College Algebra

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LSSU Math 111

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- Solution of Polynomial Equations
- Fundamental Theorem of Algebra
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Subsection 1

Higher Degree Polynomial Functions

Higher-Degree Polynomial Functions

- We have already looked at linear functions f(x) = ax + b and at quadratic functions f(x) = ax² + bx + c. Both are types of polynomial functions.
- **Higher-degree polynomial functions** are those with degree higher than 2.

Example:
$$f(x) = x^3 - 16x^2 - 2x$$
, $g(x) = 3x^4 - x^3 + 2$,
 $h(x) = 2x^5 - x^3$.

• A cubic function is one of the form

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0.$$

• A quartic function is one of the form

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0.$$

Example: The function f above is a cubic, the function g is a quartic and h is a **quintic**.

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Graphs of Polynomials of Odd Degree

• The graph of a polynomial of odd degree *n* has:

- at most n-1 turning points;
- at most *n x*-intercepts;
- asymptotically one end opening up and one end opening down.

Example: Sketch the graph of $f(x) = 2x^3 - 3x^2 - 6x$ and discuss the points above.

The two turning points are $x = \frac{1-\sqrt{5}}{2}$ and $x = \frac{1+\sqrt{5}}{2}$. The three zeros are x = 0 and $x = \frac{3\pm\sqrt{57}}{4}$. The graph opens down on the left and opens up on the right.



Graphs of Polynomials of Even Degree

• The graph of a polynomial of even degree *n* has:

- at most n-1 turning points;
- at most n x-intercepts;
- asymptotically both ends opening up or both ends opening down.

Example: Sketch the graph of $f(x) = 3x^4 - 12x^2$ and discuss the points above.

The three turning points are x = 0 and $x = \pm \sqrt{2}$. The three zeros are x = 0 and $x = \pm 2$. The graph opens up both on the left and on the right.



Graphing a Cubic

(a) Graph $y = x^3 + 4x^2 + 5$ on a window that shows a local maximum and a local minimum.



(b) At what point does a local maximum occur? It occurs at (-⁸/₃, ³⁹¹/₂₇).
(c) At what point does a local minimum occur? It occurs at (0,5).

Graphing a Quartic

(a) Graph $y = x^4 - 8x^2$ on a window that shows a local maximum and a local minimum.



(b) At what point does a local maximum occur? It occurs at (0,0).

(c) At what point do local minima occur? They occur at (-2, -16) and (2, -16).

Weekly Revenue

A firm has total weekly revenue in dollars for its product given by R(x) = 2800x - 8x² - x³, where x is the number of units sold.
 (a) Graph the function in [0, 50] by [0, 51000].



(b) Find the max revenue and the number of units giving the max revenue. The max revenue is \$50,176 and occurs at x = 28 units.
(c) Find a window showing the *complete graph* and the interval of x values, for x ≥ 0, over which the graph is increasing. The graph is increasing for 0 < x < 28.

Drunk Driving Fatalities

• Using data for 1982-2005, the total number of fatalities in drunk driving crashes in South Carolina can be modeled by

 $y = -0.0395x^4 + 2.101x^3 - 35.079x^2 + 194.109x + 100.148,$

where x is the number of years after 1980.

a) Graph the function in a window representing the years 1980-2005.



b) How many fatalities occurred in 2005?

y(25) = 426.94 fatalities.

(c) Which year had the maximum and which year the minimum number of fatalities?

The year 2013 had the maximum 518 fatalities and the year 1993 the minimum 183 fatalities.

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Subsection 2

Modeling with Cubic and Quartic Functions

Fitting a Cubic Model

Consider the following data:

Find the cubic function that models the data.

Input the data in your calculator by using STAT and Edit. Go to STAT CALC and do CubicReg to find the cubic that best fits the data: $3 - 2^2$

$$y = x^3 - 2x^2.$$



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• The table gives data on the number of Internet users in millions:

Year	1995	96	97	98	99	00	01	02	03	04	05	06	07
Users	16	36	70	147	248	361	536	598	719	817	1018	1093	1215

Letting x be equal to number of years after 1990 and y millions of users, find a cubic that best fits the data. We perform cubic regression to find the cubic that best fits the data: $y = -0.613x^3 + 23.835x^2 - 179.586x + 385.670.$

Use the model to predict the number of users in 2013. y(23) = 1405 million users.

(c) Compare the graphs of the data and the model to determine if the model is a good fit for the data.



Social Security Beneficiaries

• The table gives data on the number of Social Security beneficiaries in millions for selected years from 1950 to 2000, with projections through 2030:

Year	1950	1960	1970	1980	1990	2000	2010	2020	2030
Beneficiaries	2.9	14.3	25.2	35.1	39.5	44.8	53.3	68.8	82.7

(a) Letting x be equal to number of years after 1950 and y millions of beneficiaries, find a cubic that best fits the data.
 We do cubic regression to find the cubic that best fits the data:

 $y = 0.0002x^3 - 0.0264x^2 + 1.6019x + 2.1990.$

(b) Compare the graphs of the data and the model to determine if the model is a good fit for the data.



Fitting a Quartic

• Consider the following data:

(a) Find the quartic function that is the best fit for the data.
 We do Quartic Regression to find the quartic that best fits the data:

$$y = x^4 - 4x^2 - 3x + 1.$$

(b) Compare the graphs of the data and the model to determine if the model is a good fit for the data.



Cubic or Quartic Model?

(a) Make a scatter plot of the data in the table below:



(b) Does it appear that a cubic or a quartic model is the better fit? A cubic model. We draw both a cubic and a quartic model for comparison.

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Marriage Age

(a) Find the quartic function y = f(x) that is the best fit for the data with y the age at first marriage for women and x number of years after 1900.

	Year	19	00	191	0 3	1920	193	30	194() 19	50	196	0
W	omen	21	L.9	21.	6	21.2	21	.3	21.5	5 2	0.3	20.	3
	Ye	ear	197	70	1980) 19	990	200	0 2	2004	20	09	
	Wom	en	20	.8	22.0) 2	3.9	25.	1	25.8	25	.9	

We do quartic regression to find the quartic that best fits the data: $y = -0.00000313x^4 + 0.00007764x^3 - 0.00501512x^2 + 0.07115109x + 21.64611390.$

- (b) What does the unrounded model predict to be the age at first marriage for women in 2012?
 - y(112) = 26.6 years old.

Teen Alcohol Use

• The percent of U.S. 12th graders who had used alcohol in the last 12 months during the years 1991-2006 is given in

Year	s	1991	1992	1993	1994	1995	1996	1997	1998				
Percen	t	77.7	76.8	72.7	73.0	73.7	72.5	74.8	74.3				
Year	s	1999	2000	2001	2002	2003	2004	2005	2006				
Percen	t	73.8	73.2	73.3	71.5	70.1	70.6	68.6	66.5				
Find the quartic that is the best fit using $x = 0$ for 1990.													

 $y = 0.002x^4 - 0.075x^3 + 1.002x^2 - 5.177x + 82.325.$

- (b) Use the model to find a local minimum on the graph for $0 \le x \le 15$. The minimum is y = 73.1854 at x = 4.69365.
- (c) Is there an absolute minimum percent between 1990 and 2010 according to the graph?
 Yes! it is y = 65.9111 and occurs at x = 17.7232.



Computing Third Differences

- Consider the following data $\frac{x \ 1 \ 2 \ 3 \ 4 \ 5 \ 6}{y \ 3 \ 14 \ 47 \ 114 \ 227 \ 398}$
 - (a) Use third and/or fourth differences to determine if each set of data can be modeled by a cubic or a quartic function. We create a difference table:

Outputs	3		14		47		114		227		398
First Differences		11		33		67		113		171	
Second Differences			22		34		46		58		
Third Differences				12		12		12			

- Since the third differences are equal the data fit exactly the graph of a cubic function.
- (b) Find the cubic or quartic that fits the data.

$$f(x) = 2x^3 - x^2 + 2.$$



Computing Fourth Differences

- Consider the following data $\frac{x \ 1 \ 2 \ 3 \ 4 \ 5 \ 6}{y \ 4 \ 4 \ 18 \ 88 \ 280 \ 684}$
 - (a) Use third and/or fourth differences to determine if each set of data can be modeled by a cubic or a quartic function. We create a difference table:

Outputs	4		4		18		88		280		684
First Differences		0		14		70		192		404	
Second Differences			14		56		122		212		
Third Differences				42		66		90			
Fourth Differences					24		24				

Since the fourth differences are equal the data fit exactly the graph of a quartic function.

(b) Find the cubic or quartic that fits the data.

$$f(x) = x^4 - 3x^3 + 6x.$$



Subsection 3

Solution of Polynomial Equations

Factors, Zeros, Intercepts and Solutions

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial function. Then, the following four statements regarding f are equivalent:
 - (x a) is a factor of f(x).
 - a is a real zero of the function f.
 - a is a real solution of the equation f(x) = 0.
 - a is an x-intercept of the graph of y = f(x).
 - The graph crosses the x-axis at the point (a, 0).
 - Example: Suppose P(x) = (3x + 1)(2x 1)(x + 5). Find: (a) The zeros of P(x).
 - Since $P(x) = 3(x + \frac{1}{3})2(x \frac{1}{2})(x + 5)$, the nonconstant factors of P(x) are $(x + \frac{1}{3})$, $(x \frac{1}{2})$ and (x + 5). Hence, the zeros of P(x) are $x = -\frac{1}{3}$, $x = \frac{1}{2}$ and x = -5.
 - (b) The solutions of P(x) = 0. These are also the zeros of P(x). So, they are x = -5, $x = -\frac{1}{3}$ and $x = \frac{1}{2}$.
 - (c) The x-intercepts of the graph of y = P(x). Again these are identical with the zeros of P(x).

Factors, Zeros, Intercepts and Solutions: Revenue

- If the price from the sale of x units of a product is given by the function p = 100x x², the revenue function for the product is R = px = (100x x²)x.
 - (a) Find the numbers of units that must be sold to give zero revenue. We must solve the equation

 $R = 0 \Rightarrow (100x - x^2)x = 0 \Rightarrow x(100 - x)x = 0$. Since the factors of R are x and (100 - x), the solutions are x = 0 and x = 100.

(b) Does the graph of the revenue function verify this solution? We graph y = R(x) and seek the x-intercepts.



Constructing a Box

- A box can be constructed by cutting a square out od each corner of a piece of cardboard and folding the sides up. If the piece of cardboard is 12 inches by 12 inches and each side that is cut out has length x, the function giving the volume is $V(x) = 144x 48x^2 + 4x^3$.
 - (a) Find the values of x that make V = 0. We factor: $V(x) = 144x - 48x^2 + 4x^3 = 4x(x^2 - 12x + 36) = 4x(x-6)^2$. Thus, the values that zero the volume are x = 0 and x = 6.
 - (b) For each value in (a), what happens if squares of length x are cut out? If a square of length 0 is cut out, the box will have height 0. If a square of length 6 is cut out, the box will have length and width equal to 0.
 - (c) For what values of x does a meaningful box exist?

Only for 0 < x < 6 does a meaningful box exist.

(d) Graph the function giving V in terms of x, only for the interval that makes sense in this application.



Solution Using Factoring By Grouping I

• Solve the following equations using factoring by grouping: (a) $x^3 + 5x^2 - 4x - 20 = 0$

We group two pairs of terms:

$$\begin{array}{l} x^{3}+5x^{2}-4x-20=0 \quad \Rightarrow \quad x^{2}(x+5)-4(x+5)=0 \\ \Rightarrow \quad (x+5)(x^{2}-4)=0 \quad \Rightarrow \quad (x+5)(x+2)(x-2)=0. \end{array}$$

Thus, we get the solutions x = -5, x = -2 and x = 2. (b) $4x^3 + 8x^2 - 36x - 72 = 0$ We group two pairs of terms:

$$\begin{array}{l} 4x^3 + 8x^2 - 36x - 72 = 0 \quad \Rightarrow \quad 4(x^3 + 2x^2 - 9x - 18) = 0 \\ \Rightarrow \quad 4[x^2(x+2) - 9(x+2)] = 0 \quad \Rightarrow \quad 4(x+2)(x^2 - 9) = 0 \\ \Rightarrow \quad (x+2)(x+3)(x-3) = 0. \end{array}$$

Thus, we get the solutions x = -3, x = -2 and x = 3.

Factoring By Grouping: Cost

 The total cost of producing a product is given by the function C(x) = x³ - 12x² + 3x + 9 in thousands of dollars, where x is the number of hundreds of units produced. How many units must be produced to give a total cost of \$45,000? We need to solve the equation C(x) = 45 for x.

$$C(x) = 45 \implies x^3 - 12x^2 + 3x + 9 = 45$$

$$\Rightarrow x^3 - 12x^2 + 3x - 36 = 0$$

$$\Rightarrow x^2(x - 12) + 3(x - 12) = 0$$

$$\Rightarrow (x - 12)(x^2 + 3) = 0$$

$$\Rightarrow x = 12.$$

Therefore, the number of items to be produced is 1200.

The Root Method

The Root Method

The real solutions of the equation $x^n = C$ are found by taking the *n*-th root of both sides:

$$x = \sqrt[n]{C}$$
, if *n* is odd, and $x = \pm \sqrt[n]{C}$, if *n* is even and $C \ge 0$.

Example: Solve the following equations:
(a)
$$3x^3 - 81 = 0$$
.
We get
 $3x^3 - 81 = 0 \Rightarrow 3x^3 = 81 \Rightarrow x^3 = 27$
 $\Rightarrow x = \sqrt[3]{27} \Rightarrow x = 3$.
(b) $2x^4 - 162 = 0$.
We get
 $2x^4 - 162 = 0 \Rightarrow 2x^4 = 162 \Rightarrow x^4 = 81$
 $\Rightarrow x = \pm \sqrt[4]{81} \Rightarrow x = -3 \text{ or } x = 3$.

Factoring and Root Method

 Use factoring and the root method to solve the following equations:
 (a) ¹/₂x³ - ²⁵/₂x = 0 We factor:

$$\begin{array}{rl} \frac{1}{2}x^3 - \frac{25}{2}x = 0 & \Rightarrow & \frac{1}{2}x(x^2 - 25) = 0\\ \Rightarrow & x = 0 \text{ or } x^2 - 25 = 0 & \Rightarrow & x = 0 \text{ or } x^2 = 25\\ \Rightarrow & x = 0 \text{ or } x = \pm\sqrt{25} & \Rightarrow & x = 0 \text{ or } x = \pm 5. \end{array}$$

(b)
$$3x^4 - 24x^2 = 0$$

We factor
 $3x^4 - 24x^2 = 0 \Rightarrow 3x^2(x^2 - 8) = 0$
 $\Rightarrow x = 0 \text{ or } x^2 = 8 \Rightarrow x = 0 \text{ or } x = \pm\sqrt{8}$
 $\Rightarrow x = 0 \text{ or } x = \pm 2\sqrt{2}.$
(c) $0.2x^3 - 24x = 0$
We factor
 $0.2x^3 - 24x = 0 \Rightarrow 0.2x(x^2 - 120) = 0$
 $\Rightarrow x = 0 \text{ or } x^2 - 120 = 0 \Rightarrow x = 0 \text{ or } x^2 = 120$
 $\Rightarrow x = 0 \text{ or } x = \pm\sqrt{120} \Rightarrow x = 0 \text{ or } x = \pm2\sqrt{30}.$

Future Value

- The future value of \$5,000 invested for 4 years at rate r, compounded annually, is given by $S = 5000(1 + r)^4$.
 - (a) Graph the function in the window [0, 0.24] by [0, 12, 000].



(b) Use the root method to find the rate *r*, as a percent, for which the future value is \$10,368.

$$S = 10368 \Rightarrow 5000(1+r)^4 = 10368 \Rightarrow (1+r)^4 = \frac{10,368}{5000} \Rightarrow 1+r = +\sqrt[4]{\frac{10368}{5000}} \Rightarrow r = \sqrt[4]{\frac{10368}{5000}} - 1.$$
 Using a calculator: $r = 0.2$. So the percent rate is 20%.

(c) What rate as a percent gives \$2,320.50 in interest on this investment? $S - 5000 = 2320.50 \Rightarrow S = 7320.50 \Rightarrow 5000(1 + r)^4 = 7320.50 \Rightarrow$ $(1 + r)^4 = \frac{7320.50}{5000} \Rightarrow 1 + r = \sqrt[4]{\frac{7320.50}{5000}} \Rightarrow r = \sqrt[4]{\frac{7320.50}{5000}} - 1$. We find 10%.

Foreign-Born Population

• The percent of the U.S. population that was foreign born during the years 1900-2010 is given by the function

 $y = 0.0000384x^3 - 0.00397x^2 - 0.03829x + 14.58102,$

x is the number of years after 1900. Use graphical methods to find the year after 1900 when 16% of the population would be foreign born. We want to solve y = 16. Since this is equivalent to y - 16 = 0, the problem can be solved graphically in two ways:

- 1. Either graph y' = y 16 and find the x-intercept.
- 2. Or graph y and y = 16 and find the point of intersection.



We find that this happens in the year 2015.

Executions

• The numbers of executions carried out in the United States for selected years from 1990 through 2008 can be modeled by

$$y = 0.00894x^4 - 0.344x^3 + 3.653x^2 - 5.648x + 25.077,$$

x number of years after 1990. In what year does the model indicate that 85 executions would occur?

We want to solve y = 85. Since this is equivalent to y - 85 = 0, the problem can be solved graphically in two ways:

- 1. Either graph y' = y 85 and find the *x*-intercept.
- 2. Or graph y and y = 85 and find the point of intersection.



We find that this happens in the year 2012.

Subsection 4

Fundamental Theorem of Algebra

Synthetic Division: Example

Use synthetic division to find the quotient and the remainder of the division (x⁴ + 6x³ + 5x² − 4x + 2) ÷ (x + 2).
 We have

Therefore,

$$\frac{x^4 + 6x^3 + 5x^2 - 4x + 2}{x + 2} = x^3 + 4x^2 - 3x + 2 + \frac{-2}{x + 2}.$$

Synthetic Division: Example II

Use synthetic division to find the quotient and the remainder of the division (2x³ − 9x − 27) ÷ (x − 3).

We have

Therefore,

$$\frac{2x^3 - 9x - 27}{x - 3} = 2x^2 + 6x + 9.$$

Checking Solutions Using Synthetic Division

• Determine whether -5 is a solution of the polynomial equation $x^4 + 3x^3 - 10x^2 + 8x + 40 = 0$.

We divide $(x^4 + 3x^3 - 10x^2 + 8x + 40) \div (x + 5)$.

- If the remainder is = 0, then -5 is a solution.
- If the remainder is \neq 0, then -5 is not a solution.

We have

Since the remainder is 0, -5 is a solution of the given equation.

Solving Equations Using Synthetic Division

Given that −2 and 4 are solutions of x⁴ + 2x³ - 21x² - 22x = -40, use synthetic division to find all remaining solutions of the equation. First, we divide (x⁴ + 2x³ - 21x² - 22x + 40) ÷ (x + 2):

Now, we divide $(x^3 - 21x + 20) \div (x - 4)$: Putting together our work, we get:

$$\begin{aligned} x^4 + 2x^3 - 21x^2 - 22x + 40 &= (x+2)(x^3 - 21x + 20) \\ &= (x+2)(x-4)(x^2 + 4x - 5) \\ &= (x+2)(x-4)(x+5)(x-1). \end{aligned}$$

Therefore, the remaining solutions are x = -5 and x = 1.

Finding One Root Graphically I

Find one solution of x³ + 3x² - 18x - 40 = 0 graphically and the remaining using synthetic division.
 We graph the function f(x) = x³ + 3x² - 18x - 40 and observe that x = 4 may be a root of the polynomial.



We verify that this is the case using synthetic division: Thus, we get that

$$\begin{array}{rcl} x^3 + 3x^2 - 18x - 40 & = & (x - 4)(x^2 + 7x + 10) \\ & = & (x - 4)(x + 5)(x + 2). \end{array}$$

Thus, the remaining solutions are x = -5 and x = -2.

Finding One Root Graphically II

Find one solution of 4x³ + x² - 27x + 18 = 0 graphically and the remaining using synthetic division.
 We graph the function f(x) = 4x³ + x² - 27x + 18 and observe that x = 2 may be a root of the polynomial.



We verify that this is the case using synthetic division: Thus, we get that

$$\begin{array}{rcl} 4x^3 + x^2 - 27x + 18 & = & (x-2)(4x^2 + 9x - 9) \\ & = & (x-2)(4x - 3)(x + 3). \end{array}$$

Thus, the remaining solutions are x = -3 and $x = \frac{3}{4}$.

Break-Even Point

- The profit function in dollars for a product is given by $P(x) = -0.2x^3 + 66x^2 1600x 60000$, where x is the number of units produced and sold. Suppose break-even occurs when 50 units are produced and sold.
 - (a) Use synthetic division to find a quadratic factor of P(x).
 At the break-even point P(x) = 0. Thus, we perform synthetic division to get:

Thus, $P(x) = (x - 50)(-0.2x^2 + 56x + 1200)$.

(b) Use factoring to find the number of units other than 50 that give break-even for the product. We have

$$P(x) = (x-50)(-0.2x^2+56x+1200) = (x-50)(-0.2x-4)(x-300).$$

Thus, x = 300 gives another break-even point.

Revenue

• The revenue from the sale of a product is given by $R(x) = 1810x - 81x^2 - x^3$. If the sale of 9 units gives a total revenue of \$9000, use synthetic division to find another number of units that will give \$9,000 in revenue.

We know that R(9) = 9000 or, equivalently, R(9) - 9000 = 0. We are seeking an additional solution of the equation R(x) - 9000 = 0. We perform the synthetic division:

We get $R(x) - 9000 = (x - 9)(-x^2 - 90x + 1000) = (x - 9)(-x - 100)(x - 10)$. Thus, the other quantity that will give \$9,000 revenue is x = 10 units.

The Rational Solution Test

The Rational Solution Test

The rational solutions of

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0,$$

with integer coefficients must be of the form $\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of a_n , the leading coefficient.

• Example: Find all possible rational solutions of the polynomial $x^3 - 6x^2 + 5x + 12 = 0$.

The factors of the constant term 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. Moreover, the factors of the leading coefficient 1 are ± 1 . Therefore, the only possible rational solutions are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

• Example: Find all possible rational solutions of the polynomial $4x^3 + 5x^2 - 9x + 2 = 0$.

The factors of the constant term 2 are $\pm 1, \pm 2$. Moreover, the factors of the leading coefficient 4 are $\pm 1, \pm 2, \pm 4$. Therefore, the only possible rational solutions are $\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}$.

Solving Cubic and Quartic Equations of the Form f(x) = 0

• We execute the following steps:

- 1. Determine the possible rational solutions of f(x) = 0.
- 2. Graph y = f(x) to see if any of these values are solutions.
- 3. Find the factors associated with those values.
- 4. Use synthetic division to confirm that those values are zeros and find additional factors.
- 5. Use the remaining quadratic factors to find additional solutions.

Example: Solve the equation $6x^4 - x^3 - 42x^2 - 29x + 6 = 0$.

The factors of the constant term are $\pm 1, \pm 2, \pm 3, \pm 6$. Similarly, the factors of the leading coefficient 6 are $\pm 1, \pm 2, \pm 3, \pm 6$. Thus, the possible rational solutions are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6}$.

Fundamental Theorem of Algebra

Example (Cont'd)

By graphing, we may see that x = -2 and x = 3 seem to be solutions. These correspond to the factors x + 2 and x - 3. We verify using synthetic division:



Thus, we get

$$6x^4 - x^3 - 42x^2 - 29x + 6 = (x+2)(6x^3 - 13x^2 - 16x + 3)$$

= (x+2)(x-3)(6x^2 + 5x - 1)
= (x+2)(x-3)(x+1)(6x - 1).

Hence, the four zeros are x = -2, x = 3, x = -1 and $x = \frac{1}{6}$.

The Fundamental Theorem of Algebra and Complex Zeros

The Fundamental Theorem of Algebra

If f(x) is a polynomial of degree $n \ge 1$, then f has at least one complex zero.

Complex Zeros

Every polynomial function f(x) of degree $n \ge 1$ has exactly *n* complex zeros. Some may be imaginary and some may be repeated. Nonreal zeros occur in conjugate pairs a + bi and a - bi.

Example: Solve the equation $2z^3 + 3z^2 + 3z + 2 = 0$ exactly in the complex number system.

The equation has three solutions. Since the nonreal solutions come in conjugate pairs, the equation must have one real solution.

Example (Cont'd)

• By graphing, we see that z = -1 seems to be a real solution.



We verify by performing synthetic division:

Hence, we get $2z^3 + 3z^2 + 3z + 2 = (z + 1)(2z^2 + z + 2)$. We find the remaining two solutions using the quadratic formula:

$$z = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{-1 \pm i\sqrt{15}}{4}.$$

Subsection 5

Rational Functions and Rational Equations

Rational Functions and Domains

A rational function f is a function of the form f(x) = P(x)/Q(x), where P(x), Q(x) are polynomials with Q(x) ≠ 0.
 Example: The following functions are rational functions:

$$f(x) = \frac{7}{x-5}, \quad g(x) = \frac{x-9}{x^2-8x+12}, \quad h(x) = \frac{x^4+20}{x^3-16x}.$$

- The domain of a rational function f(x) = P(x)/Q(x) is the set of all real numbers that are not roots of the polynomial Q(x).
 Example: Find the domain of the functions f, g and h given above.
 - For f, we must have $x 5 \neq 0$. Note that $x 5 = 0 \Rightarrow x = 5$. Therefore, $Dom(f) = \mathbb{R} - \{5\}$.
 - For g, we must have $x^2 8x + 12 \neq 0$. Note that $x^2 8x + 12 = 0 \Rightarrow (x-2)(x-6) = 0 \Rightarrow x = 2$ or x = 6. Thus, $Dom(g) = \mathbb{R} \{2, 6\}$.
 - For h, we must have $x^3 16x \neq 0$. Note that $x^3 16x = 0 \Rightarrow x(x^2 16) = 0 \Rightarrow x(x 4)(x + 4) = 0 \Rightarrow x = 0$ or x = 4 or x = -4. Therefore, Dom $(h) = \mathbb{R} - \{-4, 0, 4\}$.

Vertical Asymptotes

- Let f(x) = P(x)/Q(x) be a rational function. A vertical asymptote of y = f(x) is a vertical line which the graph approaches infinitely close without ever touching.
- The vertical asymptotes of f(x) = P(x)/Q(x) occur at those values of x where Q(x) = 0 and P(x) ≠ 0, i.e., at the zeros of Q(x) that are not zeros of P(x).

Example: Find the equations of the vertical asymptotes to the graph of $f(x) = \frac{2x-5}{x}$.

$$3 - x$$

Set 3 - x = 0 and solve. We get x = 3. Check whether x = 3 zeros the numerator. Since it does not, the vertical line x = 3 is a vertical asymptote of $f(x) = \frac{2x-5}{3-x}$.

Vertical Asymptotes: Second Example

 Find the equations of the vertical asymptotes to the graph of f(x) = x-3/x²-25. Then use your calculator to sketch the graph of y = f(x) in the interval [-10, 10] adjusting your window accordingly. First, solve the equation x² - 25 = 0. We have (x - 5)(x + 5) = 0 ⇒ x = -5 or x = 5. Since neither value zeros the numerator, we conclude that both vertical lines x = -5 and x = 5 are vertical asymptotes to y = f(x).

The following is a sketch of the graph of y = f(x):



Horizontal Asymptotes

- Consider again the rational function f(x) = P(x)/Q(x). A horizontal line that the graph approaches infinitely close (without touching) when x → ±∞, is called a horizontal asymptote of y = f(x).
- If y approaches a as x approaches +∞ or as x approaches -∞, then the graph of y = f(x) has a horizontal asymptote y = a.
- The following rules apply in determining whether

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}, a_n, b_m \neq 0,$$

has a horizontal asymptote:

- 1. If n < m, y = 0 is a horizontal asymptote.
- 2. If n = m, $y = \frac{a_n}{b_m}$ is a horizontal asymptote.
- 3. If n > m, there is no horizontal asymptote.

Horizontal Asymptotes: Example I

• For the following rational functions, give the equations of the horizontal asymptotes, if any:

$$f(x) = \frac{2x^5 - 6x^3 + 7x}{4x^2 + 7x - 3}, \quad g(x) = \frac{x - 4}{x^7 + 9x^3 - 21}, \quad h(x) = \frac{6x^2 - 3}{2x^2 + x - 1}.$$

- In f, the degree of the numerator is greater than the degree of the denominator. Therefore, y = f(x) does not have a horizontal asymptote.
- In g, the degree of the denominator is larger than the degree of the numerator. Thus, y = g(x) has horizontal asymptote y = 0.
- In *h*, the degrees of the numerator and of the denominator are equal. Therefore, $y = \frac{6}{2} = 3$ is a horizontal asymptote.

Horizontal Asymptotes: Example II

• Consider the function $f(x) = \frac{2x-5}{x^2-1}$.

(a) Find the vertical and the horizontal asymptotes of y = f(x).

- For the vertical asymptotes, solve x² − 1 = 0. We get (x − 1)(x + 1) = 0 ⇒ x = −1 or x = 1. Neither −1 nor 1 zeros the numerator. Hence y = f(x) has vertical asymptotes x = −1 and x = 1.
- Since the denominator has higher degree than the numerator, y = 0 is a horizontal asymptote.
- (b) Use your calculator to sketch the graph of y = f(x) adjusting your window to clearly show the asymptotes.



Slant Asymptotes

- Consider again the rational function $f(x) = \frac{P(x)}{Q(x)}$.
- If the degree of the numerator is larger than that of the denominator then *f* does not have a horizontal asymptote.
- However, in the case when the degree of the numerator is one more than the degree of the denominator, the graph approaches a **slant asymptote**, i.e., a non-horizontal line.
- To find the slant asymptote:
 - 1. Divide the numerator by the denominator to get a linear function plus a rational expression with constant numerator.
 - 2. The slant asymptote is the linear function determined by the quotient of the division.

Slant Asymptotes: Example |

• Find the slant asymptote of $f(x) = \frac{-3x^2 + 2x}{x - 3}$. Then, use your calculator to sketch its graph, adding the slant asymptote also.



Slant Asymptotes: Example II

• Find the slant asymptote of $f(x) = \frac{2x^2 - 3}{x - 4}$. Then, use your calculator to sketch its graph, adding the slant asymptote also.



Holes

- We have seen that the rational function $f(x) = \frac{P(x)}{Q(x)}$ has a vertical asymptote x = a if a is a zero of Q(x) and not a zero of P(x).
- What happens if a is a zero of both P(x) and Q(x)? In that case, it turns out that the graph of f(x) has a "hole" at x = a. Example: Consider the rational function f(x) = x²-9/x+3. Note that x = -3 is a root of the denominator. However, it is also a root of the numerator. In this case, x = -3 is not a vertical asymptote.

After factoring and simplifying, we see that f(x) = x - 3, but $Dom(f) = \mathbb{R} - \{-3\}$, i.e., the graph of f is the same as that of the straight line y = x - 3 except that it has a hole at x = -3.



Another Example of Holes

• Consider the rational function $f(x) = \frac{x^2 + x - 2}{x - 1}$.

Note that x = 1 is a root of the denominator. It is also a root of the numerator. Thus, x = 1 is not a vertical asymptote.

After factoring and simplifying, we see that $f(x) = \frac{(x+2)(x-1)}{x-1} = x + 2$. But, $Dom(f) = \mathbb{R} - \{1\}$. So the graph of f is the same as that of the straight line y = x+2except that it has a hole at x = 1.



Rational Equations

Solve the Rational Equation

$$2 = \frac{2+4x}{x^2+1}.$$

Multiply both sides by $x^2 + 1$:

$$2(x^{2}+1) = (x^{2}+1)\frac{2+4x}{x^{2}+1} \quad \Rightarrow \quad 2x^{2}+2 = 2+4x.$$

Make one side zero and factor the other side:

$$2x^2-4x=0 \quad \Rightarrow \quad 2x(x-2)=0.$$

Use the zero factor property:

$$x = 0$$
 or $x - 2 = 0 \Rightarrow x = 0$ or $x = 2$.

Check the validity of the solutions:

• For
$$x = 0$$
: $2 = \frac{2+4 \cdot 0}{0^2 + 1} \checkmark$
• For $x = 2$: $2 = \frac{2+4 \cdot 2}{2^2 + 1} \checkmark$

Rational Equations II

Solve the Rational Equation

$$\frac{x}{x-2} - x = 1 + \frac{2}{x-2}$$

Multiply both sides by x - 2:

$$(x-2)(\frac{x}{x-2}-x) = (x-2)(1+\frac{2}{x-2}) \Rightarrow x-x(x-2) = x-2+2.$$

Make one side zero and factor the other side:

$$x - x^2 + 2x = x \quad \Rightarrow \quad x^2 - 2x = 0 \quad \Rightarrow \quad x(x - 2) = 0$$

Use the zero factor property:

$$x = 0$$
 or $x - 2 = 0 \Rightarrow x = 0$ or $x = 2$.

Check the validity of the solutions:

• For
$$x = 0$$
: $\frac{0}{0-2} - 0 = 1 + \frac{2}{0-2} \checkmark$
• For $x = 2$: $\frac{2}{2-2} - 2 = 1 + \frac{2}{2-2} \checkmark$

Rational Equations III

Solve the Rational Equation

$$x^2 + \frac{x}{x-1} = x + \frac{x^3}{x-1}$$

Multiply both sides by x - 1:

$$(x-1)(x^2+\frac{x}{x-1}) = (x-1)(x+\frac{x^3}{x-1}) \Rightarrow x^2(x-1)+x = x(x-1)+x^3.$$

Make one side zero and factor the other side:

$$x^{3} - x^{2} + x = x^{2} - x + x^{3} \Rightarrow 2x^{2} - 2x = 0 \Rightarrow 2x(x - 1) = 0.$$

Use the zero factor property:

$$x = 0 \text{ or } x - 1 = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 1.$$

Check the validity of the solutions:

• For x = 0: $0^2 + \frac{0}{0-1} = 0 + \frac{0^3}{0-1}$ • For x = 2: $1^2 + \frac{1}{1-1} = 1 + \frac{1^3}{1-1}$

Sales and Training

• During the first 3 months of employment, the monthly sales S is thousands of dollars for an average new salesperson depend on the number of hours of training x, according to

$$S(x) = \frac{40}{x} + \frac{x}{4} + 10$$
, for $x \ge 4$.

(a) Combine the terms to create a rational function.

$$S(x) = \frac{40}{x} + \frac{x}{4} + 10 = \frac{160}{4x} + \frac{x^2}{4x} + \frac{40x}{4x} = \frac{x^2 + 40x + 160}{4x}.$$

(b) How many hours of training result in monthly sales of \$21,000? We must solve S(x) = 21:

$$\frac{x^2 + 40x + 160}{4x} = 21 \implies x^2 + 40x + 160 = 84x$$

$$\Rightarrow x^2 - 44x + 160 = 0 \implies (x - 4)(x - 40) = 0$$

$$\Rightarrow x = 4 \text{ or } x = 40.$$

Worker Productivity

- Suppose the average time in hours that a new production team takes to assemble 1 unit of a product is given by $H(t) = \frac{5+3t}{2t+1}$, where t is the number of days of training for the team.
 - Graph the function using a window [0, 20] by [0, 8].



 What is the horizontal asymptote and what is its meaning?

It is $H = \frac{3}{2}$. It represents the optimal number of hours to assemble 1 unit after the team has become very experienced.

- Find the number of days of training necessary to reduce the production time to 1.6 hours.
 We must solve H(t) = 1.6 = ⁸/₅: We have ^{5+3t}/_{2t+1} = ⁸/₅ ⇒ 5(5+3t) =
 - $8(2t+1) \Rightarrow 25+15t = 16t+8 \Rightarrow t = 17$ days.

Subsection 6

Polynomial and Rational Inequalities

Example |

Solve the polynomial inequality

$$3x^3 + 3x^2 - 12x \ge 12.$$

By subtracting 12, we get $3x^3 + 3x^2 - 12x - 12 \ge 0$. Factor out 3 to get $3(x^3 + x^2 - 4x - 4) \ge 0$. Divide by 3 > 0: $x^3 + x^2 - 4x - 4 \ge 0$. By using grouping, $x^2(x + 1) - 4(x + 1) \ge 0$, implying $(x + 1)(x^2 - 4) \ge 0$, or $(x + 1)(x - 2)(x + 2) \ge 0$. The equation (x + 1)(x - 2)(x + 2) = 0 has roots -2, -1 and 2. Using these create the sign table for (x + 1)(x - 2)(x + 2):

Since we want $(x + 1)(x - 2)(x + 2) \ge 0$, we use the intervals that make (x + 1)(x - 2)(x + 2) positive! So the solution set is $(-2, -1) \cup (2, +\infty)$.

Example II

Solve the polynomial inequality

$$2x^2 + 12x \le x - 5.$$

Make one of the two sides zero: $2x^2 + 12x - x + 5 \le 0$. This gives $2x^2 + 11x + 5 < 0$. Factor the left hand side $(2x + 1)(x + 5) \le 0$. The equation (2x + 1)(x + 5) = 0 has two roots -5 and $-\frac{1}{2}$. Using these, create the sign table for (2x + 1)(x + 5):

Thus, since we want $(2x + 1)(x + 5) \le 0$, we must choose the interval where the expression is either negative or 0. So, the solution set is $[-5, -\frac{1}{2}]$.

The Method for Solving Polynomial Inequalities

• To solve a polynomial inequality algebraically:

- Write an equivalent inequality with zero on one side and with a function f(x) on the other side.
- Solve f(x) = 0.
- Create a sign table:
 - Use the solutions in Step 2 to divide the real line into intervals;
 - Use test points to determine the sign of f(x) in each of these intervals.
- Identify which intervals satisfy the original inequality and write your solution in interval notation.

Example III

Solve the inequality

$$x^3 + 10x^2 + 25x < 0.$$

Factor the left hand side: $x(x^2 + 10x + 25) < 0$ and, then, $x(x+5)^2 < 0$.

The equation $x(x+5)^2 < 0$ has roots -5 and 0. Using these, create the sign table for $x(x+5)^2$:

Since we want $x(x+5)^2 < 0$, we pick those intervals where the expression is negative. Thus, the solution set is $(-\infty, -5) \cup (-5, 0)$.

Constructing a Box

- A box can be formed by cutting a square out of each corner of a piece of tin and folding the sides up. If the piece of tin is 12 inches by 12 inches and each side of the square that is cut out has length x, the volume of the box is $V(x) = 144x 48x^2 + 4x^3$.
 - a) Use factoring and then find the values of x that give positive values for V(x).

We have V(x) > 0 is equivalent to $144x - 48x^2 + 4x^3 > 0$, or $4x(36 - 12x + x^2) > 0$, i.e., $4x(x - 6)^2 > 0$. Now construct the sign table for the expression $4x(x - 6)^2$.

Hence, we get that $4x(x+6)^2 > 0$, for all x in $(0,6) \cup (6,\infty)$.

(b) Which of the values of x that give positive values for V(x) result in a box?

The only values that result in a box are 0 < x < 6.

Rational Inequalities: Example I

Solve the inequality

$$\frac{x-3}{x+1} \ge 3.$$

Make one of the two sides 0: $\frac{x-3}{x+1} - 3 \ge 0$. Add or subtract with common denominators to obtain a single fraction and simplify: $\frac{x-3}{x+1} - \frac{3(x+1)}{x+1} \ge 0 \Rightarrow \frac{x-3-3(x+1)}{x+1} \ge 0 \Rightarrow \frac{x-3-3(x+1)}{x+1} \ge 0 \Rightarrow \frac{x-3-3(x+1)}{x+1} \ge 0 \Rightarrow \frac{x-3-3(x+1)}{x+1} \ge 0 \Rightarrow \frac{x+3}{x+1} \le 0$. Find the numbers that zero the numerator: x = -3. Find the numbers that zero the denominator: x = -1. Create the sign table for the expression $\frac{x+3}{x+1}$:

Thus, since we want $\frac{x+3}{x+1} \le 0$, we have to pick the intervals where the fraction is negative or zero: x in [-3, -1).

Rational Inequalities: Example II

Solve the inequality

$$\frac{x}{2} + \frac{x-2}{x+1} \le 1.$$

• Subtract 1: $\frac{x}{2} + \frac{x-2}{x+1} - 1 \le 0$. Take common denominators: $\frac{x(x+1)}{2(x+1)} + \frac{2(x-2)}{2(x+1)} - \frac{2(x+1)}{2(x+1)} \le 0$. Add: $\frac{x^2+x+2x-4-2x-2}{2(x+1)} \le 0$. Simplify: $\frac{x^2+x-6}{2(x+1)} \le 0$. Factor and multiply by 2: $\frac{(x+3)(x-2)}{x+1} \le 0$. Use the roots of the numerator and the denominator -3, -1 and 2. Create the sign table for $\frac{(x+3)(x-2)}{x+1}$:

Since we want $\frac{(x+3)(x-2)}{x+1} \leq 0$, we keep the intervals with the negative signs and the zeros. Thus x in $(-\infty, -3] \cup (-1, 2]$.

Average Cost

• The average cost per set for the production of 42 inch plasma TVs is given by $\overline{C}(x) = \frac{5000+80x+x^2}{x}$, where x is the number of hundreds of units produced. Find the number of TVs that must be produced to keep the average cost to at most \$590 per TV. We must ensure

$$\begin{array}{lll} \frac{5000+80x+x^{2}}{x} \leq 590 & \Rightarrow & \frac{5000+80x+x^{2}}{x} - 590 \leq 0\\ \Rightarrow & \frac{5000+80x+x^{2}}{x} - \frac{590x}{x} \leq 0 & \Rightarrow & \frac{5000-510x+x^{2}}{x} \leq 0\\ \Rightarrow & \frac{(x-500)(x-10)}{x} \leq 0. \end{array}$$

The roots of the numerator are 10,500 and of the denominator 0. We use this to create the sign table for $\frac{(x-500)(x-10)}{x}$:

$$\begin{array}{c|c|c|c|c|c|c|c|c|} & x < 0 & 0 < x < 10 & 10 < x < 500 & x > 500 \\ \hline \hline (x-500)(x-10) \\ x & - & + & - & + \end{array}$$

Thus, taking into account x > 0, we must have $10 \le x \le 500$.