Introduction to Combinatorial Mathematics

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Subsection 1

Introduction

Recurrence Relations or Difference Equations

Example: Consider the geometric series $(1, 3, 3^2, 3^3, \ldots, 3^n, \ldots)$.

- This sequence of numbers can be described by the expression for the general term $a_n = 3^n$, $n = 0, 1, 2, \ldots$.
- Alternatively, we can express the n-th number in terms of the $(n-1)$-st, together with the specification of the first number: $a_n = 3a_{n-1},\ a_0 = 1$.

- For a sequence of numbers $(a_0, a_1, a_2, \ldots, a_n, \ldots)$, an equation relating a number a_n to some of its predecessors in the sequence, for any n, is called a **recurrence relation**. A recurrence relation is also called a **difference equation**.

  Example: The recurrence relation above specifies that the n-th number is computed as three times the $(n-1)$-st number in the sequence.
- To initiate the computation, one must know one (or several) number(s) in the sequence, called the **boundary conditions**.

  Example: The boundary condition of the preceding example is $a_0 = 1$.

Example: The Fibonacci Sequence

- Consider the **Fibonacci sequence** of numbers:
  - It starts with the two numbers 1, 1;
  - It contains numbers which are equal to the sum of their two immediate predecessors.

  A portion of the sequence is

  $$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \ldots$$

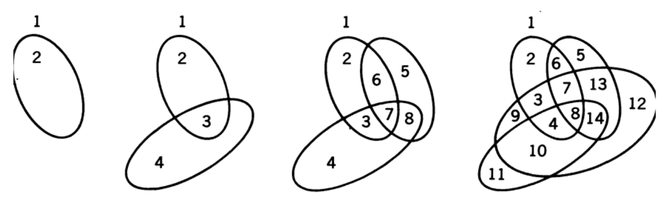
  - It is quite difficult in this case to obtain a general expression for the n-th number in the sequence by observation.
  - On the other hand, the sequence can be described by the recurrence relation

    $$a_n = a_{n-1} + a_{n-2},$$

    together with the boundary conditions $a_0 = 1$ and $a_1 = 1$.
- We are interested in the solution of a recurrence relation to obtain a general expression for the n-th number in a sequence.
- In most instances, the converse problem of obtaining a recurrence relation from a general expression for the n-th number is of less interest.

Example: Ovals and Regions of the Plane

- Let there be n ovals drawn on the plane. If an oval intersects each of the other ovals at exactly two points and no three ovals meet at the same point, into how many regions do these ovals divide the plane? Let a_n denote the number of regions into which the plane is divided by n ovals. It is clear that $a_1 = 2$. We can also see that $a_2 = 4$, $a_3 = 8$, and $a_4 = 14$.



Suppose that we have drawn $n-1$ ovals that divide the plane into a_{n-1} regions. The n-th oval will intersect these $n-1$ ovals at $2(n-1)$ points.

Ovals and Regions of the Plane (Cont'd)

- Since the n-th oval will intersect these $n-1$ ovals at $2(n-1)$ points, the n-th oval will be divided into $2(n-1)$ arcs. Each of these arcs will divide one of the a_{n-1} regions in two pieces. So, we have the recurrence relation

  $$a_n = a_{n-1} + 2(n-1).$$

  With this relation and the boundary condition $a_1 = 2$, one can compute the value of a_n for any given n simply by repeatedly applying the recurrence relation:

  $$\begin{array}{rcl} a_5 & = & a_4 + 2 \cdot (5-1) = 14 + 8 = 22; \\ a_6 & = & a_5 + 2 \cdot (6-1) = 22 + 10 = 32. \end{array}$$

- We will develop methods for solving this recurrence relation to obtain a general expression for a_n.

Subsection 2

Linear Recurrence Relations with Constant Coefficients

Linear Recurrence Relations with Constant Coefficients

- A recurrence relation of the form
$$C_0 a_n + C_1 a_{n-1} + \cdots + C_r a_{n-r} = f(n)$$
is called a **linear recurrence relation (difference equation) with constant coefficients** where all the C's are constants.
Example: $3a_n - 5a_{n-1} + 2a_{n-2} = n^2 + 5$ is a linear difference equation with constant coefficients.
- If the values of r consecutive a's in the sequence, $a_{k-r}, a_{k-r+1}, \ldots,$ a_{k-1} are known for some k, the value of a_k can be calculated by using the equation. Also, the values of $a_{k+1}, a_{k+2}, \ldots$ and the values of $a_{k-r-1}, a_{k-r-2}, \ldots$ can then be calculated recursively.
- It follows that the solution to this recurrence is determined uniquely by the values of r consecutive a's (the boundary conditions).
- The general form of the solution contains r undetermined constants. These can be determined by the values of r consecutive a's in the sequence.

Total as Sum of Homogeneous and Particular Solutions

- The (total) solution of a linear difference equation with constant coefficients is the sum of two parts:
  - The **homogeneous solution**, which satisfies the difference equation when the right-hand side of the equation is set to 0;
  - The **particular solution**, which satisfies the difference equation with $f(n)$ at the right-hand side.
- Let $a_n^{(h)}$ denote the homogeneous solution and $a_n^{(p)}$ denote the particular solution to the difference equation:

$$\begin{array}{rcl} C_0 a_n^{(h)} + C_1 a_{n-1}^{(h)} + \cdots + C_r a_{n-r}^{(h)} & = & 0 \\ C_0 a_n^{(p)} + C_1 a_{n-1}^{(p)} + \cdots + C_r a_{n-r}^{(p)} & = & f(n). \end{array}$$

  Then we have
  $C_0(a_n^{(h)} + a_n^{(p)}) + C_1(a_{n-1}^{(h)} + a_{n-1}^{(p)}) + \cdots + C_r(a_{n-r}^{(h)} + a_{n-r}^{(p)}) = f(n).$
  Thus, the total solution, $a_n = a_n^{(h)} + a_n^{(p)}$ satisfies the difference equation.

Characteristic Roots and Characteristic Equations

- The homogeneous solution of a linear difference equation $C_0 a_n + C_1 a_{n-1} + \cdots + C_r a_{n-r} = f(n)$ is of the form $a_n^{(h)} = A\alpha_1^n$, where
  - α_1 is called a **characteristic root** and
  - A is a constant determined by the boundary conditions.
- Substituting $A\alpha^n$ for a_n in the difference equation with the right-hand side of the equation set to 0, we obtain

$$C_0 A\alpha^n + C_1 A\alpha^{n-1} + C_2 A\alpha^{n-2} + \cdots + C_r A\alpha^{n-r} = 0.$$

  This equation can be simplified into the polynomial

$$C_0\alpha^r + C_1\alpha^{r-1} + C_2\alpha^{r-2} + \cdots + C_r = 0,$$

  which is called the **characteristic equation** of the difference equation.
- Thus, if α_1 is a root of the characteristic equation (a characteristic root), $A\alpha_1^n$ is a homogeneous solution to the difference equation.

Form of the Homogeneous Solution

- A characteristic equation of r-th degree has r characteristic roots.

  Suppose the roots of the characteristic equation are distinct.

  In this case it is easy to verify that the homogeneous solution is

$$a_n^{(h)} = A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_r\alpha_r^n,$$

  where
  - $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the distinct characteristic roots;
  - $A_1, A_2, \ldots, A_r$ are constants which can be determined by the boundary conditions.

Example: Fibonacci Sequence Revisited

- The recurrence relation for the Fibonacci sequence of numbers is $a_n = a_{n-1} + a_{n-2}$.
  The corresponding characteristic equation is $\alpha^2 - \alpha - 1 = 0$, which has two distinct roots $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.
  The homogeneous solution (in this case, also the total solution, since the particular solution is 0) is
  $$a_n = a_n^{(h)} = A_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + A_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$
  The two constants A_1 and A_2 can be determined from the boundary conditions $a_0 = 1$ and $a_1 = 1$ by solving the two equations
  $$\begin{cases} a_0 = 1 = A_1 + A_2 \\ a_1 = 1 = A_1 \frac{1+\sqrt{5}}{2} + A_2 \frac{1-\sqrt{5}}{2} \end{cases}$$
  These equations yield $A_1 = \frac{1}{\sqrt{5}} \frac{1+\sqrt{5}}{2}$ and $A_2 = -\frac{1}{\sqrt{5}} \frac{1-\sqrt{5}}{2}$.
  Thus, the solution is $a_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}$.

Complex Conjugate Characteristic Roots

- When the coefficients of the characteristic equation are real numbers but some of the characteristic roots are complex numbers, the homogeneous solution can be written in a different form.
- If a polynomial has real coefficients, then the complex conjugate of every root is also a root of the polynomial. Hence, complex roots always appear in pairs.
- Let $\alpha_1 = \delta + i\omega$ and $\alpha_2 = \delta - i\omega$ be a pair of complex characteristic roots. Set $\rho = |\alpha_1| = |\alpha_2| = \sqrt{\delta^2 + \omega^2}$ and $\theta = \tan^{-1} \frac{\omega}{\delta}$. Then, we have

$$\begin{aligned} A_1\alpha_1^n + A_2\alpha_2^n &= A_1(\delta + i\omega)^n + A_2(\delta - i\omega)^n \\ &= A_1\rho^n(\cos n\theta + i \sin n\theta) + A_2\rho^n(\cos n\theta - i \sin n\theta) \\ &= (A_1 + A_2)\rho^n \cos n\theta + i(A_1 - A_2)\rho^n \sin n\theta. \end{aligned}$$

  $B_1 = A_1 + A_2$ and $B_2 = i(A_1 - A_2)$ are constants determined by the boundary conditions.

Example

- Evaluate the $n \times n$ determinant $\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}$.

  Let a_k denote the value of the $k \times k$ determinant that is of this form.
  Expanding the $n \times n$ determinant with respect to the first column:

$$a_n = \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} = a_{n-1} - a_{n-2}.$$

Example (Cont'd)

- We obtained the recurrence relation

$$a_n = a_{n-1} - a_{n-2}.$$

The corresponding characteristic equation is $\alpha^2 - \alpha + 1 = 0$. The characteristic roots are $\alpha_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\alpha_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Now compute:

  - $\rho = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1$;
  - $\tan^{-1}\frac{\sqrt{3}/2}{1/2} = \frac{\pi}{3}$.

Thus, we have $a_n = B_1 \cos\frac{n\pi}{3} + B_2 \sin\frac{n\pi}{3}$.
Using the boundary conditions $a_1 = 1$ and $a_2 = 0$, the constants B_1 and B_2 are determined as $B_1 = 1$ and $B_2 = \frac{1}{\sqrt{3}}$. Therefore, the solution of the difference equation is

$$a_n = \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin\frac{n\pi}{3}.$$

Multiple Characteristic Roots

- Let α_1 be a k-multiple root.
  The corresponding homogeneous solution is
  $$(A_1 n^{k-1} + A_2 n^{k-2} + \cdots + A_{k-2} n^2 + A_{k-1} n + A_k)\alpha_1^n,$$
  where the A's are constants which are determined by the boundary conditions.
  - It is clear that $a_n^{(h)} = A_k \alpha_1^n$ is a homogeneous solution.
  - Because of multiplicity α_1 not only satisfies the equation $C_0\alpha^n + C_1\alpha^{n-1} + C_2\alpha^{n-2} + \cdots + C_r\alpha^{n-r} = 0$ but also its derivative $C_0 n\alpha^{n-1} + C_1(n-1)\alpha^{n-2} + C_2(n-2)\alpha^{n-2} + \cdots + C_r(n-r)\alpha^{n-r-1} = 0$.
    Multiplying by $A_{k-1}\alpha$ and replacing α by α_1, we obtain $C_0 A_{k-1} n\alpha_1^n + C_1 A_{k-1}(n-1)\alpha_1^{n-1} + C_2 A_{k-1}(n-2)\alpha_1^{n-2} + \cdots + C_r A_{k-1}(n-r)\alpha_1^{n-r} = 0$, which shows that $A_{k-1} n\alpha_1^n$ is indeed a homogeneous solution.
  - The fact that α_1 satisfies the second, third, ..., $(k-1)$-st derivatives enables us to prove that $A_{k-2} n^2\alpha_1^n, A_{k-3} n^3\alpha_1^n, \ldots, A_1 n^{k-1}\alpha_1^n$ are also homogeneous solutions.

Example

- Solve the difference equation

$$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0,$$

with the boundary conditions $a_0 = 1$, $a_1 = 2$, and $a_2 = 8$.
The characteristic equation is $\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$. We get

$$\begin{aligned} & \alpha^3 + 2\alpha^2 + 4\alpha^2 + 8\alpha + 4\alpha + 8 = 0 \\ & \Rightarrow \alpha^2(\alpha+2) + 4\alpha(\alpha+2) + 4(\alpha+2) = 0 \\ & \Rightarrow (\alpha+2)(\alpha^2 + 4\alpha + 4) = 0 \\ & \Rightarrow (\alpha+2)^3 = 0. \end{aligned}$$

So it has $\alpha = -2$ as a triple root. The solution is $a_n = (A_1 n^2 + A_2 n + A_3)(-2)^n$. From the boundary conditions, the constants are determined as

$$A_1 = \frac{1}{2}, \quad A_2 = -\frac{1}{2}, \quad A_3 = 1.$$

Therefore, $a_n = (\frac{1}{2}n^2 - \frac{1}{2}n + 1)(-2)^n$.

Example

- Evaluate the $n \times n$ determinant $\begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix}$.

  Let a_k denote the value of the $k \times k$ determinant that is of this form.
  Expanding the $n \times n$ determinant with respect to the first column:

$$a_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix} =$$

$$2\begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{vmatrix} = 2a_{n-1} - a_{n-2}.$$

Example (Cont'd)

- We obtained

$$a_n = 2a_{n-1} - a_{n-2}.$$

The characteristic equation is $\alpha^2 - 2\alpha + 1 = 0$. It has a double characteristic root $\alpha = 1$. So the solution is

$$a_n = (A_1 n + A_2)(1)^n = A_1 n + A_2.$$

From the boundary conditions $a_1 = 2$ and $a_2 = 3$, we obtain

$$A_1 = 1 \quad \text{and} \quad A_2 = 1.$$

Thus, the solution is

$$a_n = n + 1.$$

Finding the Particular Solution

- Solve the difference equation
$$a_n + 2a_{n-1} = n + 3, \quad a_0 = 3.$$
The homogeneous solution is $a_n^{(h)} = A(-2)^n$.
To determine the particular solution, we try a solution of the form $a_n^{(p)} = Bn + D$. Substituting this into the difference equation, we have $Bn + D + 2[B(n-1) + D] = n + 3$. Thus, $3Bn + 3D - 2B = n + 3$. Comparing the coefficients of n and the constant terms, we have $3B = 1$ and $3D - 2B = 3$, i.e., $B = \frac{1}{3}$ and $D = \frac{11}{9}$. Therefore, $a_n^{(p)} = \frac{n}{3} + \frac{11}{9}$.
The total solution of the difference equation is simply the sum of the homogeneous and particular solutions: $a_n = A(-2)^n + \frac{n}{3} + \frac{11}{9}$.
Taking into account the boundary condition, $A = \frac{16}{9}$. So we get
$$a_n = \frac{16}{9}(-2)^n + \frac{n}{3} + \frac{11}{9}.$$

Example

- Solve the difference equation

$$a_n + 2a_{n-1} + a_{n-2} = 2^n.$$

The homogeneous solution is $a_n^{(h)} = (A_1 n + A_2)(-1)^n$.
The particular solution is found by trying a solution of the form $a_n^{(p)} = B \cdot 2^n$.
Since

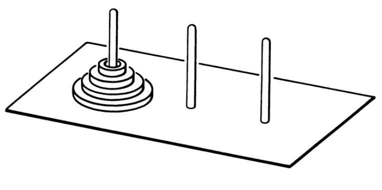
$$B \cdot 2^n + 2 \cdot B \cdot 2^{n-1} + B \cdot 2^{n-2} = 2^n.$$

B is determined as $B = \frac{4}{9}$. Hence,

$$a_n = a_n^{(h)} + a_n^{(p)} = (A_1 n + A_2)(-1)^n + \frac{4}{9} \cdot 2^n.$$

The Tower of Hanoi Problem

- n circular rings of tapering size are slipped onto a peg with the largest at the bottom:



  These rings are to be transferred one at a time onto another peg, and there is a third peg available on which rings can be left temporarily. If, during the course of transferring the rings, no ring may ever be placed on top of a smaller one, in how many moves can these rings be transferred with their relative positions unchanged?
  We transfer the n rings by:
  - first moving the top $n-1$ rings onto the third peg;
  - then we place the largest ring onto the second peg;
  - finally, move the $n-1$ rings from the third peg onto the second peg.

The Tower of Hanoi Problem: The Recurrence

- If we let a_n denote the number of moves it takes to transfer n rings from one peg to another, we have the recurrence relation

$$a_n = 2a_{n-1} + 1.$$

The homogeneous solution is $a_n^{(h)} = A \cdot 2^n$.

The particular solution is $a_n^{(p)} = -1$.

Thus $a_n = A \cdot 2^n - 1$.

Note that the boundary condition is $a_1 = 1$. So, the solution is

$$a_n = 2^n - 1.$$

Subsection 3

Solution by the Technique of Generating Functions

The Domain of a Sequence

- Consider a difference equation of the form

$$C_0 a_n + C_1 a_{n-1} + \cdots + C_r a_{n-r} = f(n).$$

  Assume it is valid only for n greater than or equal to some integer k.

  We want to determine the values of the a_n's for $n \geq k - r$.

  Among these a_n's, $a_{k-r}, a_{k-r+1}, \ldots, a_{k-1}$ are boundary conditions specified by the problem.

  We further assume that $n \geq 0$. So $k \geq r$.

- Since the values of the a_n's for $n < k - r$ are not constrained by the difference equation, they can be chosen arbitrarily.
- If we set a_n to 0 for $n < 0$ and choose some arbitrary values for the a_n's for $0 \leq n < k - r$, we can solve for the generating function of the sequence $(a_0, a_1, a_2, \ldots, a_n, \ldots)$, instead of solving for a general expression for a_n.

From a Recurrence Relation to a Generating Function

- Let $A(x)$ denote the ordinary generating function of the sequence $(a_0, a_1, \ldots, a_n, \ldots)$, i.e., $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$.

$$C_0a_n + C_1a_{n-1} + \cdots + C_ra_{n-r} = f(n)$$
$$(C_0a_n + C_1a_{n-1} + \cdots + C_ra_{n-r})x^n = f(n)x^n$$
$$\sum_{n=k}^{\infty}(C_0a_n + C_1a_{n-1} + \cdots + C_ra_{n-r})x^n = \sum_{n=k}^{\infty} f(n)x^n$$
$$\sum_{n=k}^{\infty} C_0a_nx^n + \sum_{n=k}^{\infty} C_1a_{n-1}x^n + \cdots + \sum_{n=k}^{\infty} C_ra_{n-r}x^n = \sum_{n=k}^{\infty} f(n)x^n$$
$$C_0\sum_{n=k}^{\infty} a_nx^n + C_1x\sum_{n=k}^{\infty} C_1a_{n-1}x^{n-1} + \cdots + C_rx^r\sum_{n=k}^{\infty} C_ra_{n-r}x^{n-r} = \sum_{n=k}^{\infty} f(n)x^n$$
$$C_0[A(x) - a_0 - a_1x - \cdots - a_{k-1}x^{k-1}] + C_1x[A(x) - a_0 - a_1x - \cdots - a_{k-2}x^{k-2}] + \cdots + C_rx^r[A(x) - a_0 - a_1x - \cdots - a_{k-r-1}x^{k-r-1}] = \sum_{n=k}^{\infty} f(n)x^n.$$

So $A(x) = a_0 + a_1x + \cdots + a_{k-r-1}x^{k-r-1} + \frac{1}{C_0+C_1x+\cdots+C_rx^r}[\sum_{n=k}^{\infty} f(n)x^n + C_0(a_{k-r}x^{k-r} + \cdots + a_{k-1}x^{k-1}) + C_1(a_{k-r}x^{k-r+1} + \cdots + a_{k-2}x^{k-1}) + \cdots + C_{r-1}a_{k-r}x^{k-1}]$.

Example: Two Generating Functions

- Show that
  (a) $\sum_{n=2}^{\infty}(n-1)x^n = \frac{x^2}{(1-x)^2}$;
  (b) $\frac{2x^2}{(1-x)^3} = \sum_{n=1}^{\infty} n(n-1)x^n$.

  First, since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, we obtain, by differentiation $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.
  (a) Now multiply by x^2 to get

$$\frac{x^2}{(1-x)^2} = x^2 + 2x^3 + 3x^4 + \cdots = \sum_{n=2}^{\infty}(n-1)x^n.$$

  (b) Take again the relation and differentiate once more to get $\frac{2}{(1-x)^3} = 2 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots$. Therefore, multiplying by x^2, we get

$$\begin{aligned} \frac{2x^2}{(1-x)^3} &= 2x^2 + 3 \cdot 2x^3 + 4 \cdot 3x^4 + 4 \cdot 4x^5 \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)x^n = \sum_{n=1}^{\infty} n(n-1)x^n. \end{aligned}$$

Example: Ovals and Number of Regions in the Plane

- Recall that the recurrence relation giving the number of regions in which n ovals divide the plane when the intersect at exactly two points and no three share a common point is $a_n = a_{n-1} + 2(n-1)$.

  Since a_n has physical meaning only for $n \geq 1$, the recurrence relation is valid for $n \geq 2$. Because a_0 has no physical significance, we can choose any arbitrary value for a_0. One choice is to have a value for a_0, such that the range of validity of the recurrence relation is extended. Since $a_1 = 2$, we choose $a_0 = 2$. The recurrence relation is now valid for $n \geq 1$.

Ovals and Number of Regions in the Plane (Cont'd)

- We solve the recurrence as follows:

$$\begin{aligned}
&a_n = a_{n-1} + 2(n-1)\\
&a_n x^n = (a_{n-1} + 2(n-1))x^n\\
&\textstyle\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty}(a_{n-1} + 2(n-1))x^n\\
&\textstyle\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 2(n-1)x^n\\
&\textstyle\sum_{n=1}^{\infty} a_n x^n = x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 2\sum_{n=1}^{\infty}(n-1)x^n\\
&A(x) - a_0 - xA(x) = 2\tfrac{x^2}{(1-x)^2}\\
&A(x)(1-x) = \tfrac{2x^2}{(1-x)^2} + 2\\
&A(x) = \tfrac{2x^2}{(1-x)^3} + \tfrac{2}{1-x}\\
&\textstyle A(x) = \sum_{n=1}^{\infty} n(n-1)x^n + 2\sum_{n=0}^{\infty} x^n\\
&\textstyle A(x) = 2 + \sum_{n=1}^{\infty}[n(n-1)+2]x^n.
\end{aligned}$$

It follows that $a_n = n(n-1) + 2, \ n = 0, 1, 2, \ldots$.

Example: Even Numbers of 0's

- Among the 4^n n-digit quaternary sequences, how many of them have an even number of 0's?

  Let a_{n-1} denote the number of $(n-1)$-digit quaternary sequences that have an even number of 0's.

  Then the number of $(n-1)$-digit quaternary sequences that have an odd number of 0's is $4^{n-1} - a_{n-1}$.

  - To each of the a_{n-1} sequences that have an even number of 0's, the digit 1, 2, or 3 can be appended to yield sequences of length n that contain an even number of 0's.
  - To each of the $4^{n-1} - a_{n-1}$ sequences that have an odd number of 0's, the digit 0 can be appended to yield a sequence of length n that contains an even number of 0's.

  Therefore, for $n \geq 2$, $a_n = 3a_{n-1} + 4^{n-1} - a_{n-1}$. So we get

$$a_n - 2a_{n-1} = 4^{n-1}.$$

Even Number of 0's (Cont'd)

- We discovered that
$$a_n - 2a_{n-1} = 4^{n-1}.$$
Note $a_1 = 3$. For the recurrence relation to be valid for $n = 1$, we choose $a_0 = 1$.
Multiplying both sides of the recurrence relation by x^n and summing from $n = 1$ to $n = \infty$, we obtain
$$\sum_{n=1}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} 4^{n-1}x^n.$$
$$A(x) - 1 - 2xA(x) = \frac{x}{1-4x}.$$
$$A(x) = \frac{1}{1-2x}(\frac{x}{1-4x} + 1) = \frac{1/2}{1-4x} + \frac{1/2}{1-2x}.$$
It follows that $a_n = \frac{1}{2}4^n + \frac{1}{2}2^n, \quad n \geq 0.$

Even Number of 0's and Even Number of 1's

- Among the 4^n n-digit quaternary sequences, how many of them have an even number of 0's and an even number of 1's? Denote by:
  - b_{n-1} the number of $(n-1)$-digit quaternary sequences that have an even number of 0's and an even number of 1's;
  - c_{n-1} the number of $(n-1)$-digit quaternary sequences that have an even number of 0's and an odd number of 1's;
  - d_{n-1} the number of (n-1)-digit quaternary sequences that have an odd number of 0's and an even number of 1's.

  There are $4^{n-1} - b_{n-1} - c_{n-1} - d_{n-1}$ $(n-1)$-digit quaternary sequences that have an odd number of 0's and an odd number of 1's.

	Append 0	Append 1	Append 2 or 3
Even 0's Even 1's	Odd 0's Even 1's	Even 0's Odd 1's	Even 0's Even 1's
Even 0's Odd 1's	Odd 0's Odd 1's	Even 0's Even 1's	Even 0's Odd 1's
Odd 0's Even 1's	Even 0's Even 1's	Odd 0's Odd 1's	Odd 0's Even 1's
Odd 0's Odd 1's	Even 0's Odd 1's	Odd 0's Even 1's	Odd 0's Odd 1's

The Recurrences

- We obtain, for $n \geq 2$, the recurrence relations

$$\begin{array}{rcl} b_n & = & 2b_{n-1} + c_{n-1} + d_{n-1} \\ c_n & = & b_{n-1} + 2c_{n-1} + 4^{n-1} - b_{n-1} - c_{n-1} - d_{n-1} \\ d_n & = & b_{n-1} + 2d_{n-1} + 4^{n-1} - b_{n-1} - c_{n-1} - d_{n-1} \end{array}$$

After simplification,

$$\begin{array}{rcl} b_n & = & 2b_{n-1} + c_{n-1} + d_{n-1} \\ c_n & = & c_{n-1} - d_{n-1} + 4^{n-1} \\ d_n & = & -c_{n-1} + d_{n-1} + 4^{n-1} \end{array}$$

Note $b_1 = 2$, $c_1 = 1$ and $d_1 = 1$. Therefore,
$\left\{\begin{array}{rcl} 2 & = & 2b_0 + c_0 + d_0 \\ 1 & = & c_0 - d_0 + 1 \\ 1 & = & -c_0 + d_0 + 1 \end{array}\right\}$. For the recurrence relations to be valid for $n \geq 1$, the values of b_0, c_0 and d_0 can be chosen as $b_0 = \frac{3}{4}$, $c_0 = \frac{1}{4}$, $d_0 = \frac{1}{4}$.

Solving The Recurrences

- Thus,

$$\begin{array}{rcl}\sum_{n=1}^{\infty} b_n x^n & = & 2\sum_{n=1}^{\infty} b_{n-1}x^n + \sum_{n=1}^{\infty} c_{n-1}x^n + \sum_{n=1}^{\infty} d_{n-1}x^n \\ \sum_{n=1}^{\infty} c_n x^n & = & \sum_{n=1}^{\infty} c_{n-1}x^n - \sum_{n=1}^{\infty} d_{n-1}x^n + \sum_{n=1}^{\infty} 4^{n-1}x^n \\ \sum_{n=1}^{\infty} d_n x^n & = & -\sum_{n=1}^{\infty} c_{n-1}x^n + \sum_{n=1}^{\infty} d_{n-1}x^n + \sum_{n=1}^{\infty} 4^{n-1}x^n\end{array}$$

Taking into account that $b_0 = \frac{3}{4}$, $c_0 = \frac{1}{4}$, $d_0 = \frac{1}{4}$ and using the generating function representation, these relations become

$$\begin{array}{rcl}B(x) - \frac{3}{4} & = & 2xB(x) + xC(x) + xD(x) \\ C(x) - \frac{1}{4} & = & xC(x) - xD(x) + \frac{x}{1-4x} \\ D(x) - \frac{1}{4} & = & -xC(x) + xD(x) + \frac{x}{1-4x}\end{array}$$

From the last two

$$\left\{\begin{array}{l} C(x) + D(x) - \frac{1}{2} = \frac{2x}{1-4x} \\ C(x) = D(x) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} C(x) + D(x) = \frac{1/2}{1-4x} \\ C(x) = D(x) = \frac{1/4}{1-4x} \end{array}\right\}.$$

The Generating Functions

- We got $C(x) = D(x) = \frac{1/4}{1-4x}$. By the first equation

$$B(x) - \tfrac{3}{4} = 2xB(x) + \frac{\frac{1}{2}x}{1-4x} \Rightarrow B(x)(1-2x) = \frac{\frac{1}{2}x}{1-4x} + \frac{3}{4}$$
$$\Rightarrow\ B(x) = \frac{\frac{1}{2}x}{(1-4x)(1-2x)} + \frac{3/4}{1-2x} \Rightarrow B(x) = \frac{1/4}{1-4x} - \frac{1/4}{1-2x} + \frac{3/4}{1-2x}$$
$$\Rightarrow\ B(x) = \frac{1/4}{1-4x} + \frac{1/2}{1-2x}.$$

We conclude

$$B(x) = \frac{1/4}{1-4x} + \frac{1/2}{1-2x}, \quad C(x) = D(x) = \frac{1/4}{1-4x}.$$

and

$$b_n = \frac{1}{4}4^n + \frac{1}{2}2^n, \quad c_n = d_n = \frac{1}{4}4^n, \quad n = 0, 1, 2, \ldots.$$

Subsection 4

A Special Class of Nonlinear Recurrence Relations

A Special Class of Nonlinear Recurrences

- Consider a difference equation of the form

$$a_n = a_{n-r}a_0 + a_{n-r-1}a_1 + \cdots + a_0 a_{n-r},$$

which is valid for $n \geq k$. We consider the case $k \geq r$. The value of a_n, $n \geq k$, can be computed recursively if $a_0, a_1, \ldots, a_{k-1}$ are known.

- Multiplying both sides by x^n and summing from $n = k$ to $n = \infty$, we obtain

$$\sum_{n=k}^{\infty} a_n x^n = \sum_{n=k}^{\infty} (a_{n-r}a_0 + a_{n-r-1}a_1 + \cdots + a_0 a_{n-r}) x^n.$$

Recognizing that $(a_{n-r}a_0 + a_{n-r-1}a_1 + \cdots + a_0 a_{n-r})$ is the coefficient of x^{n-r} in $A(x)A(x)$, we can write

$$\begin{aligned} & A(x) - a_0 - a_1 x - \cdots - a_{k-1}x^{k-1} \\ & = x^r [A(x)A(x) - a_0^2 - (a_1 a_0 + a_0 a_1)x - \cdots \\ & \qquad - (a_{k-r-1}a_0 + a_{k-r-2}a_1 + \cdots + a_0 a_{k-r-1})x^{k-r-1}]. \end{aligned}$$

This is a second-order algebraic equation in $A(x)$ which can be solved for $A(x)$ by the ordinary algebraic method.

Parenthesizing Expressions: The Recurrence

- Find the number of ways to parenthesize the expression $w_1 + w_2 + \cdots + w_{n-1} + w_n$ so that only two terms will be added at one time.

  Let a_i denote the number of ways of parenthesizing an expression with i terms. Consider the two subexpressions $w_1 + w_2 + \cdots + w_{n-r}$ and $w_{n-r+1} + w_{n-r+2} + \cdots + w_n$. There are:
  - a_{n-r} ways to parenthesize the first expression;
  - a_r ways to parenthesize the second expression.

  It follows that there are $a_{n-r}a_r$ ways to parenthesize the overall expression in which the last pair of parentheses added joins these two subexpressions.

  Letting r range from 1 to $n-1$, we obtain the difference equation $a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2a_{n-2} + a_1a_{n-1}$. This equation is valid for $n \geq 2$ ($a_1 = 1$).

  Since a_0 is not constrained by the difference equation, it can be chosen in an arbitrary manner.

Parenthesizing Expressions: Generating Function

- We obtained $a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2a_{n-2} + a_1a_{n-1}$, $n \geq 2$, with $a_1 = 1$.

  Letting $a_0 = 0$, we rewrite the difference equation as $a_n = a_na_0 + a_{n-1}a_1 + \cdots + a_1a_{n-1} + a_0a_n$, $n \geq 2$. It follows that

  $$\begin{array}{l} \sum_{n=2}^{\infty} a_nx^n = \sum_{n=2}^{\infty}(a_na_0 + a_{n-1}a_1 + \cdots + a_1a_{n-1} + a_0a_n)x^n \\ A(x) - a_1x - a_0 = [A(x)]^2 - a_0^2 - (a_1a_0 + a_0a_1)x \\ [A(x)]^2 - A(x) + x = 0 \ \Rightarrow \ A(x) = \frac{1 \pm \sqrt{1-4x}}{2}. \end{array}$$

  We choose the solution for $A(x)$ that generates a sequence of positive numbers. The general term in $\sqrt{1-4x}$ is
  $\frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}(-4x)^n = -\frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!}2^nx^n = -\frac{2}{n}\binom{2n-2}{n-1}$.
  Choosing the solution $A(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4x}$, we get

  $$a_n = \begin{cases} 0, & \text{if } n = 0 \\ \frac{1}{n}\binom{2n-2}{n-1}, & \text{if } n = 1, 2, \ldots \end{cases}$$

Generalizing the Preceding Setting

- Consider a difference equation of the form

$$b_n = a_{n-r}b_0 + a_{n-r-1}b_1 + \cdots + a_0 b_{n-r}, \ \ n \geq k, k \geq r.$$

Multiplying both sides by x^n and summing from $n = k$ to $n = \infty$, we obtain

$$\sum_{n=k}^{\infty} b_n x^n = \sum_{n=k}^{\infty} (a_{n-r}b_0 + a_{n-r-1}b_1 + \cdots + a_0 b_{n-r})x^n.$$

Equivalently,

$$\begin{aligned} & B(x) - b_0 - b_1 x - \cdots - b_{k-1}x^{k-1} \\ & = x^r[A(x)B(x) - a_0 b_0 - (a_1 b_0 + a_0 b_1)x - \cdots \\ & \qquad - (a_{k-r-1}b_0 + a_{k-r-2}b_1 + \cdots + a_0 b_{k-r-1})x^{k-r-1}]. \end{aligned}$$

If either $A(x)$ or $B(x)$ together with the appropriate boundary conditions are known, then the other can be obtained.

Patterns in a Binary Sequence

- A **pattern** consists of one or more consecutive binary digits like 01 and 1011.
- A pattern is said to **occur at** the k-th digit of a sequence if, in scanning the sequence from left to right, the pattern appears after the k-th digit is scanned.
- After a pattern occurs, scanning starts all over again to search for the second occurrence of the pattern that just occurred or for the occurrence of other patterns.

  Example: The pattern 010 occurs at the fifth and the ninth digits in the sequence

$$11\,\underbrace{010}_{\text{5-th}}\,1\,\underbrace{010}_{\text{9-th}}\,101,$$

  but not at the seventh and eleventh digits.

The Pattern 010: The Recurrence Relation

- Find the number of n-digit binary sequences that have the pattern 010 occurring at the n-th digit.

  Let b_n denote the number of such sequences. Among all the n-digit binary sequences, there are 2^{n-3} sequences that have 010 as the last three digits. These sequences can be divided into two groups:
  - Those that have the pattern 010 occurring at the n-th digit;
  - Those that do not have the pattern 010 occurring at the n-th digit.

  There are b_n sequences in the former group.

  The sequences in the latter group must have the pattern 010 occurring at the $(n-2)$-nd digit, since this is the only reason that the last three digits in these n-digit sequences were not accepted as a 010 pattern. It follows that there are b_{n-2} sequences in the latter group.

  Thus, $2^{n-3} = b_n + b_{n-2}$. This difference equation is valid for $n \geq 5$. The values of b_0, b_1 and b_2 are not constrained by the difference equation, so they can be chosen in an arbitrary manner.

The Pattern 010: The Generating Function

- We came up with $2^{n-3} = b_n + b_{n-2}$, $n \geq 5$.

  We set $b_0 = 1$, $b_1 = b_2 = 0$.

  For such a choice of the unconstrained values, the difference equation is valid for $n \geq 3$. We now have:

$$\sum_{n=3}^{\infty} 2^{n-3}x^n = \sum_{n=3}^{\infty} b_n x^n + \sum_{n=3}^{\infty} b_{n-2}x^n$$

$$\frac{x^3}{1-2x} = B(x) - 1 + x^2[B(x) - 1] \Rightarrow \frac{x^3}{1-2x} = (B(x)-1)(1+x^2)$$

$$\Rightarrow B(x) = 1 + \frac{x^3}{(1-2x)(1+x^2)}$$

$$\Rightarrow B(x) = 1 + x^3 \frac{1}{1-(2x - x^2 + 2x^3)}$$

$$\Rightarrow B(x) = 1 + x^3 + 2x^4 + 3x^5 + 6x^6 + \cdots.$$

Pattern Appearing for the First Time at n-th Digit

- Find the number of n-digit binary sequences that have the pattern 010 occurring for the first time at the n-th digit.
  Let a_n denote the number of such sequences. There are 2^{n-3} n-digit binary sequences that have 010 as the last three digits. These sequences can be classified according to the digit at which the pattern 010 occurs for the first time.
  - There are a_n sequences in which the first occurrence of the pattern is at the n-th digit;
  - There are a_{n-2} sequences in which the first occurrence of the pattern is at the $(n-2)$-nd digit.
  - For $3 \leq r \leq n-3$, there are $a_r 2^{n-r-3}$ sequences in which the first occurrence of the pattern is at the r-th digit, because to each of the a_r r-digit sequences that have the pattern 010 occurring for the first time at the rth digit, $n-r-3$ digits can be appended arbitrarily.

  Therefore, $2^{n-3} = a_n + a_{n-2} + a_{n-3}2^0 + a_{n-4}2^1 + \cdots + a_3 2^{n-6}$, $n \geq 6$. Because a_0, a_1, a_2 are not constrained by the difference equation, they can be chosen arbitrarily.

The Generating Function

- We found $2^{n-3} = a_n + a_{n-2} + a_{n-3}2^0 + a_{n-4}2^1 + \cdots + a_3 2^{n-6}$, $n \geq 6$. Let $a_0 = a_1 = a_2 = 0$. Let, also,

$$\begin{aligned} B(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots \\ &= 1 + x^2 + 2^0 x^3 + 2^1 x^4 + \cdots + 2^{n-3} x^n + \cdots \\ &= 1 + x^2 + \tfrac{x^3}{1-2x}. \end{aligned}$$

  We can rewrite the difference equation as $2^{n-3} = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \cdots + a_2 b_{n-2} + a_1 b_{n-1} + a_0 b_n$. The difference equation is now valid for $n \geq 3$. It follows that

$$\begin{aligned} &\textstyle\sum_{n=3}^{\infty} 2^{n-3} x^n = \sum_{n=3}^{\infty} (a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \cdots \\ &\qquad\qquad + a_2 b_{n-2} + a_1 b_{n-1} + a_0 b_n) x^n \\ &B(x) - 1 - x^2 = A(x)B(x) - a_0 b_0 - (a_1 b_0 + a_0 b_1)x - \\ &\qquad\qquad (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 \\ &A(x) = \tfrac{x^3}{1-2x+x^2-x^3} = x^3 + 2x^4 + 3x^5 + 5x^6 + 9x^7 + \cdots . \end{aligned}$$

A Second Solution

- Consider all $3 \cdot 2^{n-5}$ n-digit binary sequences, the last five digits of which are either 00010 or 10010 or 11010. Let a_n be the number of sequences in which the pattern 010 occurs for the first time at the n-th digit. We distinguish the following cases:
  - 010 occurs for the first time at the n-th digit (this excludes 01010), which can happen in a_n ways;
  - 010 occurs for the first time at the $(n-3)$rd digit, which can happen only in 10010 in a_{n-3} ways;
  - 010 occurs for the first time at the $(n-4)$th digit, which can happen only in 00010 in a_{n-4} ways;
  - 010 occurs for the fist time at the $(n-5)$th digit, which can happen in $a_{n-5} \cdot 3 \cdot 2^{5-5}$ ways;
  - 010 occurs for the first time at the $(n-6)$th digit, which can happen in $a_{n-6} \cdot 3 \cdot 2^{6-5}$ ways;

    $\vdots$
  - 010 occurs for the first time at the 3rd digit, which can happen in $a_3 \cdot 3 \cdot 2^{(n-3)-5}$ ways.

A Second Solution (Cont'd)

- Thus, for all $n \geq 8$,
$3 \cdot 2^{n-5} = a_n + a_{n-3} + a_{n-4} + a_{n-5}(3 \cdot 2^0) + a_{n-6}(3 \cdot 2^1) + \cdots + a_3(3 \cdot 2^{n-8})$.
Let $a_0 = a_1 = a_2 = 0$. Set

$$\begin{aligned} B(x) &= 1 + x^3 + x^4 + b_5x^5 + b_6x^6 + \cdots + b_nx^n + \cdots \\ &= 1 + x^3 + x^4 + 3(2^0x^5 + 2x^6 + \cdots + 2^{n-5}x^n + \cdots) \\ &= 1 + x^3 + x^4 + 3x^5(1 + 2x + \cdots + (2x)^{n-5} + \cdots) \\ &= 1 + x^3 + x^4 + \tfrac{3x^5}{1-2x}. \end{aligned}$$

The difference equation can be rewritten, for $n \geq 5$, as
$3 \cdot 2^{n-5} = a_nb_0 + a_{n-1}b_1 + a_{n-2}b_2 + \cdots + a_2b_{n-2} + a_1b_{n-1} + a_0b_n$.
Thus,
$B(x) - 1 - x^3 - x^4 = A(x)B(x) - a_3x^3 - a_4x^4 = A(x)B(x) - x^3 - 2x^4$.

A Second Solution (Conclusion)

- We obtained

$$\begin{aligned}
&B(x) - 1 - x^3 - x^4 = A(x)B(x) - x^3 - 2x^4 \\
&\tfrac{3x^5}{1-2x} = A(x)(1 + x^3 + x^4 + \tfrac{3x^5}{1-2x}) - x^3 - 2x^4 \\
&\tfrac{3x^5}{1-2x} + x^3 + 2x^4 = A(x)(1 + x^3 + x^4 \tfrac{3x^5}{1-2x}) \\
&\tfrac{3x^5+x^3-2x^4+2x^4-4x^5}{1-2x} = A(x)\tfrac{1-2x+x^3-2x^4+x^4-2x^5+3x^5}{1-2x} \\
&\tfrac{x^3-x^5}{1-2x} = A(x)\tfrac{1-2x+x^3-x^4+x^5}{1-2x} \\
&A(x) = \tfrac{x^3(1-x^2)}{1-x^2-2x+2x^3+x^2-x^4-x^3+x^5} \\
&A(x) = \tfrac{x^3(1-x^2)}{(1-x^2)-2x(1-x^2)+x^2(1-x^2)-x^3(1-x^2)} \\
&A(x) = \tfrac{x^3}{1-2x+x^2-x^3}.
\end{aligned}$$

A Third Solution

- Let b_n be the number of sequences of length n in which the pattern 010 occurs at the n-th digit.
  We again distinguish the following cases:
  - 010 occurs for the first time at the n-th digit, which happens in a_n ways;
  - 010 occurs for the first time in the $(n-3)$rd digit, which happens in $a_{n-3}b_3$ ways;
  - 010 occurs for the first time in the $(n-4)$th digit, which happens in $a_{n-4}b_4$ ways;

    $\vdots$
  - 010 occurs for the first time in the 3rd digit, which happens in a_3b_{n-3} ways.

  In this case, we have the difference equation

$$b_n = a_n + a_{n-3}b_3 + a_{n-4}b_4 + \cdots + a_3b_{n-3}, \ \ n \geq 6.$$

A Third Solution (Cont'd)

- $b_n = a_n + a_{n-3}b_3 + a_{n-4}b_4 + \cdots + a_3 b_{n-3}$, $n \geq 6$.
  Because $a_0, a_1, a_2, b_0, b_1, b_2$ are not constrained by the difference equation, we let $a_0 = a_1 = a_2 = 0$ and $b_0 = 1$, $b_1 = b_2 = 0$.
  We can rewrite the difference equation as

$$\begin{aligned} b_n &= a_n b_0 + a_{n-1}b_1 + a_{n-2}b_2 + a_{n-3}b_3 + \cdots + a_3 b_{n-3} \\ &\qquad a_2 b_{n-2} + a_1 b_{n-1} + a_0 b_n, \ n \geq 3. \end{aligned}$$

  Multiplying both sides of the difference equation by x^n and summing from $n = 3$ to $n = \infty$, we have

$$B(x) - 1 = A(x)B(x).$$

  Since $B(x)$ has been found to be $\frac{1-2x+x^2-x^3}{1-2x+x^2-2x^3}$, solving for $A(x)$, we obtain

$$A(x) = 1 - \frac{1}{B(x)} = 1 - \frac{1-2x+x^2-2x^3}{1-2x+x^2-x^3} = \frac{x^3}{1-2x+x^2-x^3}.$$

Generalizing to Arbitrary Patterns

- The preceding choice of $B(x)$ suggests a useful formula for the solution of the first-occurrence problems.
- Let a_n be the number of n-digit sequences in which a particular pattern of p digits occurs for the first time at the n-th digit.
- Let b_n be the number of n-digit sequences in which the pattern occurs at the n-th digit.
- By choosing the unconstrained values as

$$\begin{array}{l} a_0 = a_1 = a_2 = \cdots = a_{p-1} = 0, \\ b_0 = 1, \quad b_1 = b_2 = \cdots = b_{p-1} = 0, \end{array}$$

  we see that the difference equation always leads to

$$B(x) - b_0 = A(x)B(x)$$

  and

$$A(x) = 1 - \frac{1}{B(x)}.$$

Example

- Find the number of n-digit binary sequences in which an occurrence of the pattern 010 is followed by an occurrence of the pattern 110.
    - Let c_n be the number of such sequences.
    - Let a_n be the number of n-digit binary sequences in which the pattern 010 occurs for the first time at the n-th digit.
    - Let b_n be the number of n-digit binary sequences in which the pattern 110 occurs at least once.

  We distinguish the following cases:
    - 010 occurs for the first time at the 3rd digit, which can happen in $a_3 b_{n-3}$ ways;
    - 010 occurs for the first time at the 4th digit, which can happen in $a_4 b_{n-4}$ ways;

      $\vdots$
    - 010 occurs for the first time at the $(n-3)$rd digit, which can happen in $a_{n-3} b_3$ ways.

  Thus,
  $$c_n = a_3 b_{n-3} + a_4 b_{n-4} + a_5 b_{n-5} + \cdots + a_{n-3} b_3, \ \ n \geq 6.$$

Example (Cont'd)

- We found $c_n = a_3 b_{n-3} + a_4 b_{n-4} + a_5 b_{n-5} + \cdots + a_{n-3} b_3$, $n \geq 6$. Let $a_0 = a_1 = a_2 = 0$, $b_0 = b_1 = b_2 = 0$ and $c_0 = c_1 = c_2 = \cdots = c_5 = 0$, since they are not constrained by the difference equation. It follows that

$$\sum_{n=6}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (a_3 b_{n-3} + a_4 b_{n-4} + \cdots + a_{n-3} b_3) x^n.$$

Therefore, $C(x) = A(x)B(x)$.

We know $A(x) = \frac{x^3}{1-2x+x^2-x^3}$. To find $B(x)$, define d_n as the number of n-digit sequences in which the pattern 110 occurs at the n-th digit for the first time, and let $d_0 = d_1 = d_2 = 0$. Then

$$b_n = d_3 \cdot 2^{n-3} + d_4 \cdot 2^{n-4} + \cdots + d_{n-1} \cdot 2 + d_n, \ n \geq 3.$$

Consequently, $B(x) = D(x) \frac{1}{1-2x}$.

Example (Conclusion)

- Finally, top compute $D(x)$, let e_n be the number of n-digit sequences in which the pattern 110 occurs at the n-th digit.
  Then, we have (choosing $e_0 = 1, e_1 = e_2 = 0$)
  $$\begin{array}{rcl} E(x) & = & 1 + x^3 + 2x^4 + 2^2x^5 + \cdots + 2^{n-3}x^n + \cdots \\ & = & 1 + x^3(1 + 2x + 2^2x^2 + \cdots + 2^{n-3}x^{n-3} + \cdots) \\ & = & 1 + \frac{x^3}{1-2x}. \end{array}$$
  By the preceding example's relation,
  $D(x) = 1 - \frac{1}{E(x)} = 1 - \frac{1}{1+\frac{x^3}{1-2x}} = \frac{x^3}{1-2x+x^3}$. Therefore,
  $$\begin{array}{rcl} C(x) & = & \dfrac{x^3}{1-2x+x^2-x^3}\dfrac{x^3}{1-2x+x^3}\dfrac{1}{1-2x} \\ & = & \dfrac{x^6}{1-6x+13x^2-12x^3+4x^4+x^5-3x^6+2x^7} \\ & = & x^6 + 6x^7 + 23x^8 + \cdots . \end{array}$$

Subsection 5

Recurrence Relations with Two Indices

Recurrence Relations with Two Indices

- For the combinations of distinct objects, we derived the relation
$$C(n,r) = C(n-1,r-1) + C(n-1,r).$$
- This is an example of a recurrence relation with two indices.
- With the boundary conditions $C(n,0) = 1$ and $C(0,r) = 0$, for $r > 0$, the recurrence relation is valid for $n \geq 1$ and $r \geq 1$.
- The value of $C(n,r)$ can be computed recursively:
$$\begin{array}{ll} C(0,0) = 1 & \\ C(1,0) = 1; & C(1,1) = C(0,0) + C(0,1) = 1 \\ C(2,0) = 1; & C(2,1) = C(1,0) + C(1,1) = 1 + 1 = 2 \\ \ldots & \end{array}$$
- This essentially underlies the construction of the famous Pascal triangle:
$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ & & & & \ldots & & & & \end{array}$$

General Form of Recurrence with Two Indices

- The general form of a linear recurrence relation with constant coefficients that has two indices is

$$\begin{array}{ll} C_0 a_{n,r} & +C_1 a_{n,r-1} + C_2 a_{n,r-2} + \cdots \\ & +D_0 a_{n-1,r} + D_1 a_{n-1,r-1} + D_2 a_{n-1,r-2} + \cdots \\ & + \cdots \\ & +G_0 a_{n-k,r} + G_1 a_{n-k,r-1} + G_2 a_{n-k,r-2} + \cdots = f(n,r), \end{array}$$

  where the C's, D's, ..., G's are constants.
- Although it may be tedious, we can always evaluate $a_{n,r}$ using the recurrence relation and starting with the known boundary conditions.
- To solve a recurrence relation with two indices by the generating function technique, we first define a sequence of generating functions with one function for each value of one of the two indices:

$$\begin{array}{rcl} A_0(x) & = & a_{0,0} + a_{0,1}x + a_{0,2}x^2 + \cdots + a_{0,r}x^r + \cdots \\ A_1(x) & = & a_{1,0} + a_{1,1}x + a_{1,2}x^2 + \cdots + a_{1,r}x^r + \cdots \\ & \cdots & \\ A_n(x) & = & a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots + a_{n,r}x^r + \cdots \\ & \cdots & \end{array}$$

Generating Functions with Two Variables

- We can also define a generating function of the sequence $(A_0(x), A_1(x), A_2(x), \ldots)$, i.e., using powers of y:

$$\begin{aligned}\mathcal{A}(y,x) &= A_0(x) + A_1(x)y + A_2(x)y^2 + \cdots + A_n(x)y^n + \cdots \\ &= [a_{0,0} + a_{0,1}x + a_{0,2}x^2 + \cdots + a_{0,r}x^r + \cdots] \\ &\quad +[a_{1,0} + a_{1,1}x + a_{1,2}x^2 + \cdots + a_{1,r}x^r + \cdots]y \\ &\quad +[a_{2,0} + a_{2,1}x + a_{2,2}x^2 + \cdots + a_{2,r}x^r + \cdots]y^2 \\ &\quad +\cdots \\ &\quad +[a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots + a_{n,r}x^r + \cdots]y^n \\ &\quad +\cdots\end{aligned}$$

- Multiplying out, we get

$$\begin{aligned}\mathcal{A}(y,x) &= a_{0,0} + a_{0,1}x + a_{0,2}x^2 + \cdots + a_{0,r}x^r + \cdots \\ &\quad +a_{1,0}y + a_{1,1}yx + a_{1,2}yx^2 + \cdots + a_{1,r}yx^r + \cdots \\ &\quad +a_{2,0}y^2 + a_{2,1}y^2x + a_{2,2}y^2x^2 + \cdots + a_{2,r}y^2x^r + \cdots \\ &\quad +\cdots \\ &\quad +a_{n,0}y^n + a_{n,1}y^nx + a_{n,2}y^nx^2 + \cdots + a_{n,r}y^nx^r + \cdots \\ &\quad +\cdots\end{aligned}$$

- For a sequence $(a_{0,0}, a_{0,1}, a_{0,2}, \ldots, a_{0,r}, \ldots, a_{1,0}, a_{1,1}, \ldots, a_{1,r}, \ldots)$, we can use formal variables x, y and directly define the generating function $\mathcal{A}(y,x)$ so that the coefficient of y^ix^j is $a_{i,j}$.

Example: Combinations

- Find the generating function of the $C(n,r)$'s

$$F_n(x) = C(n,0) + C(n,1)x + C(n,2)x^2 + \cdots + C(n,r)x^r + \cdots .$$

From the recurrence relation $C(n,r) = C(n-1,r-1) + C(n-1,r)$, we have

$$\begin{array}{l}
\sum_{r=1}^{\infty} C(n,r)x^r = \sum_{r=1}^{\infty} C(n-1,r-1)x^r + \sum_{r=1}^{\infty} C(n-1,r)x^r \\
F_n(x) - C(n,0) = xF_{n-1}(x) + F_{n-1}(x) - C(n-1,0) \\
F_n(x) = (1+x)F_{n-1}(x).
\end{array}$$

It follows that

$$\begin{array}{rcl}
F_n(x) & = & (1+x)^2 F_{n-2}(x) = (1+x)^3 F_{n-3}(x) \\
& = & \cdots = (1+x)^n F_0(x) \\
& = & (1+x)^n C(0,0) = (1+x)^n.
\end{array}$$

Example: Combinations with Repetitions

- Find the number of r-combinations of n distinct objects with unlimited repetitions.
  Denote this number by $f(n,r)$. Let one of the n objects be labeled as a special one.
  - There are $f(n,r-1)$ r-combinations in which this special object is selected at least once.
  - There are $f(n-1,r)$ r-combinations in which this special object is not selected.

  Therefore, $f(n,r)=f(n,r-1)+f(n-1,r)$, $n\geq 1, r\geq 1$. Let $F_n(x)=f(n,0)+f(n,1)x+f(n,2)x^2+\cdots+f(n,r)x^r+\cdots$, for every $n\geq 0$. Thus,

  $$\sum_{r=1}^{\infty} f(n,r)=\sum_{r=1}^{\infty} f(n,r-1)x^r+\sum_{r=1}^{\infty} f(n-1,r)x^r$$
  $$F_n(x)-f(n,0)=xF_n(x)+F_{n-1}(x)-f(n-1,0)$$

  With $f(n,0)=1$, for $n\geq 0$, and $f(0,r)=0$, for $r\geq 0$,

  $$F_n(x)=(1-x)^{-1}F_{n-1}(x)=(1-x)^{-n}F_0(x)=(1-x)^{-n}.$$

Example

- Find the number of n-digit binary sequences that have exactly r pairs of adjacent 1's and no adjacent 0's.

  Notice that every two successive 1's are counted as a pair. E.g., there are two pairs of adjacent 1's in the sequence 111.

  Let

  - $a_{n,r} =$ number of such sequences;
  - $b_{n,r} =$ number of such sequences that have a 1 as the n-th digit;
  - $c_{n,r} =$ number of such sequences that have a 0 as the n-th digit.

  Clearly, $a_{n,r} = b_{n,r} + c_{n,r}$.

  An n-digit sequence that has r pairs of 1's, no adjacent 0's, and a 1 as the n-th digit can be formed by appending a 1 either

  - to an $(n-1)$-digit sequence that has $r-1$ pairs of 1's, no adjacent 0's, and a 1 as the $(n-1)$-st digit or
  - to an $(n-1)$-digit sequence that has r pairs of 1's, no adjacent 0's, and a 0 as the $(n-1)$-st digit.

  So we have the relation $b_{n,r} = b_{n-1,r-1} + c_{n-1,r}$.

Example (Cont'd)

- An n-digit sequence that has r pairs of 1's, no adjacent 0's, and a 0 as the n-th digit can be formed by appending a 0 to
    - an $(n-1)$-digit sequence that has r pairs of 1's, no adjacent 0's, and a 1 as the $(n-1)$-st digit.

  Hence, $c_{n,r} = b_{n-1,r}$.

  Combining with $b_{n,r} = b_{n-1,r-1} + c_{n-1,r}$, we obtain $b_{n,r} = b_{n-1,r-1} + b_{n-2,r}$.

  The value of $b_{i,j}$ has physical significance only for $i \geq 1$ and $j \geq 0$. So, the equation is valid for $n \geq 3$ and $r \geq 1$.

  As to the boundary conditions, we have $b_{n,0} = 1$, for $n \geq 1$, because there is exactly one n-digit sequence that contains neither adjacent 0's nor adjacent 1's and has a 1 as the n-th digit, namely, the sequence that consists of alternating 0's and 1's.

  We also have $b_{i,j} = 0$, for $i \leq j$. The value of $b_{0,0}$ is not constrained by the difference equation. Let it be chosen as 1.

Example (Generating Functions)

- Let $B_n(x) = b_{n,0} + b_{n,1}x + b_{n,2}x^2 + \cdots + b_{n,r}x^r + \cdots$. Multiplying both sides by x^r and summing from $r = 1$ to $r = \infty$, we obtain $\sum_{r=1}^{\infty} b_{n,r}x^r = \sum_{r=1}^{\infty} b_{n-1,r-1}x^r + \sum_{r=1}^{\infty} b_{n-2,r}x^r$ which yields $B_n(x) - b_{n,0} = xB_{n-1}(x) + B_{n-2}(x) - b_{n-2,0}$, $n \geq 3$, that is, $B_n(x) = xB_{n-1}(x) + B_{n-2}(x)$, $n \geq 3$.
  We have the following boundary conditions: $B_0(x) = b_{0,0} = 1$, $B_1(x) = b_{1,0} + b_{1,1}x = 1$, $B_2(x) = b_{2,0} + b_{2,1}x + b_{2,2}x = 1 + x$.
  The preceding equation, then, gives:

$$\begin{array}{l}
\sum_{n=3}^{\infty} B_n(x)y^n = \sum_{n=3}^{\infty} xB_{n-1}(x)y^n + \sum_{n=3}^{\infty} B_{n-2}(x)y^n \\
\mathcal{B}(y,x) - B_2(x)y^2 - B_1(x)y - B_0(x) \\
= xy[\mathcal{B}(y,x) - B_1(x)y - B_0(x)] + y^2[\mathcal{B}(y,x) - B_0(x)] \\
\mathcal{B}(y,x) - (1+x)y^2 - y - 1 = xy[\mathcal{B}(y,x) - y - 1] + y^2[\mathcal{B}(y,x) - 1] \\
\mathcal{B}(y,x) = \frac{1+(1-x)y}{1-xy-y^2} \\
= 1 + y + (1+x)y^2 + (1+x+x^2)y^3 + \\
(1+2x+x^2+x^3)y^4 + (1+2x+3x^2+x^3+x^4)y^5 + \cdots .
\end{array}$$

Example (Solving for the Generating Functions)

- Recall that $a_{n,r} = b_{n,r} + c_{n,r}$ and $c_{n,r} = b_{n-1,r}$.
  By choosing $c_{0,0} = 0$, we get $\mathcal{C}(y,x) = y\mathcal{B}(y,x)$. So
  $\mathcal{A}(y,x) = \mathcal{B}(y,x) + \mathcal{C}(y,x) = (1+y)\mathcal{B}(x,y) = (1+y)(1+y+(1+x)y^2+(1+x+x^2)y^3+(1+2x+x^2+x^3)y^4+(1+2x+3x^2+x^3+x^4)y^5+\cdots) = 1+2y+(2+x)y^2+(2+2x+x^2)y^3+(2+3x+2x^2+x^3)y^4+(2+4x+4x^2+2x^3+x^4)y^5+\cdots$.
  Equivalently,

$$
\begin{array}{rcl}
A_0(x) & = & 1; \\
A_1(x) & = & 2; \\
A_2(x) & = & 2+x; \\
A_3(x) & = & 2+2x+x^2; \\
A_4(x) & = & 2+3x+2x^2+x^3; \\
A_5(x) & = & 2+4x+4x^2+2x^3+x^4; \\
 & \cdots &
\end{array}
$$

Ordinary Generating Function for Permutations

- To arrange r out of n distinct objects, consider one of the n objects as a special object. Then we distinguish two cases:
  - Those r-permutations in which the special object does not appear, which can happen in $P(n-1,r)$ ways;
  - Those in which the special object appears, which can happen in $rP(n-1,r-1)$ ways.

  Thus, we get $P(n,r) = P(n-1,r) + rP(n-1,r-1)$, $n \geq 1, r \geq 1$.

  Let

  $$F_n(x) = P(n,0) + P(n,1)x + P(n,2)x^2 + \cdots + P(n,r)x^r + \cdots .$$

  $$\sum_{r=1}^{\infty} P(n,r)x^r = \sum_{r=1}^{\infty} P(n-1,r)x^r + \sum_{r=1}^{\infty} P(n-1,r-1)x^r$$
  $$F_n(x) - P(n,0) = F_{n-1}(x) - P(n-1,0) + x\tfrac{d}{dx}[xF_{n-1}(x)]$$
  $$F_n(x) = (1+x)F_{n-1}(x) + x^2\tfrac{d}{dx}F_{n-1}(x).$$

Ordinary Generating Function for Permutations (Cont'd)

- We obtained

$$F_n(x) = (1+x)F_{n-1}(x) + x^2 \frac{d}{dx} F_{n-1}(x).$$

With the boundary condition $F_0(x) = 1$, this recurrence yields:

$$\begin{array}{rcl}
F_1(x) & = & 1 + x; \\
F_2(x) & = & (1+x)(1+x) + x^2 \frac{d}{dx}(1+x) = 1 + 2x + 2x^2; \\
F_2(x) & = & (1+x)(1+2x+2x^2) + x^2 \frac{d}{dx}(1+2x+2x^2); \\
 & = & 1 + 3x + 6x^2 + 6x^3; \\
 & \vdots &
\end{array}$$

Exponential Generating Function for Permutations

- Define

$$G_n(x) = P(n,0) + \frac{P(n,1)}{1!}x + \frac{P(n,2)}{2!}x^2 + \cdots + \frac{P(n,r)}{r!}x^r + \cdots,$$

for every $n \geq 0$. Multiplying both sides of the equation $P(n,r) = P(n-1,r) + rP(n-1,r-1)$ by $\frac{1}{r!}x^r$ and summing both sides from $r=1$ to $r=\infty$, we obtain

$$\sum_{r=1}^{\infty} \frac{P(n,r)}{r!}x^r = \sum_{r=1}^{\infty} \frac{P(n-1,r)}{r!}x^r + \sum_{r=1}^{\infty} \frac{rP(n-1,r-1)}{r!}x^r,$$

i.e., $G_n(x) - P(n,0) = G_{n-1}(x) - P(n-1,0) + xG_{n-1}(x)$, which can be simplified to $G_n(x) = (1+x)G_{n-1}(x)$. It follows that

$$G_n(x) = (1+x)^n G_0(x) = (1+x)^n.$$