Introduction to Combinatorial Mathematics

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LSSU Math 300

1. Pólya's Theory of Counting
   - Introduction
   - Sets, Relations and Groups
   - Equivalence Classes Under a Permutation Group
   - Equivalence Classes of Functions
   - Weights and Inventories of Functions
   - Pólya's Fundamental Theorem
   - Generalization of Pólya's Theorem

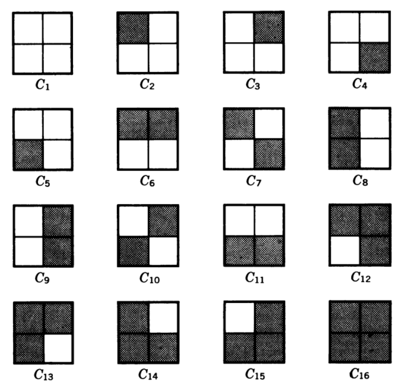
Subsection 1

Introduction

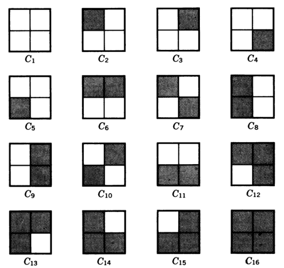
Chessboards With Black and White Cells

- Consider the problem of counting the number of 2×2 chessboards that contain black and white cells.

  Clearly, there are 2^4 such chessboards

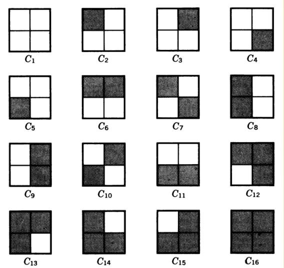


Taking Into Account Rotations

- 

If the four sides of a chessboard are not marked and one side cannot be distinguished from another, then there are chessboards that become indistinguishable from other chessboards after being rotated by $90^\circ, 180^\circ$ or 270°.
- We can check that the following groups become indistinguishable:
  - chessboards C_2, C_3, C_4 and C_5;
  - chessboards C_6, C_8, C_9 and C_{11};
  - chessboards C_7 and C_{10};
  - chessboards C_{12}, C_{13}, C_{14} and C_{15}.
- If this "indistinguishability" is termed equivalence, then, among the 16 chessboards, there are only six "nonequivalent" ones.

The Contrast Chessboard Patters

- 

  Suppose that we are not interested in the black-and-white patterns of the chessboards but are only interested in the contrast patterns.

  - Chessboards C_1 and C_{16} have the same contrast pattern.
  - Chessboards C_2 and C_{15} have the same contrast pattern. etc.

  With rotations of the chessboards also allowed, we can check that there are only four nonequivalent contrast patterns.
- We now study the theory of enumerating nonequivalent objects as first developed by Pólya in 1938.

Subsection 2

Sets, Relations and Groups

Sets

- A **set** is a collection of distinct **elements** (objects).
- We use an uppercase Roman letter to denote a set.

  Example: $S = \{a, b, c, x, z\}$ denotes a set S that contains the elements a, b, c, x and z.
  - There is no ordering among the elements in a set, i.e., $\{a, b, c\}$ and $\{c, b, a\}$ denote the same set.
  - Also, since the elements in a set are all distinct, $\{a, a, b, c\}$ is a redundant representation of the set $\{a, b, c\}$.
- The **empty** or **null set**, denoted by $\emptyset$, is a set containing no elements.

Subsets

- A set T is said to be a **subset** of another set S, written $T \subseteq S$, if every element in T is also an element in S.

  Example: $\{a, b, x\}$ is a subset of $\{a, b, c, x, z\}$, but $\{a, b, y\}$ is not.
- Every set is, trivially, a subset of itself.
- A set T is said to be a **proper subset** of S, written $T \subset S$, if T is a subset of S, but there is at least one element in S that is not in T.
- We write $a \in S$ to mean that a is an element in the set S.
- Also, $|S|$ denotes the number of elements in the set S.
- A set is said to be a k-**set** if it contains k elements.

Union, Intersection, Difference and Ring Sum

- Let A and B be two sets.
  - The **union** of A and B, denoted by $A \cup B$, is the set that contains the elements in A and the elements in B.
    Example: $\{a, b, c, d\} \cup \{a, d, e, j\} = \{a, b, c, d, e, j\}$.
  - The **intersection** of A and B, denoted by $A \cap B$, is the set that contains the elements that are in both A and B.
    Example: $\{a, b, c, d\} \cap \{a, d, e, j\} = \{a, d\}$.
  - The **difference** of A and B, denoted by $A - B$, is a set that contains the elements that are in A but not in B.
    Example: $\{a, b, c, d\} - \{a, d, e, j\} = \{b, c\}$.
  - The **ring sum** of A and B, denoted by $A \oplus B$, is the set that contains the elements in A and the elements in B which are not in the intersection of A and B.
    Example: $\{a, b, c, d\} \oplus \{a, d, e, j\} = \{b, c, e, j\}$.

Partitions and Cartesian Products

- A **partition** on a set is a subdivision of all the elements in the set into disjoint subsets. I.e., a partition on a set is a collection of subsets of the set such that every element in the set is in exactly one of the subsets.
  Example: $\{\{a, b, x\}, \{d\}, \{c, z\}\}$ is a partition on the set $\{a, b, c, d, x, z\}$.
- An **ordered pair** is an ordered arrangement of two (not necessarily distinct) elements.
  We use the notation (a, b) for an ordered pair that contains the elements a and b, arranged in that order.
  Thus, (a, b) and (b, a) are two different ordered pairs.
- The **cartesian product** of two sets S and T, denoted by $S \times T$, is the set of all ordered pairs (x, y) in which x is in S and y is in T.
  Example:
  $\{a, b, c\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

Binary Relations

- A **binary relation** between two sets S and T is a subset of the ordered pairs in the cartesian product $S \times T$.
  Example: $\{(a,1),(a,2),(c,2)\}$ is a binary relation between the sets $\{a,b,c\}$ and $\{1,2\}$.
- For a pair like $(a,2)$ in the relation, we say that a is **related to** 2.
- A binary relation between two sets can be represented in the form of a matrix: The figure shows a representation of the relation $\{(a,1),(a,3),(b,4),(d,2),(d,4)\}$ between sets $\{a,b,c,d\}$ and $\{1,2,3,4,5\}$. A checkmark in a cell indicates that the element identifying the row and the element identifying the column that contains the cell are related.

	1	2	3	4	5
a	✓		✓		
b				✓	
c					
d		✓		✓	

- A **binary relation on** a set S is a binary relation between S and itself.
  Example: $\{(a,a),(a,c),(b,a),(b,c),(c,b)\}$ is a binary relation on the set $\{a,b,c\}$.

Equivalence Relations

- A binary relation on a set is called an **equivalence relation** if the following conditions are satisfied:
  1. Every element in the set is related to itself (**reflexive law**).
  2. For any two elements a and b in the set, if a is related to b, then b is also related to a (**symmetric law**).
  3. For any three elements a, b and c in the set, if a is related to b and b is related to c, then a is also related to c (**transitive law**).
- Example: The binary relation on the left is an equivalence relation, but the binary relation on the right is not.

	a	b	c	d	e
a	✓	✓			
b	✓	✓			
c			✓	✓	✓
d			✓	✓	✓
e			✓	✓	✓

	a	b	c	d	e
a	✓	✓			
b		✓			
c				✓	✓
d			✓	✓	
e	✓		✓	✓	✓

Equivalence Classes

- Given an equivalence relation on a set S, we can divide the elements of S into classes, such that two elements are in the same class if and only if they are related.

  These classes of elements are called the **equivalence classes** into which the set S is divided by the equivalence relation.
- Notice the following:
  - Every element is in one of the equivalence classes because it can at least be in a class by itself, according to the reflexive law.
  - The symmetric law ensures that there is no ambiguity regarding membership in the equivalence classes. (If the relation is not symmetric, we might encounter the difficult situation where a is related to b but b is not related to a.)
  - Finally, because of the transitive law, no element can be in more than one equivalence class.

  Therefore, an equivalence relation on a set induces a partition on the set in which the disjoint subsets are the equivalence classes.

Equivalent Elements

Example: The partition induced by the depicted equivalence relation on the set $\{a, b, c, d, e\}$ is $\{\{a, b\}, \{c, d, e\}\}$.

	a	b	c	d	e
a	✓	✓			
b	✓	✓			
c			✓	✓	✓
d			✓	✓	✓
e			✓	✓	✓

- Two elements are said to be **equivalent** if they are in the same equivalence class.

Functions

- A (**single-valued**) **function from** a set S **to** a set T is a binary relation between the sets S and T, such that every element in S is related to exactly one element in T.

  Example: $\{(a,2),(b,1),(c,2)\}$ is a function from the set $\{a,b,c\}$ to the set $\{1,2\}$.
- The set S is called the **domain** of the function, and the set T is called the **range** of the function.
- Let f denote a function, and let $(a,2)$ be an ordered pair in the function.

  We write $f(a)=2$ to mean that a is related to 2 by the function f.

  We say that 2 is the **value**, or **image**, of a under the function f, and also that f **maps** a **into** 2.

One-to-One and Onto Functions

- An arbitrary function on the left:

	1	2	3	4
a		✓		
b		✓		
c				✓
d			✓	

	1	2	3	4	5
a			✓		
b	✓				
c				✓	
d					✓

	1	2	3	4
a	✓			
b			✓	
c		✓		
d				✓
e		✓		

- A function is a **one-to-one function** if every element in the domain has a unique image.
- A function is an **onto function** if every element in the range is the image of at least one element in the domain.

Binary Operations and Closure

- A **binary operation** on a set S is a function from the set $S \times S$ to a set T.
  Example: Both tables below describe the same binary operation $*$ on the set $S = \{a, b\}$ with $T = \{1, 2, 3\}$.

	1	2	3
(a,a)	✓		
(a,b)		✓	
(b,a)	✓		
(b,b)			✓

$*$	a	b
a	1	2
b	1	3

- Instead of the functional notation, we shall also let $*$ denote a binary operation and let $a * b$ denote the value of the ordered pair (a, b) under the binary operation.
  Example: In the operation depicted above, $a * a = 1$ and $a * b = 2$.
- A binary operation on a set S is said to be **closed** if it is a function from the set $S \times S$ to the set S.

Groups

- A set S together with a binary operation $*$ on the set S is said to form a **group** if the following conditions are satisfied:
  1. The binary operation $*$ is closed.
  2. The binary operation $*$ is **associative**, i.e., for all a, b, c in S, $(a * b) * c = a * (b * c)$.
  3. There is an element e in S, such that $a * e = a$, for every a in S. This element is called an **identity element** of the group.
  4. For any element a in S, there is another element in S, denoted by a^{-1} and called an **inverse** of a, which is such that $a * a^{-1} = e$.
- Example: The binary operation for a group consisting of the five elements 0, 1, 2, 3 and 4 is shown on the right.
  Notice that 0 is an identity element, an inverse of the element 0 is 0 itself, an inverse of the element 1 is 4, etc.

*	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Properties of Group I

1. If b is an inverse of a, then a is an inverse of b.

   If b is an inverse of a, $a * b = e$. Let b^{-1} denote an inverse of b, i.e., $b * b^{-1} = e$. Then, we have

$$\begin{array}{rcl} b * a & = & b * (a * e) = b * (a * (b * b^{-1})) \\ & = & b * ((a * b) * b^{-1}) = (b * (a * b)) * b^{-1} \\ & = & (b * e) * b^{-1} = b * b^{-1} = e. \end{array}$$

   Therefore, a is an inverse of b.

2. For every a in S, $e * a = a$.

   Using Property 1, we get

$$\begin{array}{rcl} e * a & = & (a * a^{-1}) * a = a * (a^{-1} * a) \\ & = & a * e = a. \end{array}$$

Properties of Group II

3. The identity element is unique.
   Suppose there are two elements e_1 and e_2, such that $a * e_1 = a$ and $a * e_2 = a$. Then

$$\begin{array}{rcl} a * e_1 = a * e_2 & \Rightarrow & a^{-1} * (a * e_1) = a^{-1} * (a * e_2) \\ & \Rightarrow & (a^{-1} * a) * e_1 = (a^{-1} * a) * e_2 \\ & \Rightarrow & e * e_1 = e * e_2 \\ & \Rightarrow & e_1 = e_2. \end{array}$$

4. The inverse of any element is unique.
   Suppose there are two elements b and c, such that $a * b = e$ and $a * c = e$. Then, we have

$$\begin{array}{rcl} a * b = a * c & \Rightarrow & a^{-1} * (a * b) = a^{-1} * (a * c) \\ & \Rightarrow & (a^{-1} * a) * b = (a^{-1} * a) * c \\ & \Rightarrow & e * b = e * c \\ & = & b = c. \end{array}$$

Subsection 3

Equivalence Classes Under a Permutation Group

Permutations and Composition

- A one-to-one function from a set S to itself is called a **permutation** of the set S.
- We use the notation $\binom{abcd}{bdca}$ for the permutation of the set $\{a, b, c, d\}$ that maps a into b, b into d, c into c and d into a:
  - In the upper row the elements in the set are written down in an arbitrary order;
  - In the lower row the image of an element will be written below the element itself.
- The notion of a permutation of a set is the same as an arrangement of a set of objects.
- Let π_1 and π_2 be two permutations of a set S. The **composition** of π_1 and π_2, denoted by $\pi_1\pi_2$, is the successive permutations of the set S, first according to π_2 and, then, according to π_1.

  Example: Let $\pi_1 = \binom{abcd}{adbc}$, $\pi_2 = \binom{abcd}{bacd}$ be two permutations of the set $\{a, b, c, d\}$. Then $\pi_1\pi_2 = \binom{abcd}{dabc}$. $\pi_1\pi_2$ maps a into d since π_2 maps a into b and π_1 maps b into d, and so on.

Closure of the Set of Permutations Under Composition

Claim: The composition of two permutations is also a permutation.

Let π_1 and π_2 be two permutations of the set

$$S = \{a, b, c, \ldots, x, y, z\}.$$

To show that $\pi_1\pi_2$ is also a permutation of the set S, we have only to show that no two elements in S are mapped into the same element by $\pi_1\pi_2$.

Suppose that π_2 maps the element a into b and π_1 maps the element b into c. $\pi_1\pi_2$ will then map the element a into c.

Let x be any element distinct from a. Since π_2 is a permutation of the set S, π_2 maps x into an element that is distinct from b, say y. Similarly, π_1 maps y into an element that is distinct from c, say z.

We conclude that $\pi_1\pi_2$ always maps two distinct elements (for example, a and x) into two distinct elements (for example, c and z). Thus, $\pi_1\pi_2$ is a permutation of the set S.

Non-commutativity of Composition

- The composition of permutations is noncommutative, i.e., in general, $\pi_1\pi_2 \neq \pi_2\pi_1$.

  Example: for $\pi_1 = \binom{abcd}{adbc}$, $\pi_2 = \binom{abcd}{bacd}$, we have

  $$\begin{aligned} \pi_1\pi_2 &= \binom{abcd}{dabc}, \\ \pi_2\pi_1 &= \binom{abcd}{bdac}, \end{aligned}$$

  i.e., in this case, $\pi_1\pi_2 \neq \pi_2\pi_1$.

Associativity of Composition

Claim: The composition of permutations is **associative**, i.e., for any permutations π_1, π_2 and π_3 of a set, we have $(\pi_1\pi_2)\pi_3 = \pi_1(\pi_2\pi_3)$.
Suppose:

- π_3 maps a into b;
- π_2 maps b into c;
- π_1 maps c into d.

Since $\pi_1\pi_2$ maps b into d, $(\pi_1\pi_2)\pi_3$ maps a into d.

Similarly, since $\pi_2\pi_3$ maps a into c, $\pi_1(\pi_2\pi_3)$ maps a into d.

Example: Let $\pi_1 = \binom{abcd}{adbc}$, $\pi_2 = \binom{abcd}{bacd}$ and $\pi_3 = \binom{abcd}{bdac}$. Then, we have

$$(\pi_1\pi_2)\pi_3 = \left[\begin{pmatrix} abcd \\ adbc \end{pmatrix}\begin{pmatrix} abcd \\ bacd \end{pmatrix}\right]\begin{pmatrix} abcd \\ bdac \end{pmatrix} = \begin{pmatrix} abcd \\ dabc \end{pmatrix}\begin{pmatrix} abcd \\ bdac \end{pmatrix} = \begin{pmatrix} abcd \\ acdb \end{pmatrix};$$

$$\pi_1(\pi_2\pi_3) = \begin{pmatrix} abcd \\ adbc \end{pmatrix}\left[\begin{pmatrix} abcd \\ bacd \end{pmatrix}\begin{pmatrix} abcd \\ bdac \end{pmatrix}\right] = \begin{pmatrix} abcd \\ adbc \end{pmatrix}\begin{pmatrix} abcd \\ adbc \end{pmatrix} = \begin{pmatrix} abcd \\ acdb \end{pmatrix}.$$

Permutation Groups

- Let $G = \{\pi_1, \pi_2, \ldots\}$ be a set of permutations of a set S.

  Then G is said to be a **permutation group** of S if G and the binary operation of composition of permutations form a group.
- In other words, according to the definition of a group, the following conditions should be satisfied:
  1. If π_1 and π_2 are in G, then $\pi_1\pi_2$ is also in G.
  2. The binary operation, composition of permutations, is associative. However, this is known to be true.
  3. The identity permutation that maps each element into itself is in G. This is the only permutation among all the permutations of a set that can be the identity element of the group.
  4. For every permutation π_1 in G, there is a permutation π_2, which is such that $\pi_1\pi_2$ is the identity permutation.

  Example: $G = \left\{ \binom{abc}{abc}, \binom{abc}{bca}, \binom{abc}{cab} \right\}$ is a permutation group of $\{a, b, c\}$.

The Relation Induced by a Permutation Group G

- Let G be a permutation group of a set $S = \{a, b, \ldots\}$.
- A binary relation on the set S, called the **binary relation induced by** G, is defined to be such that

  element a is related to element b if and only if
  there is a permutation in G that maps a into b.

Example: Let

$$G = \left\{ \begin{pmatrix} abcd \\ abcd \end{pmatrix}, \begin{pmatrix} abcd \\ bacd \end{pmatrix}, \begin{pmatrix} abcd \\ abdc \end{pmatrix}, \begin{pmatrix} abcd \\ badc \end{pmatrix} \right\}.$$

The binary relation induced by G is depicted on the right.

	a	b	c	d
a	✓	✓		
b	✓	✓		
c			✓	✓
d			✓	✓

The Induced Binary Relation is an Equivalence

Theorem

The binary relation on a set induced by a permutation group of the set is an equivalence relation.

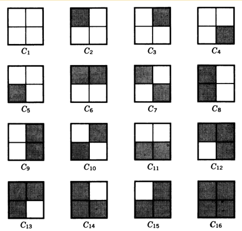
- Let G be a permutation group of the set $S = \{a, b, \ldots\}$.
  1. Since the identity permutation is in G, every element in S is related to itself in the binary relation on S induced by G. Therefore, the reflexive law is satisfied.
  2. If there is a permutation π_1 in G that maps a into b, the inverse of π_1, which is also in G, will map b into a. Therefore, the binary relation on S induced by G satisfies the symmetric law.
  3. If there is a permutation π_1 mapping a into b and a permutation π_2 mapping b into c, the permutation $\pi_2\pi_1$, which is also in G, will map a into c. Therefore, the binary relation on S induced by G satisfies the transitive law.

Invariant Elements

- Given a set S and a permutation group G of S, we wish to find the number of equivalence classes into which S is divided by the equivalence relation on S induced by G.
- The direct calculation involves finding the equivalence relation and then counting the number of equivalence classes.
- When the set S contains a large number of elements, such counting becomes prohibitively tedious.
- Burnside's Theorem enables us to find the number of equivalence classes in an alternative way by counting the number of elements that are invariant under the permutations in the group.
- An element is said to be **invariant** under a permutation, or is called an **invariance**, if the permutation maps the element into itself.

Permuting the 2×2 Squares

- Consider the example of 2×2 chessboards to see why we are interested in counting the number of equivalence classes into which a set is divided by the equivalence relation on the set induced by a permutation group.

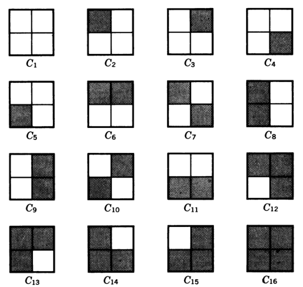
  When the chessboards are rotated clockwise by 90°, C_1 remains as C_1, C_2 becomes C_3, C_3 becomes C_4, C_4 becomes C_5, C_5 becomes C_2, C_6 becomes C_9, C_7 becomes C_{10}, and so on.
  - A 90° rotation amounts to a permutation π_1 of the chessboards:
    $\pi_1 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_3 & C_4 & C_5 & C_2 & C_9 & C_{10} & C_6 & C_{11} & C_7 & C_8 & C_{15} & C_{12} & C_{13} & C_{14} & C_{16} \end{pmatrix}$.
  - Similarly, corresponding to a 180° clockwise rotation and a 270° clockwise rotation of the chessboards, there are the permutations π_2 and π_3: $\pi_2 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_4 & C_5 & C_2 & C_3 & C_{11} & C_7 & C_9 & C_8 & C_{10} & C_6 & C_{14} & C_{15} & C_{12} & C_{13} & C_{16} \end{pmatrix}$,
    $\pi_3 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_5 & C_2 & C_3 & C_4 & C_8 & C_{10} & C_{11} & C_6 & C_7 & C_9 & C_{13} & C_{14} & C_{15} & C_{12} & C_{16} \end{pmatrix}$.
  - Let π_4 be the identity: $\pi_4 = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \end{pmatrix}$.

The Group of Rotations and Indistinguishability

- It can be shown that $G = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ is a permutation group of the set of 2×2 chessboards.

  In the equivalence relation induced by G, we see that C_2, C_3, C_4 and C_5 are in the same equivalence class, which means that they become indistinguishable when rotations of the chessboards are allowed.

  It follows that the number of equivalence classes into which the chessboards are divided by the equivalence relation induced by G is the number of "distinct" chessboards, i.e., those distinguishable through rotation.



Burnside's Theorem

Theorem (Burnside)

The number of equivalence classes into which a set S is divided by the equivalence relation induced by a permutation group G of S is given by $\frac{1}{|G|}\sum_{\pi \in G} \psi(\pi)$, where $\psi(\pi)$ is the number of elements that are invariant under the permutation π.

- For any element s in S, let $\eta(s)$ denote the number of permutations under which s is invariant. Then $\sum_{\pi \in G} \psi(\pi) = \sum_{s \in S} \eta(s)$ because both count the total number of invariances under all the permutations in G:
  - One way to count the invariances is to go through the permutations one by one and count the number of invariances under each permutation, giving $\sum_{\pi \in G} \psi(\pi)$ as the total count.
  - Another way to count the invariances is to go through the elements one by one and count the number of permutations under which an element is invariant, giving $\sum_{s \in S} \eta(s)$ as the total count.

An Auxiliary Lemma

Claim: Let a and b be two elements in S in the same equivalence class. There are exactly $\eta(a)$ permutations mapping a into b.

Since a and b are in the same equivalence class, there is at least one such permutation which we shall denote by π_x. Let $\{\pi_1, \pi_2, \pi_3, \ldots\}$ be the set of the $\eta(a)$ permutations under which a is invariant. Then, the $\eta(a)$ permutations in the set $\{\pi_x\pi_1, \pi_x\pi_2, \pi_x\pi_3, \ldots\}$ are permutations that map a into b.

- They are all distinct because, if $\pi_x\pi_1 = \pi_x\pi_2$, $\pi_x^{-1}(\pi_x\pi_1) = \pi_x^{-1}(\pi_x\pi_2)$, whence $\pi_1 = \pi_2$, which is impossible.
- No other permutation in G maps a into b: If π_y maps a into b, then $\pi_x^{-1}\pi_y$ is a permutation that maps a into a, whence it is in the set $\{\pi_1, \pi_2, \pi_3, \ldots\}$. Hence $\pi_y = \pi_x(\pi_x^{-1}\pi_y)$ is in $\{\pi_x\pi_1, \pi_x\pi_2, \pi_x\pi_3, \ldots\}$.

We conclude that there are exactly $\eta(a)$ permutations in G that map a into b.

Proof of Burnside's Theorem (Cont'd)

- Let $a, b, c, \ldots, h$ be the elements in S that are in one equivalence class. All the permutations in G can be categorized as those that map a into a, those that map a into b, those that map a into c, ..., and those that map a into h. We have shown that there are exactly $\eta(a)$ permutations in each of these categories. Thus, $\eta(a) = \eta(b) = \cdots = \eta(h) = \dfrac{|G|}{\text{number of elements in class containing } a}$. We now get:

$$\eta(a) + \eta(b) + \cdots + \eta(h) = |G|$$

$$\sum_{\text{all } s \text{ in the equivalence class}} \eta(s) = |G|$$

$$\textstyle\sum_{s \in S} \eta(s) = (\text{number of equivalence classes}) \cdot |G|$$

$$\text{Number of Classes} = \frac{1}{|G|} \textstyle\sum_{s \in S} \eta(s) = \frac{1}{|G|} \sum_{\pi \in G} \psi(\pi).$$

Example

- Let $S = \{a, b, c, d\}$, and let G be the permutation group consisting of

$$\pi_1 = \begin{pmatrix} abcd \\ abcd \end{pmatrix}, \pi_2 = \begin{pmatrix} abcd \\ bacd \end{pmatrix}, \pi_3 = \begin{pmatrix} abcd \\ abdc \end{pmatrix}, \pi_4 = \begin{pmatrix} abcd \\ badc \end{pmatrix}.$$

The equivalence relation on S induced by G is shown on the right. Clearly, S is divided into two equivalence classes, $\{a, b\}$ and $\{c, d\}$.

	a	b	c	d
a	✓	✓		
b	✓	✓		
c			✓	✓
d			✓	✓

Since

$$\psi(\pi_1) = 4,\ \psi(\pi_2) = 2,\ \psi(\pi_3) = 2,\ \psi(\pi_4) = 0,$$

according to Burnside's Theorem, the number of equivalence classes can be computed as

$$\frac{1}{|G|}(\psi(\pi_1) + \psi(\pi_2) + \psi(\pi_3) + \psi(\pi_4)) = \frac{1}{4}(4 + 2 + 2 + 0) = 2.$$

Strings of Beads of Length 2

- Find the number of distinct strings of length 2 that are made up of blue beads and yellow beads. The two ends of a string are not marked, and two strings are, therefore, indistinguishable if interchanging the ends of one will yield the other.

  Let b and y denote blue and yellow beads, respectively. Let bb, by, yb and yy denote the four different strings of length 2 when equivalence between strings is not taken into consideration. The problem is to find the number of equivalence classes into which the set $S = \{bb, by, yb, yy\}$ is divided by the equivalence relation induced by the permutation group $G = \{\pi_1, \pi_2\}$, where

$$\pi_1 = \begin{pmatrix} bb & by & yb & yy \\ bb & by & yb & yy \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} bb & by & yb & yy \\ bb & yb & by & yy \end{pmatrix}.$$

  π_1 indicates that every string is equivalent to itself, and π_2 indicates the equivalence between strings when the two ends of a string are interchanged. According to Burnside's theorem, the number of distinct strings is $\frac{1}{2}(4+2) = 3$.

Strings of Beads of Length 3

- For the case of distinct strings of length 3 made up of blue beads and yellow beads, we have $S = \{bbb, bby, byb, ybb, byy, yby, yyb, yyy\}$ and the permutation group $G = \{\pi_1, \pi_2\}$, where:
  - π_1 is the identity permutation;
  - π_2 is the permutation that maps a string into one that is obtained from the former by interchanging its ends.
    For example, *bbb* is mapped into *bbb*, *bby* is mapped into *ybb*, *byb* is mapped into *byb*, and so on.

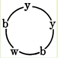
  The number of elements that are invariant under π_1 is eight.

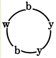
  The number of elements that are invariant under π_2 is four:

    A string will be mapped into itself under π_2 if the beads at the two ends of a string are of the same color.
    There are four such strings.

  Therefore, the number of distinct strings is equal to $\frac{1}{2}(8+4) = 6$.

Five-Bead Bracelets

- Find the number of distinct bracelets of five beads made up of yellow, blue, and white beads. Two bracelets are said to be **indistinguishable** if the rotation of one will yield another. For simplicity, we assume that the bracelets cannot be flipped over.
  Let S be the set of the $3^5 (= 243)$ distinct bracelets when rotational equivalence is not considered. Let $G = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$ be a permutation group, where:
  - π_1 is the identity permutation;
  - π_2 is the permutation that maps a bracelet into one which is the former rotated clockwise by one bead position: (e.g., 

is mapped to 

).
  - π_3, π_4 and π_5 are permutations that map a bracelet into one rotated clockwise by two, three and four bead positions, respectively.

Five-Bead Bracelets (Cont'd)

- The number of elements that are invariant under:
  - π_1 is 243;
  - π_2 is three, since only when all five beads in a bracelet have to be of the same color;
  - each of π_3, π_4 and π_5 is also three.

  Thus, the number of distinct bracelets is

$$\frac{1}{5}(\psi(\pi_1) + \psi(\pi_2) + \psi(\pi_3) + \psi(\pi_4) + \psi(\pi_5))$$
$$= \tfrac{1}{5}(243 + 3 + 3 + 3 + 3) = 51.$$

Arrangements of n People Around a Circle

- The problem of finding the number of ways to arrange n people around a circle can also be solved using Burnside's theorem.

  Let S be the set of the $n!$ distinct ways to arrange n people around a circle when rotational equivalence is not considered.

  Let $G = \{\pi_1, \pi_2, \ldots, \pi_n\}$ be a permutation group where:
  - π_1 is the identity permutation;
  - π_2 is the permutation that maps a circular arrangement into one which is the former rotated clockwise by one position;
  - π_3 is the permutation that maps a circular arrangement into one which is the former rotated clockwise by two positions;

    $\vdots$
  - π_n is the permutation that maps a circular arrangement into one which is the former rotated clockwise by $n-1$ positions.

  Since $\psi(\pi_1) = n!$ and $\psi(\pi_2) = \psi(\pi_3) = \cdots = \psi(\pi_n) = 0,$ the number of distinct circular arrangements is $\frac{1}{n}(n! + 0 + 0 + \cdots + 0) = (n-1)!$.

Printing Five-Digit Numbers

- Suppose that we are to print all the five-digit numbers on slips of paper with one number on each slip. Clearly, there are 10^5 such slips, assuming that for numbers smaller than $10,000$, leading zeros are always filled in. However, since 0, 1, 6, 8, and 9 become 0, 1, 9, 8, and 6 when they are read upside down, there are pairs of numbers that can share the same slip if the slips will be read either right side up or upside down. E.g., we can make up one slip for both the numbers 89166 and 99168. How many distinct slips will we have to make up for the 10^5 numbers?
  Let S be the set of the 10^5 numbers, and $G = \{\pi_1, \pi_2\}$ a permutation group of S, where:
  - π_1 is the identity permutation;
  - π_2 is the permutation that maps a number:
    - into itself, if it is not readable as a number when turned upside down, e.g., 13765 is mapped into 13765.
    - into the number obtained by reading the former upside down, whenever it is possible, e.g., 89166 is mapped into 99168.

Printing Five-Digit Numbers (Cont'd)

- The number of invariances under:
  - π_1 is 10^5;
  - π_2 is $(10^5 - 5^5) + 3 \cdot 5^2$:
    - there are $10^5 - 5^5$ numbers that contain one or more of the digits 2, 3, 4, 5 and 7 and cannot be read upside down;
    - there are $3 \cdot 5^2$ numbers that will read the same either right side up or upside down.

      e.g., 16891 (the center digit of these numbers must be 0 or 1 or 8, the last digit must be the first digit turned upside down, and the fourth digit must be the second digit turned upside down).

  Therefore, the number of distinct slips to be made up is

$$\frac{1}{2}(10^5 + 10^5 - 5^5 + 3 \cdot 5^2) = 10^5 - \frac{1}{2} \cdot 5^5 + \frac{3}{2} \cdot 5^2.$$

A Group Q Acting on a Set S

- Let Q be a group consisting of the elements $q_1, q_2, \ldots$, together with a binary operation $*$, and $S = \{a, b, \ldots\}$ a set.
- Suppose that every element q in Q is associated with a permutation π_q of the set S, such that, for any q_1 and q_2 in Q,

$$\pi_{q_1 * q_2} = \pi_{q_1} \pi_{q_2},$$

  i.e., the permutation associated with the element $q_1 * q_2$ is equal to the composition of the permutations π_{q_1} and π_{q_2}, the permutations associated with the elements q_1 and q_2. This condition is referred to as the **homomorphy condition**.
- Different elements in Q need not be associated with distinct permutations.

Binary Relation Induced by Q Acting on S

- We define a binary relation on the set S, called the **binary relation induced by** Q, such that elements a and b in S are related if and only if there is a permutation π_q, associated with an element q in Q, that maps a into b.

Theorem

The binary relation induced by Q is an equivalence relation.

- Proof is very similar to the one on permutation groups.

Generalization of Burnside's Theorem

Theorem (Generalization of Burnside's Theorem)

The number of equivalence classes into which S is divided by the equivalence relation induced by Q is $\frac{1}{|Q|}\sum_{q\in Q}\psi(\pi_q)$, where $\psi(\pi_q)$ is the number of elements in S that are invariant under the permutation π_q, the permutation associated with the element q in Q.

- Proof similar to Burnside's Theorem.
- The permutations associated with the elements in Q form a permutation group.
- Moreover, the binary relation on the set S induced by this permutation group is the same as that induced by Q.
- In many applications, it is more convenient to consider the structure of the group Q than the structure of the permutation group.

Subsection 4

Equivalence Classes of Functions

Introducing Pólya's Theorem

- In applying Burnside's theorem to the counting of the number of equivalence classes into which a set is divided:
  - The computation of the numbers of invariances under the permutations is still quite involved.
  - In addition to the number of equivalence classes, we may wish to have further information about the properties of the equivalence classes. E.g., in the problem of chessboards, one may wish to know the number of distinct chessboards consisting of two black cells and two white cells.
- Pólya's theory of counting offers solutions to both of these problems.
- Let f be a function from a set D, its domain, to a set R, its range.

  Since each element in D has a unique image in R, the function f corresponds to a way of distributing $|D|$ objects into $|R|$ cells.

  Therefore, the problem of enumerating the ways of distributing $|D|$ objects into $|R|$ cells is the same as that of enumerating the functions from D to R.

Equivalence Relation on Set of Functions

- Let D and R be two sets, and let G be a permutation group of D.
- We define a binary relation on the set of all the functions from D to R as follows:

  A function f_1 is related to a function f_2 if and only if there is a permutation π in G, such that $f_1(d) = f_2[\pi(d)]$, for all d in D.
- This binary relation is an equivalence relation:
  1. Because the identity permutation is in G, the reflexive law is satisfied.
  2. If $f_1(d) = f_2[\pi(d)]$, for all d in D, then $f_2(d) = f_1[\pi^{-1}(d)]$, for all d in D. Since π^{-1} is a permutation in G, the symmetric law is satisfied.
  3. If $f_1(d) = f_2[\pi_1(d)]$ and $f_2(d) = f_3[\pi_2(d)]$, for all d in D, where π_1 and π_2 are permutations in G, then $f_1(d) = f_3[\pi_2\pi_1(d)]$, for all d in D. Since $\pi_2\pi_1$ is a permutation in G, the transitive law is satisfied.
- It follows that the functions from D to R are divided into equivalence classes, called **patterns**, by the equivalence relation.
- The patterns correspond to the distinct ways of distributing $|D|$ objects into $|R|$ cells when equivalence between ways of distribution is introduced by the permutation group G.

Example

- Let $D = \{a, b, c, d\}$ and $R = \{x, y\}$. Let G be the permutation group $\{\pi_1, \pi_2, \pi_3, \pi_4\}$, where $\pi_1 = \binom{abcd}{bcda}$, $\pi_2 = \binom{abcd}{cdab}$, $\pi_3 = \binom{abcd}{dabc}$ and $\pi_4 = \binom{abcd}{abcd}$. There are 16 functions $f_1, f_2, \ldots . f_{16}$ from D to R:

	$f(a)$	$f(b)$	$f(c)$	$f(d)$
f_1	x	x	x	x
f_2	y	x	x	x
f_3	x	y	x	x
f_4	x	x	y	x
f_5	x	x	x	y
f_6	y	y	x	x
f_7	y	x	y	x
f_8	y	x	x	y
f_9	x	y	y	x
f_{10}	x	y	x	y
f_{11}	x	x	y	y
f_{12}	y	y	y	x
f_{13}	y	y	x	y
f_{14}	y	x	y	y
f_{15}	x	y	y	y
f_{16}	y	y	y	y

Note that:

$$f_3[\pi_1(a)] = f_3(b) = y = f_2(a),$$
$$f_3[\pi_1(b)] = f_3(c) = x = f_2(b),$$
$$f_3[\pi_1(c)] = f_3(d) = x = f_2(c),$$
$$f_3[\pi_1(d)] = f_3(a) = x = f_2(d).$$

So, f_2 and f_3 are equivalent.

Example (Cont'd)

- G is the permutation group $\{\pi_1, \pi_2, \pi_3, \pi_4\}$, where $\pi_1 = \binom{abcd}{bcda}$, $\pi_2 = \binom{abcd}{cdab}$, $\pi_3 = \binom{abcd}{dabc}$ and $\pi_4 = \binom{abcd}{abcd}$.

	$f(a)$	$f(b)$	$f(c)$	$f(d)$
f_1	x	x	x	x
f_2	y	x	x	x
f_3	x	y	x	x
f_4	x	x	y	x
f_5	x	x	x	y
f_6	y	y	x	x
f_7	y	x	y	x
f_8	y	x	x	y
f_9	x	y	y	x
f_{10}	x	y	x	y
f_{11}	x	x	y	y
f_{12}	y	y	y	x
f_{13}	y	y	x	y
f_{14}	y	x	y	y
f_{15}	x	y	y	y
f_{16}	y	y	y	y

Note that:

$$f_7[\pi_1(a)] = f_7(b) = x = f_{10}(a),$$
$$f_7[\pi_1(b)] = f_7(c) = y = f_{10}(b),$$
$$f_7[\pi_1(c)] = f_7(d) = x = f_{10}(c),$$
$$f_7[\pi_1(d)] = f_7(a) = y = f_{10}(d).$$

So, f_7 and f_{10} are equivalent.

Similarly, it can be seen that the 16 functions are divided into six equivalence classes: $\{f_1\}$, $\{f_2, f_3, f_4, f_5\}$, $\{f_6, f_8, f_9, f_{11}\}$, $\{f_7, f_{10}\}$, $\{f_{12}, f_{13}, f_{14}, f_{15}\}$, $\{f_{16}\}$.

The Problem of 2×2 Chessboards

- Let the four cells in a 2×2 chessboard be labeled a, b, c and d:

a	b
d	c

Let the two colors, white and black, be denoted by x and y.

A function from the set $\{a, b, c, d\}$ to the set $\{x, y\}$ then corresponds to a chessboard. The permutations in the group

$$G = \left\{ \begin{pmatrix} abcd \\ bcda \end{pmatrix}, \begin{pmatrix} abcd \\ cdab \end{pmatrix}, \begin{pmatrix} abcd \\ dabc \end{pmatrix}, \begin{pmatrix} abcd \\ abcd \end{pmatrix} \right\}$$

correspond to the rotations of the chessboards.

E.g., the permutation $\pi_1 = \binom{abcd}{bcda}$ corresponds to the rotation of the chessboards in a clockwise direction by 90°.

The 16 functions in the preceding slide correspond to the 16 chessboards. We saw that they are divided into six equivalence classes by the equivalence relation induced by G.

Subsection 5

Weights and Inventories of Functions

Weights and Store Enumerators

- Let D and R be the domain and the range, respectively, of a set of $|R|^{|D|}$ functions.
- Suppose that a **weight** is assigned to each of the elements in R.

  The weights can be either numbers or symbols. Let r be in R, and let $w(r)$ denote the weight assigned to r.
- The **store enumerator** of the set R is defined to be the sum of the weights of the elements in R, i.e.,

$$\text{Store enumerator} = \sum_{r \in R} w(r).$$

- The term "store enumerator" is actually very descriptive: The elements in the set R are the values that the elements in the set D can assume under functions from D to R. Thus, the store enumerator is a description of what is "in the store".

Examples and Comparison with Generating Functions

Example: Let $R = \{r_1, r_2, r_3\}$ and $w(r_1) = r_1, w(r_2) = r_2$ and $w(r_3) = r_3$. Then, the store enumerator is $r_1 + r_2 + r_3$. It indicates that the value that an element in D can assume is either r_1 or r_2 or r_3.

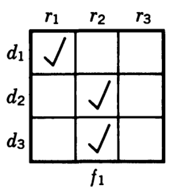
Example: Suppose we let $w(r_1) = u$, $w(r_2) = v$ and $w(r_3) = u$. The store enumerator is $2u + v$. It means that there are two elements of type u and one element of type v in the set R from which the value for an element in D can be chosen.

- The notion of store enumerator is just a generalization of the notion of generating functions.
  - For the selection of one object from the three objects r_1, r_2 and r_3, the generating function is $r_1x + r_2x + r_3x$. x is just the indicator which can be omitted when it is understood that exactly one of the three objects is selected.
  - When objects r_1 and r_3 are of the same kind u and object r_2 is of another kind v, the generating function becomes $2u + v$.

Weights and Inventories

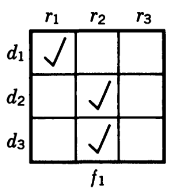
- For a function f from D to R, we define its **weight**, denoted by $W(f)$, as the product of the weights of the images of the elements in D under f, i.e.,
$$W(f) = \prod_{d \in D} w[f(d)].$$
- The **inventory** of a set of functions is defined as the sum of their weights, i.e.,
$$\text{Inventory of a set of functions} = \sum_{\text{all } f \text{ in the set}} W(f).$$

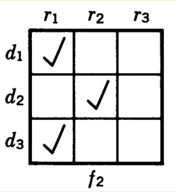
Example: Let $D = \{d_1, d_2, d_3\}$, $R = \{r_1, r_2, r_3\}$, $w(r_1) = u$, $w(r_2) = v$, and $w(r_3) = u$. The weight of the function f_1 on the right is $W(f_1) = uv^2$.

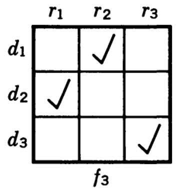


Example of Weights and Inventories

Example: Let $D = \{d_1, d_2, d_3\}$, $R = \{r_1, r_2, r_3\}$, $w(r_1) = u$, $w(r_2) = v$, and $w(r_3) = u$. The weight of the function f_1 on the left is $W(f_1) = uv^2$.







The inventory of the set of functions f_1, f_2 and f_3 is

$$W(f_1) + W(f_2) + W(f_3) = uv^2 + 2u^2v.$$

- The weight of a function is a representation of the way $|D|$ objects are distributed into $|R|$ cells as described by the function.
- The inventory of a set of functions is a representation of the ways the objects are distributed.

Weights of Patterns and Inventories of Sets of Patterns

- Let G be a permutation group of D. We saw that the $|R|^{|D|}$ functions are divided into equivalence classes by the equivalence relation induced by G.
- Let f_1 and f_2 be two functions in the same equivalence class. Since there exists a permutation π in G, such that $f_1(d) = f_2[\pi(d)]$, for all d in D, we have $\prod_{d \in D} w[f_1(d)] = \prod_{d \in D} w[f_2(\pi(d))]$. But $\prod_{d \in D} w[f_2(\pi(d))] = \prod_{d \in D} w[f_2(d)]$ because the two products contain the same factors, only in different orders. We conclude that functions in the same equivalence class have the same weight.

  This weight is called the **weight of the pattern** (**equivalence class**).
- It is important to emphasize that functions with the same weight might not be in the same equivalence class.
- The **inventory of a set of patterns** is defined as the sum of the weights of the patterns in the set.

Example

- Find all the possible ways of painting three distinct balls in solid colors when there are three kinds of paint available, an expensive kind of red paint, a cheap kind of red paint, and blue paint.

  Let D be the set of the three balls, and let R be the set of the three kinds of paint. Let r_1, r_2 and b be the weight assigned to the expensive red paint, cheap red paint, and blue paint, respectively. The store enumerator is $r_1 + r_2 + b$. So $(r_1 + r_2 + b)^3$ gives all the possible ways in which the three balls can be painted. In other words, $(r_1 + r_2 + b)^3$ is the inventory of the set of all the functions from D to R. We get

$$\begin{aligned}(r_1 + r_2 + b)^3 &= r_1^3 + r_2^3 + b^3 + 3r_1^2 r_2 + 3r_1 r_2^2 + 3r_1^2 b + 3r_2^2 b \\ &\quad + 3r_1 b^2 + 3r_2 b^2 + 6r_1 r_2 b.\end{aligned}$$

  E.g., $3r_1 r_2^2$ means there are three ways of painting the balls in which the expensive red is used for one ball and the cheap red for two balls.

Example (Cont'd)

- Suppose we let the weights of both the expensive red paint and the cheap red paint be r and let the weight of the blue paint be b. The inventory of the set of all the functions from D to R is

$$(r+r+b)^3 = (2r+b)^3 = 8r^3 + 12r^2b + 6rb^2 + b^3.$$

  The store enumerator $2r+b$ indicates that there are two ways to paint a ball red and one way to paint a ball blue.
  In the inventory $(2r+b)^3$:
  - the term $8r^3$ means that there are eight ways in which all three balls are painted red;
  - the term $12r^2b$ means that there are 12 ways in which two balls are painted red and one ball is painted blue, etc.

Important Remarks on the Example

- In the example, the two kinds of red paints are still two distinct kinds even though they are assigned the same weight.

  E.g., painting all three balls with the expensive red paint is different from painting all three balls with the cheap red paint. They are counted as two ways of painting the balls in red.
  - Assigning to the red paints the same weight indicates we wish to look at the red paints as two kinds of paint having a common property.
  - If the two kinds of red paint are indistinguishable, i.e., there is only one kind of red paint, the store enumerator should be $r + b$ instead.

Planning a Vacation

- Eight people are planning vacation trips. There are three cities they can visit. Three of these eight people are in one family, and two of them are in another family. If the people in the same family must go together, find the ways the eight people can plan their trips.

  Let $D = \{a, b, c, d, e, f, g, h\}$ be the set of the eight people. Suppose that a, b and c are in one family, and d and e are in the other family. Let $R = \{c_1, c_2, c_3\}$ be the set of the three cities. Let α, β and γ be the weights of c_1, c_2 and c_3.

  The symbolic representation of the different trips that
  - a, b and c can take is $\alpha^3 + \beta^3 + \gamma^3$ because they will either visit c_1 together, c_2 together, or c_3 together;
  - d and e can take is $\alpha^2 + \beta^2 + \gamma^2$;
  - each of f, g and h can take is $\alpha + \beta + \gamma$.

  Therefore, the different ways in which the eight people can plan their trips are $(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)(\alpha + \beta + \gamma)^3$.

Inventories with Partition Restrictions

- Let $\{D_1, D_2, \ldots, D_k\}$ be a partition on the set D, where $D_1, D_2, \ldots, D_k$ are the disjoint subsets.
- Note that the representation of the ways to distribute the objects in the subset D_i such that they will all be in the same cell is
$$\sum_{r \in R} w(r)^{|D_i|}.$$
- Therefore, the inventory of the set of all the functions from D to R, such that the elements in the same subset will have the same value is
$$\prod_{i=1}^{k}\left[\sum_{r \in R} w(r)^{|D_i|}\right].$$

Subsection 6

Pólya's Fundamental Theorem

The Setting for Pólya's Theorem

- Let D and R be two sets, and let G be a permutation group of D.
- Our problem is to find the inventory of the equivalence classes of the functions from D to R, which is also called the **pattern inventory**.

  The pattern inventory is a representation of all the distinct ways of distributing the objects in D into the cells in R.
- We categorize the $|R|^{|D|}$ functions from D to R according to their weights.
  - Let $F_1, F_2, \ldots, F_n, \ldots$ denote the sets of functions that have weights $W_1, W_2, \ldots, W_n, \ldots$, respectively.
  - Associated with each permutation π in the group G, we define a function $\pi^{(i)}$, mapping the set of functions F_i into itself, such that a function f_1 in F_i will be mapped into the function f_2, where $f_1(d) = f_2[\pi(d)]$, for all d in D.
    Notice that f_2 is, indeed, a function in the set F_i as both f_1 and f_2 have the same weight W_i.

$\pi^{(i)}$ is a permutation of F_i

Lemma

The function $\pi^{(i)}$ is a permutation of the set of functions F_i.

- We only have to prove that no two functions in F_i are mapped into the same function by $\pi^{(i)}$.

  Suppose there are two functions f_1 and f_3 both of which are mapped into f_2 under $\pi^{(i)}$, i.e.,

  $$f_1(d) = f_2[\pi(d)] \quad \text{and} \quad f_3(d) = f_2[\pi(d)], \quad \text{for all } d \text{ in } D.$$

  This means that $f_1(d) = f_3(d)$, for all d in D. Thus, f_1 and f_3 are the same function.

Homomorphy of Permutations of F_i

Lemma

For any π_1, π_2 in G,

$$(\pi_1\pi_2)^{(i)} = \pi_1^{(i)}\pi_2^{(i)}.$$

- To prove the homomorphy condition, suppose that $\pi_2^{(i)}$ maps f_1 into f_2 and $\pi_1^{(i)}$ maps f_2 into f_3. That is, for all d in D,

$$f_1(d) = f_2[\pi_2(d)] \quad \text{and} \quad f_2(d) = f_3[\pi_1(d)].$$

It follows that

$$f_1(d) = f_2[\pi_2(d)] = f_3[\pi_1\pi_2(d)], \text{ for all } d \text{ in } D.$$

Therefore, both $\pi_1^{(i)}\pi_2^{(i)}$ and $(\pi_1\pi_2)^{(i)}$ map f_1 into f_3.

Cycles

- A **cycle** in a permutation is a subset of elements that are cyclically permuted.

  Example: In the permutation $\binom{abcdef}{cedabf}$, $\{a, c, d\}$ forms a cycle:
  - a is permuted into c;
  - c is permuted into d;
  - d is permuted into a.

  Similarly, $\{b, e\}$ forms a cycle, and $\{f\}$ forms a cycle.
- The **length** of a cycle is the number of elements in the cycle.

  Example: In the permutation $\binom{abcdef}{cedabf}$, there is a cycle of length 3, a cycle of length 2, and a cycle of length 1.

Cycle Structure Representation of a Permutation

- Let π be a permutation that has
  - b_1 cycles of length 1;
  - b_2 cycles of length 2;

    $\vdots$
  - b_k cycles of length k;

    $\vdots$

  We use $x_1, x_2, \ldots, x_k, \ldots$ as formal variables and use the monomial

  $$x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k} \cdots$$

  to represent the number of cycles of various lengths in the permutation π. Such a representation is called the **cycle structure representation** of the permutation π.

Cycle Index of a Permutation Group

- Given a permutation group G, we define the **cycle index** P_G of G as the sum of the cycle structure representations of the permutations in G divided by the number of permutations in G:

$$P_G(x_1, x_2, \ldots, x_k, \ldots) = \frac{1}{|G|} \sum_{\pi \in G} x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k} \cdots .$$

Example: The cycle index of the group consisting of

$$\binom{abcd}{abcd}, \binom{abcd}{bacd} \binom{abcd}{abdc}, \binom{abcd}{badc}$$

is

$$\frac{1}{4}(x_1^4 + x_1^2 x_2 + x_1^2 x_2 + x_2^2) = \frac{1}{4}(x_1^4 + 2x_1^2 x_2 + x_2^2).$$

Pólya's Theorem

Theorem (Pólya)

The inventory of the equivalence classes of functions from domain D to range R is

$$P_G(\sum_{r\in R} w(r), \sum_{r\in R}[w(r)]^2, \ldots, \sum_{r\in R}[w(r)]^k, \ldots),$$

i.e., the pattern inventory is obtained by substituting $\sum_{r\in R} w(r)$ for x_1, $\sum_{r\in R}[w(r)]^2$ for x_2, ..., $\sum_{r\in R}[w(r)]^k$ for x_k, ... in the expression of the cycle index P_G of the permutation group G.

- Let m_i denote the number of equivalence classes of functions that have the weight W_i (in the set F_i). Clearly, the pattern inventory is equal to $\sum_i m_i W_i$. By the preceding theorem and two lemmas, $m_i = \frac{1}{|G|}\sum_{\pi\in G}\psi(\pi^{(i)})$. Therefore,

$$\sum_i m_i W_i = \sum_i \left[\frac{1}{|G|}\sum_{\pi\in G}\psi(\pi^{(i)})\right] W_i = \frac{1}{|G|}\sum_{\pi\in G}\left[\sum_i \psi(\pi^{(i)}) W_i\right].$$

Proof of Pólya's Theorem

- We obtained
$$\sum_i m_i W_i = \frac{1}{|G|} \sum_{\pi \in G} \left[\sum_i \psi(\pi^{(i)}) W_i \right].$$
The term $\sum_i \psi(\pi^{(i)}) W_i$ is the inventory of all the functions f, such that $f(d) = f[\pi(d)]$, for all d in D.
For a function f, $f(d) = f[\pi(d)]$, for all d in D, if and only if the elements in D that are in one cycle in π have the same value under f.
Therefore,
$$\sum_i \psi(\pi^{(i)}) W_i = [\sum_{r \in R} w(r)]^{b_1} [\sum_{r \in R} w(r)^2]^{b_2} \cdots [\sum_{r \in R} w(r)^k]^{b_k} \cdots,$$
where $b_1, b_2, \ldots, b_k, \ldots$ are the number of cycles of length 1, 2, ..., k, ... in π, respectively. It follows now that
$$\sum_i m_i W_i = P_G(\sum_{r \in R} w(r), \sum_{r \in R} [w(r)]^2, \ldots, \sum_{r \in R} [w(r)]^k, \ldots).$$

Counting Number of Equivalence Classes

Corollary

The number of equivalence classes of functions from D to R is

$$P_G(|R|, |R|, \ldots, |R|, \ldots).$$

- If the weight 1 is assigned to each of the elements in R, the weight of any pattern is also equal to 1. Therefore, the pattern inventory gives the number of patterns.

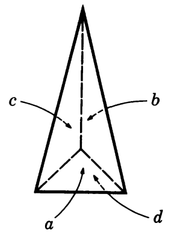
Example

- We find the number of distinct strings of three beads.
  Let $D = \{1, 2, 3\}$ be the set of the three positions, and $R = \{b, y\}$ the set of the two kinds of bead. Let $w(b) = b$ and $w(y) = y$ be the weights of the elements in R. Let $G = \{\binom{123}{123}, \binom{123}{321}\}$.
  - $\binom{123}{123}$ corresponds to leaving a string as is.
  - $\binom{123}{321}$ corresponds to interchanging the two ends of a string.

  The cycle index of the group G is
  $$P_G(x_1, x_2) = \frac{1}{2}(x_1^3 + x_1 x_2).$$
  The pattern inventory is
  $$\frac{1}{2}[(b+y)^3 + (b+y)(b^2+y^2)] = b^3 + 2b^2y + 2by^2 + y^3.$$
  We see, e.g., that there is one string that is made up of three blue beads, two strings that are made up of two blue beads and one yellow bead, and so on. By assigning $w(b) = w(y) = 1$ we find that the number of patterns is six.

Example

- Find the number of ways of painting the four faces a, b, c and d of the pyramid with two colors of paints, x and y.
  Let $D = \{a, b, c, d\}$ be the set of the four faces, and let $R = \{x, y\}$ be the set of the two colors with $w(x) = x$ and $w(y) = y$. The permutation group is $G = \{\binom{abcd}{abcd}, \binom{abcd}{bcad}, \binom{abcd}{cabd}\}$, corresponding to the identity, the counterclockwise 120° rotation and the counterclockwise 240° rotation of the pyramid around the vertical axis, respectively. Notice that in either rotation, face d remains fixed. The cycle index of the group G is $\frac{1}{3}(x_1^4 + 2x_1x_3)$. The pattern inventory is

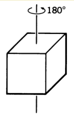


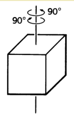
$$\frac{1}{3}[(x+y)^4 + 2(x+y)(x^3+y^3)] = x^4 + y^4 + 2x^3y + 2x^2y^2 + 2xy^3.$$

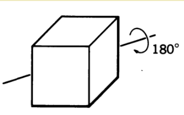
  Thus, there are eight distinct ways of painting the four faces.

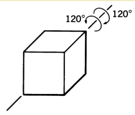
Example

- Find the distinct ways of painting the eight vertices of a cube with two colors x and y.
  Let G be the permutation group corresponding to all possible rotations of the cube. There are 24 permutations in the group:
  1. The identity permutation (cycle structure representation x_1^8).
  2. Three permutations corresponding to 180° rotations around lines connecting the centers of opposite faces (x_2^4).









  3. Six permutations corresponding to 90° rotations around lines connecting the centers of opposite faces (x_4^2).
  4. Six permutations corresponding to 180° rotations around lines connecting the midpoints of opposite edges (x_2^4).
  5. Eight permutations corresponding to 120° rotations around lines connecting opposite vertices ($x_1^2 x_3^2$).

Painting the Cube (Cont'd)

- Thus, the cycle index of the permutation group is

$$\frac{1}{24}(x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2).$$

The pattern inventory is

$$\frac{1}{24}((x+y)^8 + 9(x^2+y^2)^4 + 6(x^4+y^4)^2 + 8(x+y)^2(x^3+y^3)^2).$$

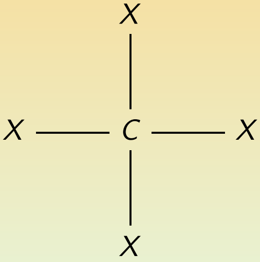
By assigning $w(x) = w(y) = 1$, we compute the number of patterns as

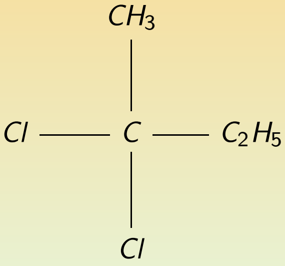
$$\frac{1}{24}[2^8 + 9 \cdot 2^4 + 6 \cdot 2^2 + 8 \cdot 2^2 \cdot 2^2] = 23.$$

This is the number of distinct ways of painting the eight vertices of a cube with two colors.

Example (Molecules)

- Consider the class of organic molecules of the form



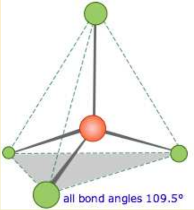


where C is a carbon atom, and each X denotes any one of the components CH_3 (methyl), C_2H_5 (ethyl), H (hydrogen), or Cl (chlorine).

For example, a typical molecule is the one on the right.

Example (Cont'd)

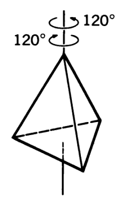
- Each such molecule can be modeled as a regular tetrahedron with the carbon atom occupying the center position and the components labeled X at the corners.

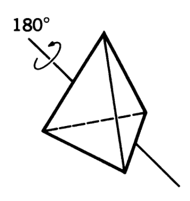


- The problem of finding the number of different molecules of this form is the same as that of finding the number of equivalence classes of functions:
  - from the domain D containing the four corners of the tetrahedron
  - to the range R containing the four components CH_3, C_2H_5, H, Cl,
  - with the permutation group G consisting of the permutations corresponding to all the possible rotations of the tetrahedron.

Example (The Cycle Index)

- To find the cycle index of the permutation group G, notice that in G:





1. There is the identity permutation (x_1^4).
2. There are eight permutations corresponding to 120° rotations around lines connecting a vertex and the center of its opposite face $(x_1 x_3)$.
3. There are three permutations corresponding to 180° rotations around lines connecting the midpoints of opposite edges (x_2^2).

It follows that $P_G = \frac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2)$.

Therefore, the number of different molecules is $P_G(4,4,4) = \frac{1}{12}(4^4 + 8 \cdot 4 \cdot 4 + 3 \cdot 4^2) = 36$.

Example (First Variation)

- Suppose we wish to find the number of molecules containing one or more hydrogen atoms.

  We assign:
  - the weight 1 to each of the components CH_3, C_2H_5, and Cl;
  - the weight 0 to the component H.

  Then, we get

$$P_G(3,3,3) = \frac{1}{12}(3^4 + 8 \cdot 3 \cdot 3 + 3 \cdot 3^2) = 15.$$

  This is the number of molecules that do not contain the hydrogen atom. Therefore, there are

$$36 - 15 = 21$$

  molecules containing the hydrogen atom.

Example (Second Variation)

- If we assign the weight 1 to each of the components CH_3, C_2H_5, Cl and the weight h to the component H, the pattern inventory is

$$\begin{aligned} P_G(h+3, h^2+3, h^3+3) \\ &= \frac{1}{12}[(h+3)^4 + 8(h+3)(h^3+3) + 3(h^2+3)^2] \\ &= h^4 + 3h^3 + 6h^2 + 11h + 15. \end{aligned}$$

  So there are:
  - one molecule containing four hydrogen atoms;
  - three molecules containing three hydrogen atoms;
  - six molecules containing two hydrogen atoms;
  - 11 molecules containing one hydrogen atom;
  - 15 molecules containing no hydrogen atoms.

Subsection 7

Generalization of Pólya's Theorem

A New Equivalence Relation on Functions

- In addition to a permutation group G of the domain D, let there be a permutation group H of the range R.
- We define a binary relation on the functions from D to R as follows:

  A function f_1 is related to a function f_2 iff there is are permutations π in G and τ in H, such that $\tau f_1(d) = f_2[\pi(d)]$, for all d in D.
- Such a binary relation is an equivalence relation:
  1. Let both π and τ be the identity permutations in G and H. It follows that each function is related to itself and the reflexive law is satisfied.
  2. Suppose that f_1 is related to f_2, i.e., $\tau f_1(d) = f_2[\pi(d)]$, for all d in D. Since π^{-1} is a permutation of D, then $f_2[\pi(\pi^{-1}(d))] = \tau f_1[\pi^{-1}(d)]$, for all d in D, i.e., $f_2(d) = \tau f_1[\pi^{-1}(d)]$, or $\tau^{-1} f_2(d) = f_1[\pi^{-1}(d)]$. Since π^{-1} is in G and τ^{-1} is in H, f_2 is related to f_1. Therefore, the symmetric law is satisfied.
  3. Suppose that f_1 is related to f_2 and f_2 to f_3. Then $\tau_1 f_1(d) = f_2[\pi_1(d)]$ and $\tau_2 f_2(d) = f_3[\pi_2(d)]$ for all d in D. Since π_1 is a permutation of D, $\tau_2 f_2(d) = f_3[\pi_2(d)]$ is the same as $\tau_2 f_2[\pi_1(d)] = f_3[\pi_2(\pi_1(d))]$, for all d in D. Thus, $\tau_2 \tau_1(d) = f_3[\pi_2 \pi_1(d)]$. Since both $\pi_2 \pi_1$ and $\tau_2 \tau_1$ are in G and H, respectively, f_1 is related to f_3, and the transitive law holds.

Equivalent Functions May Have Different Weights

- Such an equivalence relation divides the functions from D to R into equivalence classes.
- However, if we assign weights to the elements in R and compute the weights of the $|R|^{|D|}$ functions from D to R, we see that two functions in the same equivalence class may not have the same weight.

  Example: Let $D = \{a, b\}$ and $R = \{x, y\}$. Suppose that the permutation group G of the domain D contains the permutations $\pi_1 = \binom{ab}{ab}$ and $\pi_2 = \binom{ab}{ba}$. Suppose that the permutation group H of the range R contains the permutations $\tau_1 = \binom{xy}{xy}$ and $\tau_2 = \binom{xy}{yx}$. Clearly, the function f_1 with $f_1(a) = x$ and $f_1(b) = x$, and the function f_2, with $f_2(a) = y$ and $f_2(b) = y$, are equivalent because $\tau_2 f_1(d) = f_2[\pi_1(d)]$, for all d in D. However, for the assignment of weights $w(x) = x$ and $w(y) = y$, the weights of the functions f_1 and f_2 are x^2 and y^2, respectively.

Adjusting the Framework

- To be able to talk about the weight of a pattern and the pattern inventory, we must impose the additional condition that weights should be assigned to the elements in the range R in such a way that functions in the same equivalence class will have the same weight.
- We limit our discussion to the counting of the number of equivalence classes of functions. So we assign the weight 1 to each element in R. Since the weight of any function is then equal to 1, the condition that the weights of the functions in the same equivalence class are the same is trivially satisfied.
- As a consequence, the pattern inventory will be the number of equivalence classes.

Number of Equivalence Classes

Theorem

The number of equivalence classes of functions from D to R is given by

$$\frac{1}{|G|}\frac{1}{|H|}\sum_{\substack{\pi\in G\\ \tau\in H}}\psi[(\pi,\tau)'],$$

where $\psi[(\pi,\tau)']$ is the number of function f which are such that $\tau f(d)=f[\pi(d)]$, for all d in D.

- Let $G\times H$ be the set of $|G||H|$ ordered pairs for (π,τ), where π is a permutation in G and τ is a permutation in H. Let a binary operation $*$ on $G\times H$ be defined such that

$$(\pi_1,\tau_1)*(\pi_2,\tau_2)=(\pi_1\pi_2,\tau_1\tau_2).$$

$G\times H$ is a group under the binary operation $*$.

Number of Equivalence Classes (Cont'd)

- Associated with each (π, τ) in $G \times H$, we define a function $(\pi, \tau)'$, mapping the set of functions from D to R into the set itself, such that a function f_1 is mapped into a function f_2, where $\tau f_1(d) = f_2[\pi(d)]$, for all d in D. Clearly, $(\pi, \tau)'$ is a permutation.

  Claim: The homomorphy condition $(\pi_1\pi_2, \tau_1\tau_2)' = (\pi_1, \tau_1)'(\pi_2, \tau_2)'$ is satisfied.

  Thus, the number of equivalence classes into which the functions from D to R are divided by the equivalence relation induced by the group $G \times H$ is

$$\frac{1}{|G|}\frac{1}{|H|}\sum_{\substack{\pi \in G \\ \tau \in H}} \psi[(\pi, \tau)'],$$

  where $\psi[(\pi, \tau)']$ is the number of invariances of the permutation $(\pi, \tau)'$. This is exactly the number of functions f, such that $\tau f(d) = f[\pi(d)]$, for all d in D.

Invariance Under $(\pi, \tau)'$ and Cycle Decompositions

Lemma

A function f from D to R is invariant under the permutation $(\pi, \tau)'$ if and only if f maps the elements of D that are in a cycle of length i in π into the elements of R that are in a cycle of length j in τ, with j being a divisor of i. Moreover, within these two cycles, there must be a cyclic correspondence between the elements, i.e., if $f(d) = r$, then

$$f[\pi(d)] = \tau(r), f[\pi^2(d)] = \tau^2(r), \ldots, f[\pi^{i-1}(d)] = \tau^{i-1}(r).$$

- That a function that satisfies these conditions is invariant under $(\pi, \tau)'$ is clear.
  If f is invariant under $(\pi, \tau)'$, then $f(d) = r$ implies $f[\pi(d)] = \tau(r)$. It follows that $f[\pi(\pi(d))] = \tau f[\pi(d)]$, which can be rewritten as $f[\pi^2(d)] = \tau^2(r)$. Similarly, $f[\pi^3(d)] = \tau^3(r)$, $f[\pi^4(d)] = \tau^4(r)$, ..., $f[\pi^{i-1}(d)] = \tau^{i-1}(r)$, and $f[\pi^i(d)] = \tau^i(r)$. Since $\pi^i(d) = d$, it follows that $\tau^i(r) = r$, whence i must be a multiple of j.

Enumeration Using Cycle Indices

Theorem

The number of equivalence classes of functions from D to R is the value of

$$P_G(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots) \times P_H[e^{z_1+z_2+\cdots}, e^{2(z_1+z_2+\cdots)}, e^{3(z_1+z_2+\cdots)}, \ldots]$$

evaluated at $z_1 = z_2 = z_3 = \cdots = 0$.

- In view of the preceding theorem, we have only to evaluate $\psi[(\pi, \tau)']$. According to the preceding lemma, for a function that is invariant under the permutation $(\pi, \tau)'$, the elements in a cycle of length i in π must be mapped into elements in a cycle of length j in τ, with j a divisor of i. Let b_i denote the number of cycles of length i in π, and let c_j denote the number of cycles of length j in τ.
  - The elements in a cycle of length i in π can be mapped into the elements in any one of the c_j cycles of length j in τ.
  - For a cycle of length j in τ, there are j different ways in which a cyclic correspondence between the i elements in π and the j elements in τ can exist.

Enumeration Using Cycle Indices (Cont'd)

- Therefore, we have $\psi[(\pi,\tau)'] = \prod_i (\sum_{j:j|i} jc_j)^{b_i} = (c_1)^{b_1}(c_1+2c_2)^{b_2}(c_1+3c_3)^{b_3}(c_1+2c_2+4c_4)^{b_4}(c_1+5c_5)^{b_5}\cdots$. But we have

$$\begin{array}{rcl}
(c_1)^{b_1} & = & (\frac{\partial}{\partial z_1})^{b_1} e^{c_1 z_1} \mid_{z_1=0}; \\
(c_1+2c_2)^{b_2} & = & (\frac{\partial}{\partial z_2})^{b_2} e^{c_1 z_2} e^{2c_2 z_2} \mid_{z_2=0}; \\
(c_1+3c_3)^{b_3} & = & (\frac{\partial}{\partial z_3})^{b_3} e^{c_1 z_3} e^{3c_3 z_3} \mid_{z_3=0}; \\
(c_1+2c_2+4c_4)^{b_4} & = & (\frac{\partial}{\partial z_4})^{b_2} e^{c_1 z_4} e^{2c_2 z_4} e^{4c_4 z_4} \mid_{z_4=0}; \\
 & \vdots &
\end{array}$$

Therefore

$$\begin{array}{l}
\psi[(\pi,\tau)'] = [(\frac{\partial}{\partial z_1})^{b_1}(\frac{\partial}{\partial z_2})^{b_2}\cdots(\frac{\partial}{\partial z_k})^{b_k}\cdots]\times \\
\qquad [e^{c_1(z_1+z_2+z_3+\cdots)} e^{2c_2(z_2+z_4+z_6+\cdots)} e^{3c_3(z_3+z_6+z_9+\cdots)}\ldots \\
\qquad\qquad e^{mc_m(z_m+z_{2m}+z_{3m}+\cdots)}\ldots]\Big|_{z_1=z_2=\cdots=z_m=\cdots=0}.
\end{array}$$

2×2 Chessboards With Permutations and Contrasts

- We revisit the example of the 2×2 chessboards.

  Let $D = \{a, b, c, d\}$ be the set of the four cells. Let $R = \{x, y\}$ be the set of the two colors white and black. Let $G = \{\binom{abcd}{abcd}, \binom{abcd}{bcda}, \binom{abcd}{cdab}, \binom{abcd}{dabc}\}$, where the permutations correspond to the rotations of the chessboards. When we are interested only in the contrast patterns of the chessboards, we also have $H = \{\binom{xy}{xy}, \binom{xy}{yx}\}$, where the permutation $\binom{xy}{yx}$ means the interchange of the two colors x and y.

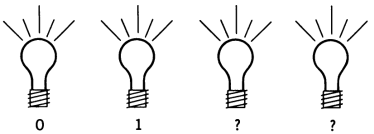
  We have $P_G = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4)$ and $P_H = \frac{1}{2}(x_1^2 + x_2)$. Thus, the number of distinct contrast patterns is

$$\frac{1}{8}\left(\frac{\partial^4}{\partial z_1^4} + \frac{\partial^2}{\partial z_2^2} + 2\frac{\partial}{\partial z_4}\right)\left[e^{2(z_1+z_2+z_3+z_4)} + e^{2(z_2+z_4)}\right]\Big|_{z_1=z_2=z_3=z_4=0}$$
$$= \frac{1}{8}[2^4 + (2^2 + 2^2) + 2 \cdot (2+2)] = 4.$$

Transmission of Messages: The Setup

- A certain number of messages are to be represented by n-digit quaternary sequences and transmitted through a communication channel. For each of the digits 0, 1, 2 and 3 received, a corresponding indicator light will be flashed so that the transmitted sequence can be recorded.

  The indicator lights for the digits 2 and 3 are not labeled and there is no way to tell which one of the two digits was transmitted.

  Therefore, we cannot expect to use all the 4^n n-digit sequences to represent 4^n distinct messages. E.g., we cannot distinguish 011023 and 011032. But 011022 and 011032 are distinguishable:
  - When the last two digits of the sequence 011022 are received, one of the two unlabeled lights will flash twice;
  - When the last two digits of the sequence 011032 are received, each of the two unlabeled lights will flash once.

Transmission of Messages: Counting Distinct Messages

- Let $D = \{a_1, a_2, a_3, \ldots, a_n\}$ be the set of the n positions in the n-digit quaternary sequences. Let $R = \{0, 1, 2, 3\}$ be the set of the four digits. Then:
  - the permutation group of D is
$$G = \{\begin{pmatrix} a_1 a_2 \cdots a_n \\ a_1 a_2 \cdots a_n \end{pmatrix}\};$$
  - the permutation group of R is
$$H = \{\begin{pmatrix} 0123 \\ 0123 \end{pmatrix}, \begin{pmatrix} 0123 \\ 0132 \end{pmatrix}\}.$$

The number of distinct messages one can transmit is
$$\frac{1}{2}(\frac{\partial^n}{\partial z_1^n})(e^{4z_1} + e^{2z_1})\Big|_{z_1=0} = \frac{1}{2}(4^n + 2^n).$$

Transmission of Messages: Errors

- Suppose that at the transmitting end a sequence occasionally will be transmitted with the first two digits interchanged.
  Since there is no way to signal the receiver when this happens, how many distinct messages can be transmitted?
  In this case,
  $$\begin{aligned} G &= \left\{\binom{a_1 a_2 \cdots a_n}{a_1 a_2 \cdots a_n}, \binom{a_1 a_2 a_3 \cdots a_n}{a_2 a_1 a_3 \cdots a_n}\right\}; \\ H &= \left\{\binom{0123}{0123}, \binom{0123}{0132}\right\}. \end{aligned}$$
  The number of distinct messages that can be transmitted is then
  $$\begin{aligned} &\frac{1}{4}\left(\frac{\partial^n}{\partial z_1^n} + \frac{\partial^{n-2}}{\partial z_1^{n-2}}\frac{\partial}{\partial z_2}\right)[e^{4(z_1+z_2)} + e^{2(z_1+z_2)}e^{2z_2}]\Big|_{z_1=z_2=0} \\ &= \frac{1}{4}[(4^n + 2^n) + 4 \cdot (4^{n-2} + 2^{n-2})] \\ &= \frac{1}{4}(4^n + 2^n + 4^{n-1} + 2^n) \\ &= \frac{1}{4}(4^n + 4^{n-1} + 2^{n+1}) \end{aligned}$$

Distributing Books to Children

- In how many ways can five books, two of which are the same, be distributed to four children, if among them there is a set of identical twins?
  Let $D = \{a, b, c, d, e\}$ be the set of the five books with a and b being the two copies of the same book. Then the permutation group of D is $G = \{\binom{abcde}{abcde}, \binom{abcde}{bacde}\}$.
  Let $R = \{u, v, x, y\}$ be the set of the four children with u and v being the twins. Then the permutation group of R is $H = \{\binom{uvxy}{uvxy}, \binom{uvxy}{vuxy}\}$.
  The number of distinct patterns from D to R is

$$\begin{aligned}
&\frac{1}{4}\left(\frac{\partial^5}{\partial z_1^5} + \frac{\partial^3}{\partial z_1^3}\frac{\partial}{\partial z_2}\right)[e^{4(z_1+z_2)} + e^{2(z_1+z_2)}e^{2z_2}]\Big|_{z_1=z_2=0} \\
&= \frac{1}{4}[(4^5 + 2^5) + 4 \cdot (4^3 + 2^3)] \\
&= \frac{1}{4}(4^5 + 2^5 + 4^4 + 4 \cdot 2^3) \\
&= 336.
\end{aligned}$$

Interchangeability versus Indistinguishability

- A opposed to the case in which objects are either distinct or totally indistinguishable, Pólya's theory allows interchangeability under permutations, which does not always mean indistinguishability.
- Only when G is the group that contains all the possible permutations of certain elements do the elements become totally indistinguishable:
- Let $D = \{a, b, c\}$, $G = \{\binom{abc}{abc}, \binom{abc}{bca}, \binom{abc}{cab}, \binom{abc}{bac}, \binom{abc}{cba}, \binom{abc}{acb}\}$. Let $R = \{x, y\}$ and $H = \{\binom{xy}{xy}, \binom{xy}{yx}\}$. The number of equivalence classes of functions from D to R is

$$\frac{1}{12}\left(\frac{\partial^3}{\partial z_1^3} + 2\frac{\partial}{\partial z_3} + 3\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}\right)[e^{2(z_1+z_2+z_3)} + e^{2z_2}]\Big|_{z_1=z_2=z_3=0}$$
$$= \frac{1}{12}(2^3 + 2 \cdot 2 + 3 \cdot 2 \cdot 2) = 2.$$

  This was expected. The number of ways of distributing three indistinguishable objects into two indistinguishable cells is two:
  - three in one cell, none in the other;
  - two in one cell, one in the other.