16 t^{2}+4 t-156=0 \quad \Rightarrow \quad 4 t^{2}+t-39=0 \\
& \Rightarrow \quad(t-3)(4 t+13)=0 \quad \Rightarrow \quad t-3=0 \text { or } 4 t+13=0 \\
& \Rightarrow \quad t=3 \text { or } t=-\frac{13}{4} .
\end{aligned}
$$

Thus, it will take 3 seconds for the ball to reach a height of 44 feet above the pool.

## Using a Graphical Method

- To solve $f(x)=a x^{2}+b x+c=0$ using the graph, recall that the following three statements are equivalent:
- $a$ is a real solution to the equation $f(x)=0$.
- $a$ is a zero of the function $f(x)$.
- $a$ is an $x$-intercept of the graph $y=f(x)$.

Example: Use a graphing utility to find the roots of $2 x^{2}+8 x-10=0$


$$
\begin{aligned}
& \text { We graph } y=2 x^{2}+8 x-10 \\
& \text { We find } x=-5 \text { or } x=1
\end{aligned}
$$

## Break-Even for Coffee Business

- The profit for Coffee Exchange coffee beans is given by

$$
P(x)=-15 x^{2}+180 x-405 \text { thousand dollars, }
$$

where $x$ is the number of tons of coffee beans produced and sold. How many tons give breakeven (that is, give zero profit) for this product?
We need to solve the equation $P(x)=0$.

$$
\begin{aligned}
& P(x)=0 \quad \Rightarrow \quad-15 x^{2}+180 x-405=0 \\
& \Rightarrow \quad x^{2}-12 x+27=0 \quad \Rightarrow \quad(x-3)(x-9)=0 \\
& \Rightarrow \quad x-3=0 \text { or } x-9=0 \quad \Rightarrow \quad x=3 \text { or } x=9
\end{aligned}
$$

Thus, the company needs to produce and sell 3 or 9 tons of coffee to break even.

## Break-Even for Home Theaters

- The total revenue function for a home theater system is given by $R=266 x$, and the total cost function is $C=2000+46 x+2 x^{2}$, where $R$ and $C$ are each measured in dollars and $x$ is the number of units produced and sold.
(a) Form the profit function for this product from the two given functions.

$$
\begin{aligned}
P(x) & =R(x)-C(x) \\
& =266 x-\left(2000+46 x+2 x^{2}\right) \\
& =-2000+220 x-2 x^{2}
\end{aligned}
$$

(b) What is the profit when 55 systems are produced and sold?

$$
P(55)=-2000+220 \cdot 55-2 \cdot 55^{2}=\$ 4,050 .
$$

(c) How many systems must be sold to break even on this product?

$$
\begin{aligned}
& P(x)=0 \quad \Rightarrow \quad-2000+220 x-2 x^{2}=0 \\
& \Rightarrow \quad x^{2}-110 x+1000=0 \quad \Rightarrow \quad(x-10)(x-100)=0 \\
& \Rightarrow \quad x-10=0 \text { or } x-100=0 \quad \Rightarrow \quad x=10 \text { or } x=100 .
\end{aligned}
$$

Thus, the company needs to sell 10 or 100 systems.

## Market Equilibrium

- The demand for diamond-studded watches is given by $p=7000-2 x$ dollars, and the supply of watches is given by $p=0.01 x^{2}+2 x+1000$ dollars, where $x$ is the number of watches demanded and supplied when the price per watch is $p$ dollars. Find the equilibrium quantity and the equilibrium price.
We must have identical prices for supply and demand at equilibrium:

$$
\begin{aligned}
& 7000-2 x=0.01 x^{2}+2 x+1000 \\
& \Rightarrow \quad 0.01 x^{2}+4 x-6000=0 \\
& \Rightarrow \quad x^{2}+400 x-600000=0 \\
& \Rightarrow \quad(x+1000)(x-600)=0 \\
& \Rightarrow \quad x+1000=0 \text { or } x-600=0 \\
& \Rightarrow \quad x=-1000 \text { or } x=600
\end{aligned}
$$

Thus, the quantity is $x=600$ watches and the price must be $p=7000-2 \cdot 600=\$ 5,800$.

## The Factor Theorem

## Factor Theorem

The polynomial function $f$ has a factor $(x-a)$ if and only if $f(a)=0$, i.e., $(x-a)$ is a factor of $f(x)$ if and only if $x=a$ is a solution to $f(x)=0$.

- If one solution can be found exactly from the graph, it can be used to find one of the factors of the function. By finding the second factor, we get the second solution.
Example: Solve $3 x^{2}-4 x-4=0$ using the following steps:
(a) Graphically find one of the intercepts

$$
\text { of } y=3 x^{2}-4 x-4
$$

Graph $y=3 x^{2}-4 x-4$. We get
$x=2$.
(b) Verify that this is a zero.

$$
y(2)=3 \cdot 2^{2}-4 \cdot 2-4=0 .
$$

(c) Factor to find the remaining solution.

$$
3 x^{2}-4 x-4=0 \Rightarrow(x-2)(3 x+2)=0 \Rightarrow x=2 \text { or } x=-\frac{2}{3} \text {. }
$$

## Another Application of the Factor Theorem

- Solve $3 x^{2}-130 x=-1000$ using the following steps:
(a) Graphically find one of the intercepts of $y=3 x^{2}-130 x+1000$.

Graph $y=3 x^{2}-130 x+1000$. We get $x=10$.
(b) Verify that this is a zero.

$$
y(10)=3 \cdot 10^{2}-130 \cdot 10+1000=0 .
$$

(c) Factor to find the remaining solution.


$$
\begin{aligned}
& 3 x^{2}-130 x+1000=0 \quad \Rightarrow \quad(x-10)(3 x-100)=0 \\
& \Rightarrow \quad x-10=0 \text { or } 3 x-100=0 \\
& \Rightarrow \quad x=10 \text { or } x=\frac{100}{3} .
\end{aligned}
$$

## U.S. Energy Consumption

- Energy consumption in the United States in quadrillion BTUs can be modeled by

$$
C(x)=-0.013 x^{2}+1.281 x+67.147
$$

where $x$ is the number of years after 1970 .
(a) One solution to the equation $87.567=-0.013 x^{2}+1.281 x+67.147$ is $x=20$. What does this mean?
In 1990 the energy consumption was 87.567 quadrillion BTUs.
(b) Graphically verify that $x=20$ is a solution to $87.567=$
$-0.013 x^{2}+1.281 x+67.147$.
(c) To find when after 2020 U.S. energy consumption will be 87.567 quadrillion BTUs according to the model, do we need to find the second
 solution to this equation? Why or why not?
Yes, this happens for $x=78.5$, i.e., during 2049.

## High School Smokers

- The percent of high school students who smoked cigarettes on 1 or more of the 30 days preceding the Youth Risk Behavior Survey can be modeled by $\quad y=-0.061 x^{2}+0.275 x+33.698$, where $x$ is the number of years after 1990 .
(a) How would you set the viewing window to show a graph of the model for the years 1990 to 2015? Graph the function.

(b) Graphically estimate this model's prediction for when the percent of high school students who smoke would be 12 .
This happens for $x \approx 21.25$, i.e., in year 2012.


## The Square Root Method

## Square Root Method

The solutions of the quadratic equation $x^{2}=C$ are $x= \pm \sqrt{C}$. Note that, when we take the square root of both sides, we use a $\pm$ symbol because there are both positive and negative values that, when squared, give $C$.

Example: Solve the equation $2 x^{2}-16=0$.

$$
\begin{gathered}
2 x^{2}-16=0 \quad \Rightarrow \quad 2 x^{2}=16 \quad \Rightarrow \quad x^{2}=8 \\
\Rightarrow \quad x= \pm \sqrt{8} \quad \Rightarrow \quad x=-2 \sqrt{2} \text { or } x=2 \sqrt{2} .
\end{gathered}
$$

Example: Solve the equation $(x-6)^{2}=18$.

$$
\begin{aligned}
& (x-6)^{2}=18 \quad \Rightarrow \quad x-6= \pm \sqrt{18} \\
& \Rightarrow \quad x-6= \pm 3 \sqrt{2} \\
& \Rightarrow \quad x=6-3 \sqrt{2} \text { or } x=6+3 \sqrt{2}
\end{aligned}
$$

## More on the Square Root Method

- Solve the following equations:
(a) $x^{2}-20=0$.

$$
\begin{aligned}
& x^{2}-20=0 \quad \Rightarrow \quad x^{2}=20 \quad \Rightarrow \quad x= \pm \sqrt{20} \\
& \Rightarrow \quad x= \pm 2 \sqrt{5} \quad \Rightarrow \quad x=-2 \sqrt{5} \text { or } x=2 \sqrt{5} .
\end{aligned}
$$

(b) $5 x^{2}-25=0$.

$$
\begin{gathered}
5 x^{2}-25=0 \Rightarrow 5 x^{2}=25 \quad \Rightarrow \quad x^{2}=5 \\
\Rightarrow \quad x= \pm \sqrt{5} \Rightarrow x=-\sqrt{5} \text { or } x=\sqrt{5} .
\end{gathered}
$$

## Completing the Square

## Completing the Square

Completing the square entails the following transformation:

$$
x^{2}+b x=x^{2}+b x+\left(\frac{b}{2}\right)^{2}-\left(\frac{b}{2}\right)^{2}=\left(x+\frac{b}{2}\right)^{2}-\frac{b^{2}}{4}
$$

Example: Solve $x^{2}-12 x+7=0$ by completing the square.

$$
\begin{aligned}
& x^{2}-12 x+7=0 \quad \Rightarrow \quad x^{2}-12 x+36-36+7=0 \\
& \Rightarrow \quad(x-6)^{2}-36+7=0 \quad \Rightarrow \quad(x-6)^{2}=29 \\
& \Rightarrow \quad x-6= \pm \sqrt{29} \quad \Rightarrow \quad x=6 \pm \sqrt{29} \\
& \Rightarrow \quad x=6-\sqrt{29} \text { or } x=6+\sqrt{29} .
\end{aligned}
$$

## More Completions of the Square

- Complete the square to solve the following quadratic equations:
(a) $x^{2}-6 x+1=0$

$$
\begin{aligned}
& x^{2}-6 x+1=0 \quad \Rightarrow \quad x^{2}-6 x+9-9+1=0 \\
& \Rightarrow \quad(x-3)^{2}-9+1=0 \quad \Rightarrow \quad(x-3)^{2}=8 \\
& \Rightarrow \quad x-3= \pm \sqrt{8} \Rightarrow \quad \Rightarrow \quad x=3 \pm 2 \sqrt{2} \\
& \Rightarrow \quad x=3-2 \sqrt{2} \text { or } x=3+2 \sqrt{2}
\end{aligned}
$$

(b) $2 x^{2}-9 x+8=0$

$$
\begin{aligned}
& 2 x^{2}-9 x+8=0 \Rightarrow x^{2}-\frac{9}{2} x+4=0 \\
& \Rightarrow \quad x^{2}-\frac{9}{2} x+\frac{81}{16}-\frac{81}{16}+4=0 \\
& \Rightarrow \quad\left(x-\frac{9}{4}\right)^{2}-\frac{81}{16}+4=0 \\
& \Rightarrow \quad\left(x-\frac{9}{4}\right)^{2}=\frac{17}{16} \Rightarrow x-\frac{9}{4}= \pm \frac{\sqrt{17}}{4} \\
& \Rightarrow \quad x=\frac{9}{4} \pm \frac{\sqrt{17}}{4} \Rightarrow x=\frac{9-\sqrt{17}}{4} \text { or } x=\frac{9+\sqrt{17}}{4} .
\end{aligned}
$$

## Discriminant and Quadratic Formula

- Consider the equation

$$
a x^{2}+b x+c=0, \quad a \neq 0
$$

The quantity

$$
D=b^{2}-4 a c
$$

is called the discriminant of $a x^{2}+b x+c$. It is used to discriminate the following cases:

- If $D>0$, then the equation has two different real solutions.
- If $D=0$, then the equation has one real solution.
- If $D<0$, then the equation has no real solutions.
- In the first two cases, the quadratic formula gives the solutions:

$$
x=\frac{-b \pm \sqrt{D}}{2 a}=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} .
$$

## Using the Quadratic Formula

- Use the quadratic formula to solve the following equations:
(a) $3 x^{2}+5 x-8=0$

Compute the discriminant $D=b^{2}-4 a c=5^{2}-4 \cdot 3 \cdot(-8)=$ $25+96=121$. We are expecting two real roots:

$$
x=\frac{-b \pm \sqrt{D}}{2 a}=\frac{-5 \pm \sqrt{121}}{2 \cdot 3}=\frac{-5 \pm 11}{6}=\left\{\begin{array}{l}
-\frac{8}{3} \\
1
\end{array}\right.
$$

(b) $3 x^{2}-6 x-12=0$

Simplify $x^{2}-2 x-4=0$. Now, compute the discriminant
$D=b^{2}-4 a c=(-2)^{2}-4 \cdot 1 \cdot(-4)=4+16=20$. We are expecting two real roots:

$$
x=\frac{-b \pm \sqrt{D}}{2 a}=\frac{2 \pm \sqrt{20}}{2 \cdot 1}=\frac{2 \pm 2 \sqrt{5}}{2}=1 \pm \sqrt{5}
$$

(c) $3 x^{2}-30 x-180=0$

Simplify $x^{2}-10 x-60=0$. Now, compute the discriminant
$D=b^{2}-4 a c=(-10)^{2}-4 \cdot 1 \cdot(-60)=100+240=340$. We are expecting two real roots:

$$
x=\frac{-b \pm \sqrt{D}}{2 a}=\frac{10 \pm \sqrt{340}}{2 \cdot 1}=\frac{10 \pm 2 \sqrt{85}}{2}=5 \pm \sqrt{85}
$$

## Complex Roots

## The Imaginary Unit

The imaginary unit $i$ is defined as a square root of -1 :

$$
i=\sqrt{-1} \quad \text { or } \quad i^{2}=-1
$$

Therefore, the equation $x^{2}=a$ has solutions $x= \begin{cases} \pm \sqrt{a}, & \text { if } a \geq 0 \\ \pm i \sqrt{-a}, & \text { if } a<0\end{cases}$
Example: Solve the equation $x^{2}+9=0$.

$$
\begin{aligned}
& x^{2}+9=0 \Rightarrow \quad x^{2}=-9 \quad \Rightarrow \quad x= \pm \sqrt{-9} \\
& \Rightarrow \quad x= \pm i \sqrt{9} \quad \Rightarrow \quad x=-3 i \text { or } x=3 i .
\end{aligned}
$$

Example: Solve the equation $3 x^{2}+24=0$.

$$
\begin{aligned}
& 3 x^{2}+24=0 \quad \Rightarrow \quad 3 x^{2}=-24 \\
& \Rightarrow \quad x^{2}=-8 \quad \Rightarrow \quad x= \pm \sqrt{-8} \quad \Rightarrow \quad x= \pm i \sqrt{8} \\
& \Rightarrow \quad x= \pm i 2 \sqrt{2} \quad \Rightarrow \quad x=-2 i \sqrt{2} \text { or } x=2 i \sqrt{2}
\end{aligned}
$$

## More on Complex Roots

- Solve the following equations
(a) $x^{2}-3 x+5=0$

Compute the discriminant
$D=b^{2}-4 a c=(-3)^{2}-4 \cdot 1 \cdot 5=9-20=-11$. We are expecting
two complex roots:

$$
x=\frac{-b \pm \sqrt{D}}{2 a}=\frac{3 \pm \sqrt{-11}}{2 \cdot 1}=\frac{3 \pm i \sqrt{11}}{2} .
$$

(b) $3 x^{2}+4 x+3=0$.

Compute the discriminant
$D=b^{2}-4 a c=4^{2}-4 \cdot 3 \cdot 3=16-36=-20$. We are expecting two complex roots:

$$
x=\frac{-b \pm \sqrt{D}}{2 a}=\frac{-4 \pm \sqrt{-20}}{2 \cdot 3}=\frac{-4 \pm 2 i \sqrt{5}}{6}=\frac{-2 \pm i \sqrt{5}}{3} .
$$

## Gold Prices

- The price of an ounce of gold in U.S. dollars for the years 1997-2011 can be modeled by the function

$$
G(x)=11.532 x^{2}-259.978 x+1666.555
$$

where $x$ is the number of years after 1990 .
(a) Graph this function for values of $x$ representing 1997-2011.


(b) According to the model, what will the price of gold be in 2020? $G(30)=11.532 \cdot 30^{2}-259.978 \cdot 30+1666.555 \approx \$ 4246.02$.
(c) Use graphical or numerical methods to estimate when the price of gold will reach $\$ 2702.80$ per ounce. $x \approx 26$, i.e., around 2016.

## Quitting Smoking

- The percent of people over 19 years of age who have ever smoked and quit is given by the equation

$$
y=0.010 x^{2}+0.344 x+28.74
$$

where $x$ is the number of years since 1960 .
(a) Graph this function for values of $x$ representing 1960-2010.


(b) Assuming that the pattern indicated by this model continues through 2015, what would be the percent in 2015? $y(55)=0.010 \cdot 55^{2}+0.344 \cdot 55+28.74 \approx 77.91 \%$.
(c) When does this model indicate that the percent reached $40 \%$ ? $x \approx 21$, i.e., at around 1981.

## Cell Phones

- Using the CTIA Wireless Survey for 1985-2009, the number of U.S. cell phone subscribers in millions can be modeled by

$$
y=0.632 x^{2}-2.651 x+1.209
$$

where $x$ is the number of years after 1985 .
(a) Graphically find when the number of U.S. subscribers was $301,617,000$.


$$
x \approx 24 \text {, i.e., around } 2009
$$

(b) When does the model estimate that the number of
U.S. subscribers would reach 359,515,000?
$x \approx 26$, i.e., around 2011.

## Subsection 3

## Piece-Wise Defined and Power Functions

## Graphing a Piece-Wise Defined Function

- Consider the piece-wise defined function $f(x)= \begin{cases}x^{2}-1, & \text { if } x \leq 0 \\ x^{3}+2, & \text { if } x>0\end{cases}$
(a) Compute the values $f(-1)$ and $f(2)$.

$$
\begin{aligned}
f(-1) & =(-1)^{2}-1=1-1=0 \\
f(2) & =2^{3}+2=8+2=10
\end{aligned}
$$

(b) Construct the graph of $y=f(x)$.




## Another Graph of a Piece-Wise Defined Function

- Consider the piece-wise defined function $f(x)= \begin{cases}3 x+1, & \text { if } x<3 \\ x^{2}, & \text { if } x \geq 3\end{cases}$
(a) Compute the values $f(-1)$ and $f(4)$.

$$
\begin{aligned}
& f(-1)=3(-1)+1 \\
& =-3+1=-2 \\
& f(4)=4^{2}=16 .
\end{aligned}
$$

(b) Construct the graph of $y=f(x)$.


## Electric Charges

- For the nonextreme weather months, Palmetto Electric charges $\$ 7.10$ plus 6.747 cents per kilowatt-hour (kWh) for the first 1200 kWh and $\$ 88.06$ plus 5.788 cents for all kilowatt-hours above 1200.
(a) Write the function that gives the monthly charge in dollars as a function of the kilowatt-hours used.

$$
\begin{aligned}
C(x) & = \begin{cases}7.10+0.06747, & \text { if } x \leq 1200 \\
88.06+0.05788(x-1200), & \text { if } x>1200\end{cases} \\
& = \begin{cases}7.10+0.06747, & \text { if } x \leq 1200 \\
18.604+0.05788 x, & \text { if } x>1200\end{cases}
\end{aligned}
$$

(b) What is the monthly charge if 960 kWh are used?

$$
C(960)=7.10+0.06747 \cdot 960=\$ 71.87 .
$$

(c) What is the monthly charge if 1580 kWh are used?
$C(1580)=18.604+0.05788 \cdot 1580=\$ 110.05$.

## First-Class Postage

- The postage charged for first-class mail is a function of its weight. The U.S. Postal Service uses the following table to describe the rates for 2011:

| Weight Increment $x(\mathrm{oz})$ | First Class Postage $P(x)$ |
| :--- | :---: |
| First ounce or fraction of an ounce | $44 \Phi$ |
| Each additional ounce or fraction | $20 \Phi$ |

(a) Find a function that represents first-class postage for letters weighing up to 4 ounces, using $x$ as the weight in ounces and $P$ as the postage in cents.

$$
P(x)= \begin{cases}44, & \text { if } 0<x \leq 1 \\ 64, & \text { if } 1<x \leq 2 \\ 84, & \text { if } 2<x \leq 3 \\ 104, & \text { if } 3<x \leq 4\end{cases}
$$

(b) Find $P(1.2)$ and explain what it means.
$P(1.2)=64$. The postage for a 1.2 ounce letter is 64 q .
(c) Find $P(2)$ and $P(2.01)$.
$P(2)=64$ and $P(2.01)=84$.
(d) Find the postage for a 2 ounce letter and for a 2.01 ounce letter. The postage is $64 \Phi$ for a 2 -ounce and $84 \Phi$ for a 2.01 ounce letter.

## The Wind Chill

- The formula that gives the wind chill factor for a $60^{\circ} \mathrm{F}$ temperature and a wind with velocity $V$ in miles per hour is

$$
W= \begin{cases}60, & \text { if } 0 \leq V<4 \\ 0.644 V-9.518 \sqrt{V}+76.495, & \text { if } 4 \leq V \leq 55.9 \\ 41, & \text { if } V>55.9\end{cases}
$$

(a) Find the wind chill factor for the $60^{\circ}$ temperature if the wind is 20 mph . $W(20)=$
$0.644 \cdot 0-9.518 \sqrt{20}+76.495 \approx 47^{\circ}$.
(b) Find the wind chill factor for the $60^{\circ}$ temperature if the wind is 65 mph . $W(65)=41^{\circ}$.
(b) Graph the function for $0 \leq V \leq 80$.
 What are the domain and the range?
The domain is $0 \leq V \leq 80$ and the range $\{41\} \cup[41.332,60.035]$.

## Absolute Value

- The absolute value $|x|$ of a number $x$ is the distance of $x$ from 0 on the real line and is formally defined by

$$
|x|=\left\{\begin{array}{rl}
x, & \text { if } x \geq 0 \\
-x, & \text { if } x<0
\end{array} .\right.
$$

- The absolute value function is defined by

$$
f(x)=|x|=\left\{\begin{array}{rl}
x, & \text { if } x \geq 0 \\
-x, & \text { if } x<0
\end{array} .\right.
$$

The graph is constructed as for any other piece-wise defined function:


## Absolute Value Equations

## Absolute Value Equation

- If $|x|=a$ and $a>0$, then $x=a$ or $x=-a$.
- If $|x|=0$, then $x=0$.
- $|x|=a$, with $a<0$, has no solutions for $x$.

Example: Solve the absolute value equations:
(a) $|x-3|=9$.

We must have $x-3=-9$ or $x-3=9$. Thus, we get $x=-6$ or
$x=12$.
(b) $|2 x-4|=8$

We must have $2 x-4=-8$ or $2 x-4=8$. Thus, $2 x=-4$ or $2 x=12$.
Hence, $x=-2$ or $x=6$.
(c) $\left|x^{2}-5 x\right|=6$.

We must have $x^{2}-5 x=-6$ or $x^{2}-5 x=6$. These give $x^{2}-5 x+6=0$ or $x^{2}-5 x-6=0$. The first is equivalent to
$(x-3)(x-2)=0$ and the second to $(x+1)(x-6)=0$. Thus, we get the solutions $x=3$ or $x=2$ or $x=-1$ or $x=6$.

## Additional Absolute Value Equations

- Solve the following absolute value equations:
(a) $|2 x-7|=15$

$$
\begin{aligned}
& |2 x-7|=15 \quad \Rightarrow \quad 2 x-7=-15 \text { or } 2 x-7=15 \\
& \Rightarrow \quad 2 x=-8 \text { or } 2 x=22 \quad \Rightarrow \quad x=-4 \text { or } x=11 .
\end{aligned}
$$

(b) $|x|=x^{2}+4 x$.

$$
\begin{aligned}
& x=-\left(x^{2}+4 x\right) \text { or } x=x^{2}+4 x \\
& \Rightarrow \quad-x^{2}-5 x=0 \text { or } x^{2}+3 x=0 \\
& \Rightarrow \quad-x(x+5)=0 \text { or } x(x+3)=0 \\
& \Rightarrow \quad x=0 \text { or } x=-5 \text { or } x=-3 .
\end{aligned}
$$

Only $x=0$ and $x=-5$ are admissible!
(c) $|x-5|=x^{2}-5 x$.

$$
\begin{aligned}
& x-5=-\left(x^{2}-5 x\right) \text { or } x-5=x^{2}-5 x \\
& \Rightarrow \quad x^{2}-4 x-5=0 \text { or } x^{2}-6 x+5=0 \\
& \Rightarrow \quad(x-5)(x+1)=0 \text { or }(x-5)(x-1)=0 \\
& \Rightarrow \quad x=5 \text { or } x=-1 \text { or } x=1
\end{aligned}
$$

Only $x=5$ and $x=-1$ are admissible.

## Power Functions

## Power Functions

A power function is a function of the form $y=a x^{b}$, where $a$ and $b$ are real numbers, with $b \neq 0$.

Examples: We sketch $y=x^{2}, y=x^{3}, y=x^{2 / 3}$ and $y=x^{3 / 2}$ :



## Female Physicians

- The number of female physicians can be modeled by

$$
F(x)=0.623 x^{1.552}
$$

where $x$ is the number of years after 1960 and $F(x)$ is the number of female physicians in thousands.
(a) What type of function is this?

It is a power function since it has the form $y=a x^{b}$.
(b) What is $F(35)$ ? What does this mean?
$F(35)=0.623 \cdot 35^{1.552} \approx 155.196$. There were 155,196 female physicians in 1995.
(c) How many female physicians will there be in 2020, according to the model?
We must compute $F(60): F(60)=0.623 \cdot 60^{1.552} \approx 358.244$. Thus, there will be 358,244 female physicians in 2020.

## Taxi Miles

- The Inner City Taxi Company estimated that the number of taxi miles driven each day can be modeled by the function $Q=489 L^{0.6}$, when they employ $L$ drivers per day.
(a) Graph this function for $0 \leq L \leq 35$.

(b) How many taxi miles are driven each day if there are 32 drivers employed?
$Q(32)=489 \cdot 32^{0.6}=3912$.
(c) Does this model indicate that the number of taxi miles increases or decreases as the number of drivers increases?
The number of miles increases as the number of drivers increases.


## Personal Expenditures

- Personal consumption expenditures for durable goods in the United States, in billions of dollars, can be modeled by the function $P(x)=306.472 x^{0.464}$, where $x$ is the number of years after 1990 .
(a) Is this function increasing or decreasing for the years 1990-2010?

It is increasing.
(b) Is this function concave up or concave down during this period
 of time?

It is concave down.
(c) Use numerical or graphical methods to find when the model predicts that personal consumption expenditures will reach $\$ 1532.35$ billion. This occurs when $x \approx 32$, i.e., around 2022.

## Root Functions

## Root Functions

A root function is a function of the form $y=a x^{1 / n}$, or $y=a \sqrt[n]{x}$, where $n$ is an integer $n \geq 2$.

Example: We sketch the graphs of $y=\sqrt{x}$ and $y=\sqrt[3]{x}$.



## Production Output

- The monthly output of a product (in units) is given by

$$
P=1200 x^{5 / 2},
$$

where $x$ is the capital investment in thousands of dollars.
(a) Graph this function for $x$ from 0 to 10 and $P$ from 0 to 200,000.

(b) Is the graph concave up or concave down?

Concave up.

## Harvesting

- A farmer's main cash crop is tomatoes, and the tomato harvest begins in the month of May. The number of bushels of tomatoes harvested on the $x$-th day of May is given by the equation

$$
B(x)=6(x+1)^{3 / 2}
$$

How many bushels did the farmer harvest on May 8?

$B(8)=6(8+1)^{3 / 2}=6 \cdot 27=127$.

## Reciprocal Function

## Reciprocal Function

The function $f(x)=x^{-1}=\frac{1}{x}$ is called the reciprocal function.


- It has domain $\mathbb{R}-\{0\}$;
- It has range $\mathbb{R}-\{0\}$;
- It has vertical asymptote $x=0$;
- It has horizontal asymptote $y=0$.


## Direct Variation

## Direct Variation as the $n$-th Power

A quality $y$ varies directly as the $n$-th power of $x$ if there is a constant $k$, such that

$$
y=k x^{n}
$$

The constant $k$ is called the constant of variation or the constant of proportionality.

Example: The number of units of raw material Y required varies as the cube of the number of units of a raw material $X$ that is required to produce a certain item. Suppose 500 units of $Y$ and 5 units of $X$ are required to produce 100 units of the item. How many units of $Y$ are required if the number of units produced requires 10 units of $X$ ?
If $x$ is the number of units of $X$ required and $y$ is the number of units of $Y$ required, then $y=k x^{3}$. Because $y=500$ when $x=5$, we have $500=k \cdot 5^{3}$, or $k=4$. Then $y=4 x^{3}$ and $y(10)=4\left(10^{3}\right)=4000$.

## Inverse Variation

A quantity $y$ is inversely proportional to $x$ (or $y$ varies inversely as $x$ ) if there exists a nonzero number $k$, such that $y=\frac{k}{x}$, or $y=k x^{-1}$. Also, $y$ is inversely proportional to the $n$-th power of $x$ if there exists a nonzero number $k$, such that $y=\frac{k}{x^{n}}$, or $y=k x^{-n}$.
We can also say that $y$ varies inversely as the $n$-th power of $x$.
Example: The illumination produced by a light varies inversely as the square of the distance from the source of the light. If the illumination 30 feet from a light source is 60 candela, what is the illumination 20 feet from the source?
If $L$ represents the illumination and $d$ represents the distance, the relation is $L=\frac{k}{d^{2}}$. Substituting for $L$ and $d$ and solving for $k$ gives $60=\frac{k}{30^{2}}$ or $k=54,000$. Thus, the relation is $L=\frac{54,000}{d^{2}}$. Then, when $d=20$ feet, $L=\frac{54,000}{20^{2}}=135$ candela.

## Two More Examples

- Suppose that $y$ varies directly as the square root of $x$. If when $y=16$, we have $x=4$, find $x$ when $y=24$.
The basic relation is $y=k \sqrt{x}$. Since $y=16$ corresponds to $x=4$, we get $16=k \sqrt{4}$, whence $k=8$. Now, we have $y=8 \sqrt{x}$, whence, for $y=24,24=8 \sqrt{x} \Rightarrow \sqrt{x}=3 \Rightarrow x=9$.
- If $S$ varies inversely as the square root of $T$ and $S=4$ when $T=4$, what is $S$ when $T=16$ ?
The basic relation is $S=\frac{k}{\sqrt{T}}$. Since $S=4$ when $T=4$, we get $4=\frac{k}{\sqrt{4}}$, i.e., $k=8$. Now we have $S=\frac{8}{\sqrt{T}}$, and, therefore, for $T=16, S=\frac{8}{\sqrt{16}}=2$.


## Investing

- If money is invested for 3 years with interest compounded annually, the future value of the investment varies directly as the cube of $1+r$, where $r$ is the annual interest rate. If the future value of the investment is $\$ 6298.56$ when the interest rate is $8 \%$, what rate gives
a future value of $\$ 5955.08$ ?
The basic relation is

$$
A=k(1+r)^{3}
$$

Hence, since the future value is $\$ 6298.56$ when the interest rate is $8 \%$, we get

$$
6298.56=k(1+0.08)^{3} \Rightarrow k=\frac{6298.56}{1.08^{3}}=5000
$$

Hence

$$
A=5000(1+r)^{3}
$$

For $A=5955.08$, we get

$$
5955.08=5000(1+r)^{3} \Rightarrow r=\sqrt[3]{\frac{5955.08}{5000}}-1=0.06
$$

Thus the rate required would be $6 \%$.

## Body Weight

- The weight of a body varies inversely as the square of its distance from the center of Earth. If the radius of Earth is 4000 miles, how much would a 180-pound man weigh 1000 miles above the surface of Earth?
The basic relation is $W=\frac{k}{R^{2}}$. Since at $R=4000, W=180$, we get $180=\frac{k}{4000^{2}}$, whence $k=288 \cdot 10^{7}$.
Therefore, 1000 miles from the surface of the earth, where the radius is $R=5000$ miles, we would have

$$
W=\frac{288 \cdot 10^{7}}{5000^{2}}=115.2 \text { pounds }
$$

## Subsection 4

## Quadratic and Power Models

## Parabola Passing Through Three Given Points I

- Write an equation for the quadratic function passing through the points $(0,-3),(4,37)$ and $(-3,30)$.
Suppose $y=a x^{2}+b x+c$. Since $(0,-3)$ is on the graph, $-3=a 0^{2}+b 0+c$, i.e., $c=-3$. Therefore, since $(4,37)$ and $(-3,30)$ are on the graph,

$$
\begin{aligned}
& \left\{\begin{array}{r}
16 a+4 b-3=37 \\
9 a-3 b-3=30
\end{array}\right\} \Rightarrow\left\{\begin{array}{r}
16 a+4 b=40 \\
9 a-3 b=33
\end{array}\right\} \\
& \Rightarrow\left\{\begin{array}{l}
4 a+b=10 \\
3 a-b=11
\end{array}\right\} \Rightarrow\left\{\begin{array}{r}
4 a+b=10 \\
7 a=21
\end{array}\right\} \\
& \Rightarrow\left\{\begin{array}{l}
a=3 \\
b=-2
\end{array}\right\}
\end{aligned}
$$

Hence, we obtain $y=3 x^{2}-2 x-3$.

## Parabola Passing Through Three Given Points II

- Write an equation for the quadratic function passing through the points $(-1,9),(2,6)$ and $(3,13)$.
Suppose $y=a x^{2}+b x+c$. Using the three given points on the graph, we obtain

$$
\begin{aligned}
& \left\{\begin{aligned}
& a-b+c= \\
& 4 a+2 b+c= \\
& 9 a \\
& 9 a+3 b+c= \\
& \hline
\end{aligned}\right\} \Rightarrow\left\{\begin{aligned}
&-8 a-4 b= \\
&-4 a-b= \\
&-7 \\
& 9 a+3 b+c= \\
& 9 a
\end{aligned}\right\} \\
& \Rightarrow\left\{\begin{array}{rll}
12 a & = & 24 \\
-5 a-b & = & -7 \\
9 a+3 b+c & = & 13
\end{array}\right\} \Rightarrow\left\{\begin{array}{lll}
a= & 2 \\
b & = & -3 \\
c & = & 4
\end{array}\right\}
\end{aligned}
$$

Hence, we obtain $y=2 x^{2}-3 x+4$.

## National Health Care

- The table shows the national expenditures for health care in the U.S. for selected years, with projections to 2015:

| Year | 1960 | 1970 | 1980 | 1990 | 1995 | 2000 | 2005 | 2010 | 2015 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Billions | 28 | 75 | 255 | 717 | 1020 | 1359 | 2016 | 2879 | 4032 |

(a) Use a scatter plot with $x$ number of years after 1950 and $y$ expenditures in billions to identify what type of function would make a good fit for the data.

(b) Find a power model and a quadratic model for the data.
$y=0.0324 * x^{2.737}$ and $y=$ $2.0254 x^{2}-86.722 x+853.890$.
(c) Which model more accurately estimates the 2010 expenditures?
The quadratic model.
(d) Use the better model from (c) to estimate the 2020 expenditures. $y(70)=2.0254 \cdot 70^{2}-86.722 \cdot 70+853.890=\$ 4707.81$ billion.

## Unemployment

- The percent of unemployment in the U.S. for 2004-2010 is given by the data shown below:

| Year | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Percent | 5.4 | 4.9 | 4.4 | 5.0 | 7.3 | 9.9 | 9.4 |

(a) Create a scatter plot for the data with $x$ equal to the number of years after 2004.

(b) Does it appear that a quadratic model will fit the data? If so, find the best fitting quadratic model.

Yes!
$y=0.225 x^{2}-0.461 x+5.071$.
(c) Does the $y$ intercept have a meaning in the context of the problem. Interpret its value.
In 2004, the unemployment rate was approximately 5\%.

## Linear Versus Quadratic Models

- When the changes in input are constant and the first differences of the outputs are constant or nearly constant, a linear model gives a good fit for the data.
- If the second differences, i.e., differences of the first differences, for equally spaced inputs are constant or nearly constant, a quadratic model is an exact or good fit for the data, respectively.

$$
\begin{aligned}
& \begin{array}{ccc}
a & b & c \\
b-a & c-b & d-c^{d} \ldots
\end{array} \\
& (c-b)-(b-a) \quad(d-c)-(c-b)
\end{aligned}
$$

## Example of Linear Versus Quadratic Models

- Test whether the following data follow a linear or quadratic pattern and find a model that best fits them:

| $x$ | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | 16 | 5 | 0 | 1 | 8 | 21 | 40 | 65 |

We compute first and second differences:

| 16 |  | 5 |  | 0 |  | 1 |  | 8 |  | 21 |  | 40 |  | 65 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| -11 |  | -5 |  | 1 |  | 7 |  | 13 |  | 19 |  | 25 |  |  |

Since the second order differences are constant, the data are quadratic. Assume the model $y=a x^{2}+b x+c$. Since $(0,0)$ is a data point, $c=0$. Since $(-1,5)$ and $(1,1)$ are data points,
$\left\{\begin{array}{l}a-b=5 \\ a+b=1\end{array}\right\}$, whence $a=3$ and $b=-2$. Thus, the model is $y=3 x^{2}-2 x$.

## Foreign-Born Population

- The following table gives the percent of the U.S. population that is foreign born.

| Year | 1900 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 2000 | 05 | 07 | 08 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Percent | 13.6 | 14.7 | 13.2 | 11.6 | 8.8 | 6.9 | 5.4 | 4.7 | 6.2 | 8.0 | 10.4 | 11.7 | 12.6 | 12.2 |

(a) Create a scatter plot for the data, with $x$ equal to the number of years after 1900 and $y$ equal to the percent.

(b) Find the best fitting quadratic function for the data.
$y=$
$0.0026 x^{2}-0.3217 x+16.6572$.
(c) Use the function to estimate the percent in 2015.
$y(115)=13.7 \%$.

## Super-Bowl Ads

- The table gives the cost in thousands of a 30 -second ad during Super Bowls for selected years from 1967 to 2010.

| Year | 1967 | 1973 | 1990 | 1996 | 2000 | 2004 | 2006 | 2008 | 2010 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cost | 42.5 | 100 | 700 | 1100 | 2100 | 2250 | 2500 | 2700 | 2974 |

(a) Create a scatter plot for the data, with $x$ equal to the number of years after 1960 and $y$ equal to the number of millions of dollars.


Find the best fitting quadratic function for the data.

```
y =
0.0019x 2 - 0.0361x+0.2121.
```

Use the function to estimate when the cost of a $30-\mathrm{sec}$ ad will be 4 million dollars.
When $x \approx 56$, i.e., at round 2016.

## Cohabiting Households

- The table gives the number of unmarried cohabiting households in thousands for selected years.

| Year | 1960 | 1970 | 1980 | 1985 | 1990 | 1995 | 2002 | 2004 | 2007 | 2008 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Households | 439 | 523 | 1589 | 1983 | 2856 | 3668 | 4898 | 5080 | 6209 | 6214 |

(a) Create a scatter plot for the data, with $x$ equal to the number of years after 1950 and $y$ equal to the number of thousands of households.

(b) Find the power model that best fits the data.
$y=6.057 x^{1.675}$.
(c) How many households does the model predict for 2020?
$y=6.057 \cdot 70^{1.675} \approx 7461$.

## Volume of a Pyramid

- The measured volume of a pyramid with each edge of the page equal to $x$ units and with its altitude equal to $x$ units is given in the following table:

| Edge Length $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Volume | $\frac{1}{3}$ | $\frac{8}{3}$ | 9 | $\frac{64}{3}$ | $\frac{125}{3}$ | 72 |

(a) Determine if the second differences of the outputs are constant.


The second differences are not constant.
(b) If yes, find a quadratic model that fits the data. If not, find a power model the best fits the data.
$y=\frac{1}{3} x^{3}$.

## Personal Savings

- The following table gives Americans' personal savings rate for selected years from 1980 to 2009.

| Year | 1980 | 1990 | 2000 | 2002 | 2004 | 2006 | 2008 | 2009 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Rate | 9.4 | 6.0 | 3.4 | 3.26 | 3.23 | 3.39 | 3.76 | 4.03 |

(a) Find a quadratic function that models the personal savings rate as a function of the number of years after 1960 .
$y=0.011 x^{2}-0.957 x+24.346$.


Find and interpret the vertex of the graph of the rounded model from Part (a).

The vertex is (43.5, 3.531).
Savings rate reached a minimum of $3.5 \%$ during
2004 for the period 1980-2009.
(c) When after 1990 will the personal savings rate reach $6 \%$ ?

During the year 2019.

## Travel and Tourism Spending

- Global spending on travel and tourism in billions of dollars for the years 1991-2009 is given in the table.

| Year | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 00 | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\$ \mathrm{~B}$ | 278 | 317 | 323 | 356 | 413 | 439 | 443 | 445 | 455 | 483 | 472 | 487 | 533 | 634 | 679 | 744 | 859 | 942 | 852 |

(a) Write down an equation of a power function that models the data, letting your input represent the number of years since 1990.
$y=222.434 x^{0.3943}$.

(b) Find the best fitting quadratic model for the data, with $x=0$ in 1990. $y=1.689 x^{2}-0.923 x+324.097$.
(c) Compare the two models by graphing on the same axes as the data points. Which model appears to be the best fit?
The quadratic model fits better.

## Auto Noise

- The noise level of a Volvo S 60 increases as the speed of the car increases. The table gives noise in decibels ( db ) at different speeds:

| Speed | 10 | 30 | 50 | 70 | 100 |
| ---: | :---: | :---: | :---: | :---: | :---: |
| Noise Level | 42 | 57 | 64 | 66 | 71 |

(a) Fit a power function model to the data.
$y=25.425 x^{0.028}$.
(b) Graph the data points and model on the same axes.
(c) Use the result to estimate the noise level at 80 mph .


$$
y(80)=25.425 \cdot 80^{0.228} \approx 69 \mathrm{db} .
$$

