Introduction to Complex Analysis

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Subsection 1

Real Integrals

Definite Integrals

- If $F(x)$ is an antiderivative of a continuous function f, i.e., F is a function for which $F'(x) = f(x)$, then the definite integral of f on the interval $[a, b]$ is the number

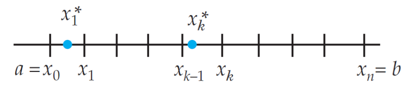
$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

- Example: $\int_{-1}^2 x^2 dx = \frac{1}{3}x^3\Big|_{-1}^2 = \frac{8}{3} - \frac{-1}{3} = 3.$
- The fundamental theorem of calculus is a method of evaluating $\int_a^b f(x)dx$; it is not the definition of $\int_a^b f(x)dx$.
- We next define:
  - The definite (or Riemann) integral of a function f;
  - Line integrals in the Cartesian plane.

  Both definitions rest on the limit concept.

Steps Leading to the Definition of the Definite Integral

1. Let f be a function of a single variable x defined at all points in a closed interval $[a, b]$.
2. Let P be a partition:
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
of $[a, b]$ into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$.
3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, i.e., the length of the longest subinterval.
4. Choose a number x_k^* in each subinterval $[x_{k-1}, x_k]$ of $[a, b]$.



5. Form n products $f(x_k^*)\Delta x_k$, $k = 1, 2, \ldots, n$, and then sum these products:
$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

The Definition of the Definite Integral

Definition (Definite Integral)

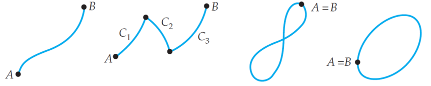
The **definite integral** of f on $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

- Whenever the limit exists we say that f is **integrable** on the interval $[a, b]$ or that the definite integral of f **exists**.
- It can be proved that if f is continuous on $[a, b]$, then the integral exists.

Terminology About Curves

- Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the initial and terminal points of C $(x(a), y(a))$, $(x(b), y(b))$ be denoted by A, B. We say that:
  - (i) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).
  - (ii) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, i.e., the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.
  - (iii) C is a **simple curve** if the curve C does not cross itself except possibly at $t = a$ and $t = b$.
  - (iv) C is a **closed curve** if $A = B$.
  - (v) C is a **simple closed curve** if the curve C does not cross itself and $A = B$, i.e., C is simple and closed.



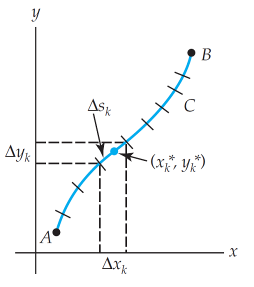
(a) Smooth curve and simple (b) Piecewise smooth curve and simple (c) Closed but not simple (d) Simple closed curve

Steps Leading to the Definition of Line Integrals

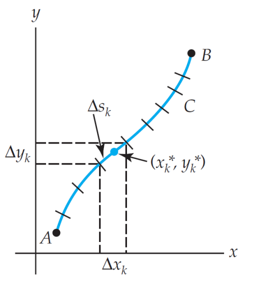
1. Let G be a function of two real variables x and y, defined at all points on a smooth curve C that lies in some region of the xy-plane. Let C be defined by the parametrization $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.
2. Let P be a partition of the parameter interval $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

The partition P induces a partition of the curve C into n subarcs of length Δs_k. Let the projection of each subarc onto the x- and y-axes have lengths Δx_k and Δy_k, respectively.



Steps Leading to the Definition of Line Integrals (Cont'd)



3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, that is, the length of the longest subinterval.
4. Choose a point (x_k^*, y_k^*) on each subarc of C.
5. Form n products $G(x_k^*, y_k^*)\Delta x_k$, $G(x_k^*, y_k^*)\Delta y_k$, $G(x_k^*, y_k^*)\Delta s_k$, $k = 1, 2, \ldots, n$, and then sum these products
$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k, \quad \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

The Definition of Line Integrals

Definition (Line Integrals in the Plane)

(i) The **line integral of G along C with respect to x** is

$$\int_C G(x,y)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k.$$

(ii) The **line integral of G along C with respect to y** is

$$\int_C G(x,y)dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k.$$

(iii) The **line integral of G along C with respect to arc length s** is

$$\int_C G(x,y)ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k.$$

- If G is continuous on C, then the three types of line integrals exist.
- The curve C is referred to as the **path of integration**.

Method of Evaluation: C Defined Parametrically

- Convert a line integral to a definite integral in a single variable.
- If C is a smooth curve parametrized by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, then replace
  - x and y in the integral by the functions $x(t)$ and $y(t)$;
  - the appropriate differential dx, dy, or ds by

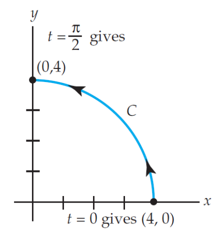
$$x'(t)dt, \quad y'(t)dt, \quad \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

- The term $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$ is called the **differential of the arc length**.
- The line integrals become definite integrals in which the variable of integration is the parameter t:

$$\begin{aligned}
\int_C G(x,y)dx &= \int_a^b G(x(t),y(t))x'(t)dt, \\
\int_C G(x,y)dy &= \int_a^b G(x(t),y(t))y'(t)dt, \\
\int_C G(x,y)ds &= \int_a^b G(x(t),y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}dt.
\end{aligned}$$

Evaluation of a Line Integral I

- Evaluate $\int_C xy^2 dx$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.



We have

$$dx = -4\sin t dt.$$

Thus,

$$\begin{aligned} \int_C xy^2 dx &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t dt) \\ &= -256\int_0^{\pi/2} \sin^3 t \cos t dt \\ &= -256[\tfrac{1}{4}\sin^4 t]_0^{\pi/2} \\ &= -64. \end{aligned}$$

Evaluation of a Line Integral II

- Evaluate $\int_C xy^2 dy$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.

  We have
  $$dy = 4\cos t dt.$$
  Thus,
  $$\begin{aligned} \int_C xy^2 dy &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2 (4\cos t dt) \\ &= 256 \int_0^{\pi/2} \sin^2 t \cos^2 t dt \\ &= 256 \int_0^{\pi/2} \frac{1}{4} \sin^2 2t dt \\ &= 64 \int_0^{\pi/2} \frac{1}{2}(1 - \cos 4t) dt \\ &= 32[t - \frac{1}{4}\sin 4t]_0^{\pi/2} = 16\pi. \end{aligned}$$

Evaluation of a Line Integral III

- Evaluate $\int_C xy^2 ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t$, $0 \leq t \leq \frac{\pi}{2}$.

  We have
  $$ds = \sqrt{16(\sin^2 t + \cos^2 t)}dt = 4dt.$$
  Therefore,
  $$\begin{aligned} \int_C xy^2 ds &= \int_0^{\pi/2} (4\cos t)(4\sin t)^2 (4dt) \\ &= 256 \int_0^{\pi/2} \sin^2 t \cos t dt \\ &= 256[\tfrac{1}{3}\sin^3 t]_0^{\pi/2} \\ &= \tfrac{256}{3}. \end{aligned}$$

Method of Evaluation: C Defined by a Function

- If the path of integration C is the graph of an explicit function $y = f(x)$, $a \leq x \leq b$, then we can use x as a parameter:
- The differential of y is $dy = f'(x)dx$, and the differential of arc length is $ds = \sqrt{1+[f'(x)]^2}dx$.
- We, thus, obtain the definite integrals:

$$\begin{aligned} \int_C G(x,y)dx &= \int_a^b G(x,f(x))dx, \\ \int_C G(x,y)dy &= \int_a^b G(x,f(x))f'(x)dx, \\ \int_C G(x,y)ds &= \int_a^b G(x,f(x))\sqrt{1+[f'(x)]^2}dx. \end{aligned}$$

- A line integral along a piecewise smooth curve C is defined as the sum of the integrals over the various smooth pieces.
- Example: To evaluate $\int_C G(x,y)ds$ when C is composed of two smooth curves C_1 and C_2, we write

$$\int_C G(x,y)ds = \int_{C_1} G(x,y)ds + \int_{C_2} G(x,y)ds.$$

Notation for Line Integrals

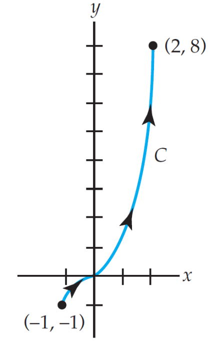
- In many applications, line integrals appear as a sum
$$\int_C P(x,y)dx + \int_C Q(x,y)dy.$$
- It is common practice to write this sum as one integral without parentheses as
$$\int_C P(x,y)dx + Q(x,y)dy$$
or simply
$$\int_C Pdx + Qdy.$$
- A line integral along a closed curve C is usually denoted by
$$\oint_C Pdx + Qdy.$$

C Defined by an Explicit Function

- Evaluate $\int_C xydx + x^2dy$, where C is the graph of $y = x^3$, $-1 \le x \le 2$.

  We have $dy = 3x^2dx$. Therefore,

$$\begin{aligned}\int_C xydx + x^2dy &= \int_{-1}^2 xx^3dx + x^2 3x^2dx \\ &= \int_{-1}^2 (x^4 + 3x^4)dx \\ &= \int_{-1}^2 4x^4dx \\ &= \frac{4}{5}x^5\Big|_{-1}^2 \\ &= \frac{4}{5}(32 - (-1)) = \frac{132}{5}.\end{aligned}$$



C a Closed Curve

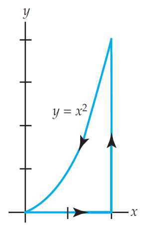
- Evaluate $\oint_C x dx$, where C is the circle defined by $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$.

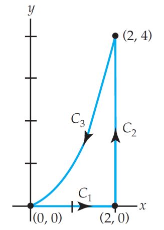
  We have $dx = -\sin t dt$, whence:

$$\begin{aligned} \oint_C x dx &= \int_0^{2\pi} \cos t(-\sin t dt) \\ &= \frac{1}{2}\cos^2 t\Big|_0^{2\pi} \\ &= \frac{1}{2}(1-1) \\ &= 0. \end{aligned}$$

C Another Closed Curve

- Evaluate $\oint_C y^2dx - x^2dy$, where C is the closed curve shown on the left.





C is piecewise smooth. So, the given integral is expressed as a sum of integrals, i.e., we write $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$, with C_1, C_2, C_3 as shown on the right.

  - On C_1, with x as a parameter: $\int_{C_1} y^2dx - x^2dy = \int_0^2 0dx - x^2(0) = 0$.
  - On C_2, with y as a parameter: $\int_{C_2} y^2dx - x^2dy = \int_0^4 y^2(0) - 4dy = -\int_0^4 4dy = -16$.
  - On C_3, we again use x as a parameter. From $y = x^2$, we get $dy = 2xdx$. Thus, $\int_{C_3} y^2dx - x^2dy = \int_2^0 (x^2)^2dx - x^2(2xdx) = \int_2^0 (x^4 - 2x^3)dx = (\frac{1}{5}x^5 - \frac{1}{2}x^4)\Big|_2^0 = \frac{8}{5}$.

Hence, $\oint_C y^2dx - x^2dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + (-16) + \frac{8}{5} = -\frac{72}{5}$.

Orientation of a Curve

- If C is not a closed curve, then we say the **positive direction** on C, or that C has **positive orientation**, if we traverse C from its initial point A to its terminal point B, i.e., if $x = x(t), y = y(t)$, $a \leq t \leq b$, are parametric equations for C, then the positive direction on C corresponds to increasing values of the parameter t.
- If C is traversed in the sense opposite to that of the positive orientation, then C is said to have **negative orientation**.
- If C has an orientation (positive or negative), then the **opposite curve**, the curve with the opposite orientation, will be denoted $-C$.
- Then
$$\int_{-C} Pdx + Qdy = -\int_C Pdx + Qdy,$$
or, equivalently
$$\int_{-C} Pdx + Qdy + \int_C Pdx + Qdy = 0.$$
- A line integral is independent of the parametrization of C, provided C is given the same orientation.

Subsection 2

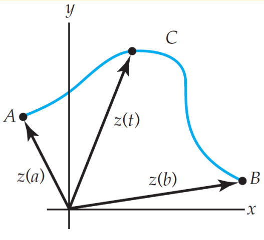
Complex Integrals

Curves Revisited

- Suppose the continuous real-valued functions $x = x(t), y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the complex plane.
- By considering $z = x + iy$, we can describe the points z on C by means of a complex-valued function of a real variable t, called a **parametrization** of C: $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.

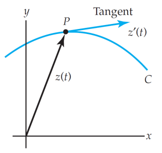
  Example: The parametric equations $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$, describe a unit circle centered at the origin. A parametrization of this circle is $z(t) = \cos t + i \sin t$, or $z(t) = e^{it}$, $0 \leq t \leq 2\pi$.
- The point $z(a) = x(a) + iy(a)$ or $A = (x(a), y(a))$ is called the **initial point** of C. and $z(b) = x(b) + iy(b)$ or $B = (x(b), y(b))$ the **terminal point**.

  As t varies from $t = a$ to $t = b$, C is being traced out by the moving arrowhead of the vector corresponding to $z(t)$.



Smooth Curves and Contours

- Suppose the derivative of $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is $z'(t) = x'(t) + iy'(t)$.
- We say C is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$.

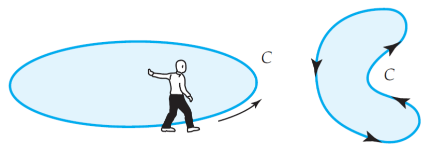


Since the vector $z'(t)$ is not zero at any point P on C, the vector $z'(t)$ is tangent to C at P. In other words, a smooth curve has a continuously turning tangent.

- A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together.
- A curve C in the complex plane is **simple** if $z(t_1) \neq z(t_2)$, for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$.
- C is a **closed curve** if $z(a) = z(b)$.
- C is a **simple closed curve** if it is simple and closed.
- A piecewise smooth curve C is also called a **contour** or **path**.

Positive and Negative Directions

- We define the **positive direction** on a contour C to be the direction on the curve corresponding to increasing values of the parameter t. It is also said that the curve C has **positive orientation**.
- In the case of a *simple closed curve* C, the **positive direction** roughly corresponds to the counterclockwise direction or the direction that a person must walk on C in order to keep the interior of C to the left.



- The **negative direction** on a contour C is the direction opposite the positive direction.
- If C has an orientation, the **opposite curve**, that is, a curve with opposite orientation, is denoted by $-C$.
- On a *simple closed curve*, the **negative direction** corresponds to the clockwise direction.

Steps Leading to the Definition of the Complex Integral I

1. Let f be a function of a complex variable z defined at all points on a smooth curve C that lies in some region of the plane. Suppose C is defined by the parametrization $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.
2. Let P be a partition of the parameter interval $[a, b]$ into n subintervals $[t_{k-1}, t_k]$ of length $\Delta t_k = t_k - t_{k-1}$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

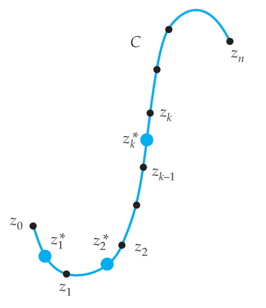
The partition P induces a partition of the curve C into n subarcs whose initial and terminal points are the pairs of numbers

$$\begin{array}{ll} z_0 = x(t_0) + iy(t_0), & z_1 = x(t_1) + iy(t_1), \\ z_1 = x(t_1) + iy(t_1), & z_2 = x(t_2) + iy(t_2), \\ \vdots & \vdots \\ z_{n-1} = x(t_{n-1}) + iy(t_{n-1}), & z_n = x(t_n) + iy(t_n). \end{array}$$

Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, \ldots, n$.

Steps Leading to the Definition of the Complex Integral II

3. Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, i.e., the length of the longest subinterval.
4. Choose a point $z_k^* = x_k^* + iy_k^*$ on each subarc of C.



5. Form n products $f(z_k^*)\Delta z_k$, $k = 1, 2, \ldots, n$, and then sum these products: $\sum_{k=1}^n f(z_k^*)\Delta z_k$.

The Definition of the Complex Integral

Definition (Complex Integral)

The **complex integral** of f on C is

$$\int_C f(z)dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k.$$

- If the limit exists, f is said to be **integrable** on C.
- The limit exists whenever f is continuous at all points on C and C is either smooth or piecewise smooth.
- Thus, we always assume that these conditions are fulfilled.
- By convention, we will use the notation $\oint_C f(z)dz$ to represent a complex integral around a *positively oriented closed curve* C.
- The notations $\ointctrclockwise_C f(z)dz$, $\ointclockwise_C f(z)dz$ denote more explicitly integration in the positive and negative directions, respectively.
- We shall refer to $\int_C f(z)dz$ as a **contour integral**.

Complex-Valued Function of a Real Variable

- Example: If t represents a real variable, then $f(t) = (2t + i)^2$ is a complex number. For $t = 2$, $f(2) = (4 + i)^2 = 16 + 8i + i^2 = 15 + 8i$.
- If f_1 and f_2 are real-valued functions of a real variable t, then $f(t) = f_1(t) + if_2(t)$ is a complex-valued function of a real variable t.
- We are interested in integration of a complex-valued function $f(t) = f_1(t) + if_2(t)$ of a real variable t carried out over a real interval.
- Example: On the interval $0 \leq t \leq 1$, it seems reasonable for $f(t) = (2t + i)^2$ to write

$$\int_0^1 (2t + i)^2 dt = \int_0^1 (4t^2 - 1 + 4ti) dt = \int_0^1 (4t^2 - 1) dt + i \int_0^1 4t dt.$$

The integrals $\int_0^1 (4t^2 - 1) dt$ and $\int_0^1 4t dt$ are real, and could be called the real and imaginary parts of $\int_0^1 (2t + i)^2 dt$. Each can be evaluated using the fundamental theorem of calculus to get:
$\int_0^1 (2t + i)^2 dt = \left(\frac{4}{3}t^3 - t\right)\Big|_0^1 + i\, 2t^2\Big|_0^1 = \frac{1}{3} + 2i.$

Integral of Complex Valued Function of a Real Variable

- If f_1 and f_2 are real-valued functions of a real variable t continuous on a common interval $a \leq t \leq b$, then we define the **integral** of the complex-valued function $f(t) = f_1(t) + if_2(t)$ on $a \leq t \leq b$ by

$$\int_a^b f(t)dt = \int_a^b f_1(t)dt + i\int_a^b f_2(t)dt.$$

- The continuity of f_1 and f_2 on $[a, b]$ guarantees that both integrals on the right exist.
- If $f(t) = f_1(t) + if_2(t)$ and $g(t) = g_1(t) + ig_2(t)$, are complex-valued functions of a real variable t continuous on $a \leq t \leq b$, then
  - $\int_a^b kf(t)dt = k\int_a^b f(t)dt$, k a complex constant;
  - $\int_a^b (f(t) + g(t))dt = \int_a^b f(t)dt + \int_a^b g(t)dt$;
  - $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$, if $c \in [a, b]$;
  - $\int_b^a f(t)dt = -\int_a^b f(t)dt$.

Evaluation of Contour Integrals

- If we use $u+iv$ for f, $\Delta x + i\Delta y$ for Δz, lim for $\lim_{\|P\|\to 0}$ and $\sum$ for $\sum_{k=1}^n$, we get $\int_C f(z)dz = \lim \sum (u+iv)(\Delta x + i\Delta y) = \lim [\sum (u\Delta x - v\Delta y) + i\sum (v\Delta x + u\Delta y)]$.
- Thus, we have
$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy.$$
- If $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, are parametric equations of C, then $dx = x'(t)dt$, $dy = y'(t)dt$.
- Now we obtain $\int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt + i\int_a^b [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]dt$.
- This is the same as $\int_a^b f(z(t))z'(t)dt$ when the integrand $f(z(t))z'(t) = [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)]$ is multiplied out and $\int_a^b f(z(t))z'(t)dt$ is expressed in terms of its real and imaginary parts.

Evaluating of a Contour Integral

Theorem (Evaluation of a Contour Integral)

If f is continuous on a smooth curve C given by $z(t)=x(t)+iy(t)$, $a\leq t\leq b$, then

$$\int_C f(z)dz=\int_a^b f(z(t))z'(t)dt.$$

- Example: Evaluate $\int_C \overline{z}dz$, where C is given by $x=3t$, $y=t^2$, $-1\leq t\leq 4$.

  A parametrization of the contour C is $z(t)=3t+it^2$. Thus, since $f(z)=\overline{z}$, we have $f(z(t))=\overline{3t+it^2}=3t-it^2$. Also, $z'(t)=3+2it$. Now, we have

$$\begin{aligned}\int_C \overline{z}dz &= \int_{-1}^4 (3t-it^2)(3+2it)dt\\ &= \int_{-1}^4 (2t^3+9t)dt+i\int_{-1}^4 3t^2dt\\ &= \left(\tfrac{1}{2}t^4+\tfrac{9}{2}t^2\right)\Big|_{-1}^4+i\ t^3\Big|_{-1}^4=195+65i.\end{aligned}$$

Another Evaluation of a Contour Integral

- Evaluate $\oint_C \frac{1}{z} dz$, where C is the circle $x = \cos t, y = \sin t$, $0 \leq t \leq 2\pi$.

  In this case $z(t) = \cos t + i \sin t = e^{it}, \ z'(t) = ie^{it}, \text{ and}$ $f(z(t)) = \frac{1}{z(t)} = e^{-it}$. Hence,

$$\begin{array}{rcl} \oint_c \frac{1}{z} dz & = & \int_0^{2\pi} (e^{-it}) ie^{it} dt \\ & = & i \int_0^{2\pi} dt \\ & = & 2\pi i. \end{array}$$

Using x as a Parameter

- For some curves the real variable x itself can be used as the parameter.
- Example: Evaluate $\int_C (8x^2 - iy)dz$ on the line segment $y = 5x$, $0 \le x \le 2$.

  We write $z = x + 5xi$, whence $dz = (1 + 5i)dx$. Therefore,

$$\begin{array}{rcl}\int_C (8x^2 - iy)dz & = & (1+5i)\int_0^2 (8x^2 - 5ix)dx \\ & = & (1+5i)\,\frac{8}{3}x^3\Big|_0^2 - (1+5i)i\,\frac{5}{2}x^2\Big|_0^2 \\ & = & \frac{214}{3} + \frac{290}{3}i.\end{array}$$

- If x and y are related by means of a continuous real function $y = f(x)$, then the corresponding curve C can be parametrized by $z(x) = x + if(x)$.

Properties of Contour Integrals

Theorem (Properties of Contour Integrals)

Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then:

(i) $\int_C kf(z)dz = k\int_C f(z)dz$, k a complex constant.

(ii) $\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$.

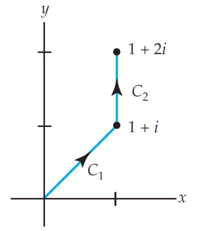
(iii) $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)dz = -\int_C f(z)dz$, where $-C$ denotes the curve having the opposite orientation of C.

- The four parts of the theorem also hold if C is a *piecewise smooth* curve in D.

C a Piecewise Smooth Curve

- Evaluate $\int_C (x^2 + iy^2)dz$, where C is the contour shown:



We write $\int_C (x^2 + iy^2)dz = \int_{C_1} (x^2 + iy^2)dz + \int_{C_2} (x^2 + iy^2)dz$.
Since the curve C_1 is defined by $y = x$, we use x as a parameter: $z(x) = x + ix$, $z'(x) = 1 + i$, $f(z) = x^2 + iy^2$, $f(z(x)) = x^2 + ix^2$,
whence, finally, $\int_{C_1} (x^2 + iy^2)dz = \int_0^1 (x^2 + ix^2)(i + 1)dx = (1 + i)^2 \int_0^1 x^2 dx = \frac{(1+i)^2}{3} = \frac{2}{3}i$.
The curve C_2 is defined by $x = 1$, $1 \leq y \leq 2$. If we use y as a parameter, then $z(y) = 1 + iy$, $z'(y) = i$, $f(z(y)) = 1 + iy^2$, and $\int_{C_2} (x^2 + iy^2)dz = \int_1^2 (1 + iy^2)idy = -\int_1^2 y^2 dy + i \int_1^2 dy = -\frac{7}{3} + i$.
Therefore $\int_C (x^2 + iy^2)dz = \frac{2}{3}i + (-\frac{7}{3} + i) = -\frac{7}{3} + \frac{5}{3}i$.

A Bounding Theorem

- We find an upper bound for the modulus of a contour integral.
- Recall the length of a plane curve $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$. If $z'(t) = x'(t) + iy'(t)$, then $|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$, whence $L = \int_a^b |z'(t)|dt$.

Theorem (A Bounding Theorem)

If f is continuous on a smooth curve C and if $|f(z)| \leq M$, for all z on C, then $|\int_C f(z)dz| \leq ML$, where L is the length of C.

- By triangle inequality, $|\sum_{k=1}^n f(z_k^*)\Delta z_k| \leq \sum_{k=1}^n |f(z_k^*)||\Delta z_k| \leq M\sum_{k=1}^n |\Delta z_k|$. Because $|\Delta z_k| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, we can interpret $|\Delta z_k|$ as the length of the chord joining the points z_k and z_{k-1} on C. Moreover, since the sum of the lengths of the chords cannot be greater than L, we get $|\sum_{k=1}^n f(z_k^*)\Delta z_k| \leq ML$. Finally, the continuity of f guarantees that $\int_C f(z)dz$ exists. Thus, letting $\|P\| \to 0$, the last inequality yields $|\int_C f(z)dz| \leq ML$.

A Bound for a Contour Integral

- Find an upper bound for the absolute value of $\int_C \frac{e^z}{z+1} dz$ where C is the circle $|z| = 4$.

  First, the length L (circumference) of the circle of radius 4 is 8π. Next, for all points z on the circle, we have that $|z+1| \geq |z| - 1 = 4 - 1 = 3$. Thus, $\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3}$. In addition, $|e^z| = |e^x(\cos y + i \sin y)| = e^x$. For points on the circle $|z| = 4$, the maximum that $x = \text{Re}(z)$ can be is 4, whence $\left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3}$. From the theorem, we have

$$\left|\int_C \frac{e^z}{z+1} dz\right| \leq \frac{8\pi e^4}{3}.$$

Single Contour: Many Parametrizations

- There is no unique parametrization for a contour C.
- Example: All of the following:

$$\begin{aligned} z(t) &= e^{it} = \cos t + i\sin t, \quad 0 \le t \le 2\pi, \\ z(t) &= e^{2\pi it} = \cos 2\pi t + i\sin 2\pi t, \quad 0 \le t \le 1, \\ z(t) &= e^{\pi it/2} = \cos \tfrac{\pi t}{2} + i\sin \tfrac{\pi t}{2}, \quad 0 \le t \le 4, \end{aligned}$$

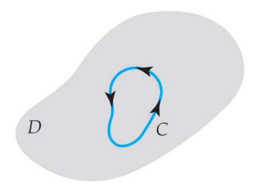
are all parametrizations, oriented in the positive direction, for the unit circle $|z| = 1$.

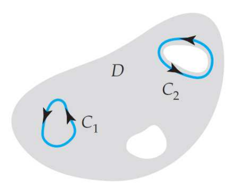
Subsection 3

Cauchy-Goursat Theorem

Simply and Multiply Connected Domains

- A **domain** is an open connected set in the complex plane.
- A domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D.





Example: The entire complex plane is a simply connected domain. The annulus defined by $1 < |z| < 2$ is not simply connected.

- A domain that is not simply connected is called a **multiply connected domain**.
  - A domain with one "hole" is **doubly connected**;
  - A domain with two "holes" **triply connected**, and so on.

Example: The open disk $|z| < 2$ is a simply connected domain. The open circular annulus $1 < |z| < 2$ is doubly connected.

Cauchy's Theorem

Cauchy's Theorem (1825)

Suppose that a function f is analytic in a simply connected domain D and that f' is continuous in D. Then, for every simple closed contour C in D,

$$\oint_C f(z)dz = 0.$$

- We apply Green's theorem and the Cauchy-Riemann equations. Recall from calculus that, if C is a positively oriented, piecewise smooth, simple closed curve forming the boundary of a region R within D, and if the real-valued functions $P(x, y)$ and $Q(x, y)$ along with their first-order partial derivatives are continuous on a domain that contains C and R, then $\oint_C Pdx + Qdy = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$. Since f' is continuous throughout D, the real and imaginary parts of $f(z) = u + iv$ and their first partial derivatives are continuous throughout D.

Proof of Cauchy's Theorem

- We have by Green's Theorem

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

By continuity of u, v and their first partial derivatives, $\oint_C f(z)dz = \oint_C u(x,y)dx - v(x,y)dy + i\oint_C v(x,y)dx + u(x,y)dy = \iint_R (-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dA + i\iint_R (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})dA$. f being analytic in D, u and v satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore,

$$\begin{aligned} \oint_C f(z)dz &= \iint_R (-\tfrac{\partial v}{\partial x} + \tfrac{\partial v}{\partial x})dA + i\iint_R (\tfrac{\partial v}{\partial y} - \tfrac{\partial v}{\partial y})dA \\ &= 0. \end{aligned}$$

The Cauchy-Goursat Theorem

- Edouard Goursat proved in 1883 that the assumption of continuity of f' is not necessary to reach the conclusion of Cauchy's theorem:

Cauchy-Goursat Theorem

Suppose that a function f is analytic in a simply connected domain D. Then, for every simple closed contour C in D,

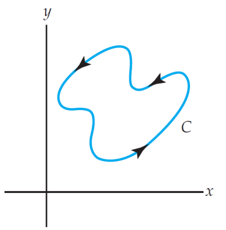
$$\oint_C f(z)dz = 0.$$

- Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat theorem can also be stated as:

  If f is analytic at all points within and on a simple closed contour C, then $\oint_C f(z)dz = 0$.

Applying the Cauchy-Goursat Theorem I

- Evaluate $\oint_C e^z dz$, where the contour C is shown below.



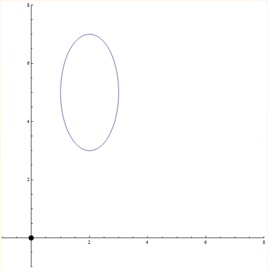
$f(z) = e^z$ is entire. Thus, it is analytic at all points within and on the simple closed contour C. It follows from the Cauchy-Goursat theorem that $\oint_C e^z dz = 0$.

- We have $\oint_C e^z dz = 0$, for any simple closed contour in the complex plane.
- Moreover, for any simple closed contour C and any entire function f, such as $f(z) = \sin z$, $f(z) = \cos z$, and $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n = 0, 1, 2, \ldots$, we also have

$$\oint_C \sin z dz = 0, \ \oint_C \cos z dz = 0, \ \oint_C p(z) dz = 0, \ \text{etc.}$$

Applying the Cauchy-Goursat Theorem II

- Evaluate $\oint_C \frac{1}{z^2} dz$, where C is the ellipse $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$. The rational function $f(z) = \frac{1}{z^2}$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the simple closed elliptical contour C.

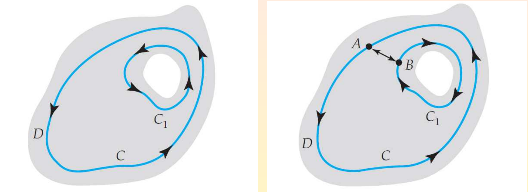


Thus, again by the Cauchy-Goursat Theorem, we get

$$\oint_C \frac{1}{z^2} dz = 0.$$

Cauchy-Goursat Theorem for Multiply Connected Domains

- If f is analytic in a multiply connected domain D, then we cannot conclude that $\oint_C f(z)dz = 0$, for every simple closed contour C in D.
- Suppose that D is a doubly connected domain and C and C_1 are simple closed contours placed as follows:



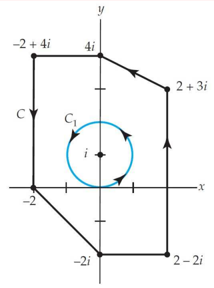
Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1.

By introducing the crosscut AB, the region bounded between the curves is now simply connected. So: $\oint_C f(z)dz + \int_{AB} f(z)dz + \oint_{-C_1} f(z)dz + \int_{-AB} f(z)dz = 0$ or $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

- This is sometimes called the **principle of deformation of contours**.
- It allows evaluation of an integral over a complicated simple closed contour C by replacing C with a more convenient contour C_1.

Applying Deformation of Contours

- Evaluate $\oint_C \frac{1}{z-i} dz$, where C is the black contour:



We choose the more convenient circular contour C_1 drawn in blue. By taking the radius of the circle to be $r = 1$, we are guaranteed that C_1 lies within C. C_1 is the circle $|z-i| = 1$. It can be parametrized by

$$z = i + e^{it},\ 0 \leq t \leq 2\pi.$$

From $z - i = e^{it}$ and $dz = ie^{it} dt$, we get:

$$\begin{aligned} \oint_C \frac{1}{z-i} dz &= \oint_{C_1} \frac{1}{z-i} dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt \\ &= i \int_0^{2\pi} dt = 2\pi i. \end{aligned}$$

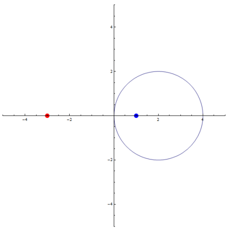
A Generalization

- This result can be generalized: If z_0 is any constant complex number interior to any simple closed contour C, and n an integer, we have
$$\oint_C \frac{1}{(z-z_0)^n} dz = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{if } n \neq 1 \end{cases}.$$
- That the integral is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem.
  - When $n = 0$ or negative, $\frac{1}{(z-z_0)^n}$ is a polynomial and therefore entire. Then, clearly, $\oint_C \frac{1}{(z-z_0)^n} dz = 0$.
  - It is not very difficult to see that the integral is still zero when n is a positive integer different from 1.
- Analyticity of the function f at all points within and on a simple closed contour C is sufficient to guarantee that $\oint_C f(z)dz = 0$.
- This result emphasizes that analyticity is not necessary, i.e., it can happen that $\oint_C f(z)dz = 0$ without f being analytic within C.
Example: If C is the circle $|z| = 1$, then $\oint_C \frac{1}{z^2} dz = 0$, but $f(z) = \frac{1}{z^2}$ is not analytic at $z = 0$ within C.

Applying the Formula for the Integral of $1/(z-z_0)^n$

- Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is circle $|z-2|=2$.

  The denominator factors as $z^2+2z-3=(z-1)(z+3)$. Thus, the integrand fails to be analytic at $z=1$ and $z=-3$.



Of these two points, only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$. By partial fractions

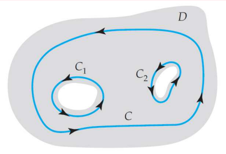
$$\frac{5z+7}{z^2+2z-3}=\frac{3}{z-1}+\frac{2}{z+3}.$$

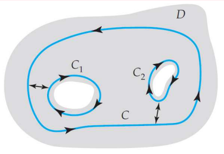
Hence, $\oint_C \frac{5z+7}{z^2+2z-3}dz=3\oint_C \frac{1}{z-1}dz+2\oint_C \frac{1}{z+3}dz$. The first integral has the value $2\pi i$, whereas the value of the second integral is 0 by the Cauchy-Goursat theorem. Hence,

$$\oint_C \frac{5z+7}{z^2+2z-3}dz=3(2\pi i)+2(0)=6\pi i.$$

Cauchy-Goursat Theorem: Multiply Connnected Domains

- If C, C_1, and C_2 are simple closed contours as shown below





and f is analytic on each of the three contours as well as at each point interior to C but exterior to both C_1 and C_2, then by introducing crosscuts between C_1 and C and between C_2 and C, we get $\oint_C f(z)dz + \oint_{-C_1} f(z)dz + \oint_{-C_2} f(z)dz = 0$, whence $\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$.

Cauchy-Goursat Theorem for Multiply Connnected Domains

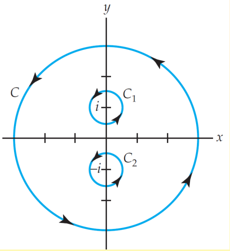
Suppose $C, C_1, \ldots, C_n$ are simple closed curves with a positive orientation, such that $C_1, C_2, \ldots, C_n$ are interior to C, but the regions interior to each C_k, $k = 1, 2, \ldots, n$, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1, 2, \ldots, n$, then $\oint_C f(z)dz = \sum_{k=1}^n \oint_{C_k} f(z)dz$.

Integrals in Multiply Connected Domains

- Evaluate $\oint_C \frac{1}{z^2+1}dz$, where C is the circle $|z|=4$.

  The denominator of the integrand factors as $z^2+1=(z-i)(z+i)$. So, the integrand $\frac{1}{z^2+1}$ is not analytic at $z=i$ and at $z=-i$. Both points lie within C. Using partial fractions, $\frac{1}{z^2+1}=\frac{1}{2i}\frac{1}{z-i}-\frac{1}{2i}\frac{1}{z+i}$. whence $\oint_C \frac{1}{z^2+1}dz=\frac{1}{2i}\oint_C(\frac{1}{z-i}-\frac{1}{z+i})dz$.

  Surround $z=i$ and $z=-i$ by circular contours C_1 and C_2, respectively, that lie entirely within C. The choice $|z-i|=\frac{1}{2}$ for C_1 and $|z+i|=\frac{1}{2}$ for C_2 will suffice.



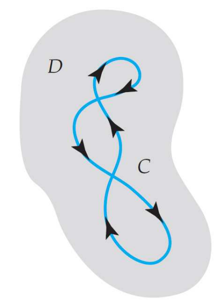
  We have $\oint_C \frac{1}{z^2+1}dz=$

  $\frac{1}{2i}\oint_{C_1}(\frac{1}{z-i}-\frac{1}{z+i})dz+\frac{1}{2i}\oint_{C_2}(\frac{1}{z-i}-\frac{1}{z+i})dz=\frac{1}{2i}\oint_{C_1}\frac{1}{z-i}dz-$
  $\frac{1}{2i}\oint_{C_1}\frac{1}{z+i}dz+\frac{1}{2i}\oint_{C_2}\frac{1}{z-i}dz-\frac{1}{2i}\oint_{C_2}\frac{1}{z+i}dz=\frac{1}{2i}2\pi i-0+0-\frac{1}{2i}2\pi i=0.$

Non-Simple Closed Contours

- Throughout the foregoing discussion we assumed that C was a simple closed contour, in other words, C did not intersect itself.
- It can be shown that the Cauchy-Goursat theorem is valid for any closed contour C in a simply connected domain D.
- For a contour C that is closed but not simple, if f is analytic in D, then

$$\oint_C f(z)dz = 0.$$



Subsection 4

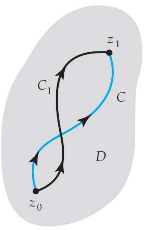
Independence of Path

Path Independence

Definition (Independence of the Path)

Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be **independent of the path** if its value is the same for all contours C in D with initial point z_0 and terminal point z_1.

- The Cauchy-Goursat theorem holds for closed contours, not just simple closed contours, in a simply connected domain D.
- Suppose that C and C_1 are two contours lying entirely in a simply connected domain D and both with initial point z_0 and terminal point z_1. C joined with $-C_1$ forms a closed contour. Thus, if f is analytic in D, $\int_C f(z)dz + \int_{-C_1} f(z)dz = 0$. Therefore, $\int_C f(z)dz = \int_{C_1} f(z)dz$.

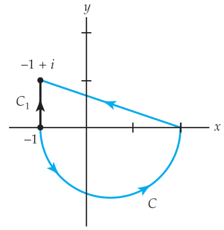


Theorem (Analyticity Implies Path Independence)

Suppose that a function f is analytic in a simply connected domain D and C is any contour in D. Then $\int_C f(z)dz$ is independent of the path C.

Choosing a Different Path

- Evaluate $\int_C 2zdz$, where C is the contour shown in blue.



The function $f(z) = 2z$ is entire. By the theorem, we can replace the piecewise smooth path C by any convenient contour C_1 joining $z_0 = -1$ and $z_1 = -1 + i$. We choose the contour C_1 to be the vertical line segment $x = -1, 0 \leq y \leq 1$.

Since $z = -1 + iy$, $dz = idy$. Therefore,

$$\begin{aligned} \int_C 2zdz &= \int_{C_1} 2zdz \\ &= \int_0^1 2(-1+iy)idy \\ &= \int_0^1 (-2i - 2y)dy \\ &= (-2iy - y^2)\Big|_0^1 \\ &= -1 - 2i. \end{aligned}$$

Antiderivatives

- A contour integral $\int_C f(z)dz$ that is independent of the path C is usually written $\int_{z_0}^{z_1} f(z)dz$, where z_0 and z_1 are the initial and terminal points of C.

Definition (Antiderivative)

Suppose that a function f is continuous on a domain D. If there exists a function F such that $F'(z) = f(z)$, for each z in D, then F is called an **antiderivative** of f.

Example: The function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$ since $F'(z) = \sin z$.

- The most general antiderivative, or **indefinite integral**, of a function $f(z)$ is written $\int f(z)dz = F(z) + C$, where $F'(z) = f(z)$ and C is some complex constant.
- Differentiability implies continuity, whence, since an antiderivative F of a function f has a derivative at each point in a domain D, it is necessarily analytic and hence continuous at each point in D.

Fundamental Theorem for Contour Integrals

Fundamental Theorem for Contour Integrals

Suppose that a function f is continuous on a domain D and F is an antiderivative of f in D. Then, for any contour C in D with initial point z_0 and terminal point z_1,

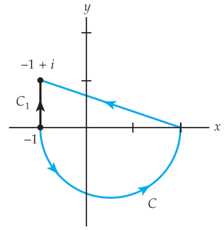
$$\int_C f(z)dz = F(z_1) - F(z_0).$$

- We prove the FTCI in the case when C is a smooth curve parametrized by $z = z(t)$, $a \leq t \leq b$. The initial and terminal points on C are $z(a) = z_0$ and $z(b) = z_1$. Since $F'(z) = f(z)$, for all z in D,

$$\begin{aligned} \int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt = \int_a^b F'(z(t))z'(t)dt \\ &= \int_a^b \frac{d}{dt}F(z(t))dt = F(z(t))\Big|_a^b \\ &= F(z(b)) - F(z(a)) \\ &= F(z_1) - F(z_0). \end{aligned}$$

Applying the Fundamental Theorem I

- The integral $\int_C 2zdz$, where C is shown



is independent of the path. Since $f(z) = 2z$ is an entire function, it is continuous. Moreover, $F(z) = z^2$ is an antiderivative of f since $F'(z) = 2z = f(z)$. Hence, by the Fundamental Theorem, we have

$$\begin{aligned} \int_{-1}^{-1+i} 2zdz &= z^2\Big|_{-1}^{-1+i} \\ &= (-1+i)^2 - (-1)^2 \\ &= -1-2i. \end{aligned}$$

Applying the Fundamental Theorem II

- Evaluate $\int_C \cos z dz$, where C is any contour with initial point $z_0 = 0$ and terminal point $z_1 = 2 + i$.

  $F(z) = \sin z$ is an antiderivative of $f(z) = \cos z$, since $F'(z) = \cos z = f(z)$. Therefore, by the Fundamental Theorem, we have

$$\begin{aligned} \int_C \cos z dz &= \int_0^{2+i} \cos z dz \\ &= \sin z|_0^{2+i} \\ &= \sin(2+i) - \sin 0 \\ &= \sin(2+i). \end{aligned}$$

Some Conclusions

- Observe that if the contour C is closed, then $z_0 = z_1$ and, consequently, $\oint_C f(z)dz = F(z_1) - F(z_0) = 0$.
- Since the value of $\int_C f(z)dz$ depends only on the points z_0 and z_1, this value is the same for any contour C in D connecting these points:

  If a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.

- Moreover, we have a sufficient condition:

  If f is continuous and $\int_C f(z)dz$ is independent of the path C in a domain D, then f has an antiderivative everywhere in D.

- Assume f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D and that F is a function defined by $F(z) = \int_{z_0}^{z} f(s)ds$, where s denotes a complex variable, z_0 is a fixed point in D, and z represents any point in D. We wish to show that $F'(z) = f(z)$, i.e., that $F(z) = \int_{z_0}^{z} f(s)ds$ is an antiderivative of f in D.

$F(z) = \int_{z_0}^{z} f(s)ds$ is an Antiderivative of f in D

- We have
$F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds = \int_{z}^{z+\Delta z} f(s)ds.$
Because D is a domain, we can choose Δz so that $z + \Delta z$ is in D. Moreover, z and $z + \Delta z$ can be joined by a straight segment. With z fixed, we can write $f(z)\Delta z = f(z) \int_{z}^{z+\Delta z} ds = \int_{z}^{z+\Delta z} f(z)ds$ or
$f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)ds$. Therefore, we have
$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds$. Since f is continuous at the point z, for any $\varepsilon > 0$, there exists a $\delta > 0$, so that $|f(s) - f(z)| < \epsilon$ whenever $|s - z| < \delta$. Consequently, if we choose Δz so that $|\Delta z| < \delta$, it follows from the ML-inequality, that
$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| = \left|\frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(s) - f(z)]ds\right| =$
$\left|\frac{1}{\Delta z}\right| \left|\int_{z}^{z+\Delta z} [f(s) - f(z)]ds\right| \leq |\frac{1}{\Delta z}|\varepsilon|\Delta z| = \varepsilon$. Hence,
$\lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z)$ or $F'(z) = f(z)$.

Existence of Antiderivative

- If f is an analytic function in a simply connected domain D, it is continuous throughout D. This implies, by the Path Independence Theorem, that path independence holds for f in D. Therefore,

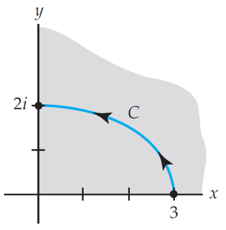
Theorem (Existence of Antiderivative)

Suppose that a function f is analytic in a simply connected domain D. Then f has an antiderivative in D, i.e., there exists a function F such that $F'(z) = f(z)$, for all z in D.

- We have seen that, for $|z| > 0$, $-\pi < \arg(z) < \pi$, $\frac{1}{z}$ is the derivative of $\text{Ln} z$. Thus, under some circumstances $\text{Ln} z$ is an antiderivative of $\frac{1}{z}$, but one must be careful!
  If D is the entire complex plane without the origin, $\frac{1}{z}$ is analytic in this multiply connected domain. If C is any simple closed contour containing the origin, it does not follow that $\oint_C \frac{1}{z} dz = 0$. In this case, $\text{Ln} z$ is not an antiderivative of $\frac{1}{z}$ in D since $\text{Ln} z$ is not analytic in D ($\text{Ln} z$ fails to be analytic on the non-positive real axis).

Using the Logarithmic Function

- Evaluate $\int_C \frac{1}{z} dz$, where C is the contour shown:



Suppose that D is the simply connected domain defined by $x > 0$, $y > 0$, i.e., the first quadrant. In this case, $\text{Ln}z$ is an antiderivative of $\frac{1}{z}$ since both these functions are analytic in D.

Therefore,

$$\int_C \frac{1}{z} dz = \int_3^{2i} \frac{1}{z} dz = \text{Ln}z\Big|_3^{2i} = \text{Ln}(2i) - \text{Ln}3.$$

Recall $\text{Ln}(2i) = \log_e 2 + \frac{\pi}{2} i$ and $\text{Ln}3 = \log_e 3$. Hence, $\int_C \frac{1}{z} dz = \log_e 2 + \frac{\pi}{2} i - \log_e 3 = \log_e \frac{2}{3} + \frac{\pi}{2} i$.

Using an Antiderivative of $z^{-1/2}$

- Evaluate $\int_C \frac{1}{z^{1/2}} dz$, where C is the line segment between $z_0 = i$ and $z_1 = 9$.

  We take $f_1(z) = z^{1/2}$ to be the principal branch of the square root function. In the domain $|z| > 0$, $-\pi < \arg(z) < \pi$, the function $\frac{1}{f_1(z)} = \frac{1}{z^{1/2}} = z^{-1/2}$ is analytic and possesses the antiderivative $F(z) = 2z^{1/2}$. Hence,

$$\begin{aligned} \int_C \frac{1}{z^{1/2}} dz &= \int_i^9 \frac{1}{z^{1/2}} dz \\ &= 2z^{1/2}\Big|_i^9 \\ &= 2[3 - (\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2})] \\ &= (6 - \sqrt{2}) - i\sqrt{2}. \end{aligned}$$

Integration-By-Parts

- In calculus indefinite integrals of certain kinds can be evaluated by integration by parts:
$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$
More compactly, $\int udv = uv - \int vdu$.
- Suppose f and g are analytic in a simply connected domain D. Then
$$\int f(z)g'(z)dz = f(z)g(z) - \int g(z)f'(z)dz.$$
- In addition, if z_0 and z_1 are the initial and terminal points of a contour C lying entirely in D, then
$$\int_{z_0}^{z_1} f(z)g'(z)dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} g(z)f'(z)dz.$$

The Mean Value Theorem for Definite Integrals

- The **Mean Value Theorem for Definite Integrals**: If f is a real function continuous on the closed interval $[a, b]$, then there exists a number c in the open interval (a, b), such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

- Let f be a complex function analytic in a simply connected domain D. Then, f is continuous at every point on a contour C in D with initial point z_0 and terminal point z_1.

  Unfortunately, no analog of the Mean Value Theorem exists for the contour integral $\int_{z_0}^{z_1} f(z)dz$.

Subsection 5

Cauchy's Integral Formulas

Cauchy's First Formula

- If f is analytic in a simply connected domain D and z_0 is a point in D, the quotient $\frac{f(z)}{z-z_0}$ is not defined at z_0 and, hence, is not analytic in D.
- Therefore, we cannot conclude that the integral of $\frac{f(z)}{z-z_0}$ around a simple closed contour C that contains z_0 is zero.
- Indeed,the integral of $\frac{f(z)}{z-z_0}$ around C has the value $2\pi i f(z_0)$.

Theorem (Cauchy's Integral Formula)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then, for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

- Let D be a simply connected domain, C a simple closed contour in D, and z_0 an interior point of C. In addition, let C_1 be a circle centered at z_0 with radius small enough so that C_1 lies within the interior of C. By the principle of deformation of contours, $\oint_C \frac{f(z)}{z-z_0} dz = \oint_{C_1} \frac{f(z)}{z-z_0} dz$.

Proof of Cauchy's Integral Formula

- From $\oint_C \frac{f(z)}{z-z_0}dz = \oint_{C_1} \frac{f(z)}{z-z_0}dz$, we get by adding and subtracting $f(z_0)$ in the numerator: $\oint_C \frac{f(z)}{z-z_0}dz = \oint_{C_1} \frac{f(z_0)-f(z_0)+f(z)}{z-z_0}dz = f(z_0)\oint_{C_1} \frac{1}{z-z_0}dz + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}dz$. We know that $\oint_{C_1} \frac{1}{z-z_0}dz = 2\pi i$, whence $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) + \oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}dz$.

  Since f is continuous at z_0, for any $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(z) - f(z_0)| < \varepsilon$, whenever $|z - z_0| < \delta$. In particular, if we choose C_1 to be $|z - z_0| = \frac{1}{2}\delta < \delta$, then by the *ML*-inequality, $\left|\oint_{C_1} \frac{f(z)-f(z_0)}{z-z_0}dz\right| \leq \frac{\varepsilon}{\delta/2}2\pi\frac{\delta}{2} = 2\pi\varepsilon$. Thus, the absolute value of the integral can be made arbitrarily small by taking the radius of the circle C_1 to be sufficiently small. This implies that the integral is 0. We conclude that $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$.

Using Cauchy's Integral Formula

- Cauchy's integral formula shows that the values of an analytic function f at points z_0 inside a simple closed contour C are determined by the values of f on the contour C.
- Since we often work problems without a simply connected domain explicitly defined, a more practical restatement is:

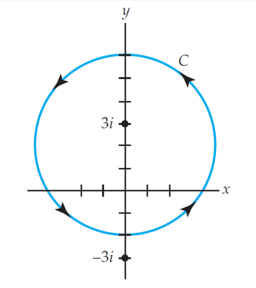
  If f is analytic at all points within and on a simple closed contour C, and z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$.

- Example: Evaluate $\oint_C \frac{z^2-4z+4}{z+i} dz$, where C is the circle $|z| = 2$.

  We identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula,
  $\oint_C \frac{z^2-4z+4}{z+i} dz = 2\pi i f(-i) = 2\pi i(3+4i) = \pi(-8+6i)$.

Another Application of Cauchys Integral Formula

- Evaluate $\oint_C \frac{z}{z^2+9} dz$, where C is the circle $|z - 2i| = 4$.



By factoring the denominator as $z^2 + 9 = (z - 3i)(z + 3i)$, we see that $3i$ is the only point within the closed contour C at which the integrand fails to be analytic. By rewriting the integrand as $\frac{z}{z^2+9} = \frac{\frac{z}{z+3i}}{z-3i}$, we identify $f(z) = \frac{z}{z+3i}$.

The function f is analytic at all points within and on the contour C. Hence, by Cauchy's integral formula

$$\oint_C \frac{z}{z^2+9} dz = \oint_C \frac{\frac{z}{z+3i}}{z-3i} dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi i.$$

Cauchy's Second Formula

- We prove that the values of the derivatives $f^{(n)}(z_0)$, $n = 1, 2, 3, \ldots$ of an analytic function are also given by an integral formula.

Theorem (Cauchy's Integral Formula for Derivatives)

Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then, for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

- Partial Proof (for $n = 1$): By the definition of the derivative and Cauchy's Integral Formula, $f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} =$
$$\lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z} \left[\oint_C \frac{f(z)}{z - (z_0 + \Delta z)} dz - \oint_C \frac{f(z)}{z - z_0} dz \right] =$$
$$\lim_{\Delta z \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} dz.$$

Prof of Cauchy's Second Formula for $n = 1$

- We work out some preliminaries:
  - Continuity of f on the contour C guarantees that f is bounded, i.e., there exists real number M, such that $|f(z)| \leq M$, for all points z on C.
  - In addition, let L be the length of C and let δ denote the shortest distance between points on C and the point z_0. Thus, for all points z on C, we have $|z - z_0| \geq \delta$, or $\frac{1}{|z-z_0|^2} \leq \frac{1}{\delta^2}$.
  - Furthermore, if we choose $|\Delta z| \leq \frac{1}{2}\delta$, then $|z - z_0 - \Delta z| \geq ||z - z_0| - |\Delta z|| \geq \delta - |\Delta z| \geq \frac{1}{2}\delta$, whence $\frac{1}{|z-z_0-\Delta z|} \leq \frac{2}{\delta}$.

Now, $\left|\oint_C \frac{f(z)}{(z-z_0)^2}dz - \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz\right| = \left|\oint_C \frac{-\Delta z\ f(z)}{(z-z_0-\Delta z)(z-z_0)^2}dz\right| \leq \frac{2ML|\Delta z|}{\delta^3}$. The last expression approaches zero as $\Delta z \to 0$, whence

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz.$$

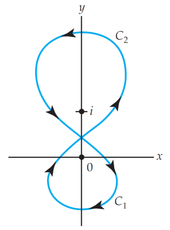
Using Cauchy's Integral Formula for Derivatives

- Evaluate $\oint_C \frac{z+1}{z^4+2iz^3}dz$, where C is the circle $|z|=1$.
  Inspection of the integrand shows that it is not analytic at $z=0$ and $z=-2i$, but only $z=0$ lies within the closed contour. By writing the integrand as $\dfrac{z+1}{z^4+2iz^3}=\dfrac{\frac{z+1}{z+2i}}{z^3}$ we can identify, $z_0=0,\ n=2$, and $f(z)=\frac{z+1}{z+2i}$. The quotient rule gives $f'(z)=\frac{-1+2i}{(z+2i)^2}$ and $f''(z)=\frac{2-4i}{(z+2i)^3}$, whence $f''(0)=\frac{2i-1}{4i}$. Therefore, we get

$$\begin{aligned}\oint_C \frac{z+1}{z^4+4z^3}dz &= \frac{2\pi i}{2!}f''(0)\\ &= \frac{2\pi i}{2!}\frac{2i-1}{4i}\\ &= -\frac{\pi}{4}+\frac{\pi}{2}i.\end{aligned}$$

Another Application of the Integral Formula for Derivatives

- Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}dz$, where C is the figure-eight contour shown below:



Although C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2. We write $\oint_C \frac{z^3+3}{z(z-i)^2}dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz = -\oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}dz + \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = -I_1 + I_2$.

  - $I_1 = \oint_{-C_1} \frac{\frac{z^3+3}{(z-i)^2}}{z}dz = 2\pi i f(0) = 2\pi i(-3) = -6\pi i$.
  - For I_2, $f(z) = \frac{z^3+3}{z}$, whence $f'(z) = \frac{2z^3-3}{z^2}$, and $f'(i) = 3+2i$. Thus, $I_2 = \oint_{C_2} \frac{\frac{z^3+3}{z}}{(z-i)^2}dz = \frac{2\pi i}{1!}f'(i) = 2\pi i(3+2i) = -4\pi + 6\pi i$.

Finally, $\oint_C \frac{z^3+3}{z(z-i)^2}dz = -I_1 + I_2 = 6\pi i + (-4\pi + 6\pi i) = -4\pi + 12\pi i$.

Subsection 6

Consequences of the Integral Formulas

The Derivatives of an Analytic Function are Analytic

Theorem (Derivative of an Analytic Function Is Analytic)

Suppose that f is analytic in a simply connected domain D. Then f possesses derivatives of all orders at every point z in D. The derivatives $f', f'', f''', \ldots$ are analytic functions in D.

- If $f(z) = u(x,y) + iv(x,y)$ is analytic in a simply connected domain D, its derivatives of all orders exist at any point z in D. Thus, f', f'', f''', $\ldots$ are continuous. From

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}, \\ f''(z) &= \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} \\ &\vdots \end{aligned}$$

  we can also conclude that the real functions u and v have continuous partial derivatives of all orders at a point of analyticity.

Cauchy's Inequality

Theorem (Cauchy's Inequality)

Suppose that f is analytic in a simply connected domain D and C is a circle defined by $|z - z_0| = r$ that lies entirely in D. If $|f(z)| \leq M$, for all points z on C, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

- From the hypothesis, $\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{r^{n+1}} \leq \frac{M}{r^{n+1}}$. Thus, by Cauchy's Formula for Derivatives and the ML-inequality,
$$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \leq \frac{n!}{2\pi}\frac{M}{r^{n+1}}2\pi r = \frac{n!M}{r^n}.$$
- The number M depends on the circle $|z - z_0| = r$. But, if $n = 0$, then $M \geq |f(z_0)|$, for any circle C centered at z_0, as long as C lies within D. Thus, an upper bound M of $|f(z)|$ on C cannot be smaller than $|f(z_0)|$.

Liouville's Theorem

- Although the next result is known as "Liouville's Theorem", it was probably first proved by Cauchy.
- The gist of the theorem is that an entire function f, one that is analytic for all z, cannot be bounded unless f itself is a constant:

Theorem (Liouville's Theorem)

The only bounded entire functions are constants.

- Suppose f is an entire bounded function, i.e., $|f(z)| \leq M$, for all z. Then, for any point z_0, by Cauchy's Inequality, $|f'(z_0)| \leq \frac{M}{r}$. By making r arbitrarily large we can make $|f'(z_0)|$ as small as we wish. This means $f'(z_0) = 0$, for all points z_0 in the complex plane. Hence, by a preceding theorem, f must be a constant.

Fundamental Theorem of Algebra

- Liouville's Theorem enables us to establish the celebrated

Fundamental Theorem of Algebra

If $p(z)$ is a nonconstant polynomial, then the equation $p(z) = 0$ has at least one root.

- Suppose that the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $n > 0$, is not 0 for any complex number z. This implies that the reciprocal of p, $f(z) = \frac{1}{p(z)}$, is an entire function. Now

$$\begin{aligned} |f(z)| &= \frac{1}{|a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|} \\ &= \frac{1}{|z|^n |a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}|}. \end{aligned}$$

Thus, $|f(z)| \to 0$ as $|z| \to \infty$. So the function f must be bounded for finite z. By Liouville's Theorem, f is a constant. Hence, p is a constant. But this contradicts p not being a constant polynomial. Therefore, there must exist at least one z for which $p(z) = 0$.

Morera's Theorem

- Morera's theorem, which gives a sufficient condition for analyticity, is often taken to be the converse of the Cauchy-Goursat Theorem:

Theorem (Morera's Theorem)

If f is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$, for every closed contour C in D, then f is analytic in D.

- By the hypotheses of continuity of f and $\oint_C f(z)dz = 0$, for every closed contour C in D, we conclude that $\int_C f(z)dz$ is independent of the path. Then, the function F, defined by $F(z) = \int_{z_0}^z f(s)ds$ (where s denotes a complex variable, z_0 is a fixed point in D, and z any point in D) is an antiderivative of f, i.e., $F'(z) = f(z)$. Hence, F is analytic in D. In addition, $F'(z)$ is analytic in view of the analyticity of the derivative of any analytic function. Since $f(z) = F'(z)$, we see that f is analytic in D.

The Maximum Modulus Theorem

- We saw that, if a function f is continuous on a closed and bounded region R, then f is bounded, i.e., there exists some constant M, such that $|f(z)| \leq M$, for z in R.
- If the boundary of R is a simple closed curve C, then the modulus $|f(z)|$ assumes its maximum value at some z on the boundary C:

Theorem (Maximum Modulus Theorem)

Suppose that f is analytic and nonconstant on a closed region R bounded by a simple closed curve C. Then the modulus $|f(z)|$ attains its maximum on C.

- If the stipulation that $f(z) \neq 0$, for all z in R, is added to the hypotheses, then the modulus $|f(z)|$ also attains its minimum on C.

Finding The Maximum Modulus

- Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$.

  We know that $|z|^2 = z \cdot \overline{z}$. By replacing z by $2z + 5i$, we have $|2z + 5i|^2 = (2z + 5i)(\overline{2z + 5i}) = (2z + 5i)(2\overline{z} - 5i) = 4z\overline{z} - 10i(z - \overline{z}) + 25$. But, $z - \overline{z} = 2i\text{Im}(z)$, whence $|2z + 5i|^2 = 4|z|^2 + 20\text{Im}(z) + 25$. Because f is a polynomial, it is analytic on the region defined by $|z| \leq 2$. Thus, $\max\limits_{|z| \leq 2} |2z + 5i|$ occurs on the boundary $|z| = 2$. There, $|2z + 5i| = \sqrt{41 + 20\text{Im}(z)}$. This attains its maximum when $\text{Im}(z)$ attains its maximum on $|z| = 2$, namely, at the point $z = 2i$. Thus, $\max_{|z| \leq 2} |2z + 5i| = \sqrt{81} = 9$.
- Note that $f(z) = 0$ only at $z = -\frac{5}{2}i$ and that this point is outside the region defined by $|z| \leq 2$. Hence we can conclude that we have a minimum when $\text{Im}(z)$ attains its minimum on $|z| = 2$ at $z = -2i$. As a result, $\min_{|z| \leq 2} |2z + 5i| = \sqrt{1} = 1$.