# Introduction to Complex Analysis 

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(1) Elementary Functions

- Exponential Functions
- Logarithmic Functions
- Complex Powers
- Complex Trigonometric Functions
- Complex Hyperbolic Functions
- Inverse Trigonometric and Hyperbolic Functions


## Subsection 1

## Exponential Functions

## Complex Exponential Function

- We repeat the definition of the complex exponential function:


## Definition (Complex Exponential Function)

The function $e^{z}$ defined by

$$
e^{z}=e^{x} \cos y+i e^{x} \sin y
$$

is called the complex exponential function.

- This function agrees with the real exponential function when $z$ is real: in fact, if $z=x+0 i$,

$$
e^{x+0 i}=e^{x}(\cos 0+i \sin 0)=e^{x}(1+i \cdot 0)=e^{x}
$$

- The complex exponential function also shares important differential properties of the real exponential function:
- $e^{x}$ is differentiable everywhere;
- $\frac{d}{d x} e^{x}=e^{x}$, for all $x$.


## Analyticity of $e^{z}$

## Theorem (Analyticity of $e^{z}$ )

The exponential function $e^{z}$ is entire and its derivative is $\frac{d}{d z} e^{z}=e^{z}$.

- We use the criterion based on the real and imaginary parts. $u(x, y)=e^{x} \cos y$ and $v(x, y)=e^{x} \sin y$ are continuous real functions and have continuous first-order partial derivatives, for all $(x, y)$. In addition, the Cauchy-Riemann equations in $u$ and $v$ are easily verified: $\frac{\partial u}{\partial x}=e^{x} \cos y=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-e^{x} \sin y=-\frac{\partial v}{\partial x}$. Therefore, the exponential function $e^{z}$ is entire. The derivative of an analytic function $f$ is given by $f^{\prime}(z)=\frac{\partial u}{\partial x}+i \frac{\partial v}{\partial x}$. So the derivative of $e^{z}$ is: $\frac{d}{d z} e^{z}=\frac{\partial u}{\partial x}+i \frac{\partial v}{\partial x}=e^{x} \cos y+i e^{x} \sin y=e^{z}$.
- Since the real and imaginary parts of an analytic function are harmonic conjugates, we can show the only entire function $f$ that agrees with the real exponential function $e^{x}$ for real input and that satisfies $f^{\prime}(z)=f(z)$ is the complex exponential function $e^{z}$.


## Derivatives of Exponential Functions

- Find the derivative of each of the following functions:
(a) $i z^{4}\left(z^{2}-e^{z}\right)$
(b) $e^{z^{2}-(1+i) z+3}$
- We use the various rules for complex derivatives:
(a)

$$
\begin{aligned}
\frac{d}{d z}\left(i z^{4}\left(z^{2}-e^{z}\right)\right) & =\frac{d}{d z}\left(i z^{4}\right)\left(z^{2}-e^{z}\right)+i z^{4} \frac{d}{d z}\left(z^{2}-e^{z}\right) \\
& =4 i z^{3}\left(z^{2}-e^{z}\right)+i z^{4}\left(2 z-e^{z}\right) \\
& =6 i z^{5}-i z^{4} e^{z}-4 i z^{3} e^{z} .
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d}{d z}\left(e^{z^{2}-(1+i) z+3}\right) & =e^{z^{2}-(1+i) z+3} \cdot \frac{d}{d z}\left(z^{2}-(1+i) z+3\right) \\
& =e^{z^{2}-(1+i) z+3} \cdot(2 z-1-i) .
\end{aligned}
$$

## Modulus, Argument, and Conjugate

- If we express the complex number $w=e^{z}$ in polar form:

$$
w=e^{x} \cos y+i e^{x} \sin y=r(\cos \theta+i \sin \theta)
$$

we see that $r=e^{x}$ and $\theta=y+2 n \pi$, for $n=0, \pm 1, \pm 2, \ldots$.

- Because $r$ is the modulus and $\theta$ is an argument of $w$, we have:

$$
\left|e^{z}\right|=e^{x}, \quad \arg \left(e^{z}\right)=y+2 n \pi, n=0, \pm 1, \pm 2, \ldots
$$

- We know from calculus that $e^{x}>0$, for all real $x$, whence $\left|e^{z}\right|>0$. This implies that $e^{z} \neq 0$, for all complex $z$, i.e., $w=0$ is not in the range of $w=e^{z}$.
- Note, however, that $e^{z}$ may be a negative real number: E.g., if $z=\pi i$, then $e^{\pi i}$ is real and $e^{\pi i}<0$.
- A formula for the conjugate of the complex exponential $e^{z}$ is found using the even-odd properties of the real cosine and sine functions: $\overline{e^{z}}=e^{x} \cos y-i e^{x} \sin y=e^{x} \cos (-y)+i e^{x} \sin (-y)=e^{x-i y}=e^{\bar{z}}$. Therefore, for all complex $z, \overline{e^{z}}=e^{\bar{z}}$.


## Algebraic Properties

## Theorem (Algebraic Properties of $e^{z}$ )

If $z_{1}$ and $z_{2}$ are complex numbers, then:
(i) $e^{0}=1$;
(ii) $e^{z_{1}} e^{z_{2}}=e^{z_{1}+z_{2}}$;
(iii) $\frac{e^{z_{1}}}{e^{z_{2}}}=e^{z_{1}-z_{2}}$.
(iv) $\left(e^{z_{1}}\right)^{n}=e^{n z_{1}}, n=0, \pm 1, \pm 2, \ldots$.
(i) Clearly, $e^{0+0 i}=e^{0}(\cos 0+i \sin 0)=1$.
(ii) Let $z_{1}=x_{1}+i y_{1}$ and $z_{2}=x_{2}+i y_{2}$. Hence
$e^{z_{1}} e^{z_{2}}=\left(e^{x_{1}} \cos y_{1}+i e^{x_{1}} \sin y_{1}\right)\left(e^{x_{2}} \cos y_{2}+i e^{x_{2}} \sin y_{2}\right)=$
$e^{x_{1}+x_{2}}\left(\cos y_{1} \cos y_{2}-\sin y_{1} \sin y_{2}\right)+i e^{x_{1}+x_{2}}\left(\sin y_{1} \cos y_{2}+\cos y_{1} \sin y_{2}\right)$.
Using the addition formulas for the real cosine and sine functions, we get $e^{z_{1}} e^{z_{2}}=e^{x_{1}+x_{2}} \cos \left(y_{1}+y_{2}\right)+i e^{x_{1}+x_{2}} \sin \left(y_{1}+y_{2}\right)$. The right-hand side is $e^{z_{1}+z_{2}}$.

- The proofs of (iii) and (iv) are similar.


## Periodicity

- The most striking difference between the real and complex exponential functions is the periodicity of $e^{z}$.
- We say that a complex function $f$ is periodic with period $T$ if

$$
f(z+T)=f(z), \quad \text { for all complex } z
$$

- The real exponential function is not periodic, but the complex exponential function is because it is defined using the real cosine and sine functions, which are periodic.
- We have

$$
e^{z+2 \pi i}=e^{z} e^{2 \pi i}=e^{z}(\cos 2 \pi+i \sin 2 \pi)=e^{z}
$$

The complex exponential function $e^{z}$ is periodic with a pure imaginary period $2 \pi i$.

That is, for $f(z)=e^{z}$, we have $f(z+2 \pi i)=f(z)$, for all $z$.

## The Fundamental Region

- We saw that, for all values of $z, e^{z+2 \pi i}=e^{z}$.
- Thus, we also have $e^{(z+2 \pi i)+2 \pi i}=e^{z+2 \pi i}=e^{z}$.
- By repeating, we find that $e^{z+2 n \pi i}=e^{z}$, for $n=0, \pm 1, \pm 2, \ldots$.
- This means that $-2 \pi i, 4 \pi i, 6 \pi i$, and so on, are also periods of $e^{z}$.
- If $e^{z}$ maps the point $z$ onto the point $w$, then it also maps the points $z \pm 2 \pi i, z \pm 4 \pi i, z \pm 6 \pi i$, and so on, onto the point $w$.
- Thus, $e^{z}$ is not one-to-one, and all values $e^{z}$ are assumed in any infinite horizontal strip of width $2 \pi$ in the $z$-plane. That is, all values are assumed in $-\infty<x<\infty, y_{0}<y \leq y_{0}+2 \pi, y_{0}$ a real constant.


The infinite horizontal strip defined by: $-\infty<x<\infty,-\pi<y \leq \pi$, is called the fundamental region of the complex exponential function.

## The Exponential Mapping

- Since all values of the complex exponential function $e^{z}$ are assumed in the fundamental region, the image of this region under the mapping $w=e^{z}$ is the same as the image of the entire complex plane.
- Note that this region consists of the collection of vertical line segments $z(t)=a+i t,-\pi<t \leq \pi$, where $a$ is any real number.
- The image of $z(t)=a+i t,-\pi<t \leq \pi$, under $w=e^{z}$ is parametrized by $w(t)=e^{z(t)}=e^{a+i t}=e^{a} e^{i t},-\pi<t \leq \pi$, and, thus, $w(t)$ defines a circle centered at the origin with radius $e^{a}$.
- Because a can be any real number, the radius $e^{a}$ of this circle can be any nonzero positive real number.
- Thus, the image of the fundamental region under the exponential mapping consists of the collection of all circles centered at the origin with nonzero radius, i.e., the image of the fundamental region $-\infty<x<\infty,-\pi<y \leq \pi$, under $w=e^{z}$ is the set of all complex $w$ with $w \neq 0$, or, equivalently, the set $|w|>0$.


## Using Horizontal Lines to Determine the Image

- The image can also be found by using horizontal lines in the fundamental region.
- Consider the horizontal line $y=b$. This line can be parametrized by $z(t)=t+i b,-\infty<t<\infty$. So its image under $w=e^{z}$ is given by $w(t)=e^{z(t)}=e^{t+i b}=e^{t} e^{i b},-\infty<t<\infty$.
- Defining a new parameter $s=e^{t}$, and observing that $0<s<\infty$, the image is given by $W(s)=e^{i b} s, 0<s<\infty$, which, is the set consisting of all points $w \neq 0$ in the ray emanating from the origin and containing the point $e^{i b}=\cos b+i \sin b$.
- Thus, the image of the horizontal line $y=b$ under the mapping $w=e^{z}$ is the set of all points $w \neq 0$ in the ray emanating from the origin and making an angle of $b$ radians with the positive $u$-axis, i.e., the set of all $w$, satisfying $\arg (w)=b$.


## Exponential Mapping Properties

## Exponential Mapping Properties

(i) $w=e^{z}$ maps the fundamental region $-\infty<x<\infty,-\pi<y \leq \pi$, onto the set $|w|>0$.
(ii) $w=e^{z}$ maps the vertical line segment $x=$ $a,-\pi<y \leq \pi$, onto the circle $|w|=e^{a}$.
(iii) $w=e^{z}$ maps the horizontal line $y=b$, $-\infty<x<\infty$, onto the ray $\arg (w)=b$.





## Exponential Mapping of a Grid

- Find the image of the grid shown below left under $w=e^{z}$.



The grid consists of the vertical line segments $x=0,1,2$, $-2 \leq y \leq 2$, and the horizontal line segments $y=$ $-2,-1,0,1,2,0 \leq x \leq 2$.

Using the properties of the exponential mapping, we have that:

- the image of the vertical line segment $x=0,-2 \leq y \leq 2$, is the circular arc $|w|=e^{0}=1,-2 \leq \arg (w) \leq 2$.
- The segments $x=1$ and $x=2,-2 \leq y \leq 2$, map onto the arcs $|w|=e$ and $|w|=e^{2},-2 \leq \arg (w) \leq 2$, respectively.
- The horizontal segment $y=0,0 \leq x \leq 2$, maps onto the portion of the ray emanating from the origin defined by $\arg (w)=0,1 \leq|w| \leq e^{2}$.
- The segments $y=-2,-1,1,2$ map onto the segments defined by $\arg (w)=-2, \arg (w)=-1, \arg (w)=1, \arg (w)=2,1 \leq|w| \leq e^{2}$.
The end result is shown on the right.


## Subsection 2

## Logarithmic Functions

## Complex Logarithm

- In real analysis, the natural logarithm function $\ln x$ is often defined as an inverse function of the real exponential function $e^{x}$. We use $\log _{e} x$ to represent the real logarithmic function.
- The situation is different in complex analysis because the complex exponential function $e^{z}$ is not a one-to-one function on its domain $\mathbb{C}$.
- Given a fixed nonzero complex number $z$, the equation $e^{w}=z$ has infinitely many solutions e.g., $\frac{1}{2} \pi i, \frac{5}{2} \pi i$, and $-\frac{3}{2} \pi i$ are all solutions to $e^{w}=i$.
- In general, if $w=u+i v$ is a solution of $e^{w}=z$, then $\left|e^{w}\right|=|z|$ and $\arg \left(e^{w}\right)=\arg (z)$. Thus, $e^{u}=|z|$ and $v=\arg (z)$, or, equivalently, $u=\log _{e}|z|$ and $v=\arg (z)$. Therefore, given a nonzero complex number $z$ we have shown that: if $e^{w}=z$, then $w=\log _{e}|z|+i \arg (z)$.
- This set of values defines a multiple-valued function $w=G(z)$, called the complex logarithm of $z$ and denoted by $\ln z$.


## Definition of the Complex Logarithmic Function

## Definition (Complex Logarithm)

The multiple-valued function $\ln z$ defined by:

$$
\ln z=\log _{e}|z|+\operatorname{iarg}(z)
$$

is called the complex logarithm.

- The notation $\ln z$ will always be used to denote the multiple valued complex logarithm.
- By switching to exponential notation $z=r e^{i \theta}$, we obtain the following alternative description of the complex logarithm:

$$
\ln z=\log _{e} r+i(\theta+2 n \pi), \quad n=0, \pm 1, \pm 2, \ldots
$$

- The complex logarithm can be used to find all solutions to the exponential equation $e^{w}=z$, when $z$ is a nonzero complex number.


## Solving Exponential Equations I

- Find all complex solutions to the equation $e^{w}=i$.

For each equation $e^{w}=z$, the set of solutions is given by $w=\ln z$, where $w=\log _{e}|z|+\operatorname{iarg}(z)$. For $z=i$, we have $|z|=1$ and $\arg (z)=\frac{\pi}{2}+2 n \pi$. Thus, we get $w=\ln i=\log _{e} 1+i\left(\frac{\pi}{2}+2 n \pi\right)$, whence

$$
w=\frac{(4 n+1) \pi}{2} i, n=0, \pm 1, \pm 2, \ldots
$$

Therefore, each of the values: $w=\ldots,-\frac{7 \pi}{2} i,-\frac{3 \pi}{2} i, \frac{\pi}{2} i, \frac{5 \pi}{2} i, \ldots$ satisfies the equation $e^{w}=i$.

## Solving Exponential Equations II

- Find all complex solutions to the equation $e^{w}=1+i$ and to the equation $e^{w}=-2$.
- For $z=1+i$, we have $|z|=\sqrt{2}$ and $\arg (z)=\frac{\pi}{4}+2 n \pi$. Thus, we get

$$
\begin{aligned}
w & =\ln (1+i)=\log _{e} \sqrt{2}+i\left(\frac{\pi}{4}+2 n \pi\right) \\
& =\frac{1}{2} \log _{e} 2+\frac{(8 n+1) \pi}{4} i, n=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

- Since $z=-2$, we have $|z|=2$ and $\arg (z)=\pi+2 n \pi$. Thus, $w=\ln (-2)=\log _{e} 2+i(\pi+2 n \pi)$. That is,

$$
w=\log _{e} 2+(2 n+1) \pi i, \quad n=0, \pm 1, \pm 2, \ldots
$$

## Logarithmic Identities

- Complex logarithm satisfies the following identities, which are analogous to identities for the real logarithm:


## Theorem (Algebraic Properties of $\ln z$ )

If $z_{1}$ and $z_{2}$ are nonzero complex numbers and $n$ is an integer, then
(i) $\ln \left(z_{1} z_{2}\right)=\ln z_{1}+\ln z_{2}$;
(ii) $\ln \frac{z_{1}}{z_{2}}=\ln z_{1}-\ln z_{2}$;
(iii) $\ln z_{1}^{n}=n \ln z_{1}$.

- $\ln z_{1}+\ln z_{2}=\log _{e}\left|z_{1}\right|+\operatorname{iarg}\left(z_{1}\right)+\log _{e}\left|z_{2}\right|+\operatorname{iarg}\left(z_{2}\right)=$ $\log _{e}\left|z_{1}\right|+\log _{e}\left|z_{2}\right|+i\left(\arg \left(z_{1}\right)+\arg \left(z_{2}\right)\right)$.
The real logarithm satisfies $\log _{e} a+\log _{e} b=\log _{e}(a b)$, for $a>0$ and $b>0$, so $\log _{e}\left|z_{1} z_{2}\right|=\log _{e}\left|z_{1}\right|+\log _{e}\left|z_{2}\right|$. Also, $\arg \left(z_{1}\right)+\arg \left(z_{2}\right)=$ $\arg \left(z_{1} z_{2}\right)$. Therefore, $\ln z_{1}+\ln z_{2}=\log _{e}\left|z_{1} z_{2}\right|+i \arg \left(z_{1} z_{2}\right)=$ $\ln \left(z_{1} z_{2}\right)$.
- Parts (ii) and (iii) are similar.


## Principal Value of Complex Logarithm

- The complex logarithm of a positive real has infinitely many values.
- Example: $\ln 5$ is the set of values $\log _{e} 5+2 n \pi i$, where $n$ is any integer, whereas $\log _{e} 5$ has a single value $\log _{e} 5=1.6094$. The unique value of $\ln 5$ corresponding to $n=0$ is the same as $\log _{e} 5$.
- In general, this value of the complex logarithm is called the principal value of the complex logarithm since it is found by using the principal argument $\operatorname{Arg}(z)$ in place of the argument $\arg (z)$.
- We denote the principal value of the logarithm by the symbol Lnz, which, thus, defines a function, whereas $\ln z$ is multi-valued.


## Definition (Principal Value of the Complex Logarithm)

The complex function Lnz defined by:

$$
\operatorname{Ln} z=\log _{e}|z|+i \operatorname{Arg}(z)
$$

is called the principal value of the complex logarithm.

## Computing the Principal Value of the Complex Logarithm

- Compute the principal value of the complex logarithm Lnz for

$$
\begin{array}{lll}
\text { (a) } z=i & \text { (b) } z=1+i & \text { (c) } z=-2
\end{array}
$$

(a) For $z=i$, we have $|z|=1$ and $\operatorname{Arg}(z)=\frac{\pi}{2}$. So we get

$$
\operatorname{Ln} i=\log _{e} 1+\frac{\pi}{2} i=\frac{\pi}{2} i
$$

(b) For $z=1+i$, we have $|z|=\sqrt{2}$ and $\operatorname{Arg}(z)=\frac{\pi}{4}$. Thus,

$$
\operatorname{Ln}(1+i)=\log _{e} \sqrt{2}+\frac{\pi}{4} i=\frac{1}{2} \log _{e} 2+\frac{\pi}{4} i
$$

(c) For $z=-2$, we have $|z|=2$ and $\operatorname{Arg}(z)=\pi$, whence

$$
\operatorname{Ln}(-2)=\log _{e} 2+\pi i
$$

- Warning! The algebraic identities for the complex logarithm are not necessarily satisfied by the principal value of the complex logarithm.


## Lnz as an Inverse Function

- Because Lnz is one of the values of the complex logarithm $\ln z$, it follows that: $e^{\operatorname{Ln} z}=z$, for all $z \neq 0$.
- This suggests that the logarithmic function Lnz is an inverse function of $e^{z}$.
- Because the complex exponential function is not one-to-one on its domain, this statement is not accurate.
- The relationship between these functions is similar to the relationship between the squaring function $z^{2}$ and the principal square root function $z^{1 / 2}=\sqrt{|z|} e^{i \operatorname{Arg}(z) / 2}$.
- The exponential function must first be restricted to a domain on which it is one-to-one in order to have a well-defined inverse function.
- In fact, $e^{z}$ is a one-to-one function on the fundamental region $-\infty<x<\infty,-\pi<y \leq \pi$.


## Lnz as an Inverse Function (Cont'd)

- We show that if the domain of $e^{z}$ is restricted to the fundamental region, then $\operatorname{Ln} z$ is its inverse function.
Consider a point $z=x+i y,-\infty<x<\infty,-\pi<y \leq \pi$. We have $\left|e^{z}\right|=e^{x}$ and $\arg \left(e^{z}\right)=y+2 n \pi, n$ an integer. Thus, $y$ is an argument of $e^{z}$. Since $z$ is in the fundamental region, we also have $-\pi<y \leq \pi$, whence $y$ is the principal argument of $e^{z}$, i.e., $\operatorname{Arg}\left(e^{z}\right)=y$. In addition, for the real logarithm we have $\log _{e} e^{x}=x$, and so $\operatorname{Ln} e^{z}=\log _{e}\left|e^{z}\right|+i \operatorname{Arg}\left(e^{z}\right)=\log _{e} e^{x}+i y=x+i y$. Thus, we have shown that $\operatorname{Ln} e^{z}=z$, if $-\infty<x<\infty$ and $-\pi<y \leq \pi$.


## Lnz as an Inverse Function of $e^{z}$

If the complex exponential $f(z)=e^{z}$ is defined on the fundamental region $-\infty<x<\infty,-\pi<y \leq \pi$, then $f$ is one-to-one and the inverse function of $f$ is the principal value of the complex logarithm $f^{-1}(z)=\operatorname{Ln} z$.

## Discontinuities of Lnz

- The principal value of the complex logarithm Lnz is discontinuous at $z=0$ since this function is not defined there.
- Lnz turns out to also be discontinuous at every point on the negative real axis.
- This may be intuitively clear since the value of $\operatorname{Ln} z$ for a point $z$ near the negative $x$-axis in the second quadrant has imaginary part close to $\pi$, whereas the value of a nearby point in the third quadrant has imaginary part close to $-\pi$.

- The function Lnz is, however, continuous on the set consisting of the complex plane excluding the non-positive real axis.


## Continuity

- Recall that a complex function $f(z)=u(x, y)+i v(x, y)$ is continuous at a point $z=x+i y$ if and only if both $u$ and $v$ are continuous real functions at $(x, y)$.
- The real and imaginary parts of $\operatorname{Lnz}$ are $u(x, y)=\log _{e}|z|=$ $\log _{e} \sqrt{x^{2}+y^{2}}$ and $v(x, y)=\operatorname{Arg}(z)$, respectively.
- From calculus, we know that the function $u(x, y)=\log _{e} \sqrt{x^{2}+y^{2}}$ is continuous at all points in the plane except $(0,0)$ and the function $v(x, y)=\operatorname{Arg}(z)$ is continuous on $|z|>0,-\pi<\arg (z)<\pi$.
- Therefore, it follows that Lnz is a continuous function on the domain $|z|>0$, $-\pi<\arg (z)<\pi$, i.e., $f_{1}$ defined by: $f_{1}(z)=\log _{e} r+i \theta$ is continuous on the domain where $r=|z|>0$ and $-\pi<\theta=$ $\arg (z)<\pi$.


## Analyticity

- Since the function $f_{1}$ agrees with the principal value of the complex logarithm Lnz where they are both defined, it follows that $f_{1}$ assigns to the input $z$ one of the values of the multiple-valued function $F(z)=\ln z$.
- I.e., we have shown that the function $f_{1}$ is a branch of the multiple-valued function $F(z)=\ln z$.
- This branch is called the principal branch of the complex logarithm. The nonpositive real axis is a branch cut for $f_{1}$ and the point $z=0$ is a branch point.
- The branch $f_{1}$ is an analytic function on its domain:


## Theorem (Analyticity of the Principal Branch of $\ln z$ )

The principal branch $f_{1}$ of the complex logarithm is an analytic function and its derivative is given by: $f_{1}^{\prime}(z)=\frac{1}{z}$.

- We prove that $f_{1}$ is analytic by using polar coordinates.


## Analyticity (Proof)

- Because $f_{1}$ is defined on the domain $r>0$ and $-\pi<\theta<\pi$, if $z$ is a point in this domain, then we can write $z=r e^{i \theta}$, with $-\pi<\theta<\pi$. Since the real and imaginary parts of $f_{1}$ are $u(r, \theta)=\log _{e} r$ and $v(r, \theta)=\theta$, respectively, we find that: $\frac{\partial u}{\partial r}=\frac{1}{r}, \frac{\partial v}{\partial \theta}=1, \frac{\partial v}{\partial r}=0$, and $\frac{\partial u}{\partial \theta}=0$. Thus, $u$ and $v$ satisfy the Cauchy-Riemann equations in polar coordinates $\frac{\partial u}{\partial r}=\frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r}=-\frac{1}{r} \frac{\partial u}{\partial \theta}$. Because $u, v$, and the first partial derivatives of $u$ and $v$ are continuous at all points in the domain, it follows that $f_{1}$ is analytic in this domain. In addition, the derivative of $f_{1}$ is given by: $f_{1}^{\prime}(z)=e^{-i \theta}\left(\frac{\partial u}{\partial r}+i \frac{\partial v}{\partial r}\right)=\frac{1}{r e^{i \theta}}=\frac{1}{z}$.
- Because $f_{1}(z)=L n z$, for each point $z$ in the domain, it follows that $\operatorname{Ln} z$ is differentiable in this domain, and that its derivative is given by $f_{1}^{\prime}$. That is, if $|z|>0$ and $-\pi<\arg (z)<\pi$ then:

$$
\frac{d}{d z} \operatorname{Ln} z=\frac{1}{z}
$$

## Derivatives of Logarithmic Functions I

- Find the derivatives of the function $z \operatorname{Lnz}$ in an appropriate domain. The function $z \operatorname{Lnz}$ is differentiable at all points where both of the functions $z$ and $L n z$ are differentiable. Because $z$ is entire and Lnz is differentiable on the domain $|z|>0,-\pi<\arg (z)<\pi, z \operatorname{Ln} z$ is differentiable on the domain defined by $|z|>0,-\pi<\arg (z)<\pi$ :

$$
\frac{d}{d z}[z \operatorname{Ln} z]=\frac{d}{d z} z \cdot \operatorname{Ln} z+z \frac{d}{d z} \operatorname{Ln} z=\operatorname{Ln} z+z \frac{1}{z}=\operatorname{Ln} z+1 .
$$

## Derivatives of Logarithmic Functions II

- Find the derivatives of the function $\operatorname{Ln}(z+1)$ in an appropriate domain.
The function $\operatorname{Ln}(z+1)$ is a composition of the functions $\operatorname{Ln} z$ and $z+1$. Because the function $z+1$ is entire, it follows from the chain rule that $\operatorname{Ln}(z+1)$ is differentiable at all points $w=z+1$ such that $|w|>0$ and $-\pi<\arg (w)<\pi$. To determine the corresponding values of $z$ for which $\operatorname{Ln}(z+1)$ is not differentiable, we first solve for $z$ in terms of $w$ to obtain $z=w-1$. The equation $z=w-1$ defines a linear mapping of the $w$-plane onto the $z$-plane given by translation by -1 . Under this mapping the non-positive real axis is mapped onto the ray emanating from $z=-1$ and containing the point $z=-2$.
Thus, $\operatorname{Ln}(z+1)$ is differentiable at all points $z$ that are not on this ray.

$$
\frac{d}{d z} \operatorname{Ln}(z+1)=\frac{1}{z+1} \cdot 1=\frac{1}{z+1} .
$$

## Logarithmic Mapping

- The complex logarithmic mapping $w=\operatorname{Lnz}$ can be understood in terms of the exponential mapping $w=e^{z}$ since these functions are inverses of each other.
- Recall $w=e^{z}$ maps the fundamental region $-\infty<x<\infty$, $-\pi<y \leq \pi$, in the $z$-plane onto the set $|w|>0$ in the $w$-plane. Hence, that inverse mapping $w=\operatorname{Lnz}$ maps the set $|z|>0$ in the $z$-plane onto the region $-\infty<u<\infty,-\pi<v \leq \pi$, in the $w$-plane.
- We summarize the relevant properties of the logarithmic mapping:


## Logarithmic Mapping Properties

(i) $w=\operatorname{Ln} z$ maps the set $|z|>0$ onto the region $-\infty<u<\infty,-\pi<v \leq \pi$.
(ii) $w=\operatorname{Ln} z$ maps the circle $|z|=r$ onto the vertical line segment $u=\log _{e} r$, $-\pi<v \leq \pi$
(iii) $w=\operatorname{Lnz}$ maps the ray $\arg (z)=\theta$ onto the horizontal line $v=\theta$, $-\infty<u<\infty$

## Example Involving the Logarithmic Mapping

- Find the image of the annulus $2 \leq|z| \leq 4$ under the logarithmic mapping $w=$ Lnz.
The boundary circles $|z|=2$ and $|z|=4$ of the annulus map onto the vertical line segments $u=\log _{e} 2$ and $u=\log _{e} 4,-\pi<v \leq \pi$. In a similar manner, each circle $|z|=r, 2 \leq r \leq 4$, maps onto a vertical line segment $u=\log _{e} r,-\pi<v \leq \pi$. Since the real log is increasing, $u=\log _{e} r$ takes on all values in $\log _{e} 2 \leq u \leq \log _{e} 4$ when $2 \leq r \leq 4$. Therefore, the image of $2 \leq|z| \leq 4$ is the rectangular region $\log _{e} 2 \leq u \leq \log _{e} 4,-\pi<v \leq \pi$ :




## Other Branches of $\ln z$

- The principal branch of the complex logarithm $f_{1}$ is just one of many possible branches of the multiple valued function $F(z)=\ln z$.
- We can define other branches of $F$ by simply changing the interval defining $\theta$ to a different interval of length $2 \pi$.
- For example, $f_{2}(z)=\log _{e} r+i \theta,-\frac{\pi}{2}<\theta<\frac{3 \pi}{2}$, defines a branch of $F$ whose branch cut is the non-positive imaginary axis.
For the branch $f_{2}$ we have $f_{2}(1)=0, f_{2}(2 i)=\log _{e} 2+\frac{1}{2} \pi i$, and $f_{2}(-1-i)=\frac{1}{2} \log _{e} 2+\frac{5}{4} \pi i$.
- It can also be shown that any branch

$$
f_{k}(z)=\log _{e} r+i \theta, \theta_{0}<\theta<\theta_{0}+2 \pi
$$

of $F(z)=\ln z$ is analytic on its domain, and its derivative is given by:

$$
f_{k}^{\prime}(z)=\frac{1}{z}
$$

## Comparisons with Real Analysis

- Although the complex exponential and logarithmic functions are similar to the real exponential and logarithmic functions in many ways, it is important to keep in mind their differences:
- The real exponential function is one-to-one, but the complex exponential is not.
- $\log _{e} x$ is a single-valued function, but $\ln z$ is multiple-valued.
- Many properties of real logarithms apply to the complex logarithm, such as $\ln \left(z_{1} z_{2}\right)=\ln z_{1}+\ln z_{2}$, but these properties do not always hold for the principal value Lnz.


## Riemann Surfaces

- Consider the mapping $w=e^{z}$ on the half-plane $x \leq 0$.
- Each half-infinite strip $S_{n}$ defined by $(2 n-1) \pi<y \leq(2 n+1) \pi$, $x \leq 0$, for $n=0, \pm 1, \pm 2, \ldots$ is mapped onto the punctured unit disk $0<|w| \leq 1$ :



Thus, $w=e^{z}$ describes an infinite-to-one covering of the punctured unit disk. To visualize this covering, we imagine there being a different image disk $B_{n}$ for each half-infinite strip $S_{n}$.

## Riemann Surfaces (Cont'd)

- Cut each disk $B_{n}$ open along the segment $-1 \leq u<0$. We construct a Riemann surface for $w=e^{z}$ by attaching, for each $n$, the cut disk $B_{n}$ to the cut disk $B_{n+1}$ along the edge that represents the image of the half-infinite line $y=(2 n+1) \pi$. In xyz-space, the images $\ldots, z_{-1}, z_{0}, z_{1}, \ldots$ of $z$ in $\ldots, B_{-1}, B_{0}, B_{1}, \ldots$, respectively, lie directly above the point $w=e^{z}$ in the $x y$-plane.

- By projecting the points of the Riemann surface vertically down onto the $x y$-plane we see the infinite-to-one nature of the mapping $w=e^{z}$.
- The multiple-valued function $F(z)=\ln z$ may be visualized by considering all points in the Riemann surface lying directly above a point in the $x y$-plane.


## Subsection 3

## Complex Powers

## Complex Powers

- Complex powers, such as $(1+i)^{i}$, are defined in terms of the complex exponential and logarithmic functions.
- Recall from that $z=e^{\ln z}$, for all nonzero complex numbers $z$.
- Thus, when $n$ is an integer, $z^{n}$ can be written as

$$
z^{n}=\left(e^{\ln z}\right)^{n}=e^{n \ln z}
$$

- This formula, which holds for integer exponents $n$, suggests the following definition for the complex power $z^{\alpha}$, for any complex exponent $\alpha$ :


## Definition (Complex Powers)

If $\alpha$ is a complex number and $z \neq 0$, then the complex power $z^{\alpha}$ is defined to be:

$$
z^{\alpha}=e^{\alpha \ln z}
$$

## Complex Power Function

- $z^{\alpha}=e^{\alpha \ln z}$ gives an infinite set of values because the complex logarithm $\ln z$ is multiple-valued.
- When $n$ is an integer, the expression is single-valued (in agreement with fact that $z^{n}$ is a function when n is an integer).
To see this, note $z^{n}=e^{n \ln z}=e^{n\left[\log _{e}|z|+i \arg (z)\right]}=e^{n \log _{e}|z|} e^{n a r g(z) i}$. If $\theta=\operatorname{Arg}(z)$, then $\arg (z)=\theta+2 k \pi$, where $k$ is an integer. So $e^{n a r g(z) i}=e^{n(\theta+2 k \pi) i}=e^{n \theta i} e^{2 n k \pi i}$. But, by definition, $e^{2 n k \pi i}=\cos (2 n k \pi)+i \sin (2 n k \pi)$. Because $n$ and $k$ are integers, we have $2 n k \pi$ is an even multiple of $\pi$, and so $\cos (2 n k \pi)=1$ and $\sin (2 n k \pi)=0$. Consequently, $e^{2 n k \pi i}=1$ and we get $z^{n}=e^{n \log _{e}|z|} e^{n \operatorname{Arg}(z) i}$, which is single-valued.
- In general, $z^{\alpha}=e^{\alpha \ln z}$ defines a multiple-valued function.
- It is called a complex power function.


## Computing Complex Powers

- Find the values of the given complex power:
(a) $i^{2 i}$
(b) $(1+i)^{i}$.
(a) We have seen that $\ln i=\frac{(4 n+1) \pi}{2} i$. Thus, we obtain:

$$
i^{2 i}=e^{2 i \ln i}=e^{2 i[(4 n+1) \pi i / 2]}=e^{-(4 n+1) \pi}
$$

for $n=0, \pm 1, \pm 2, \ldots$
(b) We have also seen that $\ln (1+i)=\frac{1}{2} \log _{e} 2+\frac{(8 n+1) \pi}{4} i$, for $n=0, \pm 1, \pm 2, \ldots$ Thus, we obtain:

$$
(1+i)^{i}=e^{i \ln (1+i)}=e^{i\left[\left(\log _{e} 2\right) / 2+(8 n+1) \pi i / 4\right]}
$$

or

$$
(1+i)^{i}=e^{-(8 n+1) \pi / 4+i\left(\log _{e} 2\right) / 2}
$$

for $n=0, \pm 1, \pm 2, \ldots$.

## Properties of Complex Powers

- Complex powers satisfy the following properties that are analogous to properties of real powers:
- $z^{\alpha_{1}} z^{\alpha_{2}}=z^{\alpha_{1}+\alpha_{2}}$;
- $\frac{z^{\alpha_{1}}}{z^{\alpha_{2}}}=z^{\alpha_{1}-\alpha_{2}}$;
- $\left(z^{\alpha}\right)^{n}=z^{n \alpha}$, for $n=0, \pm 1, \pm 2, \ldots$
- Each of these properties can be derived from the definition of complex powers and the algebraic properties of the complex exponential function $e^{z}$ :
- For example, by the definition, $z^{\alpha_{1}} z^{\alpha_{2}}=e^{\alpha_{1} \ln z} e^{\alpha_{2} \ln z}$. By using properties of the exponential, $z^{\alpha_{1}} z^{\alpha_{2}}=e^{\alpha_{1} \ln z+\alpha_{2} \ln z}=e^{\left(\alpha_{1}+\alpha_{2}\right) \ln z}$. By the definition, $e^{\left(\alpha_{1}+\alpha_{2}\right) \ln z}=z^{\alpha_{1}+\alpha_{2}}$. Thus, $z^{\alpha_{1}} z^{\alpha_{2}}=z^{\alpha_{1}+\alpha_{2}}$.


## Principal Value of a Complex Power

- The complex power $z^{\alpha}$ is, in general, multiple-valued because it is defined using the multiple-valued complex logarithm $\ln z$.
- We can assign a unique value to $z^{\alpha}$ by using the principal value of the complex logarithm $\operatorname{Ln} z$ in place of $\ln z$.
- This value of the complex power is called the principal value of $z^{\alpha}$.
- Example: Since $\operatorname{Ln} i=\frac{\pi}{2} i$, the principal value of $i^{2 i}$ is $i^{2 i}=e^{2 i \operatorname{Ln} i}=e^{2 i \frac{\pi}{2} i}=e^{-\pi}$.


## Definition (Principal Value of a Complex Power)

If $\alpha$ is a complex number and $z \neq 0$, then the function defined by:

$$
z^{\alpha}=e^{\alpha \operatorname{Ln} z}
$$

is called the principal value of the complex power $z^{\alpha}$.

- Notation: $z^{\alpha}$ will be used to denote both the multiple-valued power function $F(z)=z^{\alpha}$ and the principal value power function.


## Computing the Principal Value of a Complex Power

- Find the principal value of each complex power:

$$
\begin{array}{ll}
\text { (a) }(-3)^{i / \pi} & \text { (b) }(2 i)^{1-i}
\end{array}
$$

(a) For $z=-3$, we have $|z|=3$ and $\operatorname{Arg}(-3)=\pi$, and so $\operatorname{Ln}(-3)=\log _{e} 3+i \pi$. Thus, we obtain:

$$
(-3)^{i / \pi}=e^{(i / \pi) \operatorname{Ln}(-3)}=e^{(i / \pi)\left(\log _{e} 3+i \pi\right)}=e^{-1+i\left(\log _{e} 3\right) / \pi} .
$$

Finally, since $e^{-1+i\left(\log _{e} 3\right) / \pi}=e^{-1}\left[\cos \frac{\log _{e} 3}{\pi}+i \sin \frac{\log _{e} 3}{\pi}\right]$,
$(-3)^{i / \pi}=e^{-1}\left[\cos \frac{\log _{e} 3}{\pi}+i \sin \frac{\log _{e} 3}{\pi}\right]$.
(b) For $z=2 i$, we have $|z|=2$ and $\operatorname{Arg}(z)=\frac{\pi}{2}$, and so $\operatorname{Ln} 2 i=\log _{e} 2+i \frac{\pi}{2}$. Thus, we obtain:

$$
(2 i)^{1-i}=e^{(1-i) \operatorname{Ln} 2 i}=e^{(1-i)\left(\log _{e} 2+i \pi / 2\right)}=e^{\log _{e} 2+\pi / 2-i\left(\log _{e} 2-\pi / 2\right)}
$$

Since $(2 i)^{1-i}=e^{\log _{e} 2+\pi / 2}\left[\cos \left(\log _{e} 2-\frac{\pi}{2}\right)-i \sin \left(\log _{e} 2-\frac{\pi}{2}\right)\right]$, we finally get $(2 i)^{1-i}=e^{\log _{e} 2+\pi / 2}\left[\cos \left(\log _{e} 2-\frac{\pi}{2}\right)-i \sin \left(\log _{e} 2-\frac{\pi}{2}\right)\right]$.

## Analyticity

- In general, the principal value of a complex power $z^{\alpha}$ is not a continuous function on the complex plane because the function Lnz is not continuous on the complex plane.
- The function $e^{\alpha z}$ is continuous on the entire complex plane and the function $\operatorname{Ln} z$ is continuous on the domain $|z|>0,-\pi<\arg (z)<\pi$, so $z^{\alpha}$ is continuous on the domain $|z|>0,-\pi<\arg (z)<\pi$.
- Using polar coordinates $r=|z|$ and $\theta=\arg (z)$, we have found that $f_{1}(z)=e^{\alpha\left(\log _{e} r+i \theta\right)},-\pi<\theta<\pi$ is a branch of $F(z)=z^{\alpha}=e^{\alpha \ln z}$.
- It is called the principal branch of the complex power $z^{\alpha}$. Its branch cut is the non-positive real axis, and $z=0$ is a branch point.
- The branch $f_{1}$ agrees with the principal value $z^{\alpha}$ on the domain $|z|>0,-\pi<\arg (z)<\pi$. Consequently, the derivative of $f_{1}$ can be found using the chain rule:

$$
f_{1}^{\prime}(z)=\frac{d}{d z} e^{\alpha \operatorname{Ln} z}=e^{\alpha \operatorname{Ln} z} \frac{d}{d z}[\alpha \operatorname{Ln} z]=e^{\alpha \operatorname{Ln} z} \frac{\alpha}{z} .
$$

Using the principal value $z^{\alpha}=e^{\alpha \operatorname{Ln} z}$, we find $f_{1}^{\prime}(z)=\frac{\alpha z^{\alpha}}{z}=\alpha z^{\alpha-1}$.

## Derivative of a Power Function

- Find the derivative of the principal value $z^{i}$ at the point $z=1+i$. Because the point $z=1+i$ is in the domain $|z|>0$, $-\pi<\arg (z)<\pi$, it follows that $\frac{d}{d z} z^{i}=i z^{i-1}$, and so, $\left.\frac{d}{d z} z^{i}\right|_{z=1+i}=\left.i z^{i-1}\right|_{z=1+i}=i(1+i)^{i-1}$. We can rewrite this value as:

$$
i(1+i)^{i-1}=i(1+i)^{i}(1+i)^{-1}=i(1+i)^{i} \frac{1}{1+i}=\frac{1+i}{2}(1+i)^{i}
$$

Moreover, the principal value of $(1+i)^{i}$ is:

$$
(1+i)^{i}=e^{-\pi / 4+i\left(\log _{e} 2\right) / 2}, \text { and so }
$$

$$
\left.\frac{d}{d z} z^{i}\right|_{z=1+i}=\frac{1+i}{2} e^{-\pi / 4+i\left(\log _{e} 2\right) / 2}
$$

## Remarks

(i) There are some properties of real powers that are not satisfied by complex powers. One example of this is that for complex powers, $\left(z^{\alpha_{1}}\right)^{\alpha_{2}} \neq z^{\alpha_{1} \alpha_{2}}$ unless $\alpha_{2}$ is an integer.
(ii) As with complex logarithms, some properties that hold for complex powers do not hold for principal values of complex powers.
For example, we can prove that $\left(z_{1} z_{2}\right)^{\alpha}=z_{1}^{\alpha} z_{2}^{\alpha}$, for any nonzero complex numbers $z_{1}$ and $z_{2}$. However, this property does not hold for principal values of these complex powers:
If $z_{1}=-1, z_{2}=i$, and $\alpha=i$, then the principal value of $(-1 \cdot i)^{i}$ is $e^{i \operatorname{Ln}(-i)}=e^{\pi / 2}$. On the other hand, the product of the principal values of $(-1)^{i}$ and $i^{i}$ is $e^{i \operatorname{Ln}(-1)} e^{i \operatorname{Ln} i}=e^{-\pi} e^{-\pi / 2}=e^{-3 \pi / 2}$.

## Subsection 4

## Complex Trigonometric Functions

## Complex Sine and Cosine Functions

- If $x$ is a real variable, then

$$
e^{i x}=\cos x+i \sin x \quad \text { and } \quad e^{-i x}=\cos x-i \sin x
$$

- By adding these equations and simplifying, we get:

$$
\cos x=\frac{e^{i x}+e^{-i x}}{2}
$$

- If we subtract the two equations, then we obtain

$$
\sin x=\frac{e^{i x}-e^{-i x}}{2 i}
$$

- The formulas for the real cosine and sine functions can be used to define the complex sine and cosine functions.


## Definition (Complex Sine and Cosine Functions)

The complex sine and cosine functions are defined by:

$$
\sin z=\frac{e^{i z}-e^{-i z}}{2 i} \quad \text { and } \quad \cos z=\frac{e^{i z}+e^{-i z}}{2}
$$

## The Complex Tangent, Cotangent, Secant, and Cosecant

- The complex sine and cosine functions agree, by definition, with the real sine and cosine functions for real input.
- Analogous to real trigonometric functions, we next define the complex tangent, cotangent, secant, and cosecant functions using the complex sine and cosine:

$$
\begin{array}{ll}
\tan z=\frac{\sin z}{\cos z}, & \cot z=\frac{\cos z}{\sin z} \\
\sec z=\frac{1}{\cos z}, & \csc z=\frac{1}{\sin z}
\end{array}
$$

- These functions also agree with their real counterparts for real input.


## Values of Complex Trigonometric Functions

- Express the value of the given trigonometric function in the form $a+i b$.
(a) $\cos i$
(b) $\sin (2+i)$
(c) $\tan (\pi-2 i)$.
(a) $\cos i=\frac{e^{i \cdot i}+e^{-i \cdot i}}{2}=\frac{e^{-1}+e}{2}$.
(b) $\sin (2+i)=\frac{e^{i(2+i)}-e^{-i(2+i)}}{2 i}=\frac{e^{-1+2 i}-e^{1-2 i}}{2 i}=$
$e^{-1}(\cos 2+i \sin 2)-e(\cos (-2)+i \sin (-2))$
(c) $\tan (\pi-2 i)=\frac{\left(e^{i(\pi-2 i)}-e^{-i(\pi-2 i)}\right) / 2 i}{\left(e^{i(\pi-2 i)}+e^{-i(\pi-2 i)}\right) / 2}=\frac{e^{i(\pi-2 i)}-e^{-i(\pi-2 i)}}{\left(e^{i(\pi-2 i)}+e^{-i(\pi-2 i)}\right) i}=$

$$
\frac{e^{2}-e^{-2}}{\left(e^{2}+e^{-2}\right) i}=-\frac{e^{2}-e^{-2}}{e^{2}+e^{-2}} i
$$

## Identities

- We now list some of the more useful of the trigonometric identities:
- $\sin (-z)=-\sin z$ and $\cos (-z)=\cos z ;$
- $\cos ^{2} z+\sin ^{2} z=1$;
- $\sin \left(z_{1} \pm z_{2}\right)=\sin z_{1} \cos z_{2} \pm \cos z_{1} \sin z_{2}$;
- $\cos \left(z_{1} \pm z_{2}\right)=\cos z_{1} \cos z_{2} \mp \sin z_{1} \sin z_{2}$.
- Because of the sum/difference formulas, we also have the double-angle formulas:

$$
\sin 2 z=2 \sin z \cos z \quad \text { and } \quad \cos 2 z=\cos ^{2} z-\sin ^{2} z
$$

- We only verify $\cos ^{2} z+\sin ^{2} z=1$ :

$$
\begin{aligned}
\cos ^{2} z+\sin ^{2} z & =\left(\frac{e^{i z}+e^{-i z}}{2}\right)^{2}+\left(\frac{e^{i z}-e^{-i z}}{2}\right)^{2} \\
& =\frac{e^{2 i z}+2+e^{-2 i z}}{4}-\frac{e^{2 i z}-2+e^{-2 i z}}{4}=1
\end{aligned}
$$

- Some properties of the real trigonometric functions are not satisfied by their complex counterparts:
E.g., $|\sin x| \leq 1$ and $|\cos x| \leq 1$, for all real $x$, but $|\cos i|>1$ and $|\sin (2+i)|>1$.


## Periodicity

- We know that the complex exponential function is periodic with a pure imaginary period of $2 \pi i$, i.e., $e^{z+2 \pi i}=e^{z}$, for all complex $z$.
- Replacing $z$ with iz, we get $e^{i z+2 \pi i}=e^{i(z+2 \pi)}=e^{i z}$.
- Thus, $e^{i z}$ is a periodic function with real period $2 \pi$.
- Similarly, $e^{-i(z+2 \pi)}=e^{-i z}$, i.e., $e^{-i z}$ is periodic with period of $2 \pi$.
- It now follows that:

$$
\sin (z+2 \pi)=\frac{e^{i(z+2 \pi)}-e^{-i(z+2 \pi)}}{2 i}=\frac{e^{i z}-e^{-i z}}{2 i}=\sin z
$$

- A similar statement also holds for the complex cosine function.
- Thus, the complex sine and cosine are periodic functions with a real period of $2 \pi$.
- The periodicity of the secant and cosecant functions follows immediately from the definitions.
- Moreover, the complex tangent and cotangent are periodic with a real period of $\pi$.


## Trigonometric Equations

- Since the complex sine and cosine functions are periodic, there are always infinitely many solutions to equations of the form $\sin z=w$ or $\cos z=w$.
- One approach to solving such equations is to use the definition in conjunction with the quadratic formula:
Example: Find all solutions to the equation $\sin z=5$.
$\sin z=5$ is equivalent to the equation $\frac{e^{i z}-e^{-i z}}{2 i}=5$. By multiplying this equation by $e^{i z}$ and simplifying we obtain $e^{2 i z}-10 i e^{i z}-1=0$. This equation is quadratic in $e^{i z}$, i.e., $\left(e^{i z}\right)^{2}-10 i\left(e^{i z}\right)-1=0$. By the quadratic formula that the solutions are given by $e^{i z}=\frac{10 i+(-96)^{1 / 2}}{2}=5 i \pm 2 \sqrt{6} i=(5 \pm 2 \sqrt{6}) i$. In order to find the values of $z$, we must solve the two resulting exponential equations using the complex logarithm.


## Trigonometric Equations (Cont'd)

- We must solve $e^{i z}=(5 \pm 2 \sqrt{6}) i$ using the complex logarithm.
- If $e^{i z}=(5+2 \sqrt{6}) i$, then $i z=\ln (5 i+2 \sqrt{6} i)$ or $z=-i \ln [(5+2 \sqrt{6}) i]$. Because $(5+2 \sqrt{6}) i$ is a pure imaginary number and $5+2 \sqrt{6}>0$, we have $\arg [(5+2 \sqrt{6}) i]=\frac{1}{2} \pi+2 n \pi$. Thus,

$$
\begin{aligned}
& z=-i \ln [(5+2 \sqrt{6}) i]=-i\left[\log _{e}(5+2 \sqrt{6})+i\left(\frac{\pi}{2}+2 n \pi\right)\right] \text { or } \\
& z=\frac{(4 n+1) \pi}{2}-i \log _{e}(5+2 \sqrt{6}), \text { for } n=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

- Similarly, if $e^{i z}=(5-2 \sqrt{6}) i$, then $z=-i \ln [(5-2 \sqrt{6}) i]$. Since $(5-2 \sqrt{6}) i$ is a pure imaginary number and $5-2 \sqrt{6}>0$, it has an argument of $\frac{\pi}{2}$, and so:

$$
\begin{aligned}
& z=-i \ln [(5-2 \sqrt{6}) i]=-i\left[\log _{e}(5-2 \sqrt{6})+i\left(\frac{\pi}{2}+2 n \pi\right)\right] \text { or } \\
& z=\frac{(4 n+1) \pi}{2}-i \log _{e}(5-2 \sqrt{6}) \text { for } n=0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

## $\sin z$ and $\cos z$ in terms of $x$ and $y$

- To find a formula in terms of $x$ and $y$ for the modulus of the sine and cosine functions, we replace $z$ by $x+i y$ in $\sin z$ :

$$
\begin{aligned}
\sin z & =\frac{e^{-y+i x}-e^{y-i x}}{2 i} \\
& =\frac{e^{-y}(\cos x+i \sin x)-e^{y}(\cos x-i \sin x)}{2 i} \\
& =\sin x \frac{e^{y}+e^{-y}}{2}+i \cos x \frac{e^{y}-e^{-y}}{2}
\end{aligned}
$$

- Since the real hyperbolic sine and cosine functions are defined by $\sinh y=\frac{e^{y}-e^{-y}}{2}$ and cosh $y=\frac{e^{y}+e^{-y}}{2}$, we can rewrite as $\sin z=\sin x \cosh y+i \cos x \sinh y$.
- A similar computation gives

$$
\cos z=\cos x \cosh y-i \sin x \sinh y
$$

## Modulus of Sine and Cosine

- By the expression for $\sin z$ :

$$
|\sin z|=\sqrt{\sin ^{2} x \cosh ^{2} y+\cos ^{2} x \sinh ^{2} y}
$$

- Recall $\cos ^{2} x+\sin ^{2} x=1$ and $\cosh ^{2} y=1+\sinh ^{2} y$ :

$$
\begin{aligned}
|\sin z|= & \sqrt{\sin ^{2} x\left(1+\sinh ^{2} y\right)+\cos ^{2} x \sinh ^{2} y} \\
= & \sqrt{\sin ^{2} x+\left(\cos ^{2} x+\sin ^{2} x\right) \sinh ^{2} y} \\
& |\sin z|=\sqrt{\sin ^{2} x+\sinh ^{2} y}
\end{aligned}
$$

- Similarly, for the modulus of the complex cosine function:

$$
|\cos z|=\sqrt{\cos ^{2} x+\sinh ^{2} y}
$$

- Since $\sinh x$ is unbounded, the complex sine and cosine functions are not bounded on the complex plane, i.e., there does not exist a real constant $M$ so that $|\sin z|<M$, for all $z$ in $\mathbb{C}$, nor does there exist a real constant $M$ so that $|\cos z|<M$, for all $z$ in $\mathbb{C}$.


## Zeros

- The zeros of the real sine occur at integer multiples of $\pi$ and the zeros of the real cosine occur at odd integer multiples of $\frac{\pi}{2}$.
- These zeros of the real sine and cosine functions are also zeros of the complex sine and cosine, respectively.
- To find all zeros, we must solve $\sin z=0$ and $\cos z=0$.
- $\sin z=0$ is equivalent to $|\sin z|=0$, i.e., $\sqrt{\sin ^{2} x+\sinh ^{2} y}=0$, which is equivalent to: $\sin ^{2} x+\sinh ^{2} y=0$.
- Since $\sin ^{2} x$ and $\sinh ^{2} y$ are nonnegative real numbers, we must have $\sin x=0$ and $\sinh y=0$.
- $\sin x=0$ occurs when $x=n \pi, n=0, \pm 1, \pm 2, \ldots$.
- $\sinh y=0$ occurs only when $y=0$.

So, the only solutions of $\sin z=0$ in the complex plane are the real numbers $z=n \pi, n=0, \pm 1, \pm 2, \ldots$, i.e., the zeros of the complex sine function are the same as the zeros of the real sine.

- Similarly, the only zeros of the complex cosine function are the real numbers $z=\frac{(2 n+1) \pi}{2}, n=0, \pm 1, \pm 2, \ldots$


## Analyticity

- The derivatives of the complex sine and cosine functions are found using the chain rule:

$$
\begin{aligned}
\frac{d}{d z} \sin z & =\frac{d}{d z} \frac{e^{i z}-e^{-i z}}{2 i}=\frac{i e^{i z}+i e^{-i z}}{2 i} \\
& =\frac{e^{i z}+e^{-i z}}{2}=\cos z
\end{aligned}
$$

Since this derivative is defined for all complex $z, \sin z$ is entire.

- Similarly, $\frac{d}{d z} \cos z=-\sin z$.
- The derivatives of $\sin z$ and $\cos z$ can then be used to compute the derivatives of all of the complex trigonometric functions:

$$
\begin{array}{lll}
\frac{d}{d z} \sin z=\cos z & \frac{d}{d z} \cos z=-\sin z & \frac{d}{d z} \tan z=\sec ^{2} z \\
\frac{d}{d z} \cot z=-\csc ^{2} z & \frac{d}{d z} \sec z=\sec z \tan z & \frac{d}{d z} \csc z=-\csc z \cot z
\end{array}
$$

- The sine and cosine functions are entire, but the tangent, cotangent, secant, and cosecant functions are only analytic at those points where the denominator is nonzero.


## Trigonometric Mapping

- Since $\sin z$ is periodic with a real period of $2 \pi$, it takes on all values in any infinite vertical strip $x_{0}<x \leq x_{0}+2 \pi,-\infty<y<\infty$.
- This allows us to study the mapping $w=\sin z$ on the entire complex plane by analyzing it on any one of these strips.
- Consider the strip $-\pi<x \leq \pi,-\infty<y<\infty$.
- Observe that $\sin z$ is not one-to-one on this region, e.g., $z_{1}=0$ and $z_{2}=\pi$ are in this region and $\sin 0=\sin \pi=0$.
- From $\sin (-z+\pi)=\sin z$, it follows that the image of the strip $-\pi<x \leq-\frac{\pi}{2},-\infty<y<\infty$, is the same as the image of the strip $\frac{\pi}{2}<x \leq \pi,-\infty<y<\infty$, under $w=\sin z$.
- Therefore, we need only consider the mapping $w=\sin z$ on the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2},-\infty<y<\infty$, to gain an understanding of this mapping on the entire $z$-plane.
- One can show that the complex sine function is one-to-one on the domain $-\frac{\pi}{2}<x<\frac{\pi}{2},-\infty<y<\infty$.


## The Mapping $w=\sin z$

- Describe the image of the region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2},-\infty<y<\infty$, under the complex mapping $w=\sin z$.
We determine the image of vertical lines $x=a$ with $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$.
- Assume that $a \neq-\frac{\pi}{2}, 0, \frac{\pi}{2}$. The image of the vertical line $x=a$ is given by: $u=\sin a \cosh y, v=\cos a \sinh y,-\infty<y<\infty$. Since $-\frac{\pi}{2}<a<\frac{\pi}{2}$ and $a \neq 0$, it follows that $\sin a \neq 0$ and $\cos a \neq 0$, whence $\cosh y=\frac{u}{\sin a}$ and $\sinh y=\frac{v}{\cos a}$. The identity $\cosh ^{2} y-\sinh ^{2} y=1$ gives: $\left(\frac{\left.L^{\sin }\right)^{2}}{\sin a}-\left(\frac{v}{\cos a}\right)^{2}=1\right.$. It represents a hyperbola with vertices at $( \pm \sin a, 0)$ and slant asymptotes $v= \pm\left(\frac{\cos a}{\sin a}\right) u$.
Because the point $(a, 0)$ is on the line $x=a$, the point $(\sin a, 0)$ must be on the image of the line. Therefore, the image of the vertical line $x=a$, with $-\frac{\pi}{2}<a<\frac{\pi}{2}$ and $a \neq 0$, under $w=\sin z$ is the branch of the hyperbola that contains the point ( $\sin a, 0$ ).
Because $\sin (-z)=-\sin z$, for all $z$, the image of the line $x=-a$ is a branch of the hyperbola containing the point $(-\sin a, 0)$.
Therefore, the pair $x=a$ and $x=-a$, with $-\frac{\pi}{2}<a<\frac{\pi}{2}$ and $a \neq 0$, are mapped onto the full hyperbola.


## The Mapping $w=\sin z$ (Cont'd)

- Summary:


- The image of the line $x=-\frac{\pi}{2}$ is the set of points $u \leq-1$ on the negative real axis.
- The image of the line $x=\frac{\pi}{2}$ is the set of points $u \geq 1$ on the positive real axis.
- The image of the line $x=0$ is the imaginary axis $u=0$.

In summary, the image of the infinite vertical strip $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$,
$-\infty<y<\infty$, under $w=\sin z$, is the entire $w$-plane.

## Following Horizontal Line Segments

- The image could also be found using horizontal line segments $y=b$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, instead of vertical lines.
- The images are: $u=\sin x \cosh b, v=\cos x \sinh b,-\frac{\pi}{2}<x<\frac{\pi}{2}$.
- When $b \neq 0$, we get $\left(\frac{u}{\cosh b}\right)^{2}+\left(\frac{v}{\operatorname{sinhb} b}\right)^{2}=1$, which is an ellipse with $u$-intercepts at ( $\pm \cosh b, 0$ ) and $v$-intercepts at ( $0, \pm \sinh b$ ).
- If $b>0$, then the image of the segment $y=b$ is the upper-half of the ellipse and the image of the segment $y=-b$ the bottom-half.


- Observe that if $b=0$, then the image of the line segment $y=0$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$, is the line segment $-1 \leq u \leq 1, v=0$ on the real axis.


## Riemann Surface I

- Since the complex sine function is periodic, the mapping $w=\sin z$ is not one-to-one on the complex plane. A Riemann surface helps visualize $w=\sin z$.
- Consider the mapping on the square $S_{0}$ defined by $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.






$S_{0}$ is mapped onto the elliptical region $E$.
- Similarly, the adjacent square $S_{1}$ defined by $\frac{\pi}{2} \leq x \leq \frac{3 \pi}{2}$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, also maps onto $E$.


## Riemann Surface II

- A Riemann surface is constructed by starting with two copies of $E, E_{0}$ and $E_{1}$, representing the images of $S_{0}$ and $S_{1}$, respectively. We cut $E_{0}$ and $E_{1}$ open along the line segments in the real axis from 1 to $\cosh \left(\frac{\pi}{2}\right)$ and from -1 to $-\cosh \left(\frac{\pi}{2}\right)$.
- Part of the Riemann surface consists of the two elliptical regions $E_{0}$ and $E_{1}$ with the black segments glued together and the dashed segments glued together.
- To complete the Riemann surface, we take for every integer $n$ an elliptical region $E_{n}$ representing the image of the square $S_{n}$ defined by $\frac{(2 n-1) \pi}{2} \leq x \leq \frac{(2 n+1) \pi}{2},-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Each region $E_{n}$ is cut open, as $E_{0}$ and $E_{1}$ were, and $E_{n}$ is glued to $E_{n+1}$ along their boundaries in a manner analogous to that used for $E_{0}$ and $E_{1}$.



## Subsection 5

## Complex Hyperbolic Functions

## Complex Hyperbolic Sine and Cosine

- The real hyperbolic sine and hyperbolic cosine functions are defined using the real exponential by $\sinh x=\frac{e^{x}-e^{-x}}{2}$ and $\cosh x=\frac{e^{x}+e^{-x}}{2}$.


## Definition (Complex Hyperbolic Sine and Cosine)

The complex hyperbolic sine and hyperbolic cosine functions are defined by:

$$
\sinh z=\frac{e^{z}-e^{-z}}{2} \quad \text { and } \quad \cosh z=\frac{e^{z}+e^{-z}}{2}
$$

- These agree with the real hyperbolic functions for real input.
- Unlike the real hyperbolic functions, the complex hyperbolic functions are periodic and have infinitely many zeros.
- The complex hyperbolic tangent, cotangent, secant, and cosecant:

$$
\begin{array}{ll}
\tanh z=\frac{\sinh z}{\cosh z}, & \operatorname{coth} z=\frac{\cosh z}{\sinh z} \\
\operatorname{sech} z=\frac{1}{\cosh z}, & \operatorname{csch} z=\frac{1}{\sinh z}
\end{array}
$$

## Derivatives of Complex Hyperbolic Functions

- The hyperbolic sine and cosine functions are entire because the functions $e^{z}$ and $e^{-z}$ are entire.
- Moreover, we have:

$$
\frac{d}{d z} \sinh z=\frac{d}{d z} \frac{e^{z}-e^{-z}}{2}=\frac{e^{z}+e^{-z}}{2}=\cosh z
$$

- A similar computation for cosh z yields

$$
\frac{d}{d z} \cosh z=\sinh z
$$

- Derivatives of Complex Hyperbolic Functions

$$
\begin{array}{lll}
\frac{d}{d z} \sinh z=\cosh z, & \frac{d}{d z} \cosh z=\sinh z, & \frac{d}{d z} \tanh z=\operatorname{sech}^{2} z \\
\frac{d}{d z} \operatorname{coth} z=-\operatorname{csch}^{2} z, & \frac{d}{d z} \operatorname{sech} z=-\operatorname{sech} z \tanh z, & \frac{d}{d z} \operatorname{csch} z=-\operatorname{csch} z \operatorname{coth} z
\end{array}
$$

## Relation To Sine and Cosine

- The real trigonometric and the real hyperbolic functions share many similar properties, e.g., $\frac{d}{d x} \sin x=\cos x$ and $\frac{d}{d x} \sinh x=\cosh x$.
- There is a simple connection between the complex trigonometric and hyperbolic functions: Replace $z$ with $i z$ in the definition of $\sinh z$ :

$$
\sinh (i z)=\frac{e^{i z}-e^{-i z}}{2}=i \frac{e^{i z}-e^{-i z}}{2 i}=i \sin z
$$

$$
\text { or }-i \sinh (i z)=\sin z
$$

- Substituting iz for $z$ in $\sin z$, we find $\sinh z=-i \sin (i z)$.
- After repeating this process for $\cos z$ and $\cosh z$, we obtain:

$$
\begin{array}{lll}
\sin z=-i \sinh (i z) & \text { and } & \cos z=\cosh (i z), \\
\sinh z=-i \sin (i z) & \text { and } & \cosh z=\cos (i z) .
\end{array}
$$

- Other relations can be similarly derived:

$$
\tan (i z)=\frac{\sin (i z)}{\cos (i z)}=i \frac{\sinh z}{\cosh z}=i \tanh z
$$

## Obtaining Hyperbolic Identities

- Some of the more commonly used hyperbolic identities:
- $\sinh (-z)=-\sinh z$ and $\cosh (-z)=\cosh z$;
- $\cosh ^{2} z-\sinh ^{2} z=1$;
- $\sinh \left(z_{1} \pm z_{2}\right)=\sinh z_{1} \cosh z_{2} \pm \cosh z_{1} \sinh z_{2}$;
- $\cosh \left(z_{1} \pm z_{2}\right)=\cosh z_{1} \cosh z_{2} \pm \sinh z_{1} \sinh z_{2}$.
- Example: Verify that $\cosh \left(z_{1}+z_{1}\right)=\cosh z_{1} \cosh z_{2}+\sinh z_{1} \sinh z_{2}$, for all complex $z_{1}$ and $z_{2}$.
We have $\cosh \left(z_{1}+z_{2}\right)=\cos \left(i z_{1}+i z_{2}\right)$. So by a trigonometric identity and additional applications of the preceding identities,

$$
\begin{aligned}
\cosh \left(z_{1}+z_{2}\right) & =\cos \left(i z_{1}+i z_{2}\right) \\
& =\cos i z_{1} \cos i z_{2}-\sin i z_{1} \sin i z_{2} \\
& =\cos i z_{1} \cos i z_{2}+\left(-i \sin i z_{1}\right)\left(-i \sin i z_{2}\right) \\
& =\cosh z_{1} \cosh z_{2}+\sinh z_{1} \sinh z_{2}
\end{aligned}
$$

## Real versus Complex Trig and Hyperbolic Trig Functions

(i) In real analysis, the exponential function was just one of a number of apparently equally important elementary functions. In complex analysis, however, the complex exponential function assumes a much greater role: All of the complex elementary functions can be defined solely in terms of the complex exponential and logarithmic functions. The exponential and logarithmic functions can be used to evaluate, differentiate, integrate, and map using elementary functions.
(ii) As functions of a real variable $x, \sinh x$ and $\cosh x$ are not periodic. In contrast, the complex functions $\sinh z$ and $\cosh z$ are periodic. Moreover, $\cosh x$ has no zeros and $\sinh x$ has a single zero at $x=0$. The complex functions $\sinh z$ and $\cosh z$, on the other hand, both have infinitely many zeros.

## Subsection 6

## Inverse Trigonometric and Hyperbolic Functions

## Inverse Sine

- The complex sine function is periodic with a real period of $2 \pi$.
- Moreover, the sine function maps the complex plane onto the complex plane: Range $(\sin z)=\mathbb{C}$.
- Thus, for any complex number $z$ there exists infinitely many solutions $w$ to the equation $\sin w=z$. Rewrite as $\frac{e^{i w}-e^{-i w}}{2 i}=z$ or $e^{2 i w}-2 i z e^{i w}-1=0$. Use the quadratic formula to solve $e^{i w}=i z+\left(1-z^{2}\right)^{1 / 2}$.. This expression involves the two square roots of $1-z^{2}$. We solve for $w$ using the complex logarithm:
$i w=\ln \left[i z+\left(1-z^{2}\right)^{1 / 2}\right]$ or $w=-i \ln \left[i z+\left(1-z^{2}\right)^{1 / 2}\right]$.


## Definition (Inverse Sine)

The multiple-valued function $\sin ^{-1} z$ defined by:

$$
\sin ^{-1} z=-i \ln \left[i z+\left(1-z^{2}\right)^{1 / 2}\right]
$$

is called the inverse sine or arcsine, sometimes written $\arcsin z$.

## Values of Inverse Sine

- Find all values of $\sin ^{-1} \sqrt{5}$.

Set $z=\sqrt{5}$ in the formula defining $\sin ^{-1} z$ : $\sin ^{-1} \sqrt{5}=-i \ln \left[i \sqrt{5}+\left(1-(\sqrt{5})^{2}\right)^{1 / 2}\right]=-i \ln \left[i \sqrt{5}+(-4)^{1 / 2}\right]$.
The two square roots $(-4)^{1 / 2}$ of -4 are found to be $\pm 2 i$. So $\sin ^{-1} \sqrt{5}=-i \ln [i \sqrt{5} \pm 2 i]=-i \ln [(\sqrt{5} \pm 2) i]$. Because $(\sqrt{5} \pm 2) i$ is a pure imaginary number with positive imaginary part, we have $|(\sqrt{5} \pm 2) i|=\sqrt{5} \pm 2$ and $\arg ([(\sqrt{5} \pm 2) i])=\frac{\pi}{2}$. Thus, we have $\ln [(\sqrt{5} \pm 2) i]=\log _{e}(\sqrt{5} \pm 2)+i\left(\frac{\pi}{2}+2 n \pi\right)$ for $n=0, \pm 1, \pm 2, \ldots$. Observe that $\log _{e}(\sqrt{5}-2)=\log _{e} \frac{1}{\sqrt{5}+2}=\log _{e} 1-\log _{e}(\sqrt{5}+2)=$ $0-\log _{e}(\sqrt{5}+2)$, and so $\log _{e}(\sqrt{5} \pm 2)= \pm \log _{e}(\sqrt{5}+2)$. Therefore, $-i \ln [(\sqrt{5} \pm 2) i]=-i\left[\log _{e}(\sqrt{5} \pm 2)+i\left(\frac{\pi}{2}+2 n \pi\right)\right]=$
$-i\left[ \pm \log _{e}(\sqrt{5}+2)+i \frac{(4 n+1) \pi}{2}\right]$, and so $\sin ^{-1} \sqrt{5}=\frac{(4 n+1) \pi}{2} \pm i \log _{e}(\sqrt{5}+2)$, for $n=0, \pm 1, \pm 2, \ldots$.

## Inverse Cosine and Tangent

- Similarly, we may solve the equations $\cos w=z$ and $\tan w=z$.


## Definition (Inverse Cosine and Inverse Tangent)

The multiple-valued function $\cos ^{-1} z$ defined by:

$$
\cos ^{-1} z=-i \ln \left[z+i\left(1-z^{2}\right)^{1 / 2}\right]
$$

is called the inverse cosine. The multiple-valued function $\tan ^{-1} z$ defined by:

$$
\tan ^{-1} z=\frac{i}{2} \ln \left(\frac{i+z}{i-z}\right)
$$

is called the inverse tangent.

- The inverse cosine and inverse tangent are multiple-valued functions since they are defined in terms of the complex logarithm $\ln z$.
- The expression $\left(1-z^{2}\right)^{1 / 2}$ represents the two square roots of the complex number $1-z^{2}$.
- Every value of $w=\cos ^{-1} z$ satisfies the equation $\cos w=z$, and, similarly, every value of $w=\tan ^{-1} z$ satisfies the equation $\tan w=z$.


## Defining a Univalued Inverse Function

- The inverse sine and inverse cosine are multiple-valued functions that can be made single-valued by specifying a single value of the square root to use for the expression $\left(1-z^{2}\right)^{1 / 2}$ and a single value of the complex logarithm.
- The inverse tangent can be made single-valued by just specifying a single value of $\ln z$.
- Example: If $z=\sqrt{5}$, then the principal square root of $1-(\sqrt{5})^{2}=-4$ is $2 i$, and $\operatorname{Ln}(i \sqrt{5}+2 i)=\log _{e}(\sqrt{5}+2)+\frac{\pi i}{2}$. Using the definition, we get $f(\sqrt{5})=\frac{\pi}{2}-i \log _{e}(\sqrt{5}+2)$. Thus, the value of the function $f$ at $z=\sqrt{5}$ is the value of $\sin ^{-1} \sqrt{5}$ associated to $n=0$ and the square root $2 i$ in the preceding example.


## Branches and Analyticity

- Determining domains of branches of inverse trigonometric functions is complicated and not discussed.
- The derivatives of branches can be found using implicit differentiation: Suppose that $f_{1}$ is a branch of $F(z)=\sin ^{-1} z$. If $w=f_{1}(z)$, then $z=\sin w$. By differentiating both sides with respect to $z$ and applying the chain rule, $1=\cos w \cdot \frac{d w}{d z}$, or $\frac{d w}{d z}=\frac{1}{\cos w}$. From the trigonometric identity $\cos ^{2} w+\sin ^{2} w=1, \cos w=\left(1-\sin ^{2} w\right)^{1 / 2}$, and, since $z=\sin w, \cos w=\left(1-z^{2}\right)^{1 / 2}$. After substituting this expression for $\cos w$, we obtain $f_{1}^{\prime}(z)=\frac{d w}{d z}=\frac{1}{\left(1-z^{2}\right)^{1 / 2}}$. If we let $\sin ^{-1} z$ denote the branch $f_{1}$, then this formula may be restated as:

$$
\frac{d}{d z} \sin ^{-1} z=\frac{1}{\left(1-z^{2}\right)^{1 / 2}}
$$

We must use the same branch of the square root function that defined $\sin ^{-1} z$ when finding values of its derivative.

## Derivatives of Branches $\sin ^{-1} z, \cos ^{-1} z$ and $\tan ^{-1} z$

- In a similar manner, derivatives of branches of the inverse cosine and the inverse tangent can be found.
- In the following formulas, the symbols $\sin ^{-1} z, \cos ^{-1} z$, and $\tan ^{-1} z$ represent branches of the corresponding multiple-valued functions, so, the formulas for the derivatives hold only on the domains of these branches:

Derivatives of Branches $\sin ^{-1} z, \cos ^{-1} z$ and $\tan ^{-1} z$

$$
\frac{d}{d z} \sin ^{-1} z=\frac{1}{\left(1-z^{2}\right)^{1 / 2}}, \frac{d}{d z} \cos ^{-1} z=\frac{-1}{\left(1-z^{2}\right)^{1 / 2}}, \frac{d}{d z} \tan ^{-1} z=\frac{1}{1+z^{2}}
$$

- Again, when finding the value of a derivative, we must use the same square root as is used to define the branch.
- The formulas are similar to those for the derivatives of the real inverse trigonometric functions. The difference is the specific choice of a branch of the square root function.


## Derivative of a Branch of Inverse Sine

- Let $\sin ^{-1} z$ represent a branch of the inverse sine obtained by using the principal branches of the square root and the logarithm. Find the derivative of this branch at $z=i$.
Note that this branch is differentiable at $z=i$ because $1-i^{2}=2$ is not on the branch cut of the principal branch of the square root function, and because $i(i)+\left(1-i^{2}\right)^{1 / 2}=-1+\sqrt{2}$ is not on the branch cut of the principal branch of the complex logarithm.
We have:

$$
\left.\frac{d}{d z} \sin ^{-1} z\right|_{z=i}=\left.\frac{1}{\left(1-z^{2}\right)^{1 / 2}}\right|_{z=i}=\frac{1}{\left(1-i^{2}\right)^{1 / 2}}=\frac{1}{2^{1 / 2}} .
$$

Using the principal branch of the square root, we obtain $2^{1 / 2}=\sqrt{2}$. Therefore, the derivative is $\frac{1}{\sqrt{2}}$ or $\frac{1}{2} \sqrt{2}$.

## Inverse Hyperbolic Functions

- The inverses of the hyperbolic functions are also defined in terms of the complex logarithm because the hyperbolic functions are defined in terms of the complex exponential.


## Definition (Inverse Hyperbolic Sine, Cosine, and Tangent)

The multiple-valued functions $\sinh ^{-1} z, \cosh ^{-1} z$, and $\tanh ^{-1} z$, defined by:

$$
\begin{gathered}
\sinh ^{-1} z=\ln \left[z+\left(z^{2}+1\right)^{1 / 2}\right], \quad \cosh ^{-1} z=\ln \left[z+\left(z^{2}-1\right)^{1 / 2}\right] \\
\tanh ^{-1} z=\frac{1}{2} \ln \left(\frac{1+z}{1-z}\right)
\end{gathered}
$$

are called the inverse hyperbolic sine, the inverse hyperbolic cosine, and the inverse hyperbolic tangent, respectively.

- The expressions given in the definition allow us to solve equations involving the complex hyperbolic functions.
- In particular, if $w=\sinh ^{-1} z$, then $\sinh w=z$; if $w=\cosh ^{-1} z$, then $\cosh w=z$; and if $w=\tanh ^{-1} z$, then $\tanh w=z$.


## Analyticity

- Branches of the inverse hyperbolic functions are defined by choosing branches of the square root and complex logarithm.
- The derivative of a branch can be found using implicit differentiation: Here $\sinh ^{-1} z, \cosh ^{-1} z$ and $\tanh ^{-1} z$ represent branches of the corresponding multiple-valued functions.


## Derivatives of Branches $\sinh ^{-1} z, \cosh ^{-1} z, \tanh ^{-1} z$

$$
\begin{gathered}
\frac{d}{d z} \sinh ^{-1} z=\frac{1}{\left(z^{2}+1\right)^{1 / 2}}, \quad \frac{d}{d z} \cosh ^{-1} z=\frac{1}{\left(z^{2}-1\right)^{1 / 2}}, \\
\frac{d}{d z} \tanh ^{-1} z=\frac{1}{1-z^{2}}
\end{gathered}
$$

- We must be consistent in our use of branches when evaluating derivatives.
- The formulas are the same as the ones for the derivatives of the real inverse hyperbolic functions except for the choice of branch.


## Computing Inverse Hyperbolic Cosine

- Let $\cosh ^{-1} z$ represent the branch of the inverse hyperbolic cosine obtained by using the branch $f_{2}(z)=\sqrt{r} e^{i \theta / 2}, 0<\theta<2 \pi$, of the square root and the principal branch of the complex logarithm. Find the values: (a) $\cosh ^{-1} \frac{\sqrt{2}}{2}$
(b) $\left.\frac{d}{d z} \cosh ^{-1} z\right|_{z=\sqrt{2} / 2}$.
(a) We use $\cosh ^{-1} z=\ln \left[z+\left(z^{2}-1\right)^{1 / 2}\right]$ with $z=\frac{1}{2} \sqrt{2}$ and the stated branches of the square root and logarithm. When $z=\frac{1}{2} \sqrt{2}$, we have that $z^{2}-1=-\frac{1}{2}$. Since $-\frac{1}{2}$ has exponential form $\frac{1}{2} e^{i \pi}$, the square root given by the branch $f_{2}$ is: $f_{2}\left(\frac{1}{2} e^{i \pi}\right)=\sqrt{\frac{1}{2}} e^{i \pi / 2}=\frac{1}{\sqrt{2}} i=\frac{\sqrt{2}}{2} i$.
The value of our branch of the inverse cosine is then given by: $\cosh ^{-1} \frac{\sqrt{2}}{2}=\ln \left[z+\left(z^{2}-1\right)^{1 / 2}\right]=\ln \left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2} i\right]$. Because $\left|\frac{1}{2} \sqrt{2}+\frac{1}{2} \sqrt{2} i\right|=1$ and $\operatorname{Arg}\left(\frac{1}{2} \sqrt{2}+\frac{1}{2} \sqrt{2} i\right)=\frac{\pi}{4}$, the principal branch of the logarithm is $\log _{e} 1+i \frac{\pi}{4}=\frac{\pi i}{4}$. Therefore, $\cosh ^{-1} \frac{\sqrt{2}}{2}=\frac{\pi i}{4}$.


## Computing The Derivative of Inverse Hyperbolic Cosine

(b) We have

$$
\begin{aligned}
\left.\frac{d}{d z} \cosh ^{-1} z\right|_{z=\sqrt{2} / 2} & =\left.\frac{1}{\left(z^{2}-1\right)^{1 / 2}}\right|_{z=\sqrt{2} / 2} \\
& =\frac{1}{\left[(\sqrt{2} / 2)^{2}-1\right]^{1 / 2}} \\
& =\frac{1}{(-1 / 2)^{1 / 2}}
\end{aligned}
$$

After using $f_{2}$ to find the square root in this expression we obtain:

$$
\left.\frac{d}{d z} \cosh ^{-1} z\right|_{z=\sqrt{2} / 2}=\frac{1}{\sqrt{2} i / 2}=-\sqrt{2} i
$$

## The Riemann Surface of the Sine Revisited

- The multiple-valued function $F(z)=\sin ^{-1} z$ can be visualized using the Riemann surface constructed for $\sin z$ previously.


In order to see the image of a point $z_{0}$ under the multiple-valued mapping $w=\sin ^{-1} z$, we imagine that $z_{0}$ is lying in the $x y$-plane. We then consider all points on the Riemann surface lying directly over $z_{0}$.
Each of these points on the surface corresponds to a unique point in one of the squares $S_{n}$ described previously.

Thus, this infinite set of points in the Riemann surface represents the infinitely many images of $z_{0}$ under $w=\sin ^{-1} z$.

