Introduction to Combinatorial Mathematics

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Subsection 1

Introduction

Problems of Enumeration

- When groups of dots and dashes are used to represent alphanumeric symbols in telegraph communication, a communication engineer may wish to know the total number of distinct representations consisting of a fixed number of dots and dashes.
- To study the physical properties of materials, a physicist may wish to compute the number of ways molecules can be arranged in molecular sites or to compute the number of ways electrons are distributed among different energy levels.
- A transportation engineer may wish to determine the number of different acceptable train schedules.
- A computer scientist may wish to have some idea about the number of possible moves his chess-playing program should examine in responding to each of the opponent's moves.
- Enumerating such problems is the main topic in the basic theory of **combinations** and **permutations**.

Selections and Arrangements

- The words **selection** and **arrangement** will be used in the ordinary sense: There should be no ambiguity in the meanings of statements such as

  "to select two representatives from five candidates",
  "there are 10 possible outcomes when
  two representatives are selected from five candidates",
  "the books are arranged on the shelf",
  "there are 120 ways to arrange five different books on the shelf".

- The word **combination** has the same meaning as the word selection.
- The word **permutation** has the same meaning as the word arrangement.

Combinations and Permutations

- An r-**combination** of n objects is defined as an unordered selection of r of these objects.
- An r-**permutation** of n objects is defined as an ordered arrangement of r of these objects.

  Example: To form a committee of 20 senators is an unordered selection of 20 senators from the 100 senators. It is therefore a 20-*combination* of the 100 senators.

  Example: The outcome of a horse race can be viewed as an ordered arrangement of the t horses in the race. It is therefore a t-*permutation* of the t horses.

Number of Combinations and Number of Permutations

- We are interested in enumerating the number of combinations or permutations of a given set of objects.
  - The notation $C(n, r)$ denotes the number of r-combinations of n distinct objects.
  - The notation $P(n, r)$ denotes the number of r-permutations of n distinct objects.
- We compute easily the following:
  - $C(n, n) = 1$ (there is just one way to select n objects out of n objects);
  - $C(n, 1) = n$ (there are n ways to select one object out of n objects);
  - $C(3, 2) = 3$ (for three objects A, B, and C, the selections of two objects are AB, AC, and BC);
  - $P(3, 2) = 6$ (for three objects A, B, and C, the arrangements of two objects are AB, BA, AC, CA, BC, and CB).

Subsection 2

The Rules of Sum and Product

The Rule of Product and the Rule of Sum

- Among the five Roman letters a, b, c, d and e and the three Greek letters α, β and γ, it is clear that there are $5 \times 3 = 15$ ways to select two letters, one from each alphabet.

Rule of Product

If one event can occur in m ways and another event can occur in n ways, there are $m \times n$ ways in which these two events can occur.

- On the other hand, since there are five ways to select a Roman letter and three ways to select a Greek letter, there are $5 + 3 = 8$ ways to select one letter that is either a Roman or a Greek letter.

Rule of Sum

If one event can occur in m ways and another event can occur in n ways, there are $m + n$ ways in which one of these two events can occur.

- The occurrence of an event can mean either the selection or the arrangement of a certain number of objects.

Example: Choosing Books

- We want to choose two books of different languages among 5 books in Latin, 7 books in Greek, and 10 books in French.

  There are

  - 5×7 ways to choose a book in Latin and a book in Greek;
  - 5×10 ways to choose a book in Latin and a book in French;
  - 7×10 ways to choose a book in Greek and a book in French.

  Thus, there are

  $$5 \times 7 + 5 \times 10 + 7 \times 10 = 155$$

  ways to choose two books of different languages.

- If we just want to choose two books from the twenty-two books, there are $22 \times 21 = 462$ ways.

Formula Relating Combinations and Permutations

Claim:

$$P(n,r) = P(r,r) \times C(n,r).$$

We can make an ordered arrangement of r out of n distinct objects by:

- first selecting r objects from the n objects, which can be done in $C(n,r)$ ways;
- then arranging these r objects in order, which can be done in $P(r,r)$ ways.

By the rule of product, we get

$$P(n,r) = P(r,r) \cdot C(n,r).$$

Recursive Formula for Combinations

Claim:

$$C(n, r) = C(n-1, r-1) + C(n-1, r).$$

This can be seen from the following argument: Suppose that one of the n distinct objects is marked as a special object.

We can select r objects from these n objects by either:

- selecting $r-1$ objects so that the special object is always included, which can be done in $C(n-1, r-1)$ ways, or
- selecting r objects so that the special object is always excluded, which can be done in $C(n-1, r)$ ways.

By the rule of sum, we get that

$$C(n, r) = C(n-1, r-1) + C(n-1, r).$$

Subsection 3

Permutations

Closed Formula for $P(n, r)$

- We derive an expression for $P(n, r)$, the number of ways of arranging r of n distinct objects:
  - Arranging r of n objects into some order is the same as putting r of the n objects into r distinct (marked) positions. There are:
    - n ways to fill the first position (to choose one out of the n objects);
    - $n-1$ ways to fill the second position (to choose one out of the $n-1$ remaining objects); ...
    - $n-r+1$ ways to fill the last position (to choose one out of the $n-r+1$ remaining objects).
  - According to the rule of product, we have

$$P(n, r) = n(n-1)\cdots(n-r+1).$$

- Using the notation $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$, for $n > 1$ ($n!$ is read **n factorial**),

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Inductive Proof of of $P(n, n)$

Claim: $P(n, n) = n!$.

We show $P(n, n) = n!$ using induction on n.

As the basis of induction, $P(1, 1) = 1 = 1!$.

As the induction hypothesis, suppose that $P(n-1, n-1) = (n-1)!$.

To arrange n distinct objects in order:

- We single out a special object.
- Arrange the remaining $n-1$ objects first, which can be done in $P(n-1, n-1)$ ways.
- For each ordered arrangement of these $n-1$ objects, there are n positions for the special object (the $n-2$ positions between the arranged objects and the two end positions).

Thus, by the Product Rule,

$$P(n, n) = n \cdot P(n-1, n-1) = n \cdot (n-1)! = n!.$$

Alternative Derivation of $P(n,r)$

Claim: $P(n,r) = \frac{n!}{(n-r)!}$.

Suppose that we divide n marked positions into two groups, the first r positions and the remaining $n-r$ positions. To place the n objects in these n positions, we work as follows:

- First, put r of the n objects in the first r positions, which can be done in $P(n,r)$ ways;
- Then put the remaining $n-r$ objects in the remaining $n-r$ positions, which can be done in $P(n-r,n-r)$ ways.

Thus, by the Product Rule

$$P(n,n) = P(n,r) \cdot P(n-r,n-r),$$

i.e.,

$$P(n,r) = \frac{P(n,n)}{P(n-r,n-r)} \stackrel{\text{Prev. Slide}}{=} \frac{n!}{(n-r)!}.$$

Circular Arrangements

- In how many ways can n people stand to form a ring?
  There is a difference between a linear arrangement and a circular arrangement of objects:
  - In the case of circular arrangement, the n people are not assigned to absolute positions, but are only arranged relative to one another.

  Method 1
  : If the n people are arranged linearly and then the two ends of the line are closed to form a circular arrangement, we have a total of $P(n, n)$ such arrangements. Two of the circular arrangements obtained in this manner are actually the same if one can be changed into the second by rotating by one position, or two positions, ..., or n positions. So the number of circular arrangements is equal to $\frac{P(n,n)}{n} = (n-1)!$

  Method 2
  : If we pick a particular person and let him occupy a fixed position, the remaining $n-1$ people will be arranged using this fixed position as reference in a ring. Again, there are $(n-1)!$ ways of arranging these $n-1$ people.

Arrangements of Non-distinct Objects

Claim: Let there be n objects not all distinct. Specifically, let there be: q_1 objects of the first kind; q_2 objects of the second kind; …; q_t objects of the t-th kind. Then the number of n-permutations of these n objects is given by the formula

$$P(n; q_1, q_2, \ldots, q_t) = \frac{n!}{q_1! q_2! \cdots q_t!}.$$

Imagine that the n objects are marked so that objects of the same kind become distinguishable from one another. Then to permute these n "distinct" objects we can:

- First, permute the n objects before marking then, which, can be done in $P(n; q_1, \ldots, q_t)$ ways;
- Then, permute the marked objects of each group among their respectively occupied positions, which can be done in $q_1! q_2! \cdots q_t!$ ways.

By the Product Rule, we get $n! = P(n; q_1, q_2, \ldots, q_t) \cdot q_1! q_2! \cdots q_t!$.

Example: Dashes and Dots

- In how many different ways can five dashes and eight dots be arranged?
  They can be arranged in $P(13;5,8) = \frac{13!}{5!8!} = 1,287$ different ways.
- In how many different ways can seven symbols among five dashes and eight dots be arranged?
  We can use one of the following:
  - 0 dashes and 7 dots, which can be arranged in $P(7;0,7)$ ways;
  - 1 dash and 6 dots, which can be arranged in $P(7;1,6)$ ways;
  - 2 dashes and 5 dots, which can be arranged in $P(7;2,5)$ ways;
  - 3 dashes and 4 dots, which can be arranged in $P(7;3,4)$ ways;
  - 4 dashes and 3 dots, which can be arranged in $P(7;4,3)$ ways;
  - 5 dashes and 2 dots, which can be arranged in $P(7;5,2)$ ways.

  By the Sum Rule, we have:
  $$\frac{7!}{0!7!} + \frac{7!}{1!6!} + \frac{7!}{2!5!} + \frac{7!}{3!4!} + \frac{7!}{4!3!} + \frac{7!}{5!2!} = 120$$
  distinct representations.

A Divisibility Relation

Claim: $(k!)!$ is divisible by $(k!)^{(k-1)!}$ for any integer k.

Consider a collection of $k!$ objects among which there are:

- k of the first kind;
- k of the second kind;

  $\vdots$
- k of the $(k-1)!$-th kind.

The total number of ways of permuting these objects is given by

$$P(k!; \underbrace{k, k, \ldots, k}_{(k-1)! \text{ types}}) = \frac{(k!)!}{(k!)^{(k-1)!}}.$$

Since the total number of permutations must be an integral value, $(k!)^{(k-1)!}$ must divide $(k!)!$.

Arrangements with Repetitions

Claim: The number of ways to arrange r objects when they are selected out of n distinct objects with unlimited repetitions is n^r.

There are:

- n ways to choose an object to fill the first position;
- n ways to choose an object to fill the second position;

  $\vdots$
- n ways to choose an object to fill the r-th position.

By the Product Rule there are

$$\underbrace{n \cdot n \cdots n}_{r \text{ factors}} = n^r$$

ways to arrange r objects when they are selected out of n distinct objects with unlimited repetitions.

Example

- Among the 10 billion numbers between 1 and 10,000,000,000, how many of them contain the digit 1 and how many of them do not?

  Among the 10 billion numbers between 0 and 9,999,999,999, there are 9^{10} numbers that do not contain the digit 1.

  Therefore, among the 10 billion numbers between 1 and 10,000,000,000, there are $9^{10} - 1$ numbers that do not contain the digit 1.

  Hence, there are $10^{10} - (9^{10} - 1)$ numbers that do contain the digit 1.

Binary Sequences

- A **binary sequence** is a sequence of 0's and 1's.
- What is the number of n-digit binary sequences that contain an even number of 0's (zero is considered as an even number)?

  Method 1

  The problem is immediately solved if we observe that, because of symmetry, half of the 2^n n-digit binary sequences contain an even number of 0's, and the other half of the sequences contain an odd number of 0's.

  Method 2

  Consider the 2^{n-1} $(n-1)$-digit binary sequences.

  - If an $(n-1)$-digit binary sequence contains an even number of 0's, we can append to it a 1 as the n-th digit to yield an n-digit binary sequence that contains an even number of 0's.
  - If an $(n-1)$-digit binary sequence contains an odd number of 0's, we can append to it a 0 as the n-th digit to yield an n-digit binary sequence that contains an even number of 0's.

Therefore, there are 2^{n-1} n-digit binary sequences which contain an even number of 0's.

Quaternary Sequences

- A n-**quaternary sequence** is a sequence of length n that has 0's, 1's, 2's and 3's as digits.
- How many quaternary sequences are there in each of which the total number of 0's and 1's is even?

  Because of symmetry, there are $\frac{4^n}{2}$ such sequences.

Quaternary Sequences (Cont'd)

- Find the number of quaternary sequences that contain an even number of 0's.

  We divide the 4^n sequences into two groups:
  - The 2^n sequences that contain only 2's and 3's;
  - The $4^n - 2^n$ sequences that contain one or more 0's or 1's.

  In the first group, all sequences have an even number of 0's.

  The sequences in the second group can be subdivided into categories according to the patterns of 2's and 3's in the sequences.

  E.g., sequences of the pattern $23\times\times2\times3\times\times\times$ will be in one category, where the $\times$'s are 0's and 1's.

  Half of the sequences in each category have an even number of 0's.

  So, the number of sequences with an even number of 0's in this group is $\frac{4^n-2^n}{2}$.

  Among the 4^n n-digit quaternary sequences, there are $2^n + \frac{4^n-2^n}{2}$ with an even number of 0's.

Subsection 4

Combinations

Number of Combinations

- We saw that the number of r-combinations of n objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

- This formula yields

$$C(n,r) = C(n,n-r).$$

This was expected since selecting r objects out of n objects is equivalent to picking the $n-r$ objects that are not to be selected.

Example: A Convex Decagon

- If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?
  - First, there are
    - $C(10,2) = 45$ straight lines joining the pairs of vertices;
    - Of these 45 lines 10 are the sides of the decagon.

    So the number of diagonals is $C(10,2) - 10 = 45 - 10 = 35$.
  - Since for every four vertices we can count exactly one intersection between the diagonals (the decagon is convex), there is a total of $C(10,4) = 210$ intersections between the diagonals.
- Note that:
  - a diagonal is divided into $k+1$ straight-line segments when there are k intersecting points lying along it;
  - each intersecting point lies along two diagonals.

  So the total number of straight-line segments into which the diagonals are divided is $35 + 2 \times 210 = 455$.

Example: Scientists, Locks and Keys

- Eleven scientists are working on a secret project. They wish to lock up the documents in a cabinet, such that the cabinet can be opened if and only if six or more of the scientists are present.
  - What is the smallest number of locks needed?
  - What is the smallest number of keys each scientist must carry?
- For any group of five scientists, there must be at least one lock they cannot open. For any two different groups of five scientists, there must be two different locks they cannot open, because if both groups cannot open the same lock, there is a group of six scientists among these two groups who will not be able to open the cabinet. Thus, at least $C(11,5) = 462$ locks are needed.
- Whenever scientist A is associated with a group of five others, A should have the key to the lock(s) that these five scientists were not able to open. Thus, A carries at least $C(10,5) = 252$ keys.

Example: Divisibility

- In how many ways can three numbers be selected from the numbers $1, 2, \ldots, 300$ such that their sum is divisible by 3?
  The 300 numbers $1, 2, \ldots, 300$ can be divided into three groups:
  - Those that are divisible by 3;
  - Those that yield the remainder 1 when divided by 3;
  - Those that yield the remainder 2 when divided by 3.

  Clearly, there are 100 numbers in each of these groups.
  The sum will be divisible by 3 if:
  - three numbers from the first group are selected, or
  - three numbers from the second group are selected, or
  - three numbers from the third group are selected, or
  - three numbers, one from each of the three groups, are selected.

  Thus, the total number of ways to select three desired numbers is

$$C(100, 3) + C(100, 3) + C(100, 3) + (100)^3 = 1,485,100.$$

Selections With Repetitions

Claim: When repetitions in the selection of the objects are allowed, the number of ways of selecting r objects from n distinct objects is $C(n+r-1, r)$.

Let the n objects be $1, 2, \ldots, n$. Let a selection of r objects be identified by a list of the corresponding integers $\{i, j, k, \ldots, m\}$ arranged in increasing order.

E.g., the selection with first object selected thrice, the second not selected, the third once, the fourth once, the fifth twice, etc., is represented as $\{1, 1, 1, 3, 4, 5, 5, \ldots\}$.

To the r integers in such a list we add:

- 0 to the first integer;
- 1 to the second integer; ...;
- $r-1$ to the r-th integer.

Thus, $\{i, j, k, \ldots, m\}$ becomes $\{i, j+1, k+2, \ldots, m+(r-1)\}$.

E.g., $\{1, 1, 1, 3, 4, 5, 5, \ldots\}$ becomes $\{1, 2, 3, 6, 8, 10, 11, \ldots\}$.

Each selection is identified uniquely as a selection of r distinct integers from the integers $1, 2, \ldots, n+(r-1)$, whence the formula follows.

Example: Coins and Dice

- Out of a large number of pennies, nickels, dimes, and quarters, in how many ways can six coins be selected?
  The number of ways is the same as the number of selecting six coins from a penny, a nickle, a dime, and a quarter with unlimited repetitions. So it is
  $$C(4+6-1,6) = C(9,6) = 84.$$
- When three distinct dice are rolled, the number of outcomes is $6 \times 6 \times 6 = 216$.
  If the three dice are indistinguishable, the number of outcomes is the number of selections of three numbers from the six numbers 1, 2, 3, 4, 5, 6 when repetitions are allowed. So it is
  $$C(6+3-1,3) = 56.$$

Selections of Non-Distinct Objects

Claim: When the objects are not all distinct, the number of ways to select one or more objects from them is equal to

$$(q_1+1)(q_2+1)\cdots(q_t+1)-1,$$

where there are q_1 objects of the first kind, q_2 objects of the second kind, ..., q_t objects of the t-th kind.

This result follows directly from the rule of product.

- There are q_1+1 ways of choosing the object of the first kind, namely, choosing none of them, one of them, two of them, ..., or q_1 of them;
- There are q_2+1 ways of choosing objects of the second kind;

  ⋮
- q_t+1 ways of choosing objects of the t-th kind.

We subtract 1 for the "selection" in which no object is chosen.

Example

- How many divisors does the number 1400 have?

  The number of divisors equals the number of ways to select the prime factors of 1400.

  Since $1400 = 2^3 \cdot 5^2 \cdot 7$, the number of its divisors is

$$(3+1)(2+1)(1+1) = 24.$$

Example: Weights I

- For n given weights, what is the greatest number of different amounts that can be made up by the combinations of these weights?

  Since a weight can either be selected or not be selected in a combination, there are $2^n - 1$ combinations.

  If the values of the given weights are properly chosen, we will, at the most, have $2^n - 1$ different combined weights.

Example: Weights I (Cont'd)

- Which scheme of weights achieves the upper bound?
  Consider the collection of n weights
  $$w_0 = 2^0 = 1, w_1 = 2^1 = 2, w_2 = 2^2 = 4, \ldots, w_{n-1} = 2^{n-1}.$$
  Suppose two of the $2^n - 1$ nonempty collections of weights have the same combined weight. Denote the two collections by $\{w_{i_1}, \ldots, w_{i_k}\}$ and $\{w_{j_1}, \ldots, w_{j_\ell}\}$, where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_\ell$. Then, by hypothesis, $2^{i_1} + \cdots + 2^{i_k} = 2^{j_1} + \cdots + 2^{j_\ell}$. If $i_k \neq j_\ell$, say $i_k < j_\ell$, then we have
  $$\begin{array}{rcl} 2^{i_1} + \cdots + 2^{i_k} & \leq & 2^0 + 2^1 + \cdots + 2^{i_k} = 2^{i_k+1} - 1 \\ & < & 2^{j_\ell} < 2^{j_1} + \cdots + 2^{j_\ell}. \end{array}$$
  This contradicts the hypothesis $2^{i_1} + \cdots + 2^{i_k} = 2^{j_1} + \cdots + 2^{j_\ell}$. So we must have $i_k = j_\ell$.
  Now we get $2^{i_1} + \cdots + 2^{i_{k-1}} = 2^{j_1} + \cdots + 2^{j_{\ell-1}}$.
  Repeating, we get that $k = \ell$ and $i_p = j_p$, for all $p = 1, \ldots, k$.

Example: Weights II

- What is the greatest number of different amounts that can be weighed by using a set of n weights and a balance?

  Each of the weights may be disposed of in one of three ways:
  - It can be placed in the weight pan;
  - In the pan with the substance to be weighed;
  - It can be unused.

  Therefore, there are $3^n - 1$ ways of using the n weights.

  In at least half of these $3^n - 1$ ways, the total weight placed in the weight pan is less than or equal to the total weight placed in the other pan. Hence, we can weigh at most $\frac{3^n-1}{2}$ different amounts.
- Similarly to the previous example, a scheme of weights achieving this upper bound is

$$w_0 = 3^0 = 1, w_1 = 3^1 = 3, \ldots, w_{n-1} = 3^{n-1}.$$

Subsection 5

Distributions of Distinct Objects

Distributing Distinct Objects into Distinct Positions

- In placing r distinct objects into n distinct cells, two cases must be considered:

  Claim: For $n \geq r$, there are $P(n, r)$ ways to place r distinct objects into n distinct cells, where each cell can hold only one object.

  - The first object can be placed in one of the n cells;
  - The second object can be placed in one of the $n - 1$ remaining cells;

    $\vdots$
  - The r-th object can be placed in one of the $n - r + 1$ remaining cells.

  Claim: For $r > n$, there are $P(r, n)$ ways to place n of r distinct objects into n distinct cells, where each cell can hold only one object.

  - There are r ways to select an object to be placed in the first cell;
  - There are $r - 1$ ways to select an object to be placed in the second cell;

    $\vdots$
  - There are $r - n + 1$ ways to select an object to be placed in the r-th cell.

Multiple Objects in a Cell (Unordered)

- The distribution of r distinct objects in n distinct cells, where each cell can hold any number of objects, is equivalent to the arrangement of r of the n cells when repetitions are allowed.

  Claim: The number of ways, regardless of whether n is larger or smaller than r, is n^r.

  - The first object can be placed in one of the n cells;
  - The second object can again be placed in one of the n cells;

    $\vdots$
  - The r-th object can be placed in one of the n cells.
- Note that, here, when more than one object is placed in the same cell, the objects are not ordered inside the cell.

Multiple Objects in a Cell (Ordered) Method I

Claim: The number of ways of distributing r distinct objects in n distinct cells, where each cell can hold any number of objects, and the order of objects in a cell is also considered, is

$$\frac{(n+r-1)!}{(n-1)!} = (n+r-1)(n+r-2)\cdots(n+1)n.$$

Method 1: Imagine such a distribution as an ordered arrangement of the r (distinct) objects and the $n-1$ (nondistinct) intercell partitions. Using the previously derived formula for the permutation of $r+n-1$ objects where $n-1$ of them are of the same kind, we obtain the result

$$P(n+r-1, n-1) = \frac{(n+r-1)!}{(n-1)!}.$$

Multiple Objects in a Cell (Ordered) Method II

- The distribution of r distinct objects in n distinct cells, where each cell can hold any number of objects and the objects in a cell are ordered, is $\frac{(n+r-1)!}{(n-1)!} = (n+r-1)(n+r-2)\cdots(n+1)n$.

  Method 2:
  - There are n ways to distribute the first object.
  - After the first object is placed in a cell, it can be considered as an added partition that divides the cell into two cells.
    Therefore, there are $n+1$ ways to distribute the second object.
  - Similarly, there are $n+2$ ways to distribute the third object.

    $\vdots$
  - Finally, there are $n+r-1$ ways to distribute the r-th object.

  By the Product Rule, we get a total of
  $(n+r-1)(n+r-2)\cdots(n+1)n$ ways.

Example: Flags on Masts

- The number of ways of arranging seven flags on five masts when all the flags must be displayed but not all the masts have to be used is

$$5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11.$$

  The argument is as follows:
  - If there is a single flag on a mast, we assume that it is raised to the top of the mast.
  - However, if there is more than one flag on a mast, the order of the flags on the mast is important.

  So the number is equal to the number of ways of placing 7 distinct objects in 5 distinct cells with the objects in a single cell ordered, which is equal to $P(7+5-1, 5-1) = \frac{(7+5-1)!}{(5-1)!}$.
- Similarly, seven cars can go through five toll booths in $5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11$ ways

Distributing Objects of Various Kinds

- The distribution of n objects,
  - q_1 of one kind;
  - q_2 of another kind;

    $\vdots$
  - q_t of a t-th kind

  into n distinct cells (each of which can hold only one object) is equivalent to the permutation of these objects.

  So, the number of ways of distribution is

$$\frac{n!}{q_1!q_2!\cdots q_t!}.$$

Distributing Objects of Various Kinds (Cont'd)

- Alternatively, among the n distinct cells, we have:
  - $C(n, q_1)$ ways to pick q_1 cells for the objects of the first kind;
  - $C(n - q_1, q_2)$ ways to pick q_2 cells for the objects of the second kind;

    $\vdots$
  - $C(n - q_1 - \cdots - q_{t-1}, q_t)$ ways to pick q_t cells for the objects of the t-th kind;
  - $P(n - q_1 - \cdots - q_t, n - q_1 \cdots - q_t)$ ways of permuting those objects that are one of a kind.

  The number of ways of distribution is, therefore,
  $C(n, q_1)C(n - q_1, q_2)C(n - q_1 - q_2, q_3) \cdots C(n - q_1 - \cdots - q_{t-1}, q_t)P(n - q_1 - \cdots - q_t, n - q_1 \cdots - q_t) = \frac{n!}{q_1!(n-q_1)!} \frac{(n-q_1)!}{q_2!(n-q_1-q_2)!} \cdots \frac{(n-q_1-\cdots-q_{t-1})!}{q_t!(n-q_1-\cdots-q_t)!}(n - q_1 - \cdots - q_t)!$.
- The number of ways of distributing the r objects ($r \leq n$), into n distinct cells is $\frac{n!}{q_1!q_2!\cdots q_t!} \frac{1}{(n-r)!}$.

Subsection 6

Distributions of Non-Distinct Objects

Non-Distinct Objects into Non-Distinct Cells

Claim: There are $C(n,r)$ ways of placing r nondistinct objects into n distinct cells with at most one object in each cell $(n \geq r)$.

This follows because the distribution can be visualized as the selection of r cells from the n cells for the r nondistinct objects.

Claim: The number of ways to place r nondistinct objects into n distinct cells where a cell can hold more than one object is $C(n+r-1,r)$.

Method 1: Distributing the r nondistinct objects is equivalent to selecting r of the n cells for the r objects with repeated selections of cells allowed.

Method 2: Imagine the distribution of the r objects into n cells as an arrangement of the r objects and the $n-1$ intercell partitions. Since both the objects and the partitions are nondistinct, the number of ways of arrangement is $\frac{(n-1+r)!}{(n-1)!r!} = C(n+r-1,r)$.

Non-Distinct Objects into Non-Distinct Nonempty Cells

Claim: The number of ways to place r nondistinct objects into n distinct cells where a cell can hold more than one object and where none of the n cells can be left empty ($r \geq n$) is $C(r-1, n-1)$.

Distribute:

- First, one object in each of the n cells;
- Then, the remaining $r-n$ objects arbitrarily.

The number of ways of distribution is

$C((r-n)+n-1, r-n) = C(r-1, r-n) = C(r-1, n-1)$.

Claim: The number of ways of distributing r nondistinct objects into n distinct cells with each cell containing at least q objects ($r \geq nq$) is $C(n-nq+r-1, n-1)$.

- First, place q objects in each of the n cells;
- Then, the remaining $r-nq$ objects arbitrarily.

Number of ways is

$C((r-nq)+n-1, r-nq) = C(n-nq+r-1, n-1)$.

Example: Letters and Blanks

- Five distinct letters are to be transmitted through a communications channel. A total of 15 blanks are to be inserted between the letters with at least three blanks between every two letters. In how many ways can the letters and blanks be arranged?

  We do the following:
  - First arrange the 5 letters in 5! ways;
  - For each arrangement of the letters, consider the insertion of the blanks as placing 15 nondistinct objects into four distinct interletter positions with at least three objects in each interletter position, which can be done in $C(4-12+15-1, 4-1)$ ways.

  By the Product Rule, the total number of ways of arranging the letters and blanks is:

$$5! \cdot C(4-12+15-1, 4-1) = 5! \cdot C(6,3) = 2,400.$$

Example: Congressional Seats

- In how many ways can $2n+1$ seats in a congress be divided among three parties so that the coalition of any two parties will ensure them of a majority?
  This is a problem of distributing $2n+1$ nondistinct objects into three distinct cells.
  - Without any restriction on the number of seats each party can have, there are $C(3+(2n+1)-1, 2n+1) = C(2n+3, 2n+1) = C(2n+3, 2)$ ways of distributing the seats.
  - Among these distributions, there are some in which a party gets $n+1$ or more seats: For a party to have $n+1$ or more seats,
    - we choose a particular party and assign it $n+1$ seats in 3 ways;
    - then divide the remaining n seats among the three parties arbitrarily, which can be done in $C(3+n-1, n) = C(n+2, n) = C(n+2, 2)$ ways.

    So there are $3 \cdot C(n+2, 2)$ ways of a party having $n+1$ or more seats.

  The total number of ways to divide the seats so that no party alone will have a majority is $C(2n+3, 2) - 3 \cdot C(n+2, 2) = \frac{1}{2}(2n+3)(2n+2) - \frac{3}{2}(n+2)(n+1) = \frac{n}{2}(n+1)$.

Subsection 7

Stirling's Formula

Stirling's Formula and its Error

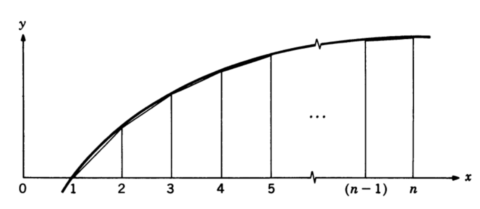
- Even for a moderately large n the evaluation of $n!$ is trying.
- We derive an approximation formula for the value of $n!$, called **Stirling's formula**:
$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$
- The absolute error of such an approximation increases as n increases:
$$\lim_{n\to\infty}\left[n! - \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right] = \infty.$$
- However, the percentage error decreases monotonically:
$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1.$$

Example: Stirling's formula approximates:

- 1! by 0.9221 with an 8% error;
- 2! by 1.919 with a 4% error;
- 5! by 118.019 with a 2% error.
- 100! is approximated with only 0.08% error.
  However, the absolute error in this case is about 1.7×10^{155}.

Proof of Stirling's Formula

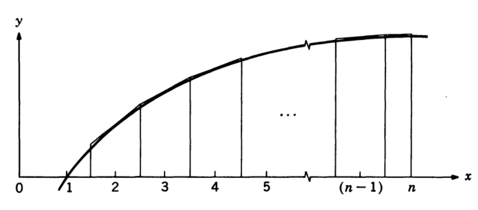
- We let
$$a_n = \log(n!) - \frac{1}{2}\log n = \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n.$$
Consider the curve $y = \log x$. The area under the curve and between the two lines $x = 1$ and $x = n$ is $\int_1^n \log x dx$. This area can be approximated by the sum of the areas of n trapezoids:



$\frac{1}{2}(\log 1 + \log 2) + \frac{1}{2}(\log 2 + \log 3) + \cdots + \frac{1}{2}[\log(n-1) + \log n] = \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n = \log(n!) - \frac{1}{2}\log n = a_n$.
This is smaller than the exact value of the area, because the curve $y = \log x$ is convex, i.e., $a_n < \int_1^n \log x dx$.

Proof of Stirling's Formula (Cont'd)

- The area under the curve $y = \log x$ and between the two lines $x = \frac{3}{2}$ and $x = n$ is $\int_{3/2}^{n} \log x dx$. This can be approximated by the sum of the areas of the $n-1$ trapezoids bounded by the tangent at the point $(k, \log k)$ and the lines $x = k - \frac{1}{2}$ and $x = k + \frac{1}{2}$ for $k = 2, \ldots, n-1$, together with the area of the rectangle bounded by the horizontal line at the point $(n, \log n)$ and the two lines $x = n - \frac{1}{2}$ and $x = n$:



The approximated area is
$\log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n = a_n$. Because the curve $y = \log x$ is convex, we have $\int_{3/2}^{n} \log x dx < a_n$.

Combining the Inequalities

- We set $a_n = \log(n!) - \frac{1}{2}\log n = \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n$ and showed that
$$a_n < \int_1^n \log x dx \quad \text{and} \quad \int_{3/2}^n \log x dx < a_n.$$
Combining the inequalities, we write
$$\int_{3/2}^n \log x dx < a_n < \int_1^n \log x dx.$$
After evaluating the integrals, we get
$$n\log n - n - \frac{3}{2}\log\frac{3}{2} + \frac{3}{2} < a_n < n\log n - n + 1.$$
But $\log(n!) = a_n + \frac{1}{2}\log n$, whence
$$(n+\frac{1}{2})\log n - n + \frac{3}{2}(1 - \log\frac{3}{2}) < \log(n!) < (n+\frac{1}{2})\log n - n + 1.$$

Finishing the Proof of Stirling's Formula

- $\log(n!) = (n+\frac{1}{2})\log n - n + \delta_n$, $\frac{3}{2}(1-\log\frac{3}{2}) = 0.893 < \delta_n < 1$.
  It follows that
  $$\begin{aligned} \delta_n &= \log(n!) - (n+\tfrac{1}{2})\log n + n \\ &= \log(n!) - \tfrac{1}{2}\log n - (n\log n - n + 1) + 1 \\ &= a_n - \int_1^n \log x dx + 1 = 1 - (\int_1^n \log x dx - a_n). \end{aligned}$$
  $(\int_1^n \log x dx - a_n)$ increases monotonically when n increases since $(\int_1^n \log x dx - a_n)$ represents the difference between the area under the curve $y = \log x$ and the sum of the areas of the trapezoids.
  Therefore, δ_n decreases monotonically as n increases. However, since δ_n has a lower bound (0.893), the limit of δ_n as n approaches ∞, denoted by δ, is a constant having a value between 0.893 and 1.
  Using δ to approximate δ_n: $\log(n!) \approx (n+\frac{1}{2})\log n - n + \delta$ or
  $n! \approx e^{(n+\frac{1}{2})\log n} e^{-n} e^{\delta} = n^{(n+\frac{1}{2})} e^{-n} e^{\delta} = e^{\delta}\sqrt{n}(n/e)^n$.
  It turns out, the value of e^{δ} is equal to $\sqrt{2\pi} = 2.507$.