Introduction to Combinatorial Mathematics

George Voutsadakis¹

¹Mathematics and Computer Science Lake Superior State University

LSSU Math 300

George Voutsadakis (LSSU)

The Principle of Inclusion and Exclusion

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Subsection 1

Introduction

Example

- In a group of ten girls:
 - six have blond hair;
 - five have blue eyes;
 - three have blond hair and blue eyes.

How many girls are there in the group who have neither blond hair nor blue eyes?

Clearly the answer is 10 - 6 - 5 + 3 = 2.

The three blondes with blue eyes are included in the count of the six blondes and are again included in the count of the five with blue eyes. Thus, they are subtracted twice in the expression 10 - 6 - 5.



Therefore, 3 should be added to the expression 10 - 6 - 5 to give the correct count of girls who have neither blond hair nor blue eyes.

Illustration of the Example



The area inside the large circle represents the total number of girls. The areas inside the two small circles represent the number of girls who have blond hair and the number of girls who have blue eyes.

The crosshatched area represents the number of girls that have both blond hair and blue eyes.

This area is subtracted twice when the areas of the two small circles are subtracted from the area of the large circle. To find the area representing the number of girls who neither are blondes nor have blue eyes, we should, therefore, compensate the oversubtraction by adding back the crosshatched area.

• This reasoning leads to the Principle of Inclusion and Exclusion:

To count the number of a certain class of objects, we exclude those that should not be included in the count and, in turn, compensate the count by including those that have been excluded incorrectly.

Subsection 2

The Principle of Inclusion and Exclusion

Properties and Counting Objects Possessing Properties

- Consider a set of *N* objects.
- Let $a_1, a_2, ..., a_r$ be a set of properties that these objects may have. In general, these properties are not mutually exclusive, i.e., an object can have one or more of these properties.

Let

- *N*(*a*₁) denote the number of objects that have the property *a*₁;
- $N(a_2)$ denote the number of objects that have the property a_2 ;
- $N(a_r)$ denote the number of objects that have the property a_r . An object having the property a_i is included in the count $N(a_i)$ regardless of the other properties it may have.
- If an object has both the properties a_i and a_j , it will contribute a count in $N(a_i)$ as well as a count in $N(a_j)$.

More Counting of Objects

- Let
 - $N(a'_1)$ denote the number of objects that do not have the property a_1 ;
 - $N(a'_2)$ denote the number of objects that do not have the property a_2 ;
 - $N(a'_r)$ denote the number of objects that do not have the property a_r .

Let

- N(a_ia_j) denote the number of objects that have both the properties a_i and a_j;
- N(a'_ia'_j) denote the number of objects that have neither the property a_i nor the property a_j;
- N(a'_ia_j) denote the number of objects that have the property a_j, but not the property a_i.
- Logically, we see that

$$N(a_i') = N - N(a_i)$$

because each of the N objects either has the property a_i or does not have the property a_i .

Additional Relationships

We also have

$$N(a_j) = N(a'_ia_j) + N(a_ia_j) \Rightarrow N(a'_ia_j) = N(a_j) - N(a_ia_j),$$

because for each of the $N(a_j)$ objects that have the property a_j , it either has the property a_i or does not have the property a_i .

Similarly,

$$N = N(a'_ia'_j) + N(a_ia'_j) + N(a'_ia_j) + N(a_ia_j)$$

$$\Rightarrow N(a'_ia'_j) = N - N(a_ia'_j) - N(a'_ia_j) - N(a_ia_j).$$

This can be rewritten as

$$egin{aligned} & \mathcal{N}(a_i'a_j') \ &= \mathcal{N} - [\mathcal{N}(a_ia_j') + \mathcal{N}(a_ia_j)] - [\mathcal{N}(a_i'a_j) + \mathcal{N}(a_ia_j)] + \mathcal{N}(a_ia_j) \ &= \mathcal{N} - \mathcal{N}(a_i) - \mathcal{N}(a_j) + \mathcal{N}(a_ia_j). \end{aligned}$$

The Principle of Inclusion and Exclusion

• The following extends the preceding cases:

$$N(a'_{1}a'_{2}\cdots a'_{r}) = N - N(a_{1}) - N(a_{2}) - \cdots - N(a_{r}) + N(a_{1}a_{2}) + N(a_{1}a_{3}) + \cdots + N(a_{r-1}a_{r}) - N(a_{1}a_{2}a_{3}) - N(a_{1}a_{2}a_{4}) - \cdots - N(a_{r-2}a_{r-1}a_{r}) + \cdots + (-1)^{r}N(a_{1}a_{2}\cdots a_{r}) = N - \sum_{i} N(a_{i}) + \sum_{i,j:i\neq j} N(a_{i}a_{j}) - \sum_{i,j,k:i\neq j\neq k\neq i} N(a_{i}a_{j}a_{k} + \cdots + (-1)^{r}N(a_{1}a_{2}\cdots a_{r}).$$

This will be proved by induction on the total number of properties the objects may have.

- Basis: We have already shown $N(a'_1) = N N(a_1)$.
- Induction Hypothesis: Assume the identity is true for objects having up to r - 1 properties, i.e.,

$$N(a'_{1}a'_{2}\cdots a'_{r-1}) = N - N(a_{1}) - N(a_{2}) - \cdots - N(a_{r-1}) + N(a_{1}a_{2}) + N(a_{1}a_{3}) + \cdots + N(a_{r-2}a_{r-1}) - \cdots + (-1)^{r-1}N(a_{1}a_{2}\cdots a_{r-1}).$$

The Induction Step

For a set of N objects having up to r properties, a₁, a₂,..., a_r, the N(a_r) objects that have the property a_r may have any of the r − 1 properties a₁, a₂,..., a_{r-1}. By the induction hypothesis,

$$N(a'_{1}a'_{2}\cdots a'_{r-1}a_{r}) = N(a_{r}) - N(a_{1}a_{r}) - N(a_{2}a_{r}) - \cdots - N(a_{r-1}a_{r}) + N(a_{1}a_{2}a_{r}) + N(a_{1}a_{3}a_{r}) + \cdots + N(a_{r-2}a_{r-1}a_{r}) - \cdots + (-1)^{r-1}N(a_{1}a_{2}\cdots a_{r-1}a_{r}).$$

Therefore, we get

$$\begin{split} & N(a'_{1}a'_{2}\cdots a'_{r-1}a'_{r}) \\ &= N(a'_{1}a'_{2}\cdots a'_{r-1}) - N(a'_{1}a'_{2}\cdots a'_{r-1}a_{r}) \\ &= N - N(a_{1}) - N(a_{2}) - \cdots - N(a_{r-1}) - N(a_{r}) \\ &+ N(a_{1}a_{2}) + N(a_{1}a_{3}) + \cdots + N(a_{r-2}a_{r-1}) + N(a_{1}a_{r}) + \cdots + N(a_{r-1}a_{r}) \\ &- N(a_{1}a_{2}a_{3}) - N(a_{1}a_{2}a_{4}) - \cdots - N(a_{r-2}a_{r-1}a_{r}) \\ &+ \cdots \\ &+ (-1)^{r}N(a_{1}a_{2}\cdots a_{r}). \end{split}$$

Colored Balls

- Twelve balls are painted in the following way:
 - Two are unpainted.
 - Two are painted red, one is painted blue, and one is painted white.
 - Two are painted red and blue, and one is painted red and white.
 - Three are painted red, blue, and white.

Let a_1, a_2 and a_3 denote the properties that a ball is painted red, blue, and white, respectively. Then

$$N(a_1) = 8, \quad N(a_2) = 6, \quad N(a_3) = 5,$$

 $N(a_1a_2) = 5, \quad N(a_1a_3) = 4, \quad N(a_2a_3) = 3,$
 $N(a_1a_2a_3) = 3.$

It follows that

$$\begin{array}{rcl} \mathsf{V}(a_1'a_2'a_3') &=& \mathsf{N}-\mathsf{N}(a_1)-\mathsf{N}(a_2)-\mathsf{N}(a_3) \\ && +\mathsf{N}(a_1a_2)+\mathsf{N}(a_1a_3)+\mathsf{N}(a_2a_3)-(a_1a_2a_3) \\ &=& 12-8-6-5+5+4+3-3=2. \end{array}$$

Divisibility

• Find the number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5, and 7.

Let a_1 , a_2 , a_3 and a_4 denote the properties that a number is divisible by 2, divisible by 3, divisible by 5, and divisible by 7, respectively. Among the integers 1 through 250 there are:

- 125 $\left(=\frac{250}{2}\right)$ integers that are divisible by 2, because every other integer is a multiple of 2;
- 83 (= the integral part of $\frac{250}{3}$) integers that are multiples of 3;
- 50 $\left(=\frac{250}{5}\right)$ integers that are multiples of 5;

:

Divisibility (Cont'd)

• Letting [x] denote the integral part of the number x,

$$N(a_1) = \begin{bmatrix} \frac{250}{2} \end{bmatrix} = 125, \ N(a_2) = \begin{bmatrix} \frac{250}{3} \end{bmatrix} = 83,$$

$$N(a_3) = \begin{bmatrix} \frac{250}{5} \end{bmatrix} = 50, \ N(a_4) = \begin{bmatrix} \frac{250}{7} \end{bmatrix} = 35,$$

$$N(a_1a_2) = \begin{bmatrix} \frac{250}{2\cdot3} \end{bmatrix} = 41, \ N(a_1a_3) = \begin{bmatrix} \frac{250}{2\cdot5} \end{bmatrix} = 25,$$

$$N(a_1a_4) = \begin{bmatrix} \frac{250}{2\cdot7} \end{bmatrix} = 17, \ N(a_2a_3) = \begin{bmatrix} \frac{250}{3\cdot5} \end{bmatrix} = 16,$$

$$N(a_2a_4) = \begin{bmatrix} \frac{250}{2\cdot3\cdot5} \end{bmatrix} = 11, \ N(a_3a_4) = \begin{bmatrix} \frac{250}{5\cdot7} \end{bmatrix} = 7,$$

$$N(a_1a_2a_3) = \begin{bmatrix} \frac{250}{2\cdot3\cdot5} \end{bmatrix} = 8, \ N(a_1a_2a_4) = \begin{bmatrix} \frac{250}{2\cdot3\cdot7} \end{bmatrix} = 5,$$

$$N(a_1a_3a_4) = \begin{bmatrix} \frac{250}{2\cdot5\cdot7} \end{bmatrix} = 3, \ N(a_2a_3a_4) = \begin{bmatrix} \frac{250}{3\cdot5\cdot7} \end{bmatrix} = 2,$$

$$N(a_1a_2a_3a_4) = \begin{bmatrix} \frac{250}{2\cdot3\cdot5\cdot7} \end{bmatrix} = 1.$$

Thus,

$$N(a_1a'_2a'_3a'_4) = 250 - (125 + 83 + 50 + 35) + (41 + 25 + 17 + 16 + 11 + 7) - (8 + 5 + 3 + 2) + 1 = 57.$$

Example: Quaternary Sequences

- Find the number of *r*-digit quaternary sequences in which each of the three digits 1, 2 and 3 appears at least once.
 - Let a_1 , a_2 and a_3 be the properties that the digits 1, 2 and 3 do not appear in a sequence, respectively. Then

$$N(a_1) = N(a_2) = N(a_3) = 3^r,$$

 $N(a_1a_2) = N(a_1a_3) = N(a_2a_3) = 2^r,$
 $N(a_1a_2a_3) = 1.$

Therefore,

$$N(a'_1a'_2a'_3) = 4^r - 3 \cdot 3^r + 3 \cdot 2^r - 1.$$

Example: Distinct Objects into Distinct Cells

The preceding problem is the same as that of distributing *r* distinct objects into four distinct cells with three of them never left empty. We solved this problem using the generating function technique. The formula can also be derived by the use of the principle of inclusion and exclusion as follows: Let *a*₁, *a*₂,..., *a_n* be the properties that the 1-st, 2-nd, ..., *n*-th cell is left empty, respectively. Then,

$$\begin{split} \mathcal{N}(a_1'a_2'\cdots a_n') &= n^r - \binom{n}{1}(n-1)^r + \binom{n}{2}(n-2)^r - \cdots \\ &+ (-1)^{n-1}\binom{n}{n-1}1^r + (-1)^n\binom{n}{n}0^r \\ &= \sum_{i=0}^n (-1)^i\binom{n}{i}(n-i)^r. \end{split}$$

Example

• Consider a single ball that is painted with *n* colors. Let a_1, a_2, \ldots, a_n denote the properties that a ball is painted with the 1-st, 2-nd, ..., *n*-th color, respectively. We have the following:

•
$$N(a_1) = N(a_2) = \cdots = N(a_n) = 1;$$

• $N(a_1a_2) = N(a_1a_3) = \cdots = N(a_{n-1}a_n) = 1;$

•
$$N(a_1a_2\cdots a_n)=1.$$

So, by Inclusion-Exclusion,

$$N(a'_1a'_2\cdots a'_n)=1-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}.$$

However, $N(a'_1a'_2\cdots a'_n)=0$, because there is no unpainted ball. Therefore, we have the identity

$$1-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0.$$

Subsection 3

The General Formula

The *s*i's

- In a set of N objects with properties a₁, a₂,..., a_r, the number of objects that do not have any of these properties, N(a'₁a'₂...a'_r), is given by the Inclusion-Exclusion Formula.
- We derive, next, a more general formula for the number of objects that have exactly *m* of the *r* properties for m = 0, 1, ..., r.
- We introduce the notations

$$\begin{split} s_0 &= N; \\ s_1 &= N(a_1) + N(a_2) + \dots + N(a_r) = \sum_i N(a_i); \\ s_2 &= N(a_1a_2) + N(a_1a_3) + \dots + N(a_{r-1}a_r) = \sum_{i,j:i \neq j} N(a_ia_j); \\ s_3 &= N(a_1a_2a_3) + N(a_1a_2a_4) + \dots + N(a_{r-2}a_{r-1}a_r) \\ &= \sum_{i,j,k:i \neq j \neq k \neq i} N(a_ia_ja_k); \\ \vdots \\ s_r &= N(a_1a_2 \cdots a_r). \end{split}$$

The e_i 's

• Let *e_i* be the number of objects that have exactly *i* properties:

With this notation, Inclusion-Exclusion can be rewritten as

$$e_0 = s_0 - s_1 + s_2 - \cdots + (-1)^r s_r$$

A Formula for the e_i 's

• We show that, for all
$$m = 0, 1, 2, \ldots, r$$
,

$$e_m = s_m - \binom{m+1}{1}s_{m+1} + \binom{m+2}{2}s_{m+2} - \cdots + (-1)^{r-m}\binom{r}{r-m}s_r.$$

• When m = 0, we get the Inclusion-Exclusion Formula.

- An object having less than m properties should not be included in the count e_m . It contributes no count to the expression on the right side.
- An object having exactly m properties should be included in the count e_m . It contributes a count of 1 to the expression on the right side: It is counted once in s_m and is not included in $s_{m+1}, s_{m+2}, \ldots, s_r$.
- An object having m + j properties, with 0 < j ≤ r − m should not be included in e_m either. Note that it contributes

•
$$\binom{m+j}{m}$$
 counts to s_m ,

- $\binom{m+j}{m+1}$ counts to $s_{m+1}, \ldots,$
- and $\binom{m+j}{m+j}$ counts to s_{m+j} .

A Formula for the *e*_i's (cont'd)

• Thus, an object having exactly *m* properties contributes to the expression on the right side a total count of

$$\binom{m+j}{m} - \binom{m+1}{1}\binom{m+j}{m+1} + \binom{m+2}{2}\binom{m+j}{m+2} - \dots + (-1)^{j}\binom{m+j}{j}\binom{m+j}{m+j}.$$

Notice that

$$\binom{(m+k)}{k} \binom{m+j}{m+k} = \frac{(m+k)!}{m!k!} \frac{(m+j)!}{(m+k)!(j-k)!}$$

$$= \frac{(m+j)!}{m!k!(j-k)!}$$

$$= \frac{(m+j)!}{m!j!} \frac{j!}{k!(j-k)!}$$

$$= \binom{(m+j)!}{m} \binom{j}{k}.$$

Thus, the total count is

$$\binom{\binom{m+j}{m} - \binom{m+j}{m}\binom{j}{1} + \binom{m+j}{m}\binom{j}{2} - \dots + (-1)^{j}\binom{m+j}{m}\binom{j}{j} }{= \binom{m+j}{m}[\binom{j}{0} - \binom{j}{1} + \binom{j}{2} - \dots + (-1)^{j}\binom{j}{j}] = 0.$$

Therefore, an object having more than m properties is not included in the count e_m .

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Example

- Twelve balls are painted in the following way:
 - Two are unpainted.
 - Two are painted red, one is painted blue, and one is painted white.
 - Two are painted red and blue, and one is painted red and white.
 - Three are painted red, blue, and white.

We have

$$s_1 = 19, \quad s_2 = 12, \quad s_3 = 3.$$



Therefore,

$$\begin{array}{rcl} e_1 & = & 19 - \binom{2}{1} \cdot 12 + \binom{3}{2} \cdot 3 = 19 - 24 + 9 = 4; \\ e_2 & = & 12 - \binom{3}{1} \cdot 3 = 12 - 9 = 3; \\ e_3 & = & 3. \end{array}$$

The Sum of the e_i 's equals s_0

Let E(x) be the ordinary generating function of the sequence (e₀, e₁, e₂, ..., e_m, ..., e_r).
 Then

$$\begin{split} E(x) &= e_0 + e_1 x + e_2 x^2 + \dots + e_m x^m + \dots + e_r x^r \\ &= [s_0 - s_1 + s_2 - \dots + (-1)^r s_r] \\ &+ [s_1 - \binom{2}{1} s_2 + \binom{3}{2} s_3 - \dots + (-1)^{r-1} \binom{r}{r-1} s_r] x \\ &+ [s_2 - \binom{3}{1} s_3 + \binom{4}{2} s_4 - \dots + (-1)^{r-2} \binom{r}{r-2} s_r] x^2 \\ &+ \dots \\ &+ [s_m - \binom{m+1}{1} s_{m+1} + \binom{m+2}{2} s_{m+2} - \dots + (-1)^{r-m} \binom{r}{r-m} s_r] x^m \\ &+ \dots \\ &+ s_r x^r \\ &= s_0 + s_1 [x-1] + s_2 [x^2 - \binom{2}{1} x + 1] + s_3 [x^3 - \binom{3}{1} x^2 + \binom{3}{2} x - 1] \\ &+ \dots \\ &+ s_m [x^m - \binom{m}{1} x^{m-1} + \binom{m}{2} x^{m-2} + \dots + (-1)^{m-1} \binom{m}{m-1} x + (-1)^m] \\ &+ \dots \\ &+ s_r [x^r - \binom{r}{1} x^{r-1} + \binom{r}{2} x^{r-2} + \dots + (-1)^r] \\ &= \sum_{j=0}^r s_j (x-1)^j. \end{split}$$

Setting x = 1, we obtain $E(1) = e_0 + e_1 + e_2 + \cdots + e_r = s_0$.

Counting Objects with Odd/Even Number of Properties

• We observe that

- $\frac{1}{2}[E(1) + E(-1)] = e_0 + e_2 + e_4 + \cdots = \frac{1}{2}[s_0 + \sum_{j=0}^r (-2)^j s_j]$ gives the number of objects having an even number of properties.
- $\frac{1}{2}[E(1) E(-1)] = e_1 + e_3 + e_5 + \cdots = \frac{1}{2}[s_0 \sum_{j=0}^r (-2)^j s_j]$ gives the number of objects having an odd number of properties.

Example: Find the number of *n*-digit ternary sequences that have an even number of 0's.

Let a_i be the property that the *i*-th digit of a sequence is 0, i = 1, 2, ..., n. Let e_j and s_j be defined as above with j = 0, 1, ..., n. Then since $s_j = \binom{n}{j} 3^{n-j}$ with j = 0, 1, ..., n, it follows that

$$e_{0} + e_{2} + e_{4} + \cdots = \frac{1}{2} [3^{n} + \sum_{j=0}^{n} (-2)^{j} {n \choose j} 3^{n-j}]$$

= $\frac{1}{2} [3^{n} + (3-2)^{n}]$
= $\frac{1}{2} (3^{n} - 1).$

Subsection 4

Derangements

Derangements

- Consider the permutations of the integers 1, 2, ..., n.
- A permutation of these integers is said to be a **derangement** of the integers if no integer appears in its natural position, i.e.:
 - 1 does not appear in the first position;
 - 2 does not appear in the second position,
 - *n* does not appear in the *n*-th position.
- In general, when each of a set of objects has a position that it is forbidden to occupy and no two objects have the same forbidden position, a **derangement** of these objects is a permutation of them such that no object is in its forbidden position.

Counting Derangements

• Let a_i be the property of a permutation in which the *i*-th object is placed in its forbidden position, with i = 1, 2, ..., n. Then

$$\begin{array}{rcl} \mathcal{N}(a_i) &=& (n-1)!, & i=1,2,\ldots,n; \\ \mathcal{N}(a_ia_j) &=& (n-2)!, & i,j=1,2,\ldots,n; i\neq j; \\ \mathcal{N}(a_ia_ja_k) &=& (n-3)!, & i,j,k=1,2,\ldots,n, i\neq j\neq k\neq i; \\ &\vdots \\ a_1a_2\cdots a_n) &=& 1. \end{array}$$

Moreover, $s_1 = \binom{n}{1}(n-1)!$, $s_2 = \binom{n}{2}(n-2)!$, ..., $s_n = \binom{n}{n}(n-n)!$. Therefore, the number d_n of derangements of n objects is

$$d_n = N(a'_1a'_2\cdots a'_n) = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\binom{n}{n}(n-n)! = n![1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}].$$

• The expression in the square brackets is the truncated series of the Taylor expansion of e^{-1} . Its value can be approximated very accurately by the value of e^{-1} , even for small values of n.

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The Hat Problem

• Suppose 10 gentlemen check their hats at the coatroom, and later on the hats are returned to them randomly. In how many ways can the hats be returned to them such that no gentleman will get his own hat back?

This is exactly the problem of permuting 10 objects (the hats) so that none of them will be in its forbidden position (the owner).

Therefore, the number of ways of returning the hats is $d_{10} = 10!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{10!}) = 1,334,961 \approx 10! \cdot e^{-1}.$

- It follows that the probability that none of the gentlemen will have his own hat back is $\frac{d_{10}}{101} \approx e^{-1}$.
 - This probability is essentially the same for 10 gentlemen as well as for 10,000 gentlemen.
 - Also, when there are 10 gentlemen, the probability is slightly higher than that when there are 9 or 11 gentlemen because of the alternating signs in the expression for d_n .

A Recurrence Relation for Derangements

- The number of derangements of integers can also be obtained by the solution of a recurrence relation.
- Consider the derangements of the integers 1, 2, ..., n in which the first position is occupied by the integer k ≠ 1.
 - If the integer 1 occupies the k-th position, then there are d_{n−2} ways to derange the n − 2 integers 2, 3, ..., k − 1, k + 1, ..., n.
 - If the integer 1 does not occupy the k-th position, then there are d_{n-1} ways to derange the integers 1, 2, ..., k 1, k + 1, ..., n, because in this case we can consider the k-th position as the forbidden position for the integer 1.

Since k can assume the n-1 values 2, 3, ..., n, we have the recurrence relation

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

An Ordinary Generating Function for Derangements

• We discovered $d_n = (n-1)(d_{n-1} + d_{n-2})$. The boundary conditions are $d_2 = 1$ and $d_1 = 0$. The equation is valid for $n \ge 2$ if we set $d_0 = 1$.

It can be rewritten as

$$\begin{array}{rcl} d_n - nd_{n-1} &=& -\left[d_{n-1} - (n-1)d_{n-2}\right] = -\left[-d_{n-2} + (n-2)d_{n-3}\right] \\ &=& \cdots = (-1)^{n-2}[d_2 - 2d_1] = (-1)^{n-2} = (-1)^n. \end{array}$$

i.e., $d_n - nd_{n-1} = (-1)^n$. To solve, let $D(x) = d_0 + \frac{d_1}{1!}x + \frac{d_2}{2!}x^2 + \frac{d_3}{3!}x^3 + \dots + \frac{d_r}{r!}x^r + \dots$ be the exponential generating function of the sequence $(d_0, d_1, \dots, d_r, \dots)$. Multiplying both sides by $\frac{x^n}{n!}$ and summing from n = 2 to $n = \infty$,

$$\sum_{n=2}^{\infty} \frac{d_n}{n!} x^n - \sum_{n=2}^{\infty} \frac{nd_{n-1}}{n!} = \sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n!}$$
$$D(x) - d_1 x - d_0 - x[D(x) - d_0] = e^{-x} - (1 - x)$$
$$D(x) = \frac{e^{-x}}{1 - x}.$$

Since $\frac{1}{1-x}$ is summing, $d_n = n! (1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!})$.

Distributing Books to Children

Let n books be distributed to n children. The books are returned and distributed to the children again later on. In how many ways can the books be distributed so that no child will get the same book twice?
 For the first time, the books can be distributed in n! ways.
 For the second time, the books can be distributed in d_n ways.
 Therefore, the total number of ways is given by

$$(n!)^2\left[1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^n\frac{1}{n!}\right]\approx (n!)^2e^{-1}.$$

Permuting Integers

In how many ways can the integers 1, 2, 3, 4, 5, 6, 7, 8 and 9 be permuted such that no odd integer will be in its natural position? Let a_i be the property that integer i appears in its natural position. Then, we want to compute N(a'₁a'₃a'₅a'₉).

Applying the principle of inclusion and exclusion, we have

$$N(a'_{1}a'_{3}a'_{5}a'_{9}) = N - N(a_{1}) - \dots - N(a_{9}) + N(a_{1}a_{3}) + \dots + N(a_{7}a_{9}) - N(a_{1}a_{3}a_{5}) - \dots - N(a_{5}a_{7}a_{9}) + N(a_{1}a_{3}a_{5}a_{7}) + \dots + N(a_{3}a_{5}a_{7}a_{9}) - N(a_{1}a_{3}a_{5}a_{7}a_{9}) = 9! - {\binom{5}{1}}8! + {\binom{5}{2}}7! - {\binom{5}{3}}6! + {\binom{5}{4}}5! - {\binom{5}{5}}4! = 205, 056.$$

More on Permuting Integers

• For a set of *n* objects, the number of permutations in which a subset of *r* objects are deranged is

$$n! - \binom{r}{1}(n-1)! + \binom{r}{2}(n-2)! - \cdots + (-1)^{r}\binom{r}{r}(n-r)!.$$

- The number of permutations in which all even integers are in their natural positions and none of the odd integers are in their natural positions is equal to $d_5 = 5!(1 \frac{1}{1!} + \frac{1}{2!} \frac{1}{3!} + \frac{1}{4!} \frac{1}{5!}) = 44.$
- The number of permutations in which exactly four of the nine integers are in their natural positions (exactly five integers are deranged) is $\binom{9}{5} \cdot d_5 = 5,544$.
- The number of permutations in which five or more integers are deranged is equal to

$$\binom{9}{5} \cdot d_5 + \binom{9}{6} \cdot d_6 + \binom{9}{7} \cdot d_7 + \binom{9}{8} \cdot d_8 + \binom{9}{9} \cdot d_9.$$

Subsection 5

Permutations and Restrictions on Relative Positions

Generalizing the Study of Forbidden Positions

• In our discussion about the derangement of objects:

- The forbidden positions are absolute positions in the permutations.
- Each object has only one forbidden position.
- No two objects have the same forbidden position.
- We now study the case in which the restrictions are on the relative positions of the objects.
- Later, we look at the case in which:
 - An object may have any number of forbidden positions.
 - Several objects may have the same forbidden position.

Forbidden Positions

- We find the number of permutations of 1, 2, ..., n in which no two adjacent integers are consecutive integers, i.e., the n − 1 patterns 12, 23, 34, ..., (n − 1)n should not appear in the permutations. Let a_i be the property that the pattern i(i + 1) appears in a permutation, with i = 1, 2, ..., n − 1.
 - Note that $N(a_1) = N(a_2) = \cdots = N(a_{n-1}) = (n-1)!$. So $s_1 = \binom{n-1}{1}(n-1)!$.
 - Observe that $N(a_1a_2) = (n-2)!$ because $N(a_1a_2)$ is equal to the number of permutations of the n-2 "objects" 123, 4, 5, ..., n. Similarly, for $1 \le i \le n-1$, $N(a_ia_{i+1}) = (n-2)!$.
 - Also, $N(a_1a_3) = (n-2)!$ because $N(a_1a_3)$ is equal to the number of permutations of the n-2 "objects" 12, 34, 5, 6, ..., n. Similarly, for $1 \le i < n-2$ and $i+1 < j \le n-1$, $N(a_ia_j) = (n-2)!$. So, $N(a_1a_2) = N(a_1a_3) = \cdots = N(a_{n-2}a_{n-1}) = (n-2)!$ and $s_2 = \binom{n-1}{2}(n-2)!$.
 - In general, $s_j = \binom{n-1}{j}(n-j)!, \ j = 0, 1, \dots, n-1.$ Thus, we get $N(a'_1a'_2 \cdots a'_{n-1}) =$ $n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \dots + (-1)^{n-1}\binom{n-1}{n-1}1!.$

Example

- Find the number of permutations of the letters *a*, *b*, *c*, *d*, *e* and *f* in which neither the pattern *ace* nor the pattern *fd* appears. Let
 - a_1 be the property that the pattern *ace* appears in a permutation.
 - a_2 be the property that the pattern fd appears in a permutation.

According to the principle of inclusion and exclusion,

$$N(a'_1a'_2) = N - N(a_1) - N(a_2) + N(a_1a_2)$$

= 6! - 4! - 5! + 3!
= 582.

Example

• In how many ways can the letters $\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta, \gamma$ and γ be arranged so that all the letters of the same kind are not in a single block?

For the permutations of these letters, let

- a₁ be the property that the four α's are in one block;
- a_2 be the property that the three β 's are in one block
- a_3 be the property that the two γ 's are in one block.

Then,

$$N(a'_{1}a'_{2}a'_{3}) = N - N(a_{1}) - N(a_{2}) - N(a_{3}) + N(a_{1}a_{2}) + N(a_{1}a_{3}) + N(a_{2}a_{3}) - N(a_{1}a_{2}a_{3}) = \frac{9!}{4!3!2!} - (\frac{6!}{3!2!} + \frac{7!}{4!2!} + \frac{8!}{4!3!}) + (\frac{4!}{2!} + \frac{5!}{3!} + \frac{6!}{4!}) - 3! = 871.$$

Subsection 6

The Rook Polynomials

Nontaking Rooks

- A **rook** is a chessboard piece which "captures" on both rows and columns.
- The problem of **nontaking rooks** is to enumerate the number of ways of placing *k* rooks on a chessboard such that no rook will be captured by any other rook.
 - Example: Consider a regular 8×8 chessboard.
 - There are (trivially) 64 ways to place one nontaking rook.
 - There are $\binom{8}{2}P(8,2) = 1,568$ ways to place two nontaking rooks. In fact, there are:
 - $\binom{8}{2}$ ways to choose two rows;
 - P(8,2) ways to choose two cells from the two rows for the rooks.
 - Obviously, we can put at the most eight nontaking rooks on the board and there are 8! ways to do so.

Generalization to Arbitrary Boards

- We generalize the problem in that we are interested in placing nontaking rooks not only on a regular 8 × 8 chessboard, but also on chessboards of arbitrary shapes and sizes.
 - Example: The figure shows a so-called "staircase" chessboard.
 - For such a chessboard, there are:
 - four ways to place one nontaking rook;
 - three ways to place two nontaking rooks;
 - no way to place three or more nontaking rooks.



The Rook Polynomial of a Chessboard

- For a given chessboard, let:
 - *r_k* denote the number of ways of placing *k* nontaking rooks on the board;
 - R(x) = ∑_{k=0}[∞] r_kx^k be the ordinary generating function of the sequence (r₀, r₁, r₂, ..., r_k, ...). R(x) is the **rook polynomial** of the given chessboard.
- R(x) is a finite polynomial whose degree is at most n, where n is the number of cells of the chessboard, because it is never possible to place more than n rooks on a chessboard of n cells.
 Example: For the staircase chessboard, the rook polynomial is R(x) = 1 + 4x + 3x².
- When there are several chessboards $C_1, C_2, C_3, ...$ under consideration, let:
 - r_k(C₁), r_k(C₂), r_k(C₃),... denote the numbers of ways of placing k nontaking rooks on the boards C₁, C₂, C₃,...;
 - $R(x, C_1), R(x, C_2), R(x, C_3), \ldots$ denote the rook polynomials of the boards C_1, C_2, C_3, \ldots , respectively.

The Expansion Formula

- Suppose that on a given chessboard *C*, a cell is selected and marked as a special cell.
- Let
 - *C_i* denote the chessboard obtained from *C* by deleting the row and the column that contain the special cell;
 - C_e denote the chessboard obtained from C by deleting the special cell.
- To find the value of $r_k(C)$, we observe that the ways of placing k nontaking rooks on C can be divided into two classes:
 - Those that have a rook in the special cell; The number of ways is equal to $r_{k-1}(C_i)$.
 - Those that do not have a rook in the special cell. The number of ways in the second class is equal to $r_k(C_e)$.

Thus, $r_k(C) = r_{k-1}(C_i) + r_k(C_e)$.

• Correspondingly, $R(x, C) = xR(x, C_i) + R(x, C_e)$. This is called the expansion formula.

Applying the Expansion Formula

- By a pair of parentheses around a chessboard we denote the rook polynomial of the board:
- By expansion with respect to the cell in the upper left corner, we get

$$(\square) = x(\square) + (\square).$$

• By expansion with respect to the cell in the upper right corner, we get

$$(\Box) = x() + (\Box).$$

• Note that () = 1 + x, () = 1 + 2x, () = 1, () = 1 + 2x + x^2. Therefore () = x(1 + x) + 1 + 2x = 1 + 3x + x^2.

Another Example



Disjunct Sub-boards and Rook Polynomials

• Two sub-boards C_1 and C_2 of a chessboard C are called **disjunct** if no cell of one sub-board is in the same row or in the same column of any cell of the other.

Claim: If a chessboard C consists of two disjunct sub-boards C_1 and C_2 , then

$$R(x,C) = R(x,C_1)R(x,C_2).$$

Observe that the way rooks are placed on C_1 is completely independent of the way rooks are placed on C_2 .

Therefore,

$$r_k(C) = \sum_{j=0}^k r_j(C_1) r_{k-j}(C_2).$$

The equation now follows by the definition of the product of two ordinary generating functions.

Subsection 7

Permutations with Forbidden Positions

Distinct Objects Into Distinct Positions

• Consider the distribution of four distinct objects, labeled *a*, *b*, *c* and *d*, into four distinct positions, labeled 1, 2, 3 and 4, with no two objects occupying the same position.

A distribution can be represented in the form of a matrix.

- The rows correspond to the objects.
- The columns correspond to the positions.
- A circle indicates that the object in the row occupies the position in column.

Example: In the figure a is placed in the second position; b is placed in the fourth position; c is placed in the first position; d is placed in the third position.

• Enumerating the number of ways of distributing distinct objects into distinct positions is the same as enumerating the number of ways of placing nontaking rooks on a chessboard.



From Non-Taking Rooks to Forbidden Positions

Placing non-taking rooks on a chessboard can be extended to the case where these are forbidden positions for the objects.
 Example: For the derangement of four objects, the forbidden positions are shown as dark cells:

Thus, the problem of enumerating the number of derangements of four objects is equivalent to the problem of finding the value of r_4 for the white chessboard.



Example: Consider the problem of painting four houses a, b, c and d with four different colors, green, blue, gray, and yellow, subject to:

- House a cannot be painted with yellow.
- House *b* cannot be painted with gray or yellow.
- House c cannot be painted with blue or gray.
- House *d* cannot be painted blue.

This is a problem of permutations with forbidden positions.

The Chessboard for the House Painting Problem

- The problem of painting four houses *a*, *b*, *c* and *d* with four different colors, green, blue, gray, and yellow, subject to:
 - House *a* cannot be painted with yellow.
 - House *b* cannot be painted with gray or yellow.
 - House c cannot be painted with blue or gray.
 - House *d* cannot be painted blue.

The forbidden positions are shown as dark cells in the following chessboard



Again, the number of ways of painting the houses is equal to the value of r_4 for the white chessboard.

Inclusion-Exclusion Instead of Chessboards

- Consider the permutation of *n* objects with restrictions on their positions.
 - Let a_i denote the property of a permutation in which the *i*-th object is in a forbidden position, i = 1, 2, ..., n.
 - Let r_k is the number of ways of placing k non-taking rooks on the dark chessboard.
- To compute *s_j*, note that:
 - *j* of the *n* objects can be placed in the forbidden positions in *r_j* ways;
 - the n j remaining objects can be placed in the n j remaining positions arbitrarily in (n j)! ways,

So
$$s_j = r_j \cdot (n-j)!$$
.

• Thus, by inclusion-exclusion, the number of permutations in which no object is in a forbidden position is

$$N(a'_{1}a'_{2}\cdots a'_{n}) = n! - r_{1} \cdot (n-1)! + r_{2} \cdot (n-2)! + \cdots + (-1)^{n-1} \cdot r_{n-1} \cdot 1! + (-1)^{n} \cdot r_{n} \cdot 0! \\ = \sum_{j=0}^{n} (-1)^{j} \cdot r_{j} \cdot (n-j)!$$

Painting the Housed Revisited

 Consider again the problem of painting the four houses with the four colors mentioned above. The rook polynomial for the board of forbidden positions is

$$R(x) = 1 + 6x + 10x^2 + 4x^3.$$



Thus, the preceding equation gives

$$e_0 = 4! - r_1 \cdot 3! + r_2 \cdot 2! - r_3 \cdot 1! + r_4 \cdot 0!$$

= 4! - 6 \cdot 3! + 10 \cdot 2! - 4 \cdot 1!
= 4.

Hit Polynomials

• We compute the number of permutations in which exactly *m* of the objects are in forbidden positions, i.e.,

$$e_m = r_m \cdot (n-m)! - \binom{m+1}{1} \cdot r_{m+1} \cdot (n-m-1)! + \cdots + (-1)^{n-m} \binom{n}{n-m} \cdot r_n \cdot 0!.$$

Recall the equation connecting E(x) and the s_j 's $E(x) = \sum_{j=0}^{n} s_j (x-1)^j$. Since $s_j = r_j \cdot (n-j)!$, we get

$$E(x) = \sum_{j=0}^{n} r_j \cdot (n-j)! \cdot (x-1)^j.$$

Because an object in a forbidden position is said to be a "hit", E(x) is also called the **hit polynomial**.

House Paintings

• For the problem of painting four houses with four colors, since the rook polynomial is $R(x) = 1 + 6x + 10x^2 + 4x^3$, the hit polynomial is

$$E(x) = \sum_{j=0}^{4} r_j \cdot (4-j)! \cdot (x-1)^j$$

= $r_0 4! (x-1)^0 + r_1 3! (x-1)^1 + r_2 2! (x-1)^2$
 $+ r_3 1! (x-1)^3 + r_4 4! (x-1)^4$
= $4! + 6 \cdot 3! \cdot (x-1) + 10 \cdot 2! \cdot (x-1)^2$
 $+ 4 \cdot 1! \cdot (x-1)^3 + 0 \cdot 0! \cdot (x-1)^4$
= $4 + 8x + 5x^2 + 4x^3$.

Thus, there are:

- four ways to paint the houses so that none of the houses will be painted with forbidden colors;
- eight ways to paint the houses so that exactly one of the houses will be painted with forbidden colors;
- no way to paint all four houses with forbidden colors.

Example

• Find the number of permutations of $\alpha, \alpha, \beta, \beta, \gamma$ and γ so that:

- no α appears in the first and second positions;
- no β appears in the third position;
- no γ appears in the fifth and sixth positions.

Imagine that the α 's, β 's and γ 's are marked so that they become distinguishable.

Then, the forbidden positions are:

- The rook polynomial for a 2×2 square chessboard is $1 + 4x + 2x^2$;
- The rook polynomial for a 2×1 rectangular chessboard is 1 + 2x.





Example (Cont'd)

We found

$$R(x) = 1 + 10x + 36x^2 + 56x^3 + 36x^4 + 8x^5.$$

Thus, we get

$$e_0 = \sum_{j=0}^{6} r_j (6-j)!$$

= 6! - r_15! + r_24! - r_33! + r_4! - r_51! + r_60!
= 6! - 10 \cdot 5! + 36 \cdot 4! - 56 \cdot 3! + 36 \cdot 2! - 8 \cdot 1!
= 112.

Since the objects are not all distinct, we divide by $2! \cdot 2! \cdot 2!$. Thus,

$$\frac{112}{2! \cdot 2! \cdot 2!} = 14$$

is the number of ways of distributing the objects with none of them in a forbidden position.

George Voutsadakis (LSSU)