## College Algebra

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LSSU Math 111

(1) Exponential and Logarithmic Functions

- Exponential Functions
- Logarithmic Functions
- Exponential and Logarithmic Equations
- Exponential and Logarithmic Models
- Exponential Functions and Investing
- Annuities; Loan Repayment


## Subsection 1

## Exponential Functions

## Review of Properties of Exponents

## Properties of Exponents

For real numbers $a, b$ and integers $m, n$,
$a^{m} \cdot a^{n}=a^{m+n}$
(Product Property)
2. $\frac{a^{m}}{a^{n}}=a^{m-n}, a \neq 0$
(Quotient Property)
3.
$(a b)^{m}=a^{m} b^{m} \quad$ (Power of Product Property)
4. $\left(\frac{a}{b}\right)^{m}=\frac{a^{m}}{b^{m}}, \quad b \neq 0$ (Power of Quotient Property)
5. $\left(a^{m}\right)^{n}=a^{m n} \quad$ (Power of Power Property)
6. $a^{-m}=\frac{1}{a^{m}}, a \neq 0$.

Example: Simplify the following expressions.
(a) $\frac{3^{7}}{3^{5}}=3^{7-5}=3^{2}=9$;
(b) $\frac{x^{3}}{x^{8}}=x^{3-8}=x^{-5}=\frac{1}{x^{5}}$;
(c) $\left(2 x^{2} y\right)^{5}=2^{5}\left(x^{2}\right)^{5} y^{5^{x}}=32 x^{2 \cdot 5} y^{5}=32 x^{10} y^{5}$;
(d) $\left(\frac{x^{2}}{y^{3}}\right)^{7}=\frac{\left(x^{2}\right)^{7}}{\left(y^{3}\right)^{7}}=\frac{x^{2 \cdot 7}}{y^{3 \cdot 7}}=\frac{x^{14}}{y^{21}}$;
(e) $5^{13-7 m} \cdot 5^{7 m}=5^{13-7 m+7 m}=5^{13}$.

## Additional Practice With Exponents

- Simplify the following

$$
\begin{aligned}
& \text { (a) }\left(-2 x^{-2} y\right)\left(5 x^{-2} y^{-3}\right)=-10 x^{-2-2} y^{1-3}=-10 x^{-4} y^{-2}=\frac{-10}{x^{4} y^{2}} . \\
& \text { (b) } \frac{8 x y^{-2}}{2 x^{4} y^{-6}}=4 x^{1-4} y^{-2-(-6)}=4 x^{-3} y^{4}=\frac{4 y^{4}}{x^{3}} . \\
& \text { (c) } \frac{\frac{2 x^{-1} y}{3 a}}{\frac{6 x y^{-2}}{5 a}}=\frac{2 x^{-1} y}{3 a} \cdot \frac{5 a}{6 x y^{-2}}=\frac{5 x^{-1-1} y^{1-(-2)}}{9}=\frac{5 x^{-2} y^{3}}{9}=\frac{5 y^{3}}{9 x^{2}} . \\
& \text { (d) }\left(4^{0} x^{3} y^{-2}\right)^{-3}=4^{0} x^{-9} y^{6}=1 \cdot \frac{1}{x^{9}} \cdot y^{6}=\frac{y^{6}}{x^{9}} .
\end{aligned}
$$

## Real Exponents

- Simplify the following expressions:
(a) $x^{1 / 2} \cdot x^{5 / 6}=x^{\frac{1}{2}+\frac{5}{6}}=x^{\frac{3}{6}+\frac{5}{6}}=x^{4 / 3}$.
(b) $y^{2 / 5} \cdot y^{1 / 4}=y^{\frac{2}{5}+\frac{1}{4}}=y^{\frac{8}{20}+\frac{5}{20}}=y^{13 / 20}$.
(c) $\left(c^{2 / 3}\right)^{5 / 2}=c^{\frac{2}{3} \cdot \frac{5}{2}}=c^{5 / 3}$.
(d) $\left(x^{3 / 2}\right)^{3 / 4}=x^{\frac{3}{2} \cdot \frac{3}{4}}=x^{9 / 8}$.
(e) $\frac{x^{3 / 4}}{x^{1 / 2}}=x^{\frac{3}{4}-\frac{1}{2}}=x^{1 / 4}$.
(f) $\frac{y^{3 / 8}}{y^{1 / 4}}=y^{\frac{3}{8}-\frac{1}{4}}=y^{1 / 8}$.


## Graph of an Exponential Function

(a) Graph the function $f(x)=3^{x}$ on the window $[-3,3]$ by $[-1,30]$.


| $x$ | $y=3^{x}$ |
| :---: | :---: |
| -2 | $1 / 9$ |
| -1 | $1 / 3$ |
| 0 | 1 |
| 1 | 3 |
| 2 | 9 |

(b) Fill in the table of values:
(c) What is the horizontal asymptote of the graph?

The $x$-axis $y=0$.
(d) What is the $y$-intercept?

The point $(0,1)$

## Graphs of Exponentials With Base $b>1$

- The graph of $f(x)=b^{x}$, for $b>1$, is increasing.

To graph consider a few points $x$.
Example: Sketch the graph of $f(x)=2^{x}$.

| $x$ | $y=2^{x}$ |
| :---: | :---: |
| -2 | $1 / 4$ |
| -1 | $1 / 2$ |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |



The domain is $\mathbb{R}$ and the horizontal asymptote is $y=0$.

## Graphs of Exponentials With Base $0<b<1$

- The graph of $f(x)=b^{x}$, for $0<b<1$, is decreasing.

To graph consider a few points $x$.
Example: Sketch the graph of $f(x)=\left(\frac{1}{3}\right)^{x}$.

| $x$ | $y=\left(\frac{1}{3}\right)^{x}$ |
| :---: | :---: |
| -2 | 9 |
| -1 | 3 |
| 0 | 1 |
| 1 | $1 / 3$ |
| 2 | $1 / 9$ |



The domain is $\mathbb{R}$ and the horizontal asymptote is $y=0$.

## Growth or Decay?

- Determine without graphing whether each of the following functions models exponential growth or exponential decay.
(a) $y=2^{0.1 x}$.

This models exponential growth since $y=\left(2^{0.1}\right)^{x}$ and $2^{0.1}>1$.
(b) $y=3^{-1.4 x}$

This models exponential decay since $y=\left(3^{-1.4}\right)^{x}$ and $3^{-1.4}=\frac{1}{3^{1.4}}<1$.
(c) $y=4 e^{-5 x}$

This models exponential decay, since $e^{-5 x}=\left(e^{-5}\right)^{x}, \frac{1}{e^{5}}<1$ and $4>0$.
(d) $y=0.8^{3 x}$

This models exponential decay, since $y=\left(0.8^{3}\right)^{x}$ and $0.8^{3}<1$.

## Investment

(a) Graph the function $S=56,000 \cdot 1.09^{t}$, which gives the future value of $\$ 56,000$ invested at $9 \%$, compounded annually, for $t$ years, $0 \leq t \leq 15$.

(b) Find the future value of $\$ 56,000$ invested for 13 years at $9 \%$ compounded annually.
$S(13)=56000 \cdot 1.09^{13} \approx \$ 171,685.06$.

## Inflation

- An antique table increases in value according to the function $v(x)=850 \cdot 1.04^{x}$ dollars, where $x$ is the number of years after 1990.
(a) How much was the table worth in 1990?

$$
v(0)=\$ 850 .
$$

(b) If the pattern indicated by the function remains valid, what was the value of the table in 2005 ?

$$
v(15)=850 \cdot 1.04^{15} \approx \$ 1,530.80 .
$$

(c) Use the graph to estimate the year when the table would reach double its 1990 value.


$$
\text { For } S=1700 \text {, we get }
$$

$$
850 \cdot 1.04^{x}=1700
$$

Thus, $x \approx 17.67$, or year 2008 .

## Population

- The population in a certain city was 800,000 in 2003 and its future size is predicted to be $P=800,000 e^{-0.020 t}$ people, where $t$ is the number of years after 2003.
(a) Does this model indicate that the population is increasing or decreasing?
The population is decreasing because $e^{-0.02}=\frac{1}{e^{0.02}} \approx 0.98<1$.
(b) Use the model to estimate the population in 2010.

$$
P(7)=800,000 e^{-0.02 \cdot 7} \approx 695,487 .
$$

(c) Use the model to predict the population in the city in 2020.

$$
P(17)=800,000 e^{-0.02 \cdot 17} \approx 569,416
$$

(d) What is the average rate of change in population between 2010 and 2020?

$$
\frac{P(17)-P(7)}{17-7}=\frac{569,416-695,487}{10} \approx-12,607 \text { per year. }
$$

## Sales Decay

- At the end of an advertising campaign, weekly sales of camcorders declined according to $y=10,000 \cdot 3^{-0.05 x}$ dollars, where $x$ is the number of weeks after the campaign ended.
(a) Determine the sales at the end of the ad campaign.

For $x=0, y=\$ 10,000$.
(b) Determine the sales 8 weeks after the end of the ad campaign. $y(8)=10,000 \cdot 3^{-0.05 \cdot 8} \approx \$ 6,443.94$.
(c) How do we know by inspecting the equation that the function is decreasing?

$$
3^{-0.05}=\frac{1}{3^{0.05}} \approx 0.95<1 .
$$

## Radioactive Decay

- A breeder reactor converts stable Uranium-238 into the isotope Plutonium-239. The decay of this isotope is given by $A(t)=100 e^{-0.00002876 t}$, where $A(t)$ is the amount of the isotope at time $t$ in years and 100 grams is the original amount.
(a) How many grams remain after 100 years?

$$
A(100)=100 e^{-0.00002876 \cdot 100} \approx 99.71 \text { grams. }
$$

(b) Graph the function for $0 \leq t \leq 50,000$

(c) The half-life is the time it takes for half of the initial amount to decay. Use the graph to estimate the half-life of the isotope.
The half-life is approximately 24,101 years.

## Carbon-14 Dating

- An exponential decay function can be used to model the number of grams of a radioactive material that remain after a period of time. Carbon-14 decays over time, with the amount remaining after $t$ years given by $y=100 e^{-0.00012097 t}$ if 100 grams is the original amount.
(a) How much remains after 1000 years?

$$
y(1000)=100 e^{-0.00012097 \cdot 1000} \approx 88.6 \text { grams. }
$$

(b) Use the graph to estimate the number of years until 10 grams of Carbon-14 remain.


For 10 grams of Carbon14 to remain, $\approx 19,034$ years must pass.

## Drugs in the Bloodstream

- If a drug is injected into the bloodstream, the percent of the maximum dosage that is present at time $t$ is given by $y=100\left(1-e^{-0.35(10-t)}\right)$, where $t$ is in hours, with $0 \leq t \leq 10$.
(a) What percent of the drug is present after 2 hours?
$y(2)=100\left(1-e^{-0.35(10-2)}\right) \approx 93.92 \%$.
(b) Graph this function.

(c) When is the drug totally gone from the bloodstream? After 10 hours


## Subsection 2

## Logarithmic Functions

## The Logarithmic Function

## Logarithmic Function

For $x>0,0<b \neq 1$, the logarithmic function with base $b y=\log _{b} x$ is defined by $x=b^{y}$. That is $\log _{b} x$ is the exponent to which we must raise the base $b$ to get the number $x$. This function is the inverse of the exponential function $y=b^{x}$.

- The definition gives the following equivalence:

$$
\underbrace{y=\log _{b} x}_{\text {logarithmic form }} \Longleftrightarrow \underbrace{x=b^{y}}_{\text {exponential form }}
$$

Example: Write in exponential form:

$$
\begin{aligned}
& 2 y=\log _{5} x \Leftrightarrow x=5^{2 y} \\
& y=\ln (2 x) \Leftrightarrow 2 x=e^{y} \\
& y=\log (-x) \Leftrightarrow-x=10^{y}
\end{aligned}
$$

Example: Write in logarithmic form:

$$
\begin{aligned}
& -x=4^{y} \Leftrightarrow y=\log _{4} x \\
& \bullet m=3^{p} \Leftrightarrow p=\log _{3} m \\
& \circ 9^{2 x}=y \Leftrightarrow 2 x=\log _{9} y
\end{aligned}
$$

## Evaluating Logarithms

- Evaluate the following expressions without using a calculator:
(a) $y=\log _{2} 8 \Leftrightarrow 2^{y}=8 \Leftrightarrow 2^{y}=2^{3} \Leftrightarrow y=3$
(b) $y=\log _{4} \frac{1}{16} \Leftrightarrow 4^{y}=\frac{1}{16} \Leftrightarrow 4^{y}=4^{-2} \Leftrightarrow y=-2$
(c) $y=\log _{10} 0.00001 \Leftrightarrow 10^{y}=0.00001 \Leftrightarrow 10^{y}=10^{-5} \Leftrightarrow y=-5$
(d) $y=\log _{3} 81 \Leftrightarrow 3^{y}=81 \Leftrightarrow 3^{y}=3^{4} \Leftrightarrow y=4$
(e) $y=\log _{5} \frac{1}{625} \Leftrightarrow 5^{y}=\frac{1}{625} \Leftrightarrow 5^{y}=5^{-4} \Leftrightarrow y=-4$
(f) $y=\log _{(1 / 64)} 4 \Leftrightarrow\left(\frac{1}{64}\right)^{y}=4 \Leftrightarrow\left(\frac{1}{64}\right)^{y}=\left(\frac{1}{64}\right)^{-1 / 3} \Leftrightarrow y=-\frac{1}{3}$
(g) $y=\log _{(1 / 27)} 9 \Leftrightarrow\left(\frac{1}{27}\right)^{y}=9 \Leftrightarrow\left(\frac{1}{27}\right)^{y}=\left(\frac{1}{27}\right)^{-2 / 3} \Leftrightarrow y=-\frac{2}{3}$


## Increasing Logarithmic Graphs

- The graph of $f(x)=\log _{b} x$, for $b>1$, is increasing.

To graph consider a few points $x$ that are powers of $b$, whose $\log _{b}$ is easy to compute.
Example: Sketch the graph of $f(x)=\log _{2} x$.

| $x$ | $y=\log _{2} x$ |
| :---: | :---: |
| $1 / 4$ | -2 |
| $1 / 2$ | -1 |
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |



The domain is $(0, \infty)$ and a vertical asymptote $x=0$.

## Decreasing Logarithmic Graphs

- The graph of $f(x)=\log _{b} x$, for $0<b<1$, is decreasing.

To graph consider a few points $x$ that are powers of $b$, whose $\log _{b}$ is easy to compute.
Example: Sketch the graph of $f(x)=\log _{1 / 3} x$.

| $x$ | $y=\log _{1 / 3} x$ |
| :---: | :---: |
| $1 / 9$ | 2 |
| $1 / 3$ | 1 |
| 1 | 0 |
| 3 | -1 |
| 9 | -2 |



The domain is $(0, \infty)$ and a vertical asymptote is $x=0$.

## The Common Logarithms

- The logarithm to base 10 is called the common logarithm and denoted $f(x)=\log x$ (base 10 is understood).
To graph consider a few points $x$ that are powers of 10 , whose $\log _{10}$ is easy to compute.
Example: Sketch the graph of $f(x)=\log x$.

| $x$ | $y=\log x$ |
| :---: | :---: |
| $\frac{1}{100}$ | -2 |
| $\frac{1}{10}$ | -1 |
| 1 | 0 |
| 10 | 1 |
| 100 | 2 |



The domain is $(0, \infty)$ and a vertical asymptote is $x=0$.

## The pH Scale

- The pH (hydrogen potential) scale measures the acidity or basicity of a solution. The pH is given by the formula $\mathrm{pH}=-\log \left[\mathrm{H}^{+}\right]$, where $\left[\mathrm{H}^{+}\right]$is the concentration of hydrogen ions in moles per liter in the solution. The pH scale ranges from 0 to 14 .
- A pH of 7 is neutral;
- A pH of less than 7 is acidic;
- A pH greater than 7 is basic.
(a) Bleach has concentration of hydrogen ions $10^{-13}$ moles/liter. Find the pH level of bleach.
$\mathrm{pH}=-\log \left(10^{-13}\right)=-(-13)=13$.
(b) Lemon juice has concentration of hydrogen ions $10^{-2}$ moles/liter. Find the pH level of lemon juice.
$\mathrm{pH}=-\log \left(10^{-2}\right)=-(-2)=2$.
(c) How much more acidic is lemon juice than bleach?

We have

$$
\frac{10^{-2}}{10^{-13}}=10^{11}
$$

i.e., lemon juice is $10^{11}$ times more acidic than bleach.

## The Natural Logarithms

- The logarithm to base $e \approx 2.718$ is called the natural logarithm and denoted $f(x)=\ln x$ (base $e$ is understood).
To graph consider a few points $x$ that are powers of $e$, whose $\ln$ is easy to compute.
Example: Sketch the graph of $f(x)=\ln x$.

| $x$ | $y=\ln x$ |
| :---: | :---: |
| $\frac{1}{e^{2}}$ | -2 |
| $\frac{1}{e}$ | -1 |
| 1 | 0 |
| $e$ | 1 |
| $e^{2}$ | 2 |



The domain is $(0, \infty)$ and a vertical asymptote is $x=0$.

## Diabetes

- Projections from 2010 to 2050 indicate that the percent of U.S. adults with diabetes can be modeled by $p(x)=-12.975+11.851 \ln x$, where $x$ is the number of years after 2000 .
(a) Graph the function.
(b) Is the function increasing or decreasing and what is the meaning as related to the application?

It is increasing, i.e., the percent of adults with diabetes increases over
 time.
(c) What does the model predict the percent of adults with diabetes will be in 2022?
$p(22)=-12.975+11.851 \cdot \ln 22 \approx 23.7$.
(d) Estimate the year when the percent will reach $33 \%$.

$$
\begin{aligned}
& 33=-12.975+11.851 \ln x \quad \Rightarrow \quad 11.851 \ln x=45.975 \quad \Rightarrow \quad \ln x= \\
& \frac{45.975}{11.851} \quad \Rightarrow \quad x=e^{45.975 / 11.851} \approx 48.4 .
\end{aligned}
$$

## Basic Properties of Logarithms

## Basic Properties of Logarithms

For $0<b \neq 1$,

1. $\log _{b} b=1$
2. $\log _{b} 1=0$
3. $\log _{b} b^{x}=x$
4. $b^{\log _{b} x}=x$

For $M, N>0, M=N$ implies $\log _{b} M=\log _{b} N$.

Example: Use the properties to simplify:

- $\log _{7} 7^{13}=13$
- $\log _{5} 5=1$
- $\log 1=0$
- $\ln e^{25}=25$
- $\log \left(\frac{1}{10^{9}}\right)=\log 10^{-9}=-9$


## Additional Logarithmic Properties

## Additional Logarithmic Properties

For $0<b \neq 1, k$ real and $M, N$ positive reals,
6. $\log _{b}(M N)=\log _{b} M+\log _{b} N \quad$ (Product Property)
7. $\log _{b}\left(\frac{M}{N}\right)=\log _{b} M-\log _{b} N \quad$ (Quotient Property)
8. $\log _{b} M^{k}=k \log _{b} M \quad$ (Power Property)

## Example:

(a) Estimate $\ln (5 e)$ if $\ln 5 \approx 1.61$.

$$
\ln (5 e)=\ln 5+\ln e \approx 1.61+1=2.61
$$

(b) Find $\log _{a}\left(\frac{15}{8}\right)$ if $\log _{a} 15=2.71$ and $\log _{a} 8=2.079$.

$$
\log _{a}\left(\frac{15}{8}\right)=\log _{a} 15-\log _{a} 8=2.71-2.079=0.631
$$

(c) Find $\log _{a}\left(\frac{15^{2}}{\sqrt{8}}\right)^{3}$ if $\log _{a} 15=2.71$ and $\log _{a} 8=2.079$.

$$
\begin{aligned}
& \log _{a}\left(\frac{15^{2}}{\sqrt{8}}\right)^{3}=3 \log _{a}\left(\frac{15^{2}}{\sqrt{8}}\right)=3\left[\log _{a} 15^{2}-\log _{a} 8^{1 / 2}\right]= \\
& 3\left[2 \log _{a} 15-\frac{1}{2} \log _{a} 8\right]=3\left(2 \cdot 2.71-\frac{1}{2} \cdot 2.079\right)=13.1415 .
\end{aligned}
$$

## Rewriting as a Sum/Difference and Product of Logarithms

- Rewrite each expression as a sum, difference and product of logarithms and simplify, is possible.
(a) $\ln \frac{3 x-2}{x+1}$

$$
\ln \frac{3 x-2}{x+1}=\ln (3 x-2)-\ln (x+1) .
$$

(b) $\log \left[x^{3}(3 x-4)^{5}\right]$
$\log \left[x^{3}(3 x-4)^{5}\right]=\log x^{3}+\log (3 x-4)^{5}=3 \log x+5 \log (3 x-4)$.
(c) $\log _{3} \frac{\sqrt[3]{4 x+1}}{4 x^{2}}$

$$
\begin{aligned}
\log _{3} \frac{\sqrt[3]{4 x+1}}{4 x^{2}} & =\log _{3}(4 x+1)^{1 / 3}-\log _{3} 4 x^{2} \\
& =\frac{1}{3} \log _{3}(4 x+1)-\left(\log _{3} 4+\log _{3} x^{2}\right) \\
& =\frac{1}{3} \log _{3}(4 x+1)-\log _{3} 4-2 \log _{3} x .
\end{aligned}
$$

(d) $\log _{3} \frac{\sqrt[3]{3 x-1}}{5 x^{2}}$

$$
\begin{aligned}
\log _{3} \frac{\sqrt[3]{3 x-1}}{5 x^{2}} & =\log _{3}(3 x-1)^{1 / 3}-\log _{3} 5 x^{2} \\
& =\frac{1}{3} \log _{3}(3 x-1)-\left(\log _{3} 5+\log _{3} x^{2}\right) \\
& =\frac{1}{3} \log _{3}(3 x-1)-\log _{3} 5-2 \log _{3} x
\end{aligned}
$$

## Rewriting as a Single Logarithm

- Rewrite each expression as a single logarithm:
(a) $3 \log _{2} x+\log _{2} y$

$$
3 \log _{2} x+\log _{2} y=\log _{2} x^{3}+\log _{2} y=\log _{2}\left(x^{3} y\right) .
$$

(b) $\log x-\frac{1}{3} \log y$

$$
\log x-\frac{1}{3} \log y=\log x-\log y^{1 / 3}=\log \left(\frac{x}{\sqrt[3]{y}}\right)
$$

(c) $4 \ln (2 a)-\ln b$

$$
4 \ln (2 a)-\ln b=\ln (2 a)^{4}-\ln b=\ln \left(\frac{16 a^{4}}{b}\right) .
$$

(d) $6 \ln (5 y)+2 \ln x$ $6 \ln (5 y)+2 \ln x=\ln (5 y)^{6}+\ln x^{2}=\ln \left(5^{6} x^{2} y^{6}\right)$.

## The Richter Scale

- Because of the enormity of the intensities of earthquakes, the Richter scale is used to provide a more manageable measuring system.
The Richter scale gives the magnitude $R$ of an earthquake using the formula

$$
R=\log \left(\frac{I}{I_{0}}\right)
$$

where $I$ is the intensity of the earthquake and $I_{0}$ is a certain minimum intensity used for comparison.

| Description | Richter Magnitude | Effects |
| :--- | :---: | :--- |
| Micro | $<2.0$ | Not felt |
| Very Minor | $2.0-2.9$ | Not felt, but recorded |
| Minor | $3.0-3.9$ | Felt, but rarely damaging |
| Light | $4.0-4.9$ | Noticeable shaking, No significant damage |
| Moderate | $5.0-5.9$ | Damaging to poorly constructed buildings |
| Strong | $6.0-6.9$ | Destructive in a radius of 100 miles |
| Major | $7.0-7.9$ | Serious damage over large areas |
| Great | $8.0-8.9$ | Serious damage for several hundred miles |
| Rarely, great | $9.0-9.9$ | Devastating for several thousand miles |
| Meteoric | $10.0+$ | Never recorded |

## Measuring Earthquakes Using the Richter Scale I

(a) If an earthquake has an intensity of 10,000 times $I_{0}$, what is the magnitude of the earthquake?
$R=\log \left(\frac{10000 I_{0}}{I_{0}}\right)=\log (10000)=\log \left(10^{4}\right)=4$.
(b) Show that if the Richter scale reading of an earthquake is $k$, the intensity is $I=10^{k} I_{0}$.
$k=\log \left(\frac{I}{1_{0}}\right) \quad \Rightarrow \quad \frac{I}{1_{0}}=10^{k} \quad \Rightarrow \quad I=10^{k} I_{0}$.
(c) An earthquake that measured 9.0 on the Richter scale hit the Indian Ocean in December 2004, causing a tsunami that killed thousands of people. Express the intensity in terms of $I_{0}$.

$$
9=\log \left(\frac{I}{I_{0}}\right) \quad \Rightarrow \quad \frac{I}{I_{0}}=10^{9} \quad \Rightarrow \quad I=10^{9} I_{0}
$$

## Measuring Earthquakes Using the Richter Scale II

(d) If an earthquake measures 7.0 on the Richter scale, give the intensity of this earthquake in terms of $I_{0}$. How much more intense is the earthquake in (c) that the one measuring 7.0 ?

- $7=\log \left(\frac{I}{I_{0}}\right) \quad \Rightarrow \quad \frac{I}{I_{0}}=10^{7} \quad \Rightarrow \quad I=10^{7} I_{0}$;
- $\frac{10^{9} o_{0}}{10^{7} I_{0}}=100$, i.e., it is a 100 times more intense.
(e) If one quake has intensity 320,000 times $I_{0}$ and a second intensity $3,200,000$ times $I_{0}$ what is the difference in their Richter scale measurements?

$$
\begin{aligned}
R_{s}-R_{w} & =\log \left(\frac{3,200,000 I_{0}}{I_{0}}\right)-\log \left(\frac{320,000 I_{0}}{I_{0}}\right) \\
& =\log 3,200,000-\log 320,000 \\
& =\log \left(\frac{3,200,000}{320,000}\right) \\
& =\log 10 \\
& =1
\end{aligned}
$$

## Subsection 3

## Exponential and Logarithmic Equations

## Exponential Equations Via Logarithmic Forms

## Solving Exponential Equations Using Logarithmic Forms

To solve an exponential equation using logarithmic forms:
Rewrite the equation with the term containing the exponent by itself on one side.
2. Divide both sides by the coefficient containing the exponent.
3. Change the new equation to logarithmic form.
4. Solve for the variable.

Example: Solve the equation $2 \cdot 10^{8 x}+600=5400$ both algebraically and graphically.

$$
\begin{aligned}
& 2 \cdot 10^{8 x}+600=5400 \\
& \Rightarrow 2 \cdot 10^{8 x}=4800 \\
& \Rightarrow 10^{8 x}=2400 \\
& \Rightarrow 8 x=\log 2400 \\
& \Rightarrow x=\frac{1}{8} \log 2400 \approx 0.422526 .
\end{aligned}
$$



## Examples Using Logarithmic Forms

- Solve the equation $5200=13 \cdot e^{12 x}$ both algebraically and graphically.

$$
\begin{aligned}
& 5200=13 \cdot e^{12 x} \\
& \Rightarrow e^{12 x}=\frac{5200}{13} \\
& \Rightarrow 12 x=\ln \frac{5200}{13} \\
& \Rightarrow x=\frac{1}{12} \ln \frac{5200}{13} \approx 0.4993 .
\end{aligned}
$$



- Solve the equation $2 \cdot 6^{2 x}=2592$ both algebraically and graphically.

$$
\begin{aligned}
& 2 \cdot 6^{2 x}=2592 \\
& \Rightarrow 6^{2 x}=\frac{2592}{2} \\
& \Rightarrow 2 x=\log _{6} \frac{2592}{2} \\
& \Rightarrow x=\frac{1}{2} \log _{6} \frac{2592}{2}=2 .
\end{aligned}
$$



## More Examples Using Logarithmic Forms

- Solve the equation $3^{5 x-4}=140$ algebraically.

$$
\begin{aligned}
& 3^{5 x-4}=140 \\
& \Rightarrow \quad 5 x-4=\log _{3} 140 \\
& \Rightarrow \quad 5 x=\log _{3} 140+4 \\
& \Rightarrow \quad x=\frac{1}{5}\left[\log _{3} 140+4\right] \approx 1.7
\end{aligned}
$$

- Solve the equation $5880=21 \cdot 2^{3 x+7}$ algebraically.

$$
\begin{aligned}
& 5880=21 \cdot 2^{3 x+7} \\
& \Rightarrow \quad 2^{3 x+7}=\frac{5880}{21} \\
& \Rightarrow \quad 3 x+7=\log _{2} \frac{5880}{21} \\
& \Rightarrow \quad 3 x=\log _{2} \frac{5880}{21}-7 \\
& \Rightarrow \quad x=\frac{1}{3}\left[\log _{2} \frac{5880}{21}-7\right] \approx 0.3764
\end{aligned}
$$

## Doubling Time

(a) Find the time it takes for an investment to double its value if the interest rate of the investment is $r$ compounded continuously.

$$
\begin{aligned}
& S=P e^{r t} \\
& \Rightarrow \quad 2 P=P e^{r t} \\
& \Rightarrow \quad 2=e^{r t} \\
& \Rightarrow \quad r t=\ln 2 \\
& \Rightarrow \quad t=\frac{1}{r} \ln 2 .
\end{aligned}
$$

(b) If $\$ 25,000$ is invested in an account earning $4 \%$ annual interest, compounded continuously, how long will it take for the amount to grow to $\$ 50,000$ ?

$$
t=\frac{1}{0.04} \ln 2 \approx 17.329
$$

## Sales Decay

- After the end of a TV ad campaign, the weekly sales of Korbel champaign fell rapidly, with weekly sales given by

$$
S=25,000 e^{-0.072 x}, \text { dollars, }
$$

where $x$ is the number of weeks since the end of the campaign.
(a) What were the weekly sales when the campaign ended?

$$
S(0)=25,000 e^{-0.072 \cdot 0}=\$ 25,000 .
$$

(b) How long did it take for the sales to fall by one-half the level of what they were when the campaign ended?

$$
\begin{aligned}
& \frac{25000}{2}=25000 e^{-0.072 x} \quad \Rightarrow \quad e^{-0.072 x}=\frac{1}{2} \\
& \Rightarrow \quad-0.072 x=\ln \frac{1}{2}=-\ln 2 \quad \Rightarrow \quad x=\frac{1}{0.072} \ln 2 \approx 9.627
\end{aligned}
$$

(c) After how many weeks will the sales reach the $\$ 10,000$ mark?

$$
\begin{aligned}
& 10000=25000 e^{-0.072 x} \quad \Rightarrow \quad e^{-0.072 x}=\frac{2}{5} \\
& \Rightarrow \quad-0.072 x=\ln \frac{2}{5} \quad \Rightarrow \quad x=-\frac{1}{0.072} \ln \frac{2}{5} \approx 12.726 .
\end{aligned}
$$

## The Change of Base Formula

## Change of Base Formula

If $0<b \neq 1,0<a \neq 1$, and $x>0$, then

$$
\log _{a} x=\frac{\log _{b} x}{\log _{b} a}
$$

In particular, for base 10 and base $e$,

$$
\log _{a} x=\frac{\log x}{\log a} \quad \text { and } \quad \log _{a} x=\frac{\ln x}{\ln a}
$$

Example: Use change of base to evaluate the following logarithms:
(a) $\log _{7} 215$
(b) $\log _{4} \sqrt[3]{10}$
(a) $\log _{7} 215=\frac{\ln 215}{\ln 7} \approx 2.76$.
(b) $\log _{4} \sqrt[3]{10}=\log _{4}\left(10^{1 / 3}\right)=\frac{1}{3} \log _{4} 10=\frac{1}{3} \frac{\log 10}{\log 4}=\frac{1}{3 \log 4} \approx 0.5537$.

## Investment and Annuities

- At the end of $t$ years, the future value of an investment of $\$ 20,000$ at $7 \%$, compounded annually is given by $S=20,000(1+0.07)^{t}$. In how many years will the investment grow to $\$ 48,196.90$ ?

$$
\begin{aligned}
& S=20,000(1+0.07)^{t} \Rightarrow \quad 48,196.90=20,000 \cdot 1.07^{t} \\
& \Rightarrow \quad \frac{48,196.90}{20,000}=1.07^{t} \Rightarrow t=\log _{1.07} \frac{48,196.90}{20,000} \\
& \Rightarrow \quad t=\frac{\ln (48,196.90 / 20,000)}{\ln 1.07}=13 .
\end{aligned}
$$

- If $\$ 2,000$ is invested at the end of each year in an annuity that pays $5 \%$ compounded annually, the number of years it takes for the future value to amount to $\$ 40,000$ is $t=\log _{1.05} 2$. Use change of base to evaluate the number.

$$
t=\log _{1.05} 2=\frac{\ln 2}{\ln 1.05} \approx 14.207
$$

## Exponential Equations Using Logarithmic Properties

## Exponential Equations Using Logarithmic Properties

To solve an exponential equation using logarithmic properties:
Rewrite the equation with a base raised to a power on one side. Take the logarithm, base e or base 10, of both sides.
Use a logarithmic property to remove the power from the exponent.
Solve for the variable.
Example: Solve the equation $4096=8^{2 x}$.

$$
\begin{aligned}
& 4096=8^{2 x} \quad \Rightarrow \quad \ln 4096=\ln \left(8^{2 x}\right) \\
& \Rightarrow \quad \ln 4096=2 x \ln 8 \quad \Rightarrow \quad x=\frac{1}{2} \frac{\ln 4096}{\ln 8}=2 .
\end{aligned}
$$

Example: Solve the equation $6 \cdot 4^{3 x-2}=120$.

$$
\begin{aligned}
& 6 \cdot 4^{3 x-2}=120 \quad \Rightarrow \quad 4^{3 x-2}=20 \\
& \Rightarrow \quad \ln \left(4^{3 x-2}\right)=\ln 20 \quad \Rightarrow \quad(3 x-2) \ln 4=\ln 20 \\
& \Rightarrow \quad 3 x-2=\frac{\ln 20}{\ln 4} \quad \Rightarrow \quad x=\frac{1}{3}\left(\frac{\ln 20}{\ln 4}+2\right) \approx 1.387
\end{aligned}
$$

## Drugs in the Bloodstream

- If a drug is injected into the bloodstream, the percent of the maximum dosage that is present at time $t$ is given by

$$
y=100\left(1-e^{-0.35(10-t)}\right)
$$

where $t$ is in hours, with $0 \leq t \leq 10$. In how many hours will the percent reach $65 \%$ ?

$$
\begin{aligned}
& 65=100\left(1-e^{-0.35(10-t)}\right) \\
& \Rightarrow \quad \frac{65}{100}=1-e^{-0.35(10-t)} \\
& \Rightarrow \quad e^{-0.35(10-t)}=1-\frac{65}{100} \\
& \Rightarrow \quad \ln \left[e^{-0.35(10-t)}\right]=\ln 0.35 \\
& \Rightarrow \quad-0.35(10-t)=\ln 0.35 \\
& \Rightarrow \quad 10-t=-\frac{1}{0.35} \ln 0.35 \\
& \Rightarrow \quad t=10+\frac{1}{0.35} \ln 0.35 \approx 7
\end{aligned}
$$

## Logarithmic Equations

- Solve the equation $\log _{4} x=-2$.

$$
\log _{4} x=-2 \quad \Rightarrow \quad x=4^{-2} \quad \Rightarrow \quad x=\frac{1}{16}
$$

- Solve the equation $4+3 \log x=10$.

$$
\begin{gathered}
4+3 \log x=10 \quad \Rightarrow \quad 3 \log x=6 \\
\Rightarrow \quad \log x=2 \quad \Rightarrow \quad x=10^{2}=100
\end{gathered}
$$

- Solve the equation $3 \ln x+8=\ln (3 x)+12.18$.

$$
\begin{aligned}
& 3 \ln x+8=\ln (3 x)+12.18 \quad \Rightarrow \quad 3 \ln x+8=\ln 3+\ln x+12.18 \\
& \Rightarrow \quad 2 \ln x=\ln 3+4.18 \quad \Rightarrow \quad \ln x=\frac{1}{2}(\ln 3+4.18) \approx 2.64 \\
& \Rightarrow \quad x \approx e^{2.64} \approx 14 .
\end{aligned}
$$

- Solve the equation $\ln (x-6)+4=\ln x+3$.

$$
\begin{aligned}
& \ln (x-6)+4=\ln x+3 \quad \Rightarrow \quad \ln (x-6)-\ln x=-1 \\
& \Rightarrow \quad \ln \frac{x-6}{x}=-1 \quad \Rightarrow \quad \frac{x-6}{x}=e^{-1}=\frac{1}{e} \\
& \Rightarrow \quad e(x-6)=x \quad \Rightarrow \quad e x-6 e=x \\
& \Rightarrow \quad(e-1) x=6 e \quad \Rightarrow \quad x=\frac{6 e}{e-1} \approx 9.49 .
\end{aligned}
$$

## Logarithmic Equations ||

- Solve the equation $\log _{2} x+\log _{2}(x-6)=4$.

$$
\begin{aligned}
& \log _{2} x+\log _{2}(x-6)=4 \quad \Rightarrow \quad \log _{2}(x(x-6))=4 \\
& \Rightarrow \quad x(x-6)=2^{4} \quad \Rightarrow \quad x^{2}-6 x-16=0 \\
& \Rightarrow \quad(x+2)(x-8)=0 \quad \Rightarrow \quad x=-2 \text { or } x=8
\end{aligned}
$$

Only $x=8$ is admissible.

- Solve the equation $\log _{2} x=3 \log _{2}(2 x)$.

$$
\begin{aligned}
& \log _{2} x=3 \log _{2}(2 x) \\
& \Rightarrow \quad \log _{2} x=\log _{2}\left[(2 x)^{3}\right] \\
& \Rightarrow \quad x=(2 x)^{3} \\
& \Rightarrow \quad 8 x^{3}-x=0 \\
& \Rightarrow \quad x\left(8 x^{2}-1\right)=0 \\
& \Rightarrow \quad x=0 \text { or } x= \pm \frac{\sqrt{2}}{4} .
\end{aligned}
$$

Only $x=\frac{\sqrt{2}}{4}$ is admissible.

## One More Logarithmic Equation

- Solve the equation $\ln (x+2)+\ln x=\ln (x+12)$.

$$
\begin{aligned}
& \ln (x+2)+\ln x=\ln (x+12) \\
& \Rightarrow \quad \ln (x(x+2))=\ln (x+12) \\
& \Rightarrow \quad x(x+2)=x+12 \\
& \Rightarrow \quad x^{2}+2 x=x+12 \\
& \Rightarrow \quad x^{2}+x-12=0 \\
& \Rightarrow \quad(x+4)(x-3)=0 \\
& \Rightarrow \quad x=-4 \text { or } x=3 .
\end{aligned}
$$

Only $x=3$ is admissible.

## Supply

- Suppose that the supply of a product is given by

$$
p=20+6 \ln (2 q+1)
$$

where $p$ is the price per unit and $q$ is the number of units supplied. How many units will be supplied if the price per unit is $\$ 68.04$ ? We must solve $68.04=20+6 \ln (2 q+1)$. We have:

$$
\begin{array}{ll}
68.04=20+6 \ln (2 q+1) \\
\Rightarrow & 6 \ln (2 q+1)=48.04 \\
\Rightarrow & \ln (2 q+1)=\frac{48.04}{6} \\
\Rightarrow & 2 q+1=e^{48.04 / 6} \\
\Rightarrow & 2 q=e^{48.04 / 6}-1 \\
\Rightarrow & q=\frac{1}{2}\left(e^{48.04 / 6}-1\right) \\
\Rightarrow & q \approx 1500 \text { units. }
\end{array}
$$

## Global Warming

- The cost-benefit equation for a proposed carbon dioxide emissions tax is

$$
\ln (1-P)=-0.0034-0.0053 t
$$

where $P$ is the percent reduction of emissions of carbon dioxide as a decimal and $t$ the tax in dollars per ton of carbon dioxide.
(a) Solve the equation for $t$, giving $t$ as a function of $P$, and graph the function.
$t=\frac{-0.0034-\ln (1-P)}{0.0053}$

(b) Use the equation to find what tax will give a $30 \%$ reduction in emissions.
$P=\frac{-0.0034-\ln (1-0.3)}{0.0053} \approx \$ 66.66 /$ ton.

## Exponential Inequalities

- Solve the inequality $7^{x} \geq 2401$.

$$
\begin{aligned}
& 7^{x} \geq 2401 \\
& \Rightarrow \quad x \geq \log _{7} 2401 \\
& \Rightarrow \quad x \geq \log _{7}\left(7^{4}\right) \\
& \Rightarrow \quad x \geq 4
\end{aligned}
$$

- Solve the inequality $15 \cdot 4^{x} \leq 15360$.

$$
\begin{aligned}
& 15 \cdot 4^{x} \leq 15360 \\
& \Rightarrow \quad 4^{x} \leq \frac{15360}{15}=1024 \\
& \Rightarrow \quad x \leq \log _{4} 1024 \\
& \Rightarrow \quad x \leq \log _{4}\left(4^{5}\right) \\
& \Rightarrow \quad x \leq 5 .
\end{aligned}
$$

## Sales Decay

- After the end of an advertising campaign, the daily sales of Genapet fell rapidly, with daily sales given by $S=3200 e^{-0.08 x}$ dollars, where $x$ is the number of days from the end of the campaign. For how many days after the campaign ended were the sales at least $\$ 1980$ ? We must solve the inequality $S \geq 1980$. We have:

$$
\begin{array}{ll}
S \geq 1980 \\
\Rightarrow & 3200 e^{-0.08 x} \geq 1980 \\
\Rightarrow & e^{-0.08 x} \geq \frac{1980}{3200} \\
\Rightarrow & -0.08 x \geq \ln \frac{1980}{3200} \\
\Rightarrow & x \leq-\frac{1}{0.08} \ln \frac{1980}{3200} \\
\Rightarrow & x \lesssim 6
\end{array}
$$



## Cost-Benefit

- The cost-benefit equation $\ln (1-P)=-0.0034-0.0053 t$ estimates the relationship between the percent reduction of emissions of carbon dioxide $P$ as a decimal and the tax $t$ in dollars per ton of carbon. What tax will give a reduction of at least $50 \%$
We must solve the inequality $P \geq 0.5$. Setting $P=0.5$, we obtain:

$$
\begin{aligned}
& \ln (1-0.5)=-0.0034-0.0053 t \\
& \Rightarrow \quad-0.6931=-0.0034-0.0053 t \\
& \Rightarrow \quad 0.0053 t=0.6897 \\
& \Rightarrow \quad t=130.14
\end{aligned}
$$



Using the graph of $P(t)=1-e^{-0.0034-0.0053 t}$, we see that for $P \geq 0.5, t \gtrsim \$ 130.14$.

## Subsection 4

## Exponential and Logarithmic Models

## Personal Income

- Total personal income (in billions of dollars) in the U.S. for selected years from 1960 and projected to 2018 is given below:

| Year | 1960 | 1970 | 1980 | 1990 | 2000 | 2008 | 2018 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Income | 411.5 | 838.3 | 2307.9 | 4878.6 | 8429.7 | $12,100.7$ | $19,129.6$ |

(a) Create an exponential model for the income as a function of the number $x$ of years after 1960 .
Input the list and perform exponential regression: $y=492.44 \cdot 1.07^{x}$.
(b) Plot the data points and sketch the graph in the same system of axes.


## Personal Income (Cont'd)

- We modeled personal income by $y=492.44 \cdot 1.07^{x}$.
(c) If the model is accurate, what will be the total personal income in 2015?
$y(55)=492.44 \cdot 1.07^{55}=\$ 20,345.16$ billion.
(d) In what year does the model predict that the total personal income will reach \$ 19 trillion?


$$
x \approx 53.989, \text { i.e., year } 1014
$$

## National Debt

- The table gives the U.S. national debt in billions for selected years from 1900 to 2010:

| Year | Debt | Year | Debt | Year | Debt |
| :--- | ---: | ---: | ---: | ---: | ---: |
| 1900 | 1.2 | 1985 | 1823.1 | 2002 | 6228.2 |
| 1910 | 1.1 | 1990 | 3233.3 | 2003 | 6783.2 |
| 1920 | 24.2 | 1992 | 4064.6 | 2004 | 7379.1 |
| 1930 | 16.1 | 1994 | 4692.8 | 2005 | 7932.7 |
| 1940 | 43.0 | 1996 | 5224.8 | 2006 | 8680.2 |
| 1945 | 258.7 | 1998 | 5526.2 | 2007 | 9229.2 |
| 1955 | 272.8 | 1999 | 5656.3 | 2008 | 10699.8 |
| 1965 | 313.8 | 2000 | 5674.2 | 2009 | 12311.3 |
| 1975 | 533.2 | 2001 | 5807.5 | 2010 | 14025.2 |

(a) Use a function $a \cdot b^{x}$ with $x=0$ in 1900 and $y$ the national debt in billions to model the data.
Tabulate the data and use exponential regression: $y=1.756 \cdot 1.085^{x}$.
(b) Use the model to predict the debt in 2013.
$y(113)=1.756 \cdot 1.085^{113} \approx \$ 17,749$ billion.

## National Debt (Cont'd)

- $y=1.756 \cdot 1.085^{x}$
(d) Plot the data points and graph the model on the same system of axes. Are there events that affect the accuracy of the model as a predictor of future public debt?


(c) Predict when the debt will be $\$ 25$ trillion. At around $x \approx 118$, i.e., in the year 2018.


## Constant Percent Changes

## Constant Percent Changes

If the percent change of the outputs of a set of data is constant for equally spaced inputs, an exponential function will be a perfect fit for the data. If the percent change of the outputs is approximately constant for equally spaced inputs, an exponential function will be an approximate fit.

Example: Use percent changes to test whether the following data are exponential.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ | 4 | 16 | 64 | 256 | 1024 | 4096 |

If they are find an exponential model that fits the data.

| Outputs | 4 |  | 16 |  | 64 |  | 256 |  | 1024 |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Differences | 12 |  | 48 |  | 192 |  | 768 |  | 3072 |  |
| Percent Change | 300 |  | 300 |  | 300 |  | 300 |  | 300 |  |

We have $f(x)=(1+3)^{x}=4^{x}$.

## Sales Decay

- A company releases a product with great expectations and extensive advertising, but sales differ because of bad word of mouth from dissatisfied customers.

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Sales (Thousands \$) | 780 | 608 | 475 | 370 | 289 | 225 | 176 | 137 |

(a) Use the data to determine the percent change for each month.

| 780 | 608 | 475 | 370 | 289 | 225 | 176137 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| - 172 | - 133 | - 105 | -81 | -64 | -49 | - 39 |
| - 22.1 | - 21.9 | -22.1 | - 21.9 | - 22.1 | -21.8 | - 22.2 |

(b) Find the exponential function modeling the data and plot the points and the graph on the same system of axes. $y=999.78 \cdot 0.78^{x}$.


## Exponential Mode

## Exponential Model

If a set of data has initial value $a$ and a constant percent change $r$ (in decimal) for equally spaced units $x$, the data can be modeled by the exponential function:

- $y=a(1+r)^{x}$ for exponential growth;
- $y=a(1-r)^{x}$ for exponential decay.

Example: Suppose that the present population of a city is 100,000 and that it will grow by $10 \%$ per year. Find an exponential model for the population $P$ as a function of time $t$.
By the theory above, with $r=0.1$, we get

$$
P(t)=100,000 \cdot 1.1^{t}
$$

where $t$ is the number of years from the present.

- Suppose inflation averages $4 \%$ per year for each year from 2000 to 2010. This means that an item that costs $\$ 1$ one year will cost $\$ 1.04$ one year later. In the second year, the $\$ 1.04$ cost will increase by a factor of 1.04 , to $(1.04)(1.04)=1.04^{2}$.
(a) Write an expression giving the cost $t$ years after 2000 of an item costing $\$ 1$ in 2000.
With $a=1$ and $r=0.04$, we get $1 \cdot 1.04^{t}=1.04^{t}$.
(b) Write an exponential function that models the cost of an item $t$ years from 2000 if its cost was $\$ 200$ in 2000.
As above, with $a=200$ and $r=0.04$, we get
$f(t)=200(1+0.04)^{t}=200 \cdot 1.04^{t}$.
(c) What will be the cost of the item in (b) in 2010?

$$
f(10)=200(1.04)^{10}=\$ 148.02
$$

## Comparing Models: Insurance Premiums

- The table gives monthly premiums required for a \$250,000 20-year term life insurance policy of female nonsmokers of different ages.

| Age | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Premium | 145 | 185 | 253 | 363 | 550 | 845 | 1593 | 2970 | 5820 |

(a) Find an exponential function for monthly premium versus the age of the person.
$y=4.304 \cdot 1.096^{x}$.
(b) Find a quadratic function that best fits the data.
$y=6.182 x^{2}-565.948 x+$ 12, 810.482.

(c) Graph the data and the functions in the window $[30,80]$ by $[-10,6800]$ and determine visually which model seems to be a better fit.

## Comparing Models: Facebook Users

- The table gives the number of millions of users of Facebook for the years 2004-2010.

| Year | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Users | 1 | 5 | 12 | 50 | 100 | 250 | 500 |

(a) Find an exponential function modeling the number in millions of Facebook users in terms of the number of years after 2000. $y=0.026 \cdot 2.776^{x}$.
(b) Find a power function that best fits the data.

$$
y=0.00008 x^{6.775}
$$

(c) Graph the data and the functions in the window $[4,11]$ by $[0,550]$ and determine visually which model seems to
 be a better fit.

## Logarithmic Models

## Logarithmic Model

A logarithmic model is used to model data that exhibit a rapid initial increase and, then, have a slow rate of growth:

$$
f(x)=a+b \ln x, \quad b>0
$$

If $b<0$, the graph of $f(x)=a+b \ln x$ is decreasing.
Example: Find a logarithmic function modeling the following data:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | 2 | 4.08 | 5.3 | 6.16 | 6.83 | 7.38 | 7.84 |

$$
y=3.00 \ln x+2.00
$$



## Modeling Data Using Several Models

- Make a scatter plot of the data in the table:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $y$ | 3.5 | 5.5 | 6.8 | 7.2 | 8 | 9 |

(a) Find a power function that models the data.
$y=3.671 \cdot x^{0.505}$.

(b) Find a quadratic function that models the data.
$y=-0.125 x^{2}+1.886 x+1.960$.
(c) Find a logarithmic function that models the data. $y=3.468+2.917 \ln x$.

## Diabetes

- The table shows the projected percent of U.S. adults with diabetes.

| Year | 2010 | 2015 | 2020 | 2025 | 2030 | 2035 | 2040 | 2045 | 2050 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Percent | 15.7 | 18.9 | 21.1 | 24.2 | 27.2 | 29.0 | 31.4 | 32.1 | 34.3 |

(a) Find a logarithmic model that fits the data with $x=0$ in 2000.

$$
y=-12.975+11.851 \ln x
$$

(b) Use the reported model to predict the percent of U.S. adults with diabetes in 2027.
$y(27)=-12.975+11.851 \cdot \ln 27 \approx 26.1$.
(c) In what year does this model predict the percent to be $16.9 \%$ ?

$$
\begin{aligned}
& 26.9=-12.975+11.851 \cdot \ln x \quad 3 \quad 39.875=11.851 \cdot \ln x \\
& \Rightarrow \quad 3.3647=\ln x \quad \Rightarrow \quad x=e^{3.3647} \approx 28.9 .
\end{aligned}
$$

So the percent will be $16.9 \%$ around 2029 .

## Poverty Threshold

- The following table gives the average poverty thresholds for one person for selected years from 1990 to 2005:

| Year | 1990 | 1995 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Income | 6652 | 7763 | 8316 | 8501 | 8794 | 9039 | 9182 | 9573 | 9827 | 10,160 |

(a) Use a logarithmic equation to model the data with $x$ number of years after 1980.
$y=-2179.067+3714.021 \cdot \ln x$
(b) Find an exponential model for the data.
$y=5069.388 \cdot 1.028^{x}$.
(c) Which model is the better fit for the data?


## Sexually Active Girls

- The percent of girls age $x$ or younger who have been sexually active is given in the table:

| Age | 15 | 16 | 17 | 18 | 19 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Percent | 5.4 | 12.6 | 27.1 | 44.0 | 62.9 | 73.6 |

(a) Create a logarithmic function that models the data, using an input equal to the age of the girls.
$y=-681.976+251.829 \ln x$.
(b) Use the model to estimate the percent of girls age 17 or younger who have been sexually active.
$y(17)=-681.976+$
$251.829 \ln 17 \approx 31.5$.

(c) Find a quadratic function that best fits the data.
$y=0.627 x^{2}-7.400 x-26.675$.
(d) Graph the two functions to determine which is the better model.

## Subsection 5

## Exponential Functions and Investing

## Future Value with Annual or Periodic Compounding

## Future Value of an Investment With Annual Compounding

If $P$ dollars are invested at an interest rate $r$ per year, compounded annually, the future value $S$ at the end of $t$ years is

$$
S=P(1+r)^{t}
$$

- The annual interest rate $r$ is also called the nominal interest rate or simply the rate.


## Future Value of an Investment With Periodic Compounding

If $P$ dollars are invested for $t$ years at the annual interest rate $r$, where the interest is compounded $k$ times per year, then the interest rate per period is $\frac{r}{k}$, the number of compounding periods is $k t$ and the future value that results is

$$
S=P\left(1+\frac{r}{k}\right)^{k t} \text { dollars. }
$$

## Future Value with Annual Compounding

- If $\$ 3,300$ is invested for $x$ years at $10 \%$ interest, compounded annually, the future value that results is $S=3300(1.10)^{x}$.
(a) Graph the function for $x=0$ to $x=8$.

(b) Use the graph to estimate when the money in the account will double. We want $S=6600$. We have $x \approx 7.27254$, i.e., it will take approximately 7 and a quarter years for the amount to double.


## Future Value with Quarterly Compounding

- If $\$ 10,000$ is invested at $12 \%$ interest compounded quarterly, find the future value in 10 years.
We have $P=10000, r=0.12, k=4$ and $t=10$. Therefore,

$$
S=10000\left(1+\frac{0.12}{4}\right)^{4 \cdot 10}=\$ 32,620.38
$$

- If \$ 10,000 is invested at $12 \%$ interest compounded daily, find the future value in 10 years.
We have $P=10000, r=0.12, k=365$ and $t=10$. Therefore,

$$
S=10000\left(1+\frac{0.12}{365}\right)^{365 \cdot 10}=\$ 33,194.62
$$

## Investments

- If $\$ 9,000$ is invested in an account that pays $8 \%$ interest rate compounded quarterly, find the future value of the investment:
(a) After 0.5 years.

We have $P=9000, r=0.08, k=4$ and $t=\frac{1}{2}$. Therefore,

$$
S=9000\left(1+\frac{0.08}{4}\right)^{4 \cdot(1 / 2)}=\$ 9,363.60
$$

(b) After 15 years.

We have $P=9000, r=0.08, k=4$ and $t=15$. Therefore,

$$
S=9000\left(1+\frac{0.08}{4}\right)^{4 \cdot 15}=\$ 29,529.28
$$

## Future Value With Continuous Compounding

## Future Value With Continuous Compounding

If $P$ dollars are invested for $t$ years at an annual interest rate $r$, compounded continuously, then the future value $S$ is given by

$$
S=P e^{r t} \text { dollars }
$$

Example: What is the future value of $\$ 5,000$ invested for 10 years at $7 \%$ compounded continuously? How much interest will be earned on the investment?

We apply the formula with $P=5000, r=0.07$ and $t=10$ :

$$
S=5000 e^{0.07 \cdot 10}=5000 e^{0.7}=\$ 10,068.80
$$

Thus, the interest amount is $10,068.80-5,000=\$ 5,068.80$.

## Continuous versus Annual Compounding

(a) For each of 7 years, compare the future value of an investment of $\$ 1000$ at $8 \%$, compounded annually, and of $\$ 1000$ at $8 \%$, compounded continuously.
For annual compounding: $S=1000(1+0.08)^{t}$, where $t$ is the year.
For continuous compounding: $S=1000 e^{0.08 t}$.

| Year | Annual | Continuous |
| :---: | ---: | ---: |
| 1 | 1080 | 1083.29 |
| 2 | 1166.4 | 1173.51 |
| 3 | 1259.71 | 1271.25 |
| 4 | 1360.49 | 1377.13 |
| 5 | 1469.33 | 1491.82 |
| 6 | 1586.87 | 1616.07 |
| 7 | 1713.92 | 1750.67 |


(b) Graph the functions for annual compounding and for continuous compounding for $t=30$ years on the same axes.
(c) What conclusion can be made regarding compounding annually and compounding continuously?
The investment increases faster under continuous compounding.

## Present Value of an Investment

## Present Value

The lump sum that will give future value $S$ in $n$ compounding periods at rate $i$ per period is the present value

$$
P=\frac{S}{(1+i)^{n}}=S(1+i)^{-n}
$$

Example: What lump sum must be invested at $8 \%$, compounded quarterly, for the investment to grow to $\$ 30,000$ in 12 years?
We have $S=30000, i=\frac{0.08}{4}=0.02$ and $n=4 \cdot 12=48$. Therefore,

$$
P=\frac{30000}{(1+0.02)^{48}}=\$ 11,596.13
$$

## College Tuition

- New parents want to put a lump sum into a money market fund to provide $\$ 30,000$ in 18 years to help pay for college tuition for their child. If the fund averages $10 \%$ per year, compounded monthly, how much should they invest?
We have $S=30000, i=\frac{0.1}{12}=0.0083333$ and $n=12 \cdot 18=216$.
Therefore, we have

$$
P=\frac{30000}{(1+0.0083333)^{216}}=\$ 4,996.09
$$

## Trust Fund

- Grandparents decide to put a lump sum of money into a trust fund on their granddaughter's 10th birthday so that she will have $\$ 1,000,000$ on her 60th birthday. If the fund pays $11 \%$, compounded monthly, how much money must they put in the account?
We have $S=1000000, i=\frac{0.11}{12}=0.00916667$ and $n=12 \cdot 50=600$. Therefore, we have

$$
P=\frac{1000000}{(1+0.00916667)^{600}}=\$ 4,190.46
$$

## Retirement

- George and Zoë inherit \$100,000 and plan to invest part of it for 25 years at $10 \%$, compounded monthly. If they want it to grow to $\$ 1$ million for their retirement, how much should they invest?
We have $S=1000000, i=\frac{0.10}{12}=0.0083333$ and $n=12 \cdot 25=300$. Therefore, we have

$$
P=\frac{1000000}{(1+0.0083333)^{300}}=\$ 82,939.75
$$

## Subsection 6

## Annuities; Loan Repayment

## Ordinary Annuities and Future Value

- An annuity is a financial plan characterized by regular payments.
- An ordinary annuity is a plan where equal payments are contributed at the end of each period to an account that pays a fixed rate of interest compounded at the same time as payments are made.


## Future Value of an Ordinary Annuity

If $R$ dollars are contributed at the end of each period for $n$ periods into an annuity that pays interest at rate $i$ at the end of the period, the future value of the annuity is

$$
\text { is } S=R\left(\frac{(1+i)^{n}-1}{i}\right) \text { dollars. }
$$

Example: Find the 5 -year future value of an ordinary annuity with a contribution of $\$ 500$ per quarter into an account that pays $8 \%$ per year, compounded quarterly.
The interest rate per quarter is $\frac{0.08}{4}=0.02$. The number of compounding periods is $4 \cdot 5=20$. Substituting the information into the formula, we get $S=500\left(\frac{(1+0.02)^{20}-1}{0.02}\right)=12,148.68$.

## Down Payment

- To start a new business, Beth deposits $\$ 1,000$ at the end of each 6 -month period in an account that pays $8 \%$, compounded semiannually. How much will she have at the end of 8 years?
The interest rate per quarter is $\frac{0.08}{2}=0.04$. The number of compounding periods is $2 \cdot 8=16$. Substituting the information into the formula, we get

$$
S=1000\left(\frac{(1+0.04)^{16}-1}{0.04}\right)=21,824.5
$$

## Future Value

- Find the value of an annuity of $\$ 2,600$ paid at the end of each 3 -month period for 5 years, if interest is earned at $6 \%$ compounded quarterly.
The interest rate per quarter is $\frac{0.03}{4}=0.015$. The number of compounding periods is $4 \cdot 5=20$. Substituting the information into the formula, we get

$$
S=2600\left(\frac{(1+0.015)^{20}-1}{0.015}\right)=60,121.5 .
$$

## Retirement

- Mr. Bekele invests $\$ 600$ at the end of each month in an account that pays $7 \%$ compounded monthly. How much will be in the account in 25 years?
The interest rate per quarter is $\frac{0.07}{12}=0.00583333$. The number of compounding periods is $12 \cdot 25=300$. Substituting the information into the formula, we get

$$
S=600\left(\frac{(1+0.00583333)^{300}-1}{0.00583333}\right)=486,043
$$

## Present Value of an Annuity

## Present Value of an Ordinary Annuity

If a payment of $R$ dollars is to be made at the end of each period for $n$ periods from an account that earns interest at a rate of $i$ per period, then the account is an ordinary annuity and its present value is

$$
A=R\left(\frac{1-(1+i)^{-n}}{i}\right)
$$

Example: Suppose a retiring couple establishes an annuity that will provide $\$ 2,000$ at the end of each month for 20 years. If the annuity earns $6 \%$, compounded monthly, how much did the couple put in the account to establish the annuity?
We have $R=2000, n=12 \cdot 20=240$ and $i=\frac{0.06}{12}=0.005$. Now the formula yields: $A=2000\left(\frac{1-(1+0.005)^{-240}}{0.005}\right)=279,161.54$.

## Lottery Winnings

- The winner of a "million dollar" lottery is to receive $\$ 50,000$ plus $\$ 50,000$ at the end for each year for 19 years or the present value of this annuity in cash. How much cash would she receive if money is worth $8 \%$, compounded annually?
We have $R=50,000, i=0.08$ and $n=19$, plus the original $\$ 50,000$. Therefore, the total amount in cash should be

$$
50000+50000\left(\frac{1-(1+0.08)^{-19}}{0.08}\right)=530180
$$

## College Tuition

- A couple wants to establish a fund that will provide $\$ 3,000$ for tuition at the end of each 6 -month period for 4 years. If a lump sum can be placed in an account that pays $8 \%$ compounded semiannually, what lump sum is required?
We have $R=3000, i=\frac{0.08}{2}=0.04$ and $n=2 \cdot 4=8$. Therefore, the total amount in cash should be

$$
A=3000\left(\frac{1-(1+0.04)^{-8}}{0.04}\right)=20,198.23
$$

## Disability Insurance

- A man is disabled in an accident and wants to receive an insurance payment that will provide him with $\$ 3,000$ at the end of each month for 30 years. If the payment can be placed in an account that pays $9 \%$ compounded monthly, what size payment should he seek?
We have $R=3000, i=\frac{0.09}{12}=0.0075$ and $n=12 \cdot 30=360$. Therefore, the total amount in cash should be

$$
A=3000\left(\frac{1-(1+0.0075)^{-360}}{0.0075}\right)=372,846
$$

## Loan Repayment and Amortization

- When money is borrowed, the borrower must repay the total amount that was borrowed (debt) plus the interest on the debt.
- Most loans require regular payments on the debt plus payment of interest on the unpaid balance of the loan.
- Two popular repayment plans are:
- An equal amount is applied to the debt each payment period plus the interest for the period, which is the interest for the unpaid balance. This payment method leads to decreasing payments as the unpaid balance decreases.
- All payments are of equal size, a method called amortization.


## Amortization Formula

If a debt of $A$ dollars, with interest at a rate of $i$ per period, is amortized by $n$ equal periodic payments made at the end of each period, then the size of each payment is

$$
R=A\left(\frac{i}{1-(1+i)^{-n}}\right)
$$

## Home Mortgage

- A couple that wants to purchase a home with a price of $\$ 230,000$ has $\$ 50,000$ for a down payment. Suppose they can get a 25 -year mortgage at $9 \%$ per year on the unpaid balance.
(a) What will be their equal monthly payments?

We have $A=180000, i=\frac{0.09}{12}=0.0075, n=12 \cdot 25=300$. Thus,
$R=180000\left(\frac{0.0075}{1-(1+0.0075)^{-300}}\right)=1510.55$.
(b) What is the total amount they will pay before they own the house outright?
The total amount will be the down payment plus their total amount after all monthly payments:

$$
50000+300 \cdot 1510.55=\$ 503,168
$$

(c) How much interest will they pay? $503,168-230,000=\$ 273,168$.

## Auto Loan

- A man wants to buy a car and can afford to pay $\$ 400$ per month.
(a) If he can get a loan for 48 months with interest at $12 \%$ per year on the unpaid balance and make monthly payments, how much can he pay for the car?
We have $R=400, i=\frac{0.12}{12}=0.01, n=48$. Therefore,
$A=400\left(\frac{1-(1+0.01)^{-48}}{0.01}\right)=\$ 15,189.58$.
(b) What is the total amount paid over the life of the loan?

The total amount will be $48 \cdot 400=\$ 19,200$.
(c) What is the total interest paid on the loan?

The total interest is $19200-15,189.58=\$ 4,010.42$.

## Business Loan

- Business partners want to purchase a restaurant that costs $\$ 750,000$. They have $\$ 300,000$ for a down payment, and they can get a 25 -year business loan for the remaining funds at $8 \%$ per year on the unpaid balance, with quarterly payments.
(a) What will be the payments?

We have $A=450000, i=\frac{0.08}{4}=0.02$ and $n=4 \cdot 25=100$. Therefore,

$$
R=450000\left(\frac{0.02}{\left(1-(1+0.02)^{-100}\right)}\right)=\$ 10,441.23
$$

(b) What is the total amount that they will pay over the 25 -year period?

The total amount will be

$$
300000+100 \cdot 10,441.23=\$ 1,344,123 .
$$

(c) How much interest will they pay over the life of the loan?

They will pay $1,344,123-750,000=\$ 594,123$.

