Introduction to Complex Analysis

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1. Analytic Functions
   - Differentiability and Analyticity
   - Cauchy-Riemann Equations
   - Harmonic Functions

Subsection 1

Differentiability and Analyticity

Complex versus Real Function Calculus

- The calculus of complex functions deals with the usual concepts of derivatives and integrals of these functions.
- We shall present, next, the limit definition of the derivative of a complex function $f(z)$.
- Many of the concepts seem familiar, such as the product, quotient, and chain rules of differentiation, but there are important differences between the calculus of complex and of real functions $f(x)$.
- In essence, apart for the familiarity of names and definitions, there is little similarity between the interpretations of quantities such as $f'(x)$ and $f'(z)$.

Derivative of Complex Function

- Suppose $z = x + iy$ and $z_0 = x_0 + iy_0$. Then the change in z_0 is the difference $\Delta z = z - z_0$ or $\Delta z = x - x_0 + i(y - y_0) = \Delta x + i\Delta y$.
- If a complex function $w = f(z)$ is defined at z and z_0, then the corresponding change in w is the difference $\Delta w = f(z_0 + \Delta z) - f(z_0)$.

Definition (Derivative of Complex Function)

Suppose the complex function f is defined in a neighborhood of a point z_0. The **derivative** of f at z_0, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

provided this limit exists.

- If the limit exists, then f is said to be **differentiable** at z_0.
- Two other symbols denoting the derivative of $w = f(z)$ are w' and $\frac{dw}{dz}$. In the latter notation, the value of $\frac{dw}{dz}$ at z_0 is written $\left.\frac{dw}{dz}\right|_{z=z_0}$.

Example

- Use the definition to find the derivative of $f(z) = z^2 - 5z$.

  To compute the derivative of f at any point z, we replace z_0 by the symbol z:

  $$f(z+\Delta z) = (z+\Delta z)^2 - 5(z+\Delta z) = z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z.$$

  $$\begin{aligned} f(z+\Delta z) - f(z) &= z^2 + 2z\Delta z + (\Delta z)^2 - 5z - 5\Delta z \\ &\quad - (z^2 - 5z) \\ &= 2z\Delta z + (\Delta z)^2 - 5\Delta z. \end{aligned}$$

  Finally, we get

  $$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2 - 5\Delta z}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z - 5)}{\Delta z} \\ &= \lim_{\Delta z \to 0} (2z + \Delta z - 5). \end{aligned}$$

  The limit is $f'(z) = 2z - 5$.

Differentiation Rules

Differentiation Rules

- **Constant Rules:** $\frac{d}{dz}c = 0$ and $\frac{d}{dz}cf(z) = cf'(z)$;
- **Sum Rule:** $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$;
- **Product Rule:** $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$;
- **Quotient Rule:** $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$;
- **Chain Rule:** $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$.

- The **power rule** for differentiation of powers of z is also valid:
$$\frac{d}{dz}z^n = nz^{n-1}, \ n \text{ an integer.}$$
- Therefore, we also have the **power rule for functions**:
$$\frac{d}{dz}[g(z)]^n = n[g(z)]^{n-1}g'(z), \ n \text{ an integer.}$$

Using the Rules of Differentiation

- Differentiate:
  (a) $f(z) = 3z^4 - 5z^3 + 2z$
  (b) $f(z) = \dfrac{z^2}{4z+1}$
  (c) $f(z) = (iz^2 + 3z)^5$

(a) $f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 \cdot 1 = 12z^3 - 15z^2 + 2.$

(b) $f'(z) = \dfrac{2z \cdot (4z+1) - z^2 \cdot 4}{(4z+1)^2} = \dfrac{4z^2 + 2z}{(4z+1)^2}.$

(c) $f'(z) = 5(iz^2 + 3z)^4 \dfrac{d}{dz}(iz^2 + 3z) = 5(iz^2 + 3z)^4(2iz + 3).$

Complex Differentiability

- For a complex function f to be differentiable at a point z_0, we know from the preceding chapter that the limit $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ must exist and equal the same complex number from any direction, i.e., the limit must exist regardless how Δz approaches 0.
- In complex analysis, the requirement of differentiability of a function $f(z)$ at a point z_0 is a far greater demand than in real calculus of functions $f(x)$ where we can approach a real number x_0 on the number line from only two directions.
- If a complex function is made up by specifying its real and imaginary parts u and v, such as $f(z) = x + 4iy$, there is a good chance that it is not differentiable.

A Nowhere Differentiable Complex Function

- The function $f(z) = x + 4iy$ is not differentiable at any point z.
  Let z be any point in the complex plane. With $\Delta z = \Delta x + i\Delta y$,
  $f(z+\Delta z) - f(z) = (x+\Delta x) + 4i(y+\Delta y) - x - 4iy = \Delta x + 4i\Delta y$
  and so $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$.
  - If we let $\Delta z \to 0$ along a line parallel to the x-axis, then $\Delta y = 0$, $\Delta z = \Delta x$ and $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1$.
  - If we let $\Delta z \to 0$ along a line parallel to the y-axis, then $\Delta x = 0$, and $\Delta z = i\Delta y$, so that $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{4i\Delta y}{i\Delta y} = 4$.

  Since the two values are different, $f(z) = x + 4iy$ is nowhere differentiable, i.e., f is not differentiable at any point z.

Analytic Functions

- There is an important class of functions whose members satisfy even more severe requirements than just differentiability.

Definition (Analyticity at a Point)

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

- A function f is **analytic in a domain** D if it is analytic at every point in D. Sometimes "analytic *on* a domain D" is also used.
- A function f that is analytic throughout a domain D is called **holomorphic** or **regular**.

Analyticity versus Differentiability

- It is very important to notice that analyticity at a point is not the same as differentiability at a point:
  - Analyticity at a point is a neighborhood property, i.e., analyticity is a property that is defined over an open set.
- Example: The function $f(z) = |z|^2$ is differentiable at $z = 0$ but is not differentiable anywhere else. Even though $f(z) = |z|^2$ is differentiable at $z = 0$, it is not analytic at $z = 0$ because there exists no neighborhood of $z = 0$ throughout which f is differentiable. Hence the function $f(z) = |z|^2$ is nowhere analytic.
- Example: The simple polynomial $f(z) = z^2$ is differentiable at every point z in the complex plane. Hence, $f(z) = z^2$ is analytic everywhere.

Entire Functions

- A function that is analytic at every point z in the complex plane is said to be an **entire function**.
- The differentiation rules allow us to conclude that:
  - Polynomial functions are differentiable at every point z in the complex plane;
  - Rational functions are analytic throughout any domain D that contains no points at which the denominator is zero.

Theorem (Polynomial and Rational Functions)

(i) A polynomial function

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where n is a nonnegative integer, is an entire function.

(ii) A rational function $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Singular Points

- Since the rational function
$$f(z) = \frac{4z}{z^2 - 2z + 2}$$
is discontinuous at $1+i$ and $1-i$, f fails to be analytic at $1 \pm i$.
By the preceding theorem, f is not analytic in any domain containing one or both of these points.
- In general, a point z at which a complex function $w = f(z)$ fails to be analytic is called a **singular point** of f.

Analyticity of Sum, Product, and Quotient

If the functions f and g are analytic in a domain D, then:

- The sum $f(z) + g(z)$, difference $f(z) - g(z)$, and product $f(z)g(z)$ are analytic.
- The quotient $\frac{f(z)}{g(z)}$ is analytic provided $g(z) \neq 0$ in D.

An Alternative Definition of $f'(z)$

- Since $\Delta z = z - z_0$, then $z = z_0 + \Delta z$. Thus, we get

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

- If we wish to compute f' at a general point z, then we replace z_0 by the symbol z after the limit is computed.

Theorem (Differentiability Implies Continuity)

If f is differentiable at a z_0 in a domain D, then f is continuous at z_0.

- The limits $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ and $\lim_{z \to z_0}(z - z_0)$ exist and equal $f'(z_0)$ and 0, respectively. Hence, we can write $\lim_{z \to z_0}(f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \cdot (z - z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \cdot \lim_{z \to z_0}(z - z_0) = f'(z_0) \cdot 0 = 0$. From $\lim_{z \to z_0}(f(z) - f(z_0)) = 0$, we conclude that $\lim_{z \to z_0} f(z) = f(z_0)$. Thus, f is continuous at z_0.

L'Hôpital's Rule

- The converse of the preceding theorem is not true, i.e., continuity of a function f at a point does not guarantee that f is differentiable at the point.
- Example: The simple function $f(z) = x + 4iy$ is continuous everywhere because the real and imaginary parts of f, $u(x, y) = x$ and $v(x, y) = 4y$ are continuous at any point (x, y). Yet we have seen that $f(z) = x + 4iy$ is not differentiable at any point z.
- L'Hôpital's rule for computing limits of the indeterminate form $0/0$, carries over to complex analysis:

Theorem (L'Hôpital's Rule)

Suppose f and g are functions that are analytic at a point z_0 and $f(z_0) = 0$, $g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Applying L'Hôpital's Rule I

- Compute $\lim_{z \to 2+i} \dfrac{z^2 - 4z + 5}{z^3 - z - 10i}$

  Let $f(z) = z^2 - 4z + 5$ and $g(z) = z^3 - z - 10i$. Then $f(2+i) = 0$ and $g(2+i) = 0$. Thus, the given limit has the indeterminate form $0/0$. Since f and g are polynomial functions, both functions are necessarily analytic at $z_0 = 2 + i$. We also have $f'(z) = 2z - 4$, $g'(z) = 3z^2 - 1$, $f'(2+i) = 2i$, $g'(2+i) = 8 + 12i$. Therefore, $\lim_{z \to 2+i} \dfrac{z^2 - 4z + 5}{z^3 - z - 10i} = \dfrac{f'(2+i)}{g'(2+i)} = \dfrac{2i}{8 + 12i} = \dfrac{3}{26} + \dfrac{1}{13}i.$

Applying L'Hôpital's Rule II

- In a preceding example, we used factoring and cancelation to compute the limit
$$\lim_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i}.$$
This limit also has the indeterminate form $0/0$.
With $f(z)=z^2-2z+4$, $g(z)=z-1-\sqrt{3}i$, we have $f'(z)=2z-2$, and $g'(z)=1$. L'Hôpital's Rule gives
$$\lim_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = \frac{f'(1+\sqrt{3}i)}{1} = 2(1+\sqrt{3}i-1) = 2\sqrt{3}i.$$

Interpreting the Derivative

- In real calculus the derivative of a function $y = f(x)$ at a point x has many interpretations.
  - $f'(x)$ is the slope of the tangent line to the graph of f at $(x, f(x))$. When the slope is positive, negative, or zero, the function, in turn, is increasing, decreasing, and possibly has a maximum or minimum.
  - Also, $f'(x)$ is the instantaneous rate of change of f at x. In a physical setting, this rate can be interpreted as velocity of a moving object.
- None of these interpretations carry over to complex calculus.
- In complex analysis the primary concern is not what a derivative is or represents, but rather, whether a function f has a derivative.
- The fact that a complex function f possesses a derivative tells us a lot about the function.
- E.g., in the theory of mappings by complex functions: Under a mapping defined by an analytic function f, the magnitude and sense of an angle between two curves that intersect a point z_0 in the z-plane is preserved in the w-plane at all points at which $f'(z) \neq 0$.

Some Differences With Real Analysis

- $f(z) = |z|^2$ is differentiable only at $z = 0$, but $f(x) = |x|^2$ is differentiable everywhere. $f(x) = x$ is differentiable everywhere, but $f(z) = x = \text{Re}(z)$ is nowhere differentiable.
- The differentiation formulas are important, but not as important as in real analysis. In complex analysis we deal with functions such as $f(z) = 4x^2 - iy$ and $g(z) = xy + i(x + y)$, which, even if they possess derivatives, cannot be differentiated by those formulas.
- Higher-order derivatives of complex functions are defined in exactly the same manner as in real analysis.
  - In real analysis, if a function f possesses a first derivative, there is no guarantee that f possesses any other higher derivatives.
  - In complex analysis, if a function f is analytic in a domain D, then, by assumption, f possesses a derivative at each point in D and, we will see that this fact alone guarantees that f possesses higher-order derivatives at all points in D. Indeed, an analytic function f on a domain D is infinitely differentiable in D.

Real Analyticity and L'Hôpital's Rule

- The definition of "analytic at a point a" in real analysis differs from the usual definition of that concept in complex analysis.
  - In real analysis, analyticity of a function is defined in terms of power series: A function $y = f(x)$ is analytic at a point a if f has a Taylor series at a that represents f in some neighborhood of a.
- As in real calculus, it may be necessary to apply L' Hôpital's rule several times in succession to calculate a limit. In other words, if $f(z_0)$, $g(z_0)$, $f'(z_0)$, and $g'(z_0)$ are all zero, the limit $\lim_{z \to z_0} \frac{f(z)}{g(z)}$ may still exist. In general, if f,g, and their first $n-1$ derivatives are zero at z_0 and $g^{(n)}(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)}.$$

Subsection 2

Cauchy-Riemann Equations

Revisiting Analyticity and Differentiability

- We saw that a function f of a complex variable z is analytic at a point z when f is differentiable at z and differentiable at every point in some neighborhood of z.
- We emphasized that this requirement is more stringent than just differentiability at a point because a complex function can be differentiable at a point z but yet be differentiable nowhere else.
- A function f is analytic in a domain D if f is differentiable at all points in D.
- We now present a test for analyticity of a complex function

$$f(z) = u(x, y) + iv(x, y)$$

based on partial derivatives of its real and imaginary parts u and v.

The Cauchy-Riemann Equations

Theorem (Cauchy-Riemann Equations)

Suppose $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- The derivative of f at z is given by $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$. By writing $f(z) = u(x,y) + iv(x,y)$ and $\Delta z = \Delta x + i\Delta y$, we get

  $$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}.$$

  Since the limit is assumed to exist, Δz can approach zero from any convenient direction.

The Cauchy-Riemann Equations (Cont'd)

- In particular, if we choose to let $\Delta z \to 0$ along a horizontal line, then $\Delta y = 0$ and $\Delta z = \Delta x$. We then get $f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y)-u(x,y)+i[v(x+\Delta x,y)-v(x,y)]}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y)-u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$. The existence of $f'(z)$ implies that each limit exists. These limits are the definitions of the first-order partial derivatives with respect to x of u and v, respectively. Hence, we have shown that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exist at the point z, and that the derivative of f is $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$.
- We now let $\Delta z \to 0$ along a vertical line. With $\Delta x = 0$ and $\Delta z = i\Delta y$, we get $f'(z) = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}$. In this case, we obtain that $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ exist at z and that $f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$.

Equate real and imaginary parts to obtain the Cauchy-Riemann Equations.

Application of the Equations

- The Cauchy-Riemann equations hold at z as a necessary consequence of f being differentiable at z.
- Thus, even though we cannot use the theorem to determine where f is differentiable, it can tell us where f does not possess a derivative:

  If the equations are not satisfied at a point z, then f cannot be differentiable at z.

- Example: We saw that $f(z) = x + 4iy$ is not differentiable at any point z. If we identify $u = x$ and $v = 4y$, then

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 4, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0.$$

  In view of $\frac{\partial u}{\partial x} = 1 \neq 4 = \frac{\partial v}{\partial y}$ the Cauchy-Riemann equations cannot be satisfied at any point z. Thus, f is nowhere differentiable.

- Note that, if a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic throughout a domain D, then the real functions u and v satisfy the Cauchy-Riemann equations at every point in D.

Verifying the Equations

- The polynomial function $f(z) = z^2 + z$ is analytic for all z and can be written in terms of x, y as $f(z) = x^2 - y^2 + x + i(2xy + y)$. Thus, $u(x, y) = x^2 - y^2 + x$ and $v(x, y) = 2xy + y$. For any point (x, y) in the complex plane, we see that the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Criterion for Non-analyticity

Criterion for Non-analyticity

If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x, y) + iv(x, y)$ cannot be analytic in D.

- Example: Show that the complex function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

  We identify $u(x, y) = 2x^2 + y$ and $v(x, y) = y^2 - x$. From $\frac{\partial u}{\partial x} = 4x$, $\frac{\partial v}{\partial y} = 2y$, $\frac{\partial u}{\partial y} = 1$ and $\frac{\partial v}{\partial x} = -1$. we see that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, but that the equality $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ is satisfied only on the line $y = 2x$. However, for any point z on the line, there is no neighborhood or open disk about z in which f is differentiable at every point. We conclude that f is nowhere analytic.

A Sufficient Condition for Analyticity

- The Cauchy-Riemann equations are not sufficient for analyticity of a function $f(z) = u(x, y) + iv(x, y)$ at a point $z = x + iy$: It is possible for the Cauchy-Riemann equations to be satisfied at z without $f(z)$ being differentiable at z, or, with $f(z)$ being differentiable at z, but nowhere else. In either case, f is not analytic at z.
- However, when we add the condition of continuity to u and v and to the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$, it can be shown that the Cauchy-Riemann equations are not only necessary but also sufficient to guarantee analyticity of $f(z) = u(x, y) + iv(x, y)$ at z.

Theorem (Criterion for Analyticity)

Suppose the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

- The proof is long and complicated and we omit it.

An Application of the Theorem

- For the function $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$, the real functions $u(x,y) = \frac{x}{x^2+y^2}$ and $v(x,y) = -\frac{y}{x^2+y^2}$ are continuous except at the point where $x^2+y^2=0$, i.e., at $z=0$. Moreover, the first four first-order partial derivatives $\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$, $\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$, $\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$ and $\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$ are continuous except at $z=0$. Finally, we see from $\frac{\partial u}{\partial x} = \frac{y^2x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$ that the Cauchy-Riemann equations are satisfied except at $z=0$. Thus, we conclude that f is analytic in any domain D that does not contain the point $z=0$.

Formulas for $f'(z)$

- The components of the Cauchy Riemann Equations were obtained under the assumption that f was differentiable at the point z.
- They provide a formula for computing the derivative $f'(z)$:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

- Example: We know that $f(z) = z^2$ is entire and so is differentiable for all z. With $u(x, y) = x^2 - y^2$, $\frac{\partial u}{\partial x} = 2x$, $v(x, y) = 2xy$, and $\frac{\partial v}{\partial x} = 2y$, we have $f'(z) = 2x + i2y = 2(x + iy) = 2z$.

Sufficient Conditions for Differentiability

- Recall that analyticity implies differentiability but not conversely. The following is a criterion for differentiability:

Sufficient Conditions for Differentiability

If the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point z, and if u and v satisfy the Cauchy-Riemann equations at z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f'(z)$ is given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Application of the Sufficient Conditions

- Example: We saw that the complex function

$$f(z) = 2x^2 + y + i(y^2 - x)$$

is nowhere analytic, but yet the Cauchy-Riemann equations were satisfied on the line $y = 2x$. Since the functions $u(x, y) = 2x^2 + y$, $\frac{\partial u}{\partial x} = 4x$, $\frac{\partial u}{\partial y} = 1$, $v(x, y) = y^2 - x$, $\frac{\partial v}{\partial x} = -1$ and $\frac{\partial v}{\partial y} = 2y$ are continuous at every point, it follows that f is differentiable on the line $y = 2x$. Moreover, the derivative of f at points on this line is given by $f'(z) = 4x - i = 2y - i$.

Theorem (Constant Functions)

Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D.

(i) If $|f(z)|$ is constant in D, then so is $f(z)$.

(ii) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a constant.

Polar Coordinates

- We saw that a complex function can be expressed in terms of polar coordinates in the form $f(z) = u(r,\theta) + iv(r,\theta)$.
- In polar coordinates the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

- The polar version of $f'(z)$ at a point z whose polar coordinates are (r,θ) is then

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right).$$

- Remarks: In real calculus, one of the noteworthy properties of the exponential function $f(x) = e^x$ is that $f'(x) = e^x$.
  We gave the definition of the complex exponential $f(z) = e^z$. We can now show that $f(z) = e^z$ is differentiable everywhere and shares the same derivative property $f'(z) = f(z)$.

Subsection 3

Harmonic Functions

A Preview of Harmonic Functions

- We will see that when a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point z, then all the derivatives of f : $f'(z), f''(z)$, $f'''(z)$, etc., are also analytic at z. Thus, all partial derivatives of the real functions $u(x,y)$ and $v(x,y)$ are continuous at z. So the second-order mixed partial derivatives are equal.
- This last fact, coupled with the Cauchy-Riemann equations, will be used now to demonstrate that there is a connection between the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ and the second-order partial differential equation
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$
- This equation is known as **Laplace's Equation** in two variables.
- The sum $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ of the two second partial derivatives is denoted by $\nabla^2 \phi$ and is called the **Laplacian** of ϕ.
- Thus, Laplace's equation is written $\nabla^2 \phi = 0$.

Harmonic Functions

- A solution $\phi(x, y)$ of Laplaces equation in a domain D of the plane is given a special name:

Definition (Harmonic Function)

A real-valued function ϕ of two real variables x and y that has continuous first and second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

Theorem (Harmonic Functions)

Suppose the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Then the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

- Assume $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D and that u and v have continuous second-order partial derivatives in D. Since f is analytic, the Cauchy-Riemann equations are satisfied at every point z.

Harmonic Functions(Cont'd)

- Differentiating both sides of $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ with respect to x, we get $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$. Differentiating both sides of $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with respect to y gives $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$. With the assumption of continuity, the mixed partials $\frac{\partial^2 v}{\partial x \partial y}$ and $\frac{\partial^2 v}{\partial y \partial x}$ are equal. Hence, by adding the two equations we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ or $\nabla^2 u = 0$. This shows that $u(x, y)$ is harmonic.

  Now differentiating both sides of $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ with respect to y, we get $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$. Differentiating both sides of $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with respect to x gives $\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial^2 x}$. Subtracting the last two equations yields $\nabla^2 v = 0$.
- Example: The function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is entire. Thus, the functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ are necessarily harmonic in any domain D of the complex plane.

Harmonic Conjugate Functions

- If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its real and imaginary parts u and v are necessarily harmonic in D.
- Now suppose $u(x, y)$ is a given real function that is known to be harmonic in D. If it is possible to find another real harmonic function $v(x, y)$ so that u and v satisfy the Cauchy-Riemann equations throughout the domain D, then the function $v(x, y)$ is called a **harmonic conjugate** of $u(x, y)$.
- By combining the functions as $u(x, y) + iv(x, y)$, we obtain a function that is analytic in D.

Example of Harmonic Conjugate Functions

(a) Verify that $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.

(b) Find the harmonic conjugate function of u.

(a) From the partial derivatives $\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial^2 u}{\partial x^2} = 6x$, $\frac{\partial u}{\partial y} = -6xy - 5$, $\frac{\partial^2 u}{\partial y^2} = -6x$ we see that u satisfies Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$.

(b) v must satisfy the Cauchy-Riemann equations $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, i.e., we must have $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$ and $\frac{\partial v}{\partial x} = 6xy + 5$. Partial integration of the first equation with respect to y gives $v(x,y) = 3x^2y - y^3 + h(x)$. The partial derivative with respect to x of this last equation is $\frac{\partial v}{\partial x} = 6xy + h'(x)$. When this result is substituted into the second equation we obtain $h'(x) = 5$, and so $h(x) = 5x + C$, where C is a real constant. Therefore, the harmonic conjugate of u is $v(x,y) = 3x^2y - y^3 + 5x + C$.

Using Transformations to Solve $\nabla^2\phi = 0$

- We have seen if $f(z) = u(x,y) + iv(x,y)$ is an analytic function in a domain D, then both functions u and v satisfy $\nabla^2\phi = 0$ in D.
- There is another important connection between analytic functions and Laplace's equation:
  - In applied mathematics we often wish to solve Laplace's equation $\nabla^2\phi = 0$ in a domain D in the xy-plane, and for reasons that depend on the shape of D, it simply may not be possible to determine ϕ.
  - It may be possible to devise a special analytic mapping $f(z) = u(x,y) + iv(x,y)$ or $u = u(x,y)$, $v = v(x,y)$ from the xy-plane to the uv-plane so that D', the image of D under the mapping, has a more convenient shape and the function $\phi(x,y)$ that satisfies Laplace's equation in D also satisfies Laplace's equation in D'.
  - We then solve Laplace's equation in D' (the solution Φ will be a function of u and v) and then return to the xy-plane and $\phi(x,y)$ by means of the preceding equations.