# Introduction to Combinatorial Mathematics 

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## (1) Generating Functions

- Introduction
- Generating Functions and Combinations
- Enumerators for Permutations
- Distributions of Distinct Objects Into NonDistinct Cells
- Partitions of Integers
- The Ferrers Graph
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## Subsection 1

## Introduction

## Representing Choices Using Algebraic Expressions

- From three distinct objects $a, b$ and $c$, there are three ways to choose one object, namely, to choose either $a$ or $b$ or $c$.
We represent these possible choices symbolically as $a+b+c$.
- From these three objects, there are three ways to choose two objects, namely, to choose either $a$ and $b$, or $b$ and $c$, or $c$ and $a$.
These can be represented symbolically as $a b+b c+c a$.
- There is only one way to choose three objects, which can be represented symbolically as $a b c$.
- Examining the polynomial

$$
(1+a x)(1+b x)(1+c x)=1+(a+b+c) x+(a b+b c+c a) x^{2}+(a b c) x^{3}
$$

we discover that all these possible ways of selection are exhibited as the coefficients of the powers of $x$ :

The coefficient of $x^{i}$ is the representation of the ways of selecting $i$ objects from the three objects.

## Analysis of the Interpretation

- This involves an interpretation of the polynomial according to the combinatorial rules of sum and of product.
- The factor $1+a x$ means that for the object $a$, the two ways of selection are "not to select $a$ " or "to select $a$ ". The variable $a$ is a formal variable and is used simply as an indicator. The coefficient of $x^{0}$ shows the ways no object is selected, and the coefficient of $x^{1}$ shows the ways one object is selected.
- Similar interpretation can be given to the factors $1+b x$ and $1+c x$.
- Thus, the product $(1+a x)(1+b x)(1+c x)$ indicates that for the objects $a, b$ and $c$, the ways of selection are: "to select or not to select $a$ " and "to select or not to select $b$ " and "to select or not to select $c^{\prime \prime}$.
- Therefore, the powers of $x$ in the polynomial indicate the number of objects that are selected, and the corresponding coefficients show all the possible ways of selection.


## Ordinary Generating Function of a Sequence

- This example motivates the formal definition of the generating function of a sequence.
- Let $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{r}, \ldots\right)$ be the symbolic representation of a sequence of events, or let it simply be a sequence of numbers. The function

$$
F(x)=a_{0} \mu_{0}(x)+a_{1} \mu_{1}(x)+a_{2} \mu_{2}(x)+\cdots+a_{r} \mu_{r}(x)+\cdots
$$

is called the ordinary generating function of the sequence ( $a_{0}, a_{1}$, $\left.a_{2}, \ldots, a_{r}, \ldots\right)$, where $\mu_{0}(x), \mu_{1}(x), \mu_{2}(x), \ldots, \mu_{r}(x), \ldots$ is a sequence of functions of $x$ that are used as indicators.

- Another kind of generating function called the exponential generating function will be discussed later.
- The indicator functions $\mu(x)$ 's are usually chosen in such a way that no two distinct sequences will yield the same generating function.
- The generating function of a sequence is just an alternative representation of the sequence.


## Examples of an Ordinary Generating Function

- Using 1, $\cos x, \cos 2 x, \ldots, \cos r x, \ldots$ as the indicator functions, we see that the ordinary generating function of the sequence $\left(1, \omega, \omega^{2}, \ldots, \omega^{r}, \ldots\right)$ is

$$
F(x)=1+\omega \cos x+\omega^{2} \cos 2 x+\cdots+\omega^{r} \cos r x+\cdots
$$

- Using $1,1+x, 1-x, 1+x^{2}, 1-x^{2}, \ldots, 1+x^{r}, 1-x^{r}, \ldots$ as the indicator functions, the ordinary generating function of the sequence $(3,2,6,0,0)$ is

$$
3+2(1+x)+6(1-x)=11-4 x
$$

The sequences $(1,3,7,0,0)$ and $(1,2,6,1,1)$ will also yield the same ordinary generating function:

$$
\begin{aligned}
& 1+3(1+x)+7(1-x)=11-4 x \\
& -1+2(1+x)+6(1-x)+\left(1+x^{2}\right)+\left(1-x^{2}\right)=11-4 x
\end{aligned}
$$

Thus, the functions $1,1+x, 1-x, 1+x^{2}, 1-x^{2}, \ldots$ should not be used as indicator functions.

## Committing to a Specific Family of Indicator Functions

- The most usual and useful form of $\mu_{r}(x)$ is $x^{r}$.

In that case, for the sequence $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{r}, \ldots\right)$, we have

$$
F(x)=a_{0}+a_{1} x+a_{2} x^{2}+\cdots+a_{r} x^{r}+\cdots
$$

- From now on, when we talk about the generating functions of a sequence, we shall mean the generating function of the sequence with the powers of $x$ as indicator functions.
- Notice that the sequence $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{r}, \ldots\right)$ can be an infinite sequence. Then $F(x)$ will be an infinite series. Since $x$ is just a formal variable, there is no need to question whether the series converges!


## Subsection 2

## Generating Functions and Combinations

## Enumerators

- We saw that the polynomial $(1+a x)(1+b x)(1+c x)$ is the ordinary generating function of the different ways to select the objects $a, b$ and c.
- Instead of the different ways of selection, we may only be interested in the number of ways of selection.
- By setting $a=b=c=1$, we have

$$
(1+x)(1+x)(1+x)=(1+x)^{3}=1+3 x+3 x^{2}+x^{3} .
$$

We see that there are:

- One way to select no objects from the three objects $C(3,0)$;
- Three ways to select one object out of three, $C(3,1)$; etc.
- A generating function that gives the number of combinations or permutations is called an enumerator.
- An ordinary generating function that gives the number of combinations or permutations is called an ordinary enumerator.


## Ordinary Enumerator for Combinations

- To find the number of combinations of $n$ distinct objects, we have the ordinary enumerator

$$
\begin{aligned}
(1+x)^{n}= & 1+n x+\frac{n(n-1)}{2!} x^{2}+\cdots \\
& \quad+\frac{n(n-1) \cdots(n-r+1)}{r!} x^{r}+\cdots+x^{n} \\
= & C(n, 0)+C(n, 1) x+C(n, 2) x^{2}+\cdots \\
& \quad+C(n, r) x^{r}+\cdots+C(n, n) x^{n} .
\end{aligned}
$$

In the expansion of $(1+x)^{n}$, the coefficient of the term $x^{r}$ is the number of ways the term $x^{r}$ can be formed by taking $r x^{\prime} s$ and $n-r$ 1 's among the $n$ factors $1+x$.
For this reason the $C(n, r)$ 's are called the binomial coefficients. In a binomial expansion, $\binom{n}{r}$ is a common alternative notation for $C(n, r)$.

## Some Consequences

- From $\binom{n}{0}+\binom{n}{1} x+\binom{n}{2} x^{2}+\cdots+\binom{n}{r} x^{r}+\cdots+\binom{n}{n} x^{n}=(1+x)^{n}$, by setting $x=1$, we get

$$
\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{r}+\cdots+\binom{n}{n}=2^{n} .
$$

The combinatorial significance of this identity is that both sides give the number of ways of selecting none, or one, or two, ..., or $n$ objects out of $n$ distinct objects.

- By setting $x=-1$, we also have the identity

$$
\begin{gathered}
\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{r}\binom{n}{r}+\cdots+(-1)^{n}\binom{n}{n}=0 \text {. Rewriting as } \\
\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots,
\end{gathered}
$$

we see that
the number of ways of selecting an even number of objects is equal to the number of ways of selecting an odd number of objects from $n$ distinct objects.

## Example (Algebraic Proof)

Claim: $\binom{n}{0}^{2}+\binom{n}{1}^{2}+\cdots+\binom{n}{r}^{2}+\cdots+\binom{n}{n}^{2}=\binom{2 n}{n}$.
Method 1: By the binomial theorem
$(1+x)^{n}=1+\binom{n}{1} x+\binom{n}{2} x^{2}+\cdots+\binom{n}{n} x^{n}$.
Similarly, $\left(1+x^{-1}\right)^{n}=1+\binom{n}{1} x^{-1}+\binom{n}{2} x^{-2}+\cdots+\binom{n}{n} x^{-n}$.
Therefore, the constant term in $(1+x)^{n}\left(1+x^{-1}\right)^{n}$ is

$$
\binom{n}{0}^{2}+\binom{n}{1}^{2}+\cdots+\binom{n}{r}^{2}+\cdots+\binom{n}{n}^{2} .
$$

On the other hand, we get

$$
\begin{aligned}
(1+x)^{n}\left(1+x^{-1}\right)^{n} & =(1+x)^{n}\left((1+x) x^{-1}\right)^{n} \\
& =(1+x)^{n}(1+x)^{n} x^{-n} \\
& =(1+x)^{2 n} x^{-n} \\
& =\left(1+\binom{2 n}{1} x+\binom{2 n}{2} x^{2}+\cdots+\binom{2 n}{2 n} x^{2 n}\right) x^{-n} .
\end{aligned}
$$

The constant term is therefore $\binom{2 n}{n}$.

## Example (Combinatorial Proof)

Claim: $\binom{n}{0}^{2}+\binom{n}{1}^{2}+\cdots+\binom{n}{r}^{2}+\cdots+\binom{n}{n}^{2}=\binom{2 n}{n}$.
Method 2: The identity to be proved can be rewritten as
$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-1}+\cdots+\binom{n}{r}\binom{n}{n-r}+\cdots+\binom{n}{n}\binom{n}{0}=\binom{2 n}{n}$.
The right side is the number of ways to select $n$ objects out of a total of $2 n$ objects.
The same selection can also be performed as follows:

- First divide the $2 n$ objects into two piles with $n$ objects in each pile.
- To select $n$ objects, we may select $i$ objects from the first pile and $n-i$ objects from the second pile. There are:
- ( $\binom{n}{i}$ ways to select $i$ objects from the first pile;
- ( $\left.\begin{array}{c}n \\ n-i\end{array}\right)$ ways to select $n-i$ objects from the second pile.

Hence, the total number of ways to make the selection is

$$
\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}=\binom{2 n}{n}
$$

## An Application

- Find the number of $2 n$-digit binary sequences which are such that the number of 0 's in the first $n$ digits of a sequence is equal to the number of 0 's in the last $n$ digits of the sequence.
Since the number of $n$-digit binary sequences containing $r 0$ 's is $\binom{n}{r}$, the number of $2 n$-digit binary sequences containing $r 0$ 's in the first $n$ digits as well as in the last $n$ digits is $\binom{n}{r}^{2}$. Therefore, the number of $2 n$-digit binary sequences which are such that the number of 0 's in the first $n$ digits of a sequence is equal to the number of 0 's in the last $n$ digits of the sequence is

$$
\binom{n}{0}^{2}+\binom{n}{1}^{2}+\cdots+\binom{n}{r}^{2}+\cdots+\binom{n}{n}^{2}=\binom{2 n}{n} .
$$

- What does the combinatorial argument used in the previous slide tell us in this case?


## Example

- Prove the identity

$$
\binom{n}{1}+2\binom{n}{2}+\cdots+r\binom{n}{r}+\cdots+n\binom{n}{n}=n 2^{n-1}
$$

Start with

$$
\binom{n}{0}+\binom{n}{1} x+\cdots+\binom{n}{r} x^{r}+\cdots+\binom{n}{n} x^{n}=(1+x)^{n} .
$$

Differentiate both sides with respect to $x$ :

$$
\binom{n}{1}+2\binom{n}{2} x+\cdots+r\binom{n}{r} x^{r-1}+\cdots+n\binom{n}{n} x^{n-1}=n(1+x)^{n-1} .
$$

Now set $x=1$.

## Example

- What is the coefficient of the term $x^{23}$ in $\left(1+x^{5}+x^{9}\right)^{100}$ ?
$x^{5} x^{9} x^{9}=x^{23}$ is the only way the term $x^{23}$ can be made up in the expansion of $\left(1+x^{5}+x^{9}\right)^{100}$. Moreover, there are:
- $C(100,2)$ ways to choose the two factors $x^{9}$;
- Then, there are $C(98,1)$ ways to choose the factor $x^{5}$.

Thus, the coefficient of $x^{23}$ is

$$
C(100,2) \cdot C(98,1)=\frac{100 \cdot 99}{2} \cdot 98=485,100
$$

## Example

Claim: The ordinary generating function of the sequence $\binom{0}{0},\binom{2}{1},\binom{4}{2}, \ldots,\binom{2 r}{r}, \ldots$ is $(1-4 x)^{-1 / 2}$.
Recall the binomial theorem

$$
(1+x)^{n}=1+\sum_{r=1}^{\infty} \frac{n(n-1)(n-2) \cdots(n-r+1)}{r!} x^{r}
$$

for $n$ not a positive integer.
Applying the theorem, we get

$$
\begin{aligned}
(1-4 x)^{-1 / 2} & =1+\sum_{r=1}^{\infty} \frac{(-1 / 2)(-1 / 2-1) \cdots(-1 / 2-r+1)}{1!}(-4 x)^{r} \\
& =1+\sum_{r=1}^{\infty} \frac{4^{r}(1 / 2)(3 / 2)(5 / 2) \cdots((2 r-1) / 2)}{r} x^{r} \\
& =1+\sum_{r=1}^{\infty} \frac{2^{r}(1 \cdot 3 \cdot 5 \cdots(2 r-1))}{r!} x^{r} \\
& =1+\sum_{r=1}^{\infty} \frac{(2 r \cdot r!)(1 \cdot 3 \cdot 5 \cdot \cdots(2 r-1))}{r \cdot 1} x^{r} \\
& =1+\sum_{r=1}^{\infty} \frac{(2 \cdot 4 \cdot 6 \cdots \cdot 2 r)(1 \cdot 3 \cdot 5 \cdots(2 r-1))}{r!} x^{r} \\
& =1+\sum_{r=1}^{\infty} \frac{(2 r)!}{r!r!} x^{r}=1+\sum_{r=1}^{\infty}\binom{2 r}{r} x^{r} .
\end{aligned}
$$

## An Application

- Evaluate the sum

$$
\sum_{i=0}^{t}\binom{2 i}{i}\binom{2 t-2 i}{t-i}, \text { for a given } t
$$

We know:

- $\binom{2 i}{i}$ is the coefficient of the term $x^{i}$ in $(1-4 x)^{-1 / 2}$;
- $\binom{2 t-2 i}{t-i}$ is the coefficient of the term $x^{t-i}$ in $(1-4 x)^{-1 / 2}$.

Thus, $\sum_{i=0}^{t}\binom{2 i}{i}\binom{2 t-2 i}{t-i}$ is the coefficient of the term $x^{t}$ in $(1-4 x)^{-1 / 2}(1-4 x)^{-1 / 2}$.
But, we have $(1-4 x)^{-1 / 2}(1-4 x)^{-1 / 2}=(1-4 x)^{-1}=$ $1+4 x+(4 x)^{2}+(4 x)^{3}+\cdots+(4 x)^{r}+\cdots$. Thus,

$$
\sum_{i=0}^{t}\binom{2 i}{i}\binom{2 t-2 i}{t-i}=4^{t}
$$

## Selections With Repetitions

- We can extend the framework for the case when repetitions are allowed in the selections.
Example: The ordinary generating function for the combinations of the objects $a, b$ and $c$, where $a$ can be selected twice is

$$
\begin{aligned}
& \left(1+a x+a^{2} x^{2}\right)(1+b x)(1+c x) \\
& =1+(a+b+c) x+\left(a b+b c+a c+a^{2}\right) x^{2} \\
& \quad+\left(a b c+a^{2} b+a^{2} c\right) x^{3}+\left(a^{2} b c\right) x^{4}
\end{aligned}
$$

- Notice the difference between the combinatorial significance of the preceding polynomial and that of

$$
\begin{aligned}
& (1+a x)\left(1+a^{2} x^{2}\right)(1+b x)(1+c x) \\
& =\left(1+a x+a^{2} x^{2}+a^{3} x^{3}\right)(1+b x)(1+c x)
\end{aligned}
$$

## Selections With Repetitions (Cont'd)

Example: Consider the generating function

$$
\begin{aligned}
& (1+a x)\left(1+a^{2} x\right)(1+b x)(1+c x) \\
& =1+\left(a+b+c+a^{2}\right) x+\left(d b+b c+a c+a^{3}+a^{2} b+a^{2} c\right) x^{2} \\
& +\left(a b c+a^{3} b+a^{2} b c+a^{3} c\right) x^{3}+\left(a^{3} b c\right) x^{4} .
\end{aligned}
$$

We can imagine that there are four boxes:

- one containing a;
- one containing two a's;
- one containing $b$;
- one containing $c$.

The generating function gives all possible outcomes of the selection of the boxes.

## Enumerator of Combinations With Repetitions

Example: Find the ordinary enumerator for the combinations of the objects $a, b$ and $c$, where $a$ can be selected twice.
For the object $a$,

- there is one way not to select it;
- one way to select it once;
- one way to select it twice.

This is modeled by a factor $1+x+x^{2}$.
For the objects $b$, there is one way not to select it and one way to select it. This is modeled by a factor of $(1+x)$.
Similarly for $c$.
Thus, the ordinary enumerator for the combinations of the objects $a, b$ and $c$, where $a$ can be selected twice is

$$
\left(1+x+x^{2}\right)(1+x)^{2}=1+3 x+4 x^{2}+3 x^{3}+x^{4}
$$

## Example

- Given two each of $p$ kinds of objects and one each of $q$ additional kinds of objects, in how many ways can $r$ objects be selected?
The ordinary enumerator for the combinations is

$$
\left(1+x+x^{2}\right)^{p}(1+x)^{q} .
$$

Let $\left\lfloor\frac{r}{2}\right\rfloor$ denote the integral part of $\frac{r}{2}:\left\lfloor\frac{r}{2}\right\rfloor= \begin{cases}\frac{r}{2}, & \text { if } r \text { is even } \\ \frac{r-1}{2}, & \text { if } r \text { is odd }\end{cases}$ Note that in the product above, to form the power $x^{r}$ we may select:

- $i x^{2}$ 's among the $p$ factors of the form $1+x+x^{2}$;
- $r-2 i x$ 's among the $p-i$ remaining factors of the form $1+x+x^{2}$ and the $q$ factors of the form $1+x$.
Thus, the coefficient of $x^{r}$ in the enumerator above is

$$
\sum_{i=0}^{\lfloor r / 2\rfloor}\binom{p}{i}\binom{p+q-i}{r-2 i} .
$$

## Enumerator of Selections with Unlimited Repetitions

- The ordinary enumerator for the selection of $r$ objects out of $n$ objects with unlimited repetitions is

$$
\begin{aligned}
(1+x+ & \left.x^{2}+\cdots+x^{k}+\cdots\right)^{n} \\
& =\left(\frac{1}{1-x}\right)^{n} \\
& =(1-x)^{-n} \\
& =1+\sum_{r=1}^{\infty} \frac{(-n)(-n-1) \cdots(-n-r+1)}{r!}(-x)^{r} \\
& =1+\sum_{r=1}^{\infty} \frac{(n)(n+1) \cdots(n+r-1)}{r!} x^{r} \\
& =\sum_{r=0}^{\infty}\binom{n+r-1}{r} x^{r} .
\end{aligned}
$$

This verifies the formula we developed for the number of ways to select $r$ objects from $n$ objects with unlimited repetitions.

## Selections with Unlimited Repetitions with a Restriction

- The ordinary enumerator for the selection of $r$ objects out of $n$ objects ( $r \geq n$ ), with unlimited repetitions but with each object included in each selection, is

$$
\begin{aligned}
\left(x+x^{2}\right. & \left.+\cdots+x^{k}+\cdots\right)^{n} \\
& =x^{n}\left(1+x+\cdots+x^{k-1}+\cdots\right)^{n} \\
& =x^{n}\left(\frac{1}{1-x}\right)^{n}=x^{n}(1-x)^{-n} \\
& =x^{n} \sum_{i=0}^{\infty}\binom{n+i-1}{i} x^{i}=\sum_{i=0}^{\infty}\binom{n+i-1}{i} x^{n+i} \\
& =\sum_{r=n}^{\infty}\binom{r-1}{r-n} x^{r} \\
& =\sum_{r=n}^{\infty}\binom{r-1}{n-1} x^{r} .
\end{aligned}
$$

## Example

Claim: The number of ways in which $r$ nondistinct objects can be distributed into $n$ distinct cells, with the condition that no cell contains less than $q$ nor more than $q+z-1$ objects, is the coefficient of $x^{r-q n}$ in the expansion of $\left(\frac{1-x^{z}}{1-x}\right)^{n}$.
Since the ordinary enumerator for the ways a particular cell can be filled is $x^{q}+x^{q+1}+\cdots+x^{q+z-1}$ the ordinary enumerator for the distributions is

$$
\begin{aligned}
\left(x^{q}+x^{q+1}+\right. & \left.\cdots+x^{q+z-1}\right)^{n} \\
& =x^{q n}\left(1+x+\cdots+x^{z-1}\right)^{n} \\
& =x^{q n}\left(\frac{1-x^{z}}{1-x}\right)^{n}
\end{aligned}
$$

## An Application

- Find the number of ways in which four persons, each rolling a single die once, can have a total score of 17 .
For $r=17, n=4, q=1$ and $z=6$, the ordinary enumerator is $x^{4}\left(\frac{1-x^{6}}{1-x}\right)^{4}$. We have:

$$
\begin{aligned}
\left(1-x^{6}\right)^{4} & =1-4 x^{6}+6 z^{12}-4 z^{18}+x^{24} \\
(1-x)^{-4} & =1+\frac{4}{1!} x+\frac{4 \cdot 5}{2!} x^{2}+\frac{4 \cdot 5 \cdot 6}{3!} x^{3}+\cdots
\end{aligned}
$$

Thus, the coefficient of $x^{13}$ in $\left(1-x^{6}\right)^{4}(1-x)^{-4}$ is

$$
\begin{aligned}
& \frac{4 \cdot 5 \cdot 6 \cdots 16}{13!}-4 \cdot \frac{4 \cdot 5 \cdot 6 \cdots 10}{7!}+6 \cdot \frac{4}{1!} \\
& =\frac{14 \cdot 15 \cdot 16}{3!}-4 \cdot \frac{8 \cdot 9 \cdot 10}{3!}+6 \cdot \frac{4}{1!}=104
\end{aligned}
$$

## Subsection 3

## Enumerators for Permutations

## Permutation Enumerators and Commutativity

- There is an obvious difficulty when attempting to extend the previous results to generating functions of permutations.
Multiplication in the ordinary algebra in the field of real numbers is commutative (that is, $a b=b a$ ). As a result, we cannot handle the case of permutations using ordinary algebra.
Example: Consider the permutations of the two objects $a$ and $b$.
- What we want to have as a generating function for the permutations is

$$
1+(a+b) x+(a b+b a) x^{2}
$$

- However, this polynomial is equivalent to

$$
1+(a+b) x+(2 a b) x^{2}
$$

in which the two distinct permutations $a b$ and $b a$ can no longer be recognized.

- Instead of introducing a new algebra that is noncommutative for the case of permutations, we discuss enumerators for permutations which can still be handled by the ordinary algebra in the real numbers.


## Introducing a Different Generating Function

- An enumerator for the permutations of $n$ distinct objects would have the form

$$
\begin{aligned}
F(x)= & P(n, 0) x^{0}+P(n, 1) x+P(n, 2) x^{2}+\cdots \\
& \quad+P(n, r) x^{r}+\cdots+P(n, n) x^{n} \\
= & 1+\frac{n!}{(n-1)!} x+\frac{n!}{(n-2)!} x^{2}+\cdots+\frac{n!}{(n-r)!} x^{r}+\cdots+n!x^{n} .
\end{aligned}
$$

- There is no simple compact closed-form expression for $F(x)$.
- To carry along the entire polynomial defeats the purpose of using the generating function representation.
- Recall the binomial expansion
$(1+x)^{n}=1+C(n, 1) x+C(n, 2) x^{2}+\cdots$

$$
\begin{gathered}
+C(n, r) x^{r}+\cdots+C(n, n) x^{n} \\
=1+\frac{P(n, 1)}{1!} x+\frac{P(n, 2)}{2!} x^{2}+\cdots+\frac{P(n, r)}{r!} x^{r}+\cdots+\frac{P(n, n)}{n!} x^{n} .
\end{gathered}
$$

We see that the key lies in defining another kind of generating function, the exponential generating function.

## The Exponential Generating Function

- Let $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{r}, \ldots\right)$ be the symbolic representations of a sequence of events or simply be a sequence of numbers. The function

$$
F(x)=\frac{a_{0}}{0!} \mu_{0}(x)+\frac{a_{1}}{1!} \mu_{1}(x)+\frac{a_{2}}{2!} \mu_{2}(x)+\cdots+\frac{a_{r}}{r!} \mu_{r}(x)+\cdots
$$

is called the exponential generating function of the sequence $\left(a_{0}, a_{1}, a_{2}, \ldots, a_{r}, \ldots\right)$, with $\mu_{0}(x), \mu_{1}(x), \mu_{2}(x), \ldots, \mu_{r}(x), \ldots$, as the indicator functions.
Example: $(1+x)^{n}$ is the exponential generating function of the $P(n, r)$ 's with the powers of $x$ as the indicator functions.

- An exponential generating function that gives the number of combinations or permutations is called an exponential enumerator.


## Examples of Exponential Generating Functions I

Claim: The exponential generating function of the sequence $(1,1,1, \ldots, 1, \ldots)$ is $e^{x}$.
Recall Maclaurin's series for $e^{x}: e^{x}=\sum_{n=0}^{\infty} \frac{1}{n!} x^{n}$. This shows that $e^{x}$ is the exponential generating function of $(1,1,1, \ldots, 1, \ldots)$.
Claim: The exponential generating function of the sequence $(1,1 \cdot 3,1 \cdot 3 \cdot 5, \ldots, 1 \cdot 3 \cdot 5 \cdots(2 r+1), \ldots)$ is $(1-2 x)^{-3 / 2}$.

$$
\begin{aligned}
(1-2 x)^{3 / 2} & =\sum_{n=0}^{\infty}\binom{-3 / 2}{n}(-2 x)^{n} \\
& =\sum_{n=0}^{\infty} \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) \cdots\left(-\frac{3}{2}-n+1\right)}{n!}(-2)^{n} x^{n} \\
& =\sum_{n=0}^{\infty} \frac{(-3)(-5)(-7) \cdots(-2 n-1)}{2^{n} n!}(-1)^{n} 2^{n} x^{n} \\
& =\sum_{n=0}^{\infty}(3 \cdot 5 \cdot 7 \cdots(2 n+1)) \frac{x^{n}}{n!} .
\end{aligned}
$$

This proves the claim.

## Examples of Exponential Generating Functions II

Claim: The exponential generating function of the sequence $(P(0,0), P(2,1), P(4,2), \ldots, P(2 r, r), \ldots)$ is $(1-4 x)^{-1 / 2}$.

$$
\begin{aligned}
(1-4 x)^{-1 / 2} & =\sum_{n=0}^{\infty}\binom{-1 / 2}{n}(-4 x)^{n} \\
& =\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) \cdots\left(-\frac{1}{2}-n+1\right)}{n!}(-4)^{n} x^{n} \\
& =\sum_{n=0}^{\infty} \frac{(-1)(-3)(-5) \cdots(-2 n+1)}{2^{n} n!}(-1)^{n} 2^{n} 2^{n} x^{n} \\
& =\sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots(2 n-1) n!2^{n}}{n!} \frac{x^{n}}{n!} \\
& =\sum_{n=0}^{\infty} \frac{(2 n)!}{n!} \frac{x^{n}}{n!} \\
& =\sum_{n=0}^{\infty} P(2 n, n) \frac{x^{n}}{n!} .
\end{aligned}
$$

## Permutations With Repetitions

- The exponential enumerator for the permutations of:
- a single object with no repetitions is $1+x$.
- $n$ distinct objects with no repetitions is $(1+x)^{n}$.
- When repetitions are allowed in the permutations, the extension is immediate:
- The exponential enumerator for the permutations of all $p$ of $p$ identical objects is $\frac{x^{p}}{p!}$, since there is only one way of doing so.
- The exponential enumerator for the permutations of none, one, two, $\ldots, p$ of $p$ identical objects is $1+\frac{1}{1!} x+\frac{1}{2!} x^{2}+\cdots+\frac{1}{p!} x^{p}$.
- Similarly, the exponential enumerator for the permutation of all $p+q$ of $p+q$ objects, with $p$ of them of one kind and $q$ of them of another kind, is $\frac{x^{p}}{p!} \frac{x^{q}}{q!}=\frac{x^{p+q}}{p!q!}=\frac{(p+q)!}{p!q!} \frac{x^{p+q}}{(p+q)!}$, which agrees with the known result that the number of permutations is $\frac{(p+q)!}{p!q!}$.
- The exponential enumerator for the permutations of none, one, two, $\ldots, p+q$ of $p+q$ objects, with $p$ of them of one kind and $q$ of them of another kind, is $\left(1+\frac{1}{1!} x+\cdots+\frac{1}{p!} x^{p}\right)\left(1+\frac{1}{1!} x+\cdots+\frac{1}{q!} x^{q}\right)$.


## Examples

- The exponential enumerator for the permutations of two objects of one kind and three objects of another kind is

$$
\begin{aligned}
& \left(1+\frac{1}{1!} x+\frac{1}{2!} x^{2}\right)\left(1+\frac{1}{1!} x+\frac{1}{2!} x^{2}+\frac{1}{3!} x^{3}\right) \\
& =1+\left(\frac{1}{1!}+\frac{1}{1!}\right) x+\left(\frac{1}{1!1!}+\frac{1}{2!}+\frac{1}{2!}\right) x^{2}+\left(\frac{1}{1!2}+\frac{1}{1!2!}+\frac{1}{33}\right) x^{3} \\
& \quad+\left(\frac{1}{1!3!}+\frac{1}{2!2!}\right) x^{4}+\left(\frac{1}{2!3!}\right) x^{5} .
\end{aligned}
$$

- The number of $r$-permutations of $n$ distinct objects with unlimited repetitions is given by the exponential enumerator

$$
\left(1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots\right)^{n}=e^{n x}=\sum_{r=0}^{\infty} \frac{n^{r}}{r!} x^{r}
$$

## Example

- Find the number of $r$-digit quaternary sequences in which each of the digits 1,2 , and 3 appears at least once.
This problem is the same as that of permuting four distinct objects with the restriction that three of the four objects must be included in the permutations. The exponential enumerator for the permutations:
- of the digit 0 is $\left(1+x+\frac{1}{2!} x^{2}+\cdots\right)=e^{x}$;
- of the digit 1 (or 2 , or 3 ) is $\left(x+\frac{1}{2!} x^{2}+\frac{1}{3!} x^{3}+\cdots\right)=e^{x}-1$.

It follows that the exponential enumerator for the permutations of the four digits is

$$
\begin{aligned}
e^{x}\left(e^{x}-1\right)\left(e^{x}-1\right)\left(e^{x}-1\right) & =e^{x}\left(e^{3 x}-3 e^{2 x}+3 e^{x}-1\right) \\
& =e^{4 x}-3 e^{3 x}+3 e^{2 x}-e^{x} \\
& =\sum_{r=0}^{\infty} \frac{\left(4^{r}-3 \cdot 3^{r}+3 \cdot 2^{r}-1\right)}{r!} x^{r}
\end{aligned}
$$

Therefore, the number of $r$-digit quaternary sequences in which each of the digits 1,2 , and 3 appears at least once is $4^{r}-3 \cdot 3^{r}+3 \cdot 2^{r}-1$.

## Example

- Find the number of $r$-digit quaternary sequences that contain an even number of 0 's.

The exponential enumerator for the permutations of the digit 0 is $\left(1+\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\cdots\right)=\frac{1}{2}\left(e^{x}+e^{-x}\right)$.
The exponential enumerator for the permutations of each of the digits 1,2 , and 3 is $\left(1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots\right)=e^{x}$.
It follows that the exponential enumerator for the number of quaternary sequences containing an even number of 0 's is

$$
\begin{aligned}
\frac{1}{2}\left(e^{x}+e^{-x}\right)\left(e^{x}\right)^{3} & =\frac{1}{2}\left(e^{4 x}+e^{2 x}\right) \\
& =1+\sum_{r=1}^{\infty} \frac{1}{2} \frac{\left(4^{r}+2^{r}\right)}{r!} x^{r}
\end{aligned}
$$

Therefore, the number of $r$-digit quaternary sequences that contain an even number of 0 's is $\frac{4^{r}+2^{r}}{2}$.

## Example

- To find the number of $r$-digit quaternary sequences that contain an even number of 0 's and an even number of 1 's, we have the exponential enumerator

$$
\begin{aligned}
\frac{1}{2}\left(e^{x}+e^{-x}\right) \frac{1}{2}\left(e^{x}+e^{-x}\right) e^{x} e^{x} & =\frac{1}{4}\left(e^{2 x}+2+e^{-2 x}\right) e^{2 x} \\
& =\frac{1}{4}\left(e^{4 x}+2 e^{2 x}+1\right) \\
& =1+\sum_{r=1}^{\infty} \frac{1}{4} \frac{\left(4^{r}+2 \cdot 2^{r}\right)}{r!} x^{r}
\end{aligned}
$$

Thus, the number of $r$-digit quaternary sequences that contain an even number of 0 's and an even number of 1 's is

$$
\frac{1}{4}\left(4^{r}+2 \cdot 2^{r}\right)
$$

## Example

- Find the exponential enumerator for the number of ways to choose $r$ or less objects from $r$ distinct objects and distribute them into $n$ distinct cells, with objects in the same cell ordered. There are:
- $C(r, m)$ ways to select $m$ objects out of $r$ objects;
- $n(n+1) \cdots(n+m-1)$ ways to arrange them in the $n$ distinct cells.

Since $m$ can range from 0 to $r$, the total number is

$$
\begin{aligned}
& C(r, 0)+C(r, 1) \cdot n+C(r, 2) \cdot n(n+1)+C(r, 3) \cdot n(n+1)(n+2) \\
& \quad+\cdots+C(r, r) \cdot n(n+1) \cdots(n+r-1) \\
& =r!\left[\frac{1}{r!} \cdot 1+\frac{1}{(r-1)!1!} \cdot n+\frac{1}{(r-2)!2!} \cdot n(n+1)+\right. \\
& \left.\quad \frac{1}{(r-3)!3!} \cdot n(n+1)(n+2)+\cdots+\frac{1}{r!} \cdot n(n+1) \cdots(n+r-1)\right] .
\end{aligned}
$$

## Example (Cont'd)

- We obtained

$$
\begin{aligned}
r! & {\left[\frac{1}{r!} \cdot 1+\frac{1}{(r-1)!1!} \cdot n+\frac{1}{(r-2)!2!} \cdot n(n+1)+\right.} \\
& \left.\frac{1}{(r-3)!3!} \cdot n(n+1)(n+2)+\cdots+\frac{1}{r!} \cdot n(n+1) \cdots(n+r-1)\right] .
\end{aligned}
$$

The expression in the square brackets is the coefficient of the term $x^{r}$ in the product of the two series

$$
\begin{aligned}
& -e^{x}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\cdots+\frac{x^{r}}{!}+\cdots ; \\
& \cdot(1-x)^{-n}=1+\frac{n}{1!}+\frac{n(n+1)}{2!} x^{2}+\cdots+\frac{n(n+1) \cdots(n+r-1)}{r!} x^{r}+\cdots
\end{aligned}
$$

Therefore, the sought after exponential enumerator is

$$
\frac{e^{x}}{(1-x)^{n}}
$$

## Subsection 4

## Distributions of Distinct Objects Into NonDistinct Cells

## Stirling Numbers of the Second Kind

- We consider the number of ways of distributing $r$ distinct objects into $n$ distinct cells so that no cell is empty and the order of objects within a cell is not important.
It is the same as the number of the $r$-permutations of the $n$ distinct cells with each cell included at least once in a permutation. The exponential enumerator for the permutations is

$$
\begin{aligned}
\left(x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots\right)^{n} & =\left(e^{x}-1\right)^{n}=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i} e^{(n-i) x} \\
& =\sum_{i=0}^{n}\binom{n}{i}(-1)^{i} \sum_{r=0}^{\infty} \frac{1}{r!}(n-i)^{r} x^{r} \\
& =\sum_{r=0}^{\infty} \frac{x^{r}}{r!} \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^{r} .
\end{aligned}
$$

Thus, the number of ways of placing $r$ distinct objects into $n$ distinct cells with no cell left empty is $\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^{r}=n!S(r, n)$ where $S(r, n)$ is defined as $\frac{1}{n!} \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^{r} . S(r, n)$ is called the Stirling number of the second kind.

## Allowing Empty Cells

- The number of ways of placing $r$ distinct objects into $n$ nondistinct cells with no cell left empty is equal to $S(r, n)$.
Claim: The number of ways of distributing $r$ distinct objects into $n$ nondistinct cells with empty cells allowed is:
- $S(r, 1)+S(r, 2)+\cdots+S(r, n)$, for $r \geq n$;
- $S(r, 1)+S(r, 2)+\cdots+S(r, r)$, for $r \leq n$.

To distribute $r$ distinct objects into $n$ nondistinct cells with empty cells allowed, we can do one of the following:

- Distribute them so that one cell is not empty in $S(r, 1)$ ways;
- Distribute them so that two cells are not empty in $S(r, 2)$ ways;

Thus, by the Sum Rule, the claim follows.

## Exponential Enumerator for $r \leq n$

- When $r \leq n$, i.e., there are at least as many cells as objects, there is a closed-form expression for the ordinary generating function of the numbers of ways of distributing the objects.
Since $S(i, j)=0$, for $i<j$, the count $S(r, 1)+S(r, 2)+\cdots+S(r, r)$ does not change if we add to it an infinite number of terms as follows: $S(r, 1)+S(r, 2)+\cdots+S(r, r)+S(r, r+1)+S(r, r+2)+\cdots$. But

$$
\begin{aligned}
& \frac{e^{x}-1}{1!}=S(0,1)+\frac{S(1,1)}{1!} x+\frac{S(2,1)}{2!} x^{2}+\cdots+\frac{S(r, 1)}{r!} x^{r}+\cdots \\
& \frac{\left(e^{x}-1\right)^{2}}{2!}=S(0,2)+\frac{S(1,1)}{1!} x+\frac{S(2,2)}{2!} x^{2}+\cdots+\frac{S(r, 2)}{r!} x^{r}+\cdots
\end{aligned}
$$

$$
\frac{\left(e^{x}-1\right)^{r}}{r!}=S(0, r)+\frac{S(1, r)}{1!} x+\frac{S(2, r)}{2!} x^{2}+\cdots+\frac{S(r, r)}{r!} x^{r}+\cdots
$$

$$
\frac{\left(e^{x}-1\right)^{r+1}}{(r+1)!}=S(0, r+1)+\frac{S(1, r+1)}{1!} x+\frac{S(2, r+1)}{2!} x^{2}+\cdots+\frac{S(r, r+1)}{r!} x^{r}+\cdots
$$

## Exponential Enumerator for $r \leq n$ (Cont'd)

- Therefore, the expression

$$
S(r, 1)+S(r, 2)+\cdots+S(r, r)+S(r, r+1)+S(r, r+2)+\cdots,
$$

which is the number of ways of distributing $r$ distinct objects into $r$ or more nondistinct cells, is the coefficient of $\frac{x^{r}}{r!}$, in

$$
\frac{e^{x}-1}{1!}+\frac{\left(e^{x}-1\right)^{2}}{2!}+\cdots+\frac{\left(e^{x}-1\right)^{r}}{r!}+\frac{\left(e^{x}-1\right)^{r+1}}{(r+1)!}+\cdots
$$

This generating function is

$$
e^{e^{x}-1}-1
$$

## Subsection 5

## Partitions of Integers

## Nondistinct Objects Into Nondistinct Cells and Partitions

- A partition of an integer is a division of the integer into positive integral parts, in which the order of these parts is not important.
Example: The five different partitions of the integer 4 are

$$
4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1
$$

- A partition of the integer $n$ is equivalent to a way of distributing $n$ nondistinct objects into $n$ nondistinct cells with empty cells allowed.
- We discuss these distributions in the context of the partitions of integers because this is also an important topic in number theory.


## Number of Ways to Pick 1's, 2's, etc.

- In the polynomial $1+x+x^{2}+x^{3}+\cdots+x^{n}$ the coefficient of $x^{k}$ is the number of ways of having $k 1$ 's in a partition of the integer $n$ : In a partition of $n$ there can be from no 1's to at most $n 1$ 's.
- There is one way of having $k 1$ 's, for $0 \leq k \leq n$;
- There is no way of having $k 1$ 's, for $k>n$.

Hence, in the infinite sum $1+x+x^{2}+x^{3}+\cdots+x^{r}+\cdots=\frac{1}{1-x}$, the coefficient of $x^{k}$ is the number of ways of having $k 1^{\prime}$ 's in a partition of any integer larger than or equal to $k$.

- Similarly, in the polynomial $1+x^{2}+x^{4}+\cdots+x^{\lfloor x / 2\rfloor}$ the coefficient of $x^{2 k}$ is the number of ways of having $k 2$ 's in a partition of the integer $n$.
Also, in the infinite sum $1+x^{2}+x^{4}+x^{6}+\cdots+x^{2 r}+\cdots=\frac{1}{1-x^{2}}$ the coefficient of $x^{2 k}$ is the number of ways of having $k 2$ 's in a partition of any integer larger than or equal to $2 k$.


## Partition Enumerators

- We conclude that the ordinary generating function of the sequence $(p(0), p(1), \ldots, p(n))$, where $p(i)$ denotes the number of partitions of the integer $i$, is

$$
\begin{aligned}
F(x)= & \left(1+x+x^{2}+x^{3}+\cdots+x^{r}+\cdots\right) \\
& \left(1+x^{2}+x^{4}+x^{6}+\cdots+x^{2 r}+\cdots\right) \\
& \left(1+x^{3}+x^{6}+x^{9}+\cdots+x^{3 r}+\cdots\right) \\
& \left(1+x^{4}+x^{8}+x^{12}+\cdots+x^{4 r}+\cdots\right) \\
& \cdots\left(1+x^{n}+x^{2 n}+x^{3 n}+\cdots+x^{n r}+\cdots\right) \\
= & \frac{1}{(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right)\left(1-x^{4}\right) \cdots\left(1-x^{n}\right)} .
\end{aligned}
$$

- $F(x)$ does not enumerate the $p(j)$ 's for $j>n$. It only enumerates the number of partitions of the integer $j$ that have no part exceeding $n$.


## Examples

- From $\frac{1}{(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right)}=1+x+2 x^{2}+3 x^{3}+4 x^{4}+5 x^{5}+7 x^{6}+\cdots$, we get:
- There are three ways to partition the integer 3 .
- There are seven ways to partition the integer 6 , such that the parts do not exceed 3.
- The ordinary generating function of the infinite sequence $(p(0), p(1), p(2), \ldots, p(n), \ldots)$ is

$$
F(x)=\frac{1}{(1-x)\left(1-x^{2}\right)\left(1-x^{3}\right) \cdots}
$$

$-\ln \frac{1}{(1-x)\left(1-x^{3}\right)\left(1-x^{5}\right) \cdots\left(1-x^{2 n+1}\right)}$ :

- the coefficient of $x^{k}$ for $k \leq 2 n+1$ is the number of partitions of the integer $k$ into odd parts;
- the coefficient of $x^{k}$ for $k>2 n+1$ is the number of partitions of the integer $k$ into odd parts not exceeding $2 n+1$.


## More Examples

- $\ln \frac{1}{(1-x)\left(1-x^{3}\right)\left(1-x^{5}\right) \ldots}$ the coefficient of $x^{k}$ is the number of partitions of the integer $k$ into odd parts.
- $\ln \frac{1}{\left(1-x^{2}\right)\left(1-x^{4}\right)\left(1-x^{6}\right) \cdots\left(1-x^{2 n}\right)}$ :
- the coefficient of $x^{k}$ for $k \leq 2 n$ is the number of partitions of the integer $k$ into even parts;
- the coefficient of $x^{k}$ for $k>2 n$ is the number of partitions of the integer $k$ into even parts not exceeding $2 n$.
- $\ln \frac{1}{\left(1-x^{2}\right)\left(1-x^{4}\right)\left(1-x^{6}\right) \ldots}$ the coefficient of $x^{k}$ is the number of partitions of the integer $k$ into even parts.
- The polynomial $(1+x)\left(1+x^{2}\right)\left(1+x^{3}\right) \cdots\left(1+x^{n}\right)$ enumerates:
- the partitions of integers $\leq n$ into distinct (unequal) parts;
- the partitions of integers $>n$ into distinct parts not exceeding $n$.
- $(1+x)\left(1+x^{2}\right)\left(1+x^{3}\right) \cdots\left(1+x^{n}\right) \cdots$ enumerates the partitions of the integers into distinct parts.


## Example

- Note that

$$
\begin{aligned}
(1+x) & \left(1+x^{2}\right)\left(1+x^{3}\right)\left(1+x^{4}\right) \cdots\left(1+x^{r}\right) \cdots \\
& =\frac{1-x^{2}}{1-x} \cdot \frac{1-x^{4}}{1-x^{2}} \cdot \frac{1-x^{6}}{1-x^{3}} \cdots \frac{1-x^{2 r}}{1-x^{r}} \cdots \\
& =\frac{1}{(1-x)\left(1-x^{3}\right)\left(1-x^{5}\right) \cdots}
\end{aligned}
$$

So, the number of partitions of an integer into distinct parts is equal to the number of partitions of the integer into odd parts.
Example The integer 6 can be partitioned into distinct parts in four different ways:

$$
6, \quad 5+1, \quad 4+2, \quad 3+2+1
$$

There are also exactly four different ways in which 6 can be partitioned into odd parts:

$$
5+1, \quad 3+3, \quad 3+1+1+1, \quad 1+1+1+1+1+1
$$

## Example

- Note that

$$
\begin{aligned}
& (1-x)(1+x)\left(1+x^{2}\right)\left(1+x^{4}\right)\left(1+x^{8}\right) \cdots\left(1+x^{2^{r}}\right) \cdots \\
& =\left(1-x^{2}\right)\left(1+x^{2}\right)\left(1+x^{4}\right)\left(1+x^{8}\right) \cdots\left(1+x^{2^{r}}\right) \cdots \\
& =\left(1-x^{4}\right)\left(1+x^{4}\right)\left(1+x^{8}\right) \cdots\left(1+x^{2^{r}}\right) \cdots \\
& =\cdots=1
\end{aligned}
$$

So, we have the identity

$$
\frac{1}{1-x}=(1+x)\left(1+x^{2}\right)\left(1+x^{4}\right)\left(1+x^{8}\right) \cdots\left(1+x^{2^{r}}\right) \cdots
$$

Recalling that $\frac{1}{1-x}=1+x+x^{2}+x^{3}+x^{4}+\cdots$, we conclude that any integer can be expressed as the sum of a selection of the integers $1,2,4,8, \ldots, 2^{r}, \ldots$ (without repetition) in exactly one way.
This is the well-known fact that a decimal number can be represented uniquely as a binary number.

## Example

- Note that

$$
\begin{aligned}
1-x= & \frac{1}{(1+x)\left(1+x^{2}\right)\left(1+x^{4}\right)\left(1+x^{8}\right) \cdots\left(1+x^{2}\right) \cdots} \\
= & \left(1-x+x^{2}-x^{3}+x^{4}-\cdots\right) \\
& \left(1-x^{2}+x^{4}-x^{6}+x^{8}-\cdots\right) \\
& \left(1-x^{4}+x^{8}-x^{12}+x^{16}-\cdots\right) \cdots \\
& \quad\left(1-x^{2^{r}}+x^{2 \cdot 2^{r}}-x^{3 \cdot 2^{r}}+x^{4 \cdot 2^{r}}-\cdots\right) \cdots
\end{aligned}
$$

Thus, to partition any integer $n$ larger than 1 into parts that are powers of 2 , namely, $1,2,4,8, \ldots, 2^{r}, \ldots$, the number of partitions that have an even number of parts is equal to the number of partitions that have an odd number of parts.

- The series $1-x+x^{2}-x^{3}+x^{4}-\cdots$ enumerates the number of 1 's in a partition, with terms corresponding to an even number of 1's in the partition having +1 as the coefficients and terms corresponding to an odd number of 1 's in the partition having -1 as the coefficients.


## Example (Cont'd)

- The series $1-x^{2}+x^{4}-x^{6}+x^{8}-\cdots$ enumerates the number of 2 's in a partition, with terms corresponding to an even number of 2's having positive coefficients and terms corresponding to an odd number of 2'shaving negative coefficients.
- The series $1-x^{4}+x^{8}-x^{12}+x^{16}-\cdots$ enumerates the number of 4 's in a partition, with terms corresponding to an even number of 4's having positive coefficients and terms corresponding to an odd number of 4's having negative coefficients.
Therefore, in the expansion of the product $\left(1-x+x^{2}-x^{3}+x^{4}-\cdots\right)\left(1-x^{2}+x^{4}-x^{6}+x^{8}-\cdots\right)\left(1-x^{4}+\right.$ $\left.x^{8}-x^{12}+x^{16}-\cdots\right) \cdots\left(1-x^{2^{r}}+x^{2 \cdot 2^{r}}-x^{3 \cdot 2^{r}}+x^{4 \cdot 2^{r}}-\cdots\right) \cdots$ a
term $+x^{n}$ corresponds to a partition of the integer $n$ into an even number of parts, and a term $-x^{n}$ corresponds to a partition of the integer $n$ into an odd number of parts.


## Example (Illustrated)

Example: The four partitions of the integer 5 into parts that are powers of 2 are

$$
4+1, \quad 2+1+1+1, \quad 2+2+1, \quad 1+1+1+1+1
$$

Two ( $4+1$ and $2+1+1+1$ ) have an even number of parts. The other two have an odd number of parts.

## Subsection 6

## The Ferrers Graph

## Ferrers Graphs

- A Ferrers graph consists of rows of dots.

The dots are arranged in such a way that an upper row has at least as many dots as a lower row.

- A partition of an integer can be represented by a Ferrers graph by making each row in the graph correspond to a part in the partition, with the number of dots in a row specifying the value of the corresponding part.
Example: The partition of the integer 14 into $6+3+3+2$ is represented by


## Relating Various Kinds of Partitions

Claim: The number of partitions of an integer into exactly $m$ parts is equal to the number of partitions of the integer into parts, the largest of which is $m$.
Note that the transposition of a Ferrers graph (the leftmost column becomes the uppermost row and so on) is also a Ferrers graph. It follows that the transposition of the Ferrers graph of a partition having exactly $m$ parts becomes the Ferrers graph of a partition having $m$ as the largest part.
Example: Integer 6 has exactly two partitions that have exactly four parts each $2+2+1+1$ and $3+1+1+1$. There are also two partitions that have 4 as their largest part: $4+2$ and $4+1+1$.

## Connection with Ordinary Enumerators

- The number of partitions of an integer into at most $m$ parts is equal to the number of partitions of the integer into parts not exceeding $m$.
- Therefore, the ordinary generating function of the numbers of partitions of integers into at most $m$ parts is also

$$
\frac{1}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m}\right)}
$$

- The ordinary generating function of the numbers of partitions of integers into at most $m-1$ parts is

$$
\frac{1}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m-1}\right)}
$$

- Thus, the ordinary generating function of the numbers of partitions of integers into exactly $m$ parts is

$$
\frac{1}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m}\right)}-\frac{1}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m-1}\right)}=\frac{x^{m}}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m}\right)}
$$

## Partitions into Exactly m Unequal Parts

- Find the partitions of 8 into exactly 3 unequal parts.

Consider the partitions of $8-\frac{3(3-1)}{2}=5$ into 3 (not necessarily unequal) parts.


Now do the following:

- Add 2 bullets in the first row;
- Add 1 bullet in the second row;
- Add no bullets in the last row.

We get

These are the partitions of 8 into 3 unequal parts!

## Generating Partitions into Exactly $m$ Unequal Parts

- Find the ordinary generating function of the numbers of partitions of integers into exactly $m$ unequal parts.
Consider the Ferrers graph of an m-part partition of the integer $n-\frac{m(m-1)}{2}$. Add
- $m-1$ dots to the first row;
- $m-2$ dots to the second row;
- one dot to the $(m-1)$-st row.

We get the Ferrers graph of a partition of the integer $n$ into $m$ unequal parts.
This gives a one-to-one correspondence between the $m$-part partitions of the integer $n-\frac{m(m-1)}{2}$ and the partitions of the integer $n$ into $m$ unequal parts.

## Generating Partitions into $m$ Unequal Parts (Cont'd)

- The number of partitions of the integer $n$ into exactly $m$ unequal parts equals the number of partitions of the integer $n-\frac{m(m-1)}{2}$ into exactly $m$ parts.
Therefore, the ordinary generating function of the numbers of the partitions of integers into $m$ distinct parts is

$$
\begin{gathered}
x^{m(m-1) / 2} \frac{x^{m}}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m}\right)} \\
=\frac{x^{m(m+1) / 2}}{(1-x)\left(1-x^{2}\right) \cdots\left(1-x^{m}\right)}
\end{gathered}
$$

## Subsection 7

## Elementary Relations

## Sum and Product of Generating Functions

- Let $A(x), B(x)$ and $C(x)$ be the ordinary generating functions of the sequences $\left(a_{0}, a_{1}, \ldots, a_{r}, \ldots\right),\left(b_{0}, b_{1}, \ldots, b_{r}, \ldots\right)$ and $\left(c_{0}, c_{1}, \ldots, c_{r}, \ldots\right)$, respectively.
- By definition,

$$
C(x)=A(x)+B(x)
$$

if and only if the members of the sequences are related as follows:

$$
c_{0}=a_{0}+b_{0}, c_{1}=a_{1}+b_{1}, \ldots, c_{r}=a_{r}+b_{r}, \ldots
$$

- Similarly, by definition,

$$
C(x)=A(x) \times B(x)
$$

if and only if the members of the sequences are related as follows:

$$
\begin{aligned}
& c_{0}=a_{0} b_{0}, c_{1}=a_{1} b_{0}+a_{0} b_{1}, \ldots \\
& c_{r}=a_{r} b_{0}+a_{r-1} b_{1}+a_{r-2} b_{2}+\cdots+a_{1} b_{r-1} a_{0} b_{r}, \ldots
\end{aligned}
$$

## The Summing Operator

- Let $A(x)$ be the ordinary generating function of the sequence ( $a_{0}, a_{1}, a_{2}, \ldots, a_{r}, \ldots$ ). Since $\frac{1}{1-x}$ is the ordinary generating function of the sequence $(1,1,1, \ldots, 1, \ldots), \frac{1}{1-x} A(x)$ is the ordinary generating function of the sequence

$$
\left(a_{0}, a_{0}+a_{1}, a_{0}+a_{1}+a_{2}, \ldots, a_{0}+a_{1}+a_{2}+\cdots+a_{r}, \ldots\right) .
$$

$\frac{1}{1-x}$ is, therefore, called the summing operator.
Example: Find the coefficient of the term $x^{37}$ in

$$
\frac{1-3 x^{2}+4 x^{7}+12 x^{21}-5 x^{45}}{1-x}
$$

We have as the answer $1-3+4+12=14$.

## The Generating Function of the Squares

- Evaluate the sum $1^{2}+2^{2}+3^{2}+\cdots+r^{2}$.

We first find the ordinary generating function of the sequence $\left(0^{2}, 1^{2}, 2^{2}, 3^{2}, \ldots, r^{2}, \ldots\right)$.
We have in sequence:

$$
\begin{aligned}
& \frac{1}{1-x}=1+x+x^{2}+x^{3}+z^{4}+\cdots+x^{r}+\cdots \\
& \frac{d}{d x} \frac{1}{1-x}=\frac{1}{(1-x)^{2}}=1+2 x+3 x^{2}+4 x^{3}+\cdots+r x^{r-1}+\cdots \\
& \frac{x}{(1-x)^{2}}=x+2 x^{2}+3 x^{3}+4 x^{4}+\cdots+r x^{r}+\cdots \\
& \frac{d}{d x} \frac{x}{(1-x)^{2}}=\frac{1+x}{(1-x)^{3}}=1^{2}+2^{2} x+3^{2} x^{2}+4^{2} x^{3}+\cdots+r^{2} x^{r-1}+\cdots \\
& x \frac{1+x}{(1-x)^{3}}=1^{2} x+2^{2} x^{2}+3^{2} x^{3}+4^{2} x^{4}+\cdots+r^{2} x^{r}+\cdots
\end{aligned}
$$

Therefore, the ordinary generating function of the sequence $\left(0^{2}, 1^{2}, 2^{2}, 3^{2}, \ldots, r^{2}, \ldots\right)$ is $\frac{x(1+x)}{(1-x)^{3}}$.

## Summing the Squares

- We found that $x \frac{d}{d x} \frac{x}{(x-1)^{2}}=\frac{x(1+x)}{(1-x)^{3}}$ is the ordinary generating function of the sequence $\left(0^{2}, 1^{2}, 2^{2}, 3^{2}, \ldots, r^{2}, \ldots\right)$.
Thus, the ordinary generating function of the sequence $\left(0^{2}, 0^{2}+1^{2}, 0^{2}+1^{2}+2^{2}, \ldots, 0^{2}+1^{2}+2^{2}+3^{2}+\cdots+r^{2}, \cdots\right)$ is $\frac{x(1+x)}{(1-x)^{4}}$.
By the binomial theorem, the coefficient of $x^{r}$ in $\frac{1}{(1-x)^{4}}$ is

$$
\frac{(-4)(-4-1)(-4-2) \cdots(-4-r+1)}{r!}(-1)^{r}=\frac{4 \cdot 5 \cdot 6 \cdots(r+3)}{r!}=\frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3} .
$$

Therefore, the coefficient of $x^{r}$ in the expansion of $\frac{x(1+x)}{(1-x)^{4}}$ is $\frac{r(r+1)(r+2)}{1 \cdot 2 \cdot 3}+\frac{(r-1) r(r+1)}{1 \cdot 2 \cdot 3}=\frac{r(r+1)(2 r+1)}{6}$.
We conclude

$$
1^{2}+2^{2}+3^{2}+\cdots+r^{2}+\cdots=\frac{r(r+1)(2 r+1)}{6}
$$

## Operations on Exponential Generating Functions

- Let $A(x), B(x)$ and $C(x)$ be the exponential generating functions of the sequences $\left(a_{0}, a_{1}, \ldots, a_{r}, \ldots\right),\left(b_{0}, b_{1}, \ldots, b_{r}, \ldots\right)$ and $\left(c_{0}, c_{1}, \ldots, c_{r}, \ldots\right)$, respectively.
- By definition,

$$
C(x)=A(x)+B(x)
$$

if and only if the members of the sequences are related as follows:

$$
c_{0}=a_{0}+b_{0}, c_{1}=a_{1}+b_{1}, \ldots, c_{r}=a_{r}+b_{r}, \ldots
$$

- By definition,

$$
C(x)=A(x) \times B(x)
$$

if and only if the members of the sequences are related as follows:

$$
\begin{aligned}
& c_{0}=a_{0} b_{0}, c_{1}=a_{1} b_{0}+a_{0} b_{1}, \quad c_{2}=2!\left(\frac{a_{0} b_{0}}{2!}+\frac{a_{1} b_{1}}{111!}+\frac{a_{0} b_{2}}{2!}\right), \ldots, \\
& c_{r}=r!\left[\frac{a, b b_{0}}{r!}+\frac{a_{r-1} b_{1}}{(r-1)!!!}+\cdots+\frac{a_{0} b_{r}}{r!}\right]=\sum_{i=0}^{r}\binom{r}{i} a_{r-i} b_{i}, \ldots
\end{aligned}
$$

## Example

- Evaluate the sum $\sum_{i=0}^{r} \frac{r!}{(r-i+1)!(i+1)!}$.

Note that
$\sum_{i=0}^{r} \frac{r!}{(r-i+1)!(i+1)!}=\sum_{i=0}^{r} \frac{r!}{(r-i)!i!} \frac{1}{r-i+1} \frac{1}{i+1}=\sum_{i=0}^{r}\binom{r}{i} \frac{1}{r-i+1} \frac{1}{i+1}$.
We also have

$$
\begin{aligned}
e^{x} & =1+\frac{1}{1!} x+\frac{1}{2!} x^{2}+\cdots+\frac{1}{r!} x^{r}+\cdots \\
\frac{1}{x}\left(e^{x}-1\right) & =1+\frac{1}{2!} x+\frac{1}{3!} x^{2}+\cdots+\frac{1}{r!} x^{r-1}+\cdots
\end{aligned}
$$

Thus, the exponential generating function of $\left(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{r}, \cdots\right)$ is $\frac{1}{x}\left(e^{x}-1\right)$. We conclude that the exponential generating function of the sequence
$1 \cdot 1, \frac{1}{2} \cdot 1+1 \cdot \frac{1}{2},\binom{2}{0} \cdot \frac{1}{3} \cdot 1+\binom{2}{1} \cdot \frac{1}{2} \cdot \frac{1}{2}+\binom{2}{2} \cdot 1 \cdot \frac{1}{3}, \cdots, \sum_{i=0}^{r}\binom{r}{i} \frac{1}{r-i+1} \frac{1}{i+1}, \cdots$ is

$$
\frac{1}{x^{2}}\left(e^{x}-1\right)^{2}
$$

## Example (Cont'd)

- Note that

$$
\begin{aligned}
& \frac{1}{x^{2}}\left(e^{x}-1\right)^{2}=\frac{1}{x^{2}}\left(e^{2 x}-2 e^{x}+1\right) \\
& =\frac{1}{x^{2}}\left(\left(1+\frac{(2 x)}{1!}+\frac{(2 x)^{2}}{2!}+\cdots\right)-2\left(1+\frac{x}{1!}+\frac{x^{2}}{2!}+\cdots\right)+1\right) \\
& =\frac{1}{x^{2}}\left(\left((1-2)+\frac{(2 x-2 x)}{1!}+\frac{\left(2^{2}-2\right)}{2!} x^{2}+\frac{\left(2^{3}-2\right)}{3!} x^{3}+\cdots\right)+1\right) \\
& =\left(\frac{2^{2}}{2!}-\frac{2}{2!}\right)+\left(\frac{2^{3}}{3!}-\frac{2}{3!}\right) x+\left(\frac{2^{4}}{4!}-\frac{2}{4!}\right) x^{3}+\cdots \\
& \quad+\left(\frac{2^{r+2}}{(r+2)!}-\frac{2}{(r+2)!}\right) x^{r}+\cdots .
\end{aligned}
$$

Thus, we obtain that

$$
\sum_{i=0}^{r} \frac{r!}{(r-i+1)(i+1)!}=r!\left(\frac{2^{r+2}}{(r+2)!}-\frac{2}{(r+2)!}\right)=\frac{2\left(2^{r+1}-1\right)}{(r+2)(r+1)}
$$

