## Introduction to Complex Analysis

## George Voutsadakis<sup>1</sup>

<sup>1</sup>Mathematics and Computer Science Lake Superior State University

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George Voutsadakis (LSSU)

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## Subsection 1

## Sequences and Series

## Sequences

- A sequence  $\{z_n\}$  is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers  $\mathbb{C}$ .
- Example: The sequence  $\{1 + i^n\}$  is 1 + i,  $0 \\ n=2$ , 1 i,  $2 \\ n=3$ , 1 + i, ....
- If  $\lim_{n\to\infty} z_n = L$ , we say the sequence  $\{z_n\}$  is **convergent**, i.e.,  $\{z_n\}$  converges to the number L if, for each positive real number  $\varepsilon$ , an N can be found, such that  $|z_n L| < \varepsilon$ , whenever n > N.
- Since |z<sub>n</sub> L| is distance, the terms z<sub>n</sub> of a sequence that converges to L can be made arbitrarily close to L. In a different way, when a sequence {z<sub>n</sub>} converges to L, then all but a finite number of the terms of the sequence are within every ε-neighborhood of L.



• A sequence that is not convergent is said to be **divergent**. Example: The sequence  $\{1 + i^n\}$  is divergent since the general term  $z_n = 1 + i^n$  does not approach a fixed complex number as  $n \to \infty$ .

## An Example of a Convergent Sequence

• The sequence 
$$\left\{\frac{i^{n+1}}{n}\right\}$$
 converges since  $\lim_{n \to \infty} \frac{i^{n+1}}{n} = 0$ . As we see from  $-1, -\frac{i}{2}, \frac{1}{3}, \frac{i}{4}, -\frac{1}{5}, \dots,$ 

the terms of the sequence spiral in toward the point z = 0 as n increases.



# Criterion for Convergence

## Theorem (Criterion for Convergence)

A sequence  $\{z_n\}$  converges to a complex number L = a + ib if and only if  $\operatorname{Re}(z_n)$  converges to  $\operatorname{Re}(L) = a$  and  $\operatorname{Im}(z_n)$  converges to  $\operatorname{Im}(L) = b$ .

• Example: Consider the sequence 
$$\left\{\frac{3+ni}{n+2ni}\right\}$$
.

$$z_n = \frac{3+ni}{n+2ni} = \frac{(3+ni)(n-2ni)}{n^2+4n^2} = \frac{2n^2+3n}{5n^2} + i\frac{n^2-6n}{5n^2}.$$

Thus, we get

$$\operatorname{Re}(z_n) = \frac{2n^2 + 3n}{5n^2} = \frac{2}{5} + \frac{3}{5n} \to \frac{2}{5}$$

$$\operatorname{Im}(z_n) = \frac{n^2 - 6n}{5n^2} = \frac{1}{5} - \frac{6}{5n} \to \frac{1}{5}$$

By the theorem, the given sequence converges to  $a + ib = \frac{2}{5} + \frac{1}{5}i$ .

## Series and Geometric Series

- An infinite series or series of complex numbers ∑<sub>k=1</sub><sup>∞</sup> z<sub>k</sub> = z<sub>1</sub> + z<sub>2</sub> + z<sub>3</sub> + ··· + z<sub>n</sub> + ··· is convergent if the sequence of partial sums {S<sub>n</sub>}, where S<sub>n</sub> = z<sub>1</sub> + z<sub>2</sub> + z<sub>3</sub> + ··· + z<sub>n</sub> converges. If S<sub>n</sub> → L as n → ∞, we say that the series converges to L or that the sum of the series is L.
- Geometric Series: A geometric series is any series of the form  $\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots$ . The *n*-th term of the sequence of partial sums is  $S_n = a + az + az^2 + \cdots + az^{n-1}$ . To get a formula for  $S_n$ , multiply by z:  $zS_n = az + az^2 + az^3 + \cdots + az^n$ . Subtract this from  $S_n$ :  $S_n - zS_n = (a + az + az^2 + \dots + az^{n-1}) - az^n$  $(az + az^{2} + az^{3} + \dots + az^{n-1} + az^{n}) = a - az^{n}$ . Thus,  $(1-z)S_n = a(1-z^n)$ , and, hence,  $S_n = \frac{a(1-z^n)}{1-z^n}$ . • If |z| < 1,  $z^n \to 0$  as  $n \to \infty$ . So  $S_n \to \frac{a}{1-z}$ . I.e., for |z| < 1,  $\frac{a}{1-z} = a + az + az^2 + \dots + az^{n-1} + \dots$ • If  $|z| \ge 1$ , a geometric series diverges.

## Special Geometric Series

Recall the sum formulas

$$S_n=rac{a(1-z^n)}{1-z},\quad rac{a}{1-z}=a+az+az^2+\cdots+az^{n-1}+\cdots.$$

• If we set a = 1, we get

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

• If we then replace z by -z:

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

• For the finite sum, we have  $\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \dots + z^{n-1}$ . Rewriting the left side of the above equation as  $\frac{1-z^n}{1-z} = \frac{1}{1-z} + \frac{-z^n}{1-z}$ , we get  $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots + z^{n-1} + \frac{z^n}{1-z}$ .

## A Convergent Geometric Series

### The infinite series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$$

is a geometric series.

It has the standard form, with  $a = \frac{1}{5}(1+2i)$  and  $z = \frac{1}{5}(1+2i)$ . Since  $|z| = \frac{\sqrt{5}}{5} < 1$ , the series is convergent and its sum is given by:

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1-\frac{1+2i}{5}} = \frac{1+2i}{4-2i} = \frac{1}{2}i.$$

# Necessary Condition for Convergence

• We turn to some important theorems about convergence and divergence of an infinite series:

Theorem (A Necessary Condition for Convergence)

If  $\sum_{k=1}^{\infty} z_k$  converges, then  $\lim_{n\to\infty} z_n = 0$ .

• Let *L* denote the sum of the series. Then  $S_n \to L$  and  $S_{n-1} \to L$  as  $n \to \infty$ . By taking the limit of both sides of  $S_n - S_{n-1} = z_n$  as  $n \to \infty$ , we obtain the desired conclusion.

Theorem (The *n*-th Term Test for Divergence)

If  $\lim_{n\to\infty} z_n \neq 0$ , then  $\sum_{k=1}^{\infty} z_k$  diverges.

- Example: The series  $\sum_{k=1}^{\infty} \frac{ik+5}{k}$  diverges, since  $z_n = \frac{in+5}{n} \rightarrow i \neq 0$  as  $n \rightarrow \infty$ .
  - The geometric series  $\sum_{k=1}^{\infty} az^k$  diverges if  $|z| \ge 1$  because even in the case when  $\lim_{n\to\infty} |z^n|$  exists, the limit is not zero.

# Absolute and Conditional Convergence

## Definition (Absolute and Conditional Convergence)

An infinite series  $\sum_{k=1}^{\infty} z_k$  is said to be **absolutely convergent** if  $\sum_{k=1}^{\infty} |z_k|$  converges. An infinite series  $\sum_{k=1}^{\infty} z_k$  is said to be **conditionally convergent** if it converges but  $\sum_{k=1}^{\infty} |z_k|$  diverges.

- In elementary calculus a real series of the form  $\sum_{k=1}^{\infty} \frac{1}{k^p}$  is called a *p*-series and
  - converges for p > 1;
  - diverges for  $p \leq 1$ .
- Example: The series  $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$  is absolutely convergent: The series  $\sum_{k=1}^{\infty} \left| \frac{i^k}{k^2} \right|$  is the same as the real convergent *p*-series  $\sum_{k=1}^{\infty} \frac{1}{k^2}$ .
- As in real calculus, absolute convergence implies convergence.
- Example: The series  $\sum_{k=1}^{\infty} \frac{i^k}{k^2} = i \frac{1}{2^2} \frac{i}{3^2} + \cdots$  converges, because it is was shown to be absolutely convergent.

# Tests for Convergence

## Theorem (The Ratio Test)

Let 
$$\sum_{k=1}^{\infty} z_k$$
 be a series of nonzero terms, with  $\lim_{n\to\infty} |$ 

$$\left|\frac{z_{n+1}}{z_n}\right| = L.$$

- (i) If L < 1, then the series converges absolutely.
- (ii) If L > 1 or  $L = \infty$ , then the series diverges.
- (iii) If L = 1, the test is inconclusive.

#### Theorem (The Root Test)

Let  $\sum_{k=1}^{\infty} z_k$  be a series of complex terms, with  $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$ .

- (i) If L < 1, then the series converges absolutely.
- (ii) If L > 1 or  $L = \infty$ , then the series diverges.
- (iii) If L = 1, the test is inconclusive.

• We are interested primarily in applying these tests to power series.

# Power Series and Circle of Convergence

- An infinite series of the form  $\sum_{k=0}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$ , where the coefficients  $a_k$  are complex constants, is called a **power series** in  $z z_0$ .
- The power series is said to be **centered at**  $z_0$  and the complex point  $z_0$  is referred to as the **center** of the series.
- It is also convenient to define  $(z z_0)^0 = 1$  even when  $z = z_0$ .
- Every complex power series has a radius of convergence and a circle of convergence: It is the circle centered at  $z_0$  of largest radius R > 0 for which the series converges at every point within the circle  $|z - z_0| = R$ .



A power series converges absolutely at all points z satisfying  $|z - z_0| < R$ , and diverges at all points z, with  $|z - z_0| > R$ .

# Possibilities for Radius of Convergence

- The radius of convergence can be:
  - (i) R = 0 (series converges only at its center  $z = z_0$ );
  - (ii) R a finite positive number (series converges in interior of  $|z z_0| = R$ );
  - (iii)  $R = \infty$  (series converges for all z).

A power series may converge at some, all, or at none of the points on the actual circle of convergence.  $|z^{n+2}|$ 

• Example: Consider  $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$ . By the ratio test,  $\lim_{n\to\infty} \left| \frac{z^{n+2}}{\frac{z^{n+1}}{k}} \right| =$ 

 $\lim_{n\to\infty} \frac{n}{n+1}|z| = |z|$ . Thus, the series converges absolutely for |z| < 1. The circle of convergence is |z| = 1 and the radius of convergence is R = 1. On the circle |z| = 1, the series does not converge absolutely since  $\sum_{k=1}^{\infty} \frac{1}{k}$  is the well-known divergent harmonic series. This does not mean that the series diverges on the circle of convergence. In fact, at z = -1,  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$  is the convergent alternating harmonic series. It can be shown that the series converges at all points on the circle |z| = 1 except at z = 1.

## Dependence of the Radius on the Coefficients

For a power series

$$\sum_{k=0}^{\infty}a_k(z-z_0)^k,$$

the limit depends only on the coefficients  $a_k$ . Thus:

(i) if 
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$$
, the radius of convergence is  $R = \frac{1}{L}$ ;  
(ii) if  $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ , the radius of convergence is  $R = \infty$ ;  
(iii) if  $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , the radius of convergence is  $R = 0$ .

• Similar conclusions can be made for the root test by utilizing  $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ . E.g., if  $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0$ , then  $R = \frac{1}{L}$ .

## Finding Radius of Convergence Using Ratio Test

• Consider the power series 
$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} (z - 1 - i)^k.$$
With the identification  $a_n = \frac{(-1)^{n+1}}{n!}$ , we have

$$\lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}} \right| = \lim_{n \to \infty} \frac{1}{n+1} = 0.$$

Hence, the radius of convergence is  $\infty$ . The power series with center  $z_0 = 1 + i$  converges absolutely for all z, i.e., for  $|z - 1 - i| < \infty$ .

## Finding Radius of Convergence Using Root Test

• Consider the power series 
$$\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$$
.

• With 
$$a_n = \left(\frac{6n+1}{2n+5}\right)^n$$
, the root test gives

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left(\frac{6n+1}{2n+5}\right) = 3.$$

We conclude that the radius of convergence of the series is  $R = \frac{1}{3}$ . The circle of convergence is  $|z - 2i| = \frac{1}{3}$ ; the power series converges absolutely for  $|z - 2i| < \frac{1}{3}$ .

## The Arithmetic of Power Series

- Some facts concerning power-series stated informally:
  - A power series  $\sum_{k=0}^{\infty} a_k (z z_0)^k$  can be multiplied by a nonzero complex constant *c* without affecting its convergence or divergence.
  - A power series  $\sum_{k=0}^{\infty} a_k (z z_0)^k$  converges absolutely within its circle of convergence. As a consequence, within the circle of convergence the terms of the series can be rearranged and the rearranged series has the same sum *L* as the original series.
  - Two power series  $\sum_{k=0}^{\infty} a_k(z-z_0)^k$  and  $\sum_{k=0}^{\infty} b_k(z-z_0)^k$  can be added and subtracted by adding or subtracting like terms:

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k \pm \sum_{k=0}^{\infty} b_k (z-z_0)^k = \sum_{k=0}^{\infty} (a_k \pm b_k) (z-z_0)^k.$$

- If both series have the same nonzero radius R of convergence, the radius of convergence of  $\sum_{k=0}^{\infty} (a_k \pm b_k)(z z_0)^k$  is R.
- If one series has radius of convergence r > 0 and the other R > 0, where  $r \neq R$ , then  $\sum_{k=0}^{\infty} (a_k \pm b_k)(z - z_0)^k$  has radius of convergence the smaller of r and R.
- Two power series can (with care) be multiplied and divided.

## Final Remarks on Series and Power Series

- If  $z_n = a_n + ib_n$  then the *n*-th term of the sequence of partial sums for  $\sum_{k=1}^{\infty} z_k$  is  $S_n = \sum_{k=1}^{n} (a_k + ib_k) = \sum_{k=1}^{n} a_k + i \sum_{k=1}^{n} b_k$ . Thus,  $\sum_{k=1}^{\infty} z_k$  converges to L = a + ib if and only if  $\operatorname{Re}(S_n) = \sum_{k=1}^{n} a_k$  converges to *a* and  $\operatorname{Im}(S_n) = \sum_{k=1}^{n} b_k$  converges to *b*.
- In summation notation a geometric series need not start at k = 1 nor does the general term have to appear precisely as az<sup>k-1</sup>.
- Example: Consider  $\sum_{k=3}^{\infty} 40 \frac{i^{k+2}}{2^{k-1}}$ . It does not appear to match the form  $\sum_{k=1}^{\infty} az^{k-1}$  of a geometric series. By writing out three terms,  $\sum_{k=3}^{\infty} 40 \frac{i^{k+2}}{2^{k-1}} = 40 \frac{i^5}{2^2} + 40 \frac{i^6}{2^3} + 40 \frac{i^7}{2^4} + \cdots$  we see  $a = 40 \frac{i^5}{2^2}$  and  $z = \frac{i}{2}$ . Since  $|z| = \frac{1}{2} < 1$ , the sum is  $\sum_{k=3}^{\infty} 40 \frac{i^{k+2}}{2^{k-1}} = \frac{40 \frac{i^5}{2^2}}{1 - \frac{i}{2}} = -4 + 8i$ .
- A power series  $\sum_{k=0}^{\infty} a_k (z z_0)^k$  always possesses a radius of convergence R. The ratio and root tests lead to  $\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$  and  $\frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|a_n|}$  assuming the appropriate limit exists.

## Subsection 2

Taylor Series

## Differentiation of Power Series

## Theorem (Continuity)

A power series  $\sum_{k=0}^{\infty} a_k (z - z_0)^k$  represents a continuous function f within its circle of convergence  $|z - z_0| = R$ .

### Theorem (Term-by-Term Differentiation)

A power series  $\sum_{k=0}^{\infty} a_k (z - z_0)^k$  can be differentiated term by term within its circle of convergence  $|z - z_0| = R$ .

• Differentiating a power series term-by-term gives,

$$\frac{d}{dz}\sum_{k=0}^{\infty}a_k(z-z_0)^k = \sum_{k=0}^{\infty}a_k\frac{d}{dz}(z-z_0)^k = \sum_{k=1}^{\infty}a_kk(z-z_0)^{k-1}.$$

- Using the ratio test, it can be shown that the original series and the differentiated series have the same circle of convergence.
- Since the derivative of a power series is another power series, the first series can be differentiated as many times as we wish.

# Integration of Power Series

#### Theorem (Term-by-Term Integration)

A power series  $\sum_{k=0}^{\infty} a_k (z - z_0)^k$  can be integrated term-by-term within its circle of convergence  $|z - z_0| = R$ , for every contour C lying entirely within the circle of convergence.

• The theorem states that

$$\int_C \sum_{k=0}^\infty a_k (z-z_0)^k dz = \sum_{k=0}^\infty a_k \int_C (z-z_0)^k dz,$$

whenever C lies in the interior of  $|z - z_0| = R$ .

• Indefinite integration can also be carried out term by term:

$$\int \sum_{k=0}^{\infty} a_k (z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int (z-z_0)^k dz = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (z-z_0)^{k+1} + K.$$

• The ratio test can be used to prove that both series have the same circle of convergence.

# Analyticity

• Suppose a power series represents a function f within  $|z - z_0| = R$ , i.e.,  $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + a_3 (z - z_0)^3 + \cdots$ .

• Then, the derivatives of f are the series

$$f'(z) = \sum_{\substack{k=1 \\ k=2}}^{\infty} a_k k(z-z_0)^{k-1} = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$$
  
$$f''(z) = \sum_{\substack{k=2 \\ k=2}}^{\infty} a_k k(k-1)(z-z_0)^{k-2} = 2 \cdot 1a_2 + 3 \cdot 2a_3(z-z_0) + \cdots$$
  
$$f'''(z) = \sum_{\substack{k=3 \\ k=3}}^{\infty} a_k k(k-1)(k-2)(z-z_0)^{k-3} = 3 \cdot 2 \cdot 1a_3 + \cdots$$

• Since the power series represents a differentiable function f within its circle of convergence  $|z - z_0| = R$ , it represents an analytic function within its circle of convergence.

# Taylor Series and Maclaurin Series

• Evaluating the derivatives at  $z = z_0$  gives

$$f(z_0) = a_0, \ f'(z_0) = 1!a_1, \ f''(z_0) = 2!a_2, \ f'''(z_0) = 3!a_3.$$

- In general,  $f^{(n)}(z_0) = n! a_n$ , or  $a_n = \frac{f^{(n)}(z_0)}{n!}$ ,  $n \ge 0$ .
- When n = 0, we interpret the zero-order derivative as  $f(z_0)$  and 0! = 1, so that the formula gives  $a_0 = f(z_0)$ .
- Substituting into the series yields

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

This series is called the **Taylor series** for f centered at  $z_0$ .

• A Taylor series with center  $z_0 = 0$ ,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

is referred to as a Maclaurin series.

## Taylor's Theorem

• Since a power series converges in a circular domain, and a domain *D* is generally not circular, the following question arises:

Can we expand f in one or more power series that are valid, i.e., a power series that converges at z and the number to which the series converges is f(z), in circular domains that are all contained in D?

#### Theorem (Taylor's Theorem)

Let f be analytic within a domain D and let  $z_0$  be a point in D. Then f has the series representation  $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$  valid for the largest circle C with center at  $z_0$  and radius R that lies entirely within D.

• Let z be a fixed point within the circle C and let s denote the variable of integration. The circle C is then described by  $|s-z_0| = R$ . We use the Cauchy integral formula to obtain the value of f at z:



# Proof of Taylor's Theorem I

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• 
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z} ds = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)-(z-z_0)} ds =$$
  
 $\frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z_0} \left(\frac{1}{1-\frac{z-z_0}{s-z_0}}\right) ds$ . By the power series for  $\frac{1}{1-z}$ , we get  
 $\frac{1}{1-\frac{z-z_0}{s-z_0}} = 1 + \frac{z-z_0}{s-z_0} + \left(\frac{z-z_0}{s-z_0}\right)^2 + \dots + \left(\frac{z-z_0}{s-z_0}\right)^{n-1} + \frac{(z-z_0)^n}{(s-z)(s-z_0)^{n-1}}$ , whence, we get  
 $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z_0} ds + \frac{z-z_0}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^2} ds + \frac{(z-z_0)^2}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^3} ds + \dots + \frac{(z-z_0)^{n-1}}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^n} ds + \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(s)}{(s-z)(s-z_0)^n} ds$ . By Cauchy's integral formula for derivatives,  $f(z) =$   
 $f(z_0) + \frac{f'(z_0)}{1!} (z-z_0) + \frac{f''(z_0)}{2\pi i} (z-z_0)^2 + \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!} (z-z_0)^{n-1} + R_n(z)$ , where  $R_n(z) = \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(s)}{(s-z)(s-z_0)^n} ds$ . This is called **Taylor's formula with remainder**  $R_n$ . The goal now is to show that  $R_n(z) \to 0$  as  $n \to \infty$ .

## Proof of Taylor's Theorem II

• To see that  $R_n(z) = \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^n} ds \to 0$ , it suffices to show that  $|R_n(z)| \to 0$  as  $n \to \infty$ . Since f is analytic in D, we know that |f(z)| has a maximum value M on the contour C. In addition, since z is inside C,  $|z - z_0| < R$  and, consequently,  $|s-z| = |s-z_0 - (z-z_0)| > |s-z_0| - |z-z_0| = R - d$ , where  $d = |z - z_0|$  is the distance from z to  $z_0$ . The *ML*-inequality then gives  $|R_n(z)| = \left| \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(s)}{(s-z)(s-z_0)^n} ds \right| \le \frac{d^n}{2\pi} \cdot \frac{M}{(R-d)R^n} \cdot 2\pi R = \frac{MR}{R-d} \left(\frac{d}{R}\right)^n.$ Because d < R,  $\left(\frac{d}{R}\right)^n \to 0$  as  $n \to \infty$ , we conclude that  $|R_n(z)| \to 0$ as  $n \to \infty$ . It follows that the infinite series  $f(z_0) + \frac{f'(z_0)}{1}(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots$  converges to f(z).

# Isolated Singularities and Important Maclaurin Series

• An **isolated singularity** of a function *f* is a point at which *f* fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.

Example:  $f(z) = \frac{1}{z-5i}$  has an isolated singularity at z = 5i.

- The radius of convergence *R* of a Taylor series for *f* is the distance from the center *z*<sub>0</sub> of the series to the nearest isolated singularity of *f*.
- Thus, if the function f is entire, then the radius of convergence of a Taylor series centered at any point  $z_0$  is necessarily  $R = \infty$ .
- We summarize some Important Maclaurin Series:

$$e^{z} = 1 + \frac{z}{1!} + \frac{z^{2}}{2!} + \dots = \sum_{k=0}^{\infty} \frac{z^{k}}{k!}$$
  

$$\sin z = z - \frac{z^{3}}{3!} + \frac{z^{5}}{5!} - \dots = \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k+1}}{(2k+1)}$$
  

$$\cos z = 1 - \frac{z^{2}}{2!} + \frac{z^{4}}{4!} - \dots = \sum_{k=0}^{\infty} (-1)^{k} \frac{z^{2k}}{(2k)!}$$

# Finding Radius of Convergence

Suppose the function f(z) = 3-i/(1-i+z) is expanded in a Taylor series with center z<sub>0</sub> = 4 - 2i. What is its radius of convergence R?
 Observe that the function is analytic at every point except at z = -1 + i, which is an isolated singularity of f. The distance from z = -1 + i to z<sub>0</sub> = 4 - 2i is

$$|z - z_0| = \sqrt{(-1 - 4)^2 + (1 - (-2))^2} = \sqrt{34}.$$

Thus, the radius of convergence for the Taylor series centered at 4 - 2i is  $R = \sqrt{34}$ .

## Uniqueness of the Series Expansion

• If two power series with center z<sub>0</sub>,

$$\sum_{k=0}^{\infty}a_k(z-z_0)^k$$
 and  $\sum_{k=0}^{\infty}b_k(z-z_0)^k$ 

represent the same function f and have the same nonzero radius R of convergence, then  $a_k = b_k = \frac{f^{(k)}(z_0)}{k!}$ , k = 0, 1, 2, ...

- Stated in another way, the power series expansion of a function, with center z<sub>0</sub>, is unique.
- Thus, a power series expansion of an analytic function f centered at  $z_0$ , irrespective of the method used to obtain it, is the Taylor series expansion of the function.

# Finding a Maclaurin Series

Find the Maclaurin expansion of f(z) = 1/((1-z)^2).
 Recall that for |z| < 1,</li>

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

If we differentiate both sides of the last result with respect to z,

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}1 + \frac{d}{dz}z + \frac{d}{dz}z^2 + \frac{d}{dz}z^3 + \cdots$$

or

$$\frac{1}{(1-z)^2} = 0 + 1 + 2z + 3z^2 + \dots = \sum_{k=1}^{\infty} kz^{k-1}.$$

The radius of convergence of the last power series is the same as the original series R = 1.

# Finding a Taylor Series

• Expand  $f(z) = \frac{1}{1-z}$  in a Taylor series with center  $z_0 = 2i$ . We use again  $\frac{1}{1-z} = 1 + z + z^2 + \cdots$ . By adding and subtracting 2*i* in the denominator,  $\frac{1}{1-z} = \frac{1}{1-z+2i-2i} = \frac{1}{1-2i-(z-2i)} = \frac{1}{1-2i} \cdot \frac{1}{1-\frac{z-2i}{1-z}}$ We now write  $\frac{1}{1-\frac{z-2I}{z-2}}$  as a power series:  $\frac{1}{1-z} = \frac{1}{1-2i} \left| 1 + \frac{z-2i}{1-2i} + \left(\frac{z-2i}{1-2i}\right)^2 + \left(\frac{z-2i}{1-2i}\right)^3 + \cdots \right| \text{ or }$  $\frac{1}{1-z} = \frac{1}{1-2i} + \frac{1}{(1-2i)^2}(z-2i) + \frac{1}{(1-2i)^3}(z-2i)^2 + \frac{1}{(1-2i)^4}(z-2i)^3 + \cdots$ Because the distance from the center  $z_0 = 2i$  to the nearest singularity z = 1 is  $\sqrt{5}$ , we conclude that the circle of convergence is  $|z-2i| = \sqrt{5}$ .

## Power Series for the Same Function

• We have represented the same function  $f(z) = \frac{1}{1-z}$  by two different power series; one with center  $z_0 = 0$  and radius of convergence R = 1; another with center  $z_0 = 2i$  and radius of convergence  $R = \sqrt{5}$ .



The interior of the intersection of the two circles is the region where both series converge, i.e., at a specified point  $z^*$  in this region, both series converge to same value  $f(z^*) = \frac{1}{1-z^*}$ . Outside the colored region at least one of the two series must diverge.

## Subsection 3

## Laurent Series

# Isolated Singularities

- Suppose that  $z = z_0$  is a singularity of a complex function f, i.e., a point at which f fails to be analytic.
- The point  $z = z_0$  is said to be an **isolated singularity** of the function f if there exists some deleted neighborhood, or punctured open disk,  $0 < |z z_0| < R$  of  $z_0$  throughout which f is analytic. Example: The points z = 2i and z = -2i are singularities of  $f(z) = \frac{z}{z^2+4}$ . Both 2i and -2i are isolated singularities since f is analytic at every point in the neighborhood defined by |z - 2i| < 1, except at z = 2i, and at every point in the neighborhood defined by |z - (-2i)| < 1, except at z = -2i. In other words, f is analytic in the deleted neighborhoods 0 < |z - 2i| < 1 and 0 < |z + 2i| < 1.
- A singular point  $z = z_0$  of a function f is **nonisolated** if every neighborhood of  $z_0$  contains at least one singularity of f other than  $z_0$ . Example: The branch point z = 0 is a nonisolated singularity of Lnz since every neighborhood of z = 0 contains points on the negative real axis.

# A New Kind of Series

- If  $z = z_0$  is a singularity of a function f, then certainly f cannot be expanded in a power series with  $z_0$  as its center.
- About an isolated singularity  $z = z_0$ , it is still possible to represent f by a series involving both negative and nonnegative integer powers of  $z z_0$ , i.e.,

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

• Example: Consider the function  $f(z) = \frac{1}{z-1}$ . The point z = 1 is an isolated singularity of f and, consequently, the function cannot be expanded in a Taylor series centered at that point. Nevertheless, f can expanded in a series of the previous form that is valid for all z near 1:  $f(z) = \cdots + \frac{0}{(z-1)^2} + \frac{1}{z-1} + 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots$ . This series representation is valid for  $0 < |z-1| < \infty$ .
# Principal Part and Analytic Part

• Using summation notation, we can rewrite

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

• The part with negative powers  $\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$  is called the **principal part** of the series. It converges for  $\left|\frac{1}{z-z_0}\right| < r^*$  or, equivalently, for  $|z-z_0| > \frac{1}{r^*} = r$ .

- The part consisting of the nonnegative powers  $\sum_{k=0}^{\infty} a_k (z z_0)^k$ , is called the **analytic part** of the series. It converges for  $|z z_0| < R$ .
- Thus, the sum converges when z satisfies both  $|z z_0| > r$  and  $|z z_0| < R$ , i.e., when z is a point in an annular domain defined by  $r < |z z_0| < R$ .
- By summing over negative and nonnegative integers, we can rewrite  $f(z) = \sum_{k=-\infty}^{\infty} a_k (z z_0)^k$ .

# An Example

- The function  $f(z) = \frac{\sin z}{z^4}$  is not analytic at the isolated singularity z = 0 and hence cannot be expanded in a Maclaurin series.
- However, sin z is an entire function having Maclaurin series

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots,$$

which converges for  $|z| < \infty$ .

• By dividing this power series by  $z^4$  we obtain a series for f with negative and positive integer powers of z:

$$f(z) = \frac{\sin z}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1}{3!z}}_{principal part} + \frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots$$

- The analytic part converges for  $|z| < \infty$ .
- The principal part is valid for |z| > 0.
- The series converges for all z, but z = 0, i.e., is valid for  $0 < |z| < \infty$ .

## Laurent Series and Laurent's Theorem

 A series representation of a function *f* consisting of both negative and nonnegative powers of *z* − *z*<sub>0</sub> is called a Laurent series or a Laurent expansion of *f* about *z*<sub>0</sub> on the annulus *r* < |*z* − *z*<sub>0</sub>| < *R*.

#### Theorem (Laurent's Theorem)

Let f be analytic within the annulus D defined by  $r < |z - z_0| < R$ . Then f has the series representation  $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$  valid for  $r < |z - z_0| < R$ .

The coefficients  $a_k$  are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^{k+1}} ds,$$

 $k = 0, \pm 1, \pm 2, \dots$ , where C is a simple closed curve that lies entirely within D and has  $z_0$  in its interior.



# Proof of Laurent's Theorem I

• Let  $C_1$  and  $C_2$  be concentric circles with center  $z_0$  and radii  $r_1$  and  $R_2$ , where  $r < r_1 < R_2 < R$ . Let z be a fixed point in D that satisfies  $r_1 < |z - z_0| < R_2$ . By introducing a crosscut between  $C_2$  and  $C_1$ , Cauchy's formula gives  $f(z) = \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)}{s-z} ds - \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{s-z} ds$ .



We can write  $\frac{1}{2\pi i} \oint_{C_2} \frac{f(s)}{s-z} ds = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ , where  $a_k = \frac{1}{2\pi i} \oint_{C_2} \frac{f(s)}{(s-z_0)^{k+1}} ds$ ,  $k = 0, 1, 2, \dots$  We have  $-\frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{s-z} ds = \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{(z-z_0) - (s-z_0)} ds = \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{z-z_0} \left( \frac{1}{1-\frac{s-z_0}{z-z_0}} \right) ds = \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{z-z_0} \left( 1 + \frac{s-z_0}{z-z_0} + \dots + \left( \frac{s-z_0}{z-z_0} \right)^{n-1} + \frac{(s-z_0)^n}{(z-s)(z-z_0)^{n-1}} \right) ds = \sum_{k=1}^n \frac{a_{-k}}{(z-z_0)^k} + R_n(z), \ a_{-k} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{(s-z_0)^{-k+1}} ds,$  $R_n(z) = \frac{1}{2\pi i (z-z_0)^n} \oint_{C_1} \frac{f(s)(s-z_0)^n}{z-s} ds.$ 

# Proof of Laurent's Theorem II

• Now let  $d = |z - z_0|$  and let M denote the maximum value of |f(z)|on  $C_1$ . Using  $|s - z_0| = r_1$  and  $|z - s| = |z - z_0 - (s - z_0)|$  $\geq |z - z_0| - |s - z_0| = d - r_1$ , the ML-inequality gives:  $|R_n(z)| = \left|\frac{1}{2\pi i (z - z_0)^n} \oint_{C_1} \frac{f(s)(s - z_0)^n}{z - s} ds\right| \leq \frac{1}{2\pi d^n} \frac{Mr_1^n}{d - r_1} 2\pi r_1 = \frac{Mr_1}{d - r_1} \left(\frac{r_1}{d}\right)^n$ . Because  $r_1 < d$ ,  $\left(\frac{r_1}{d}\right)^n \to 0$  as  $n \to \infty$ , and so  $|R_n(z)| \to 0$  as  $n \to \infty$ . Thus we have shown that  $-\frac{1}{2\pi i} \oint_{C_1} \frac{f(s)}{s - z} ds = \sum_{k=1}^{\infty} a_{-k} (z - z_0)^k$ . • Therefore, overall we have

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^k + \sum_{k=0}^{\infty} a_k(z-z_0)^k.$$

By summing over all integer powers,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k, \ a_k = \oint_C \frac{f(z)}{(z-z_0)^{k+1}} dz, \ k = 0, \pm 1, \pm 2, \dots$$

# Remarks

- In the case when  $a_{-k} = 0$  for k = 1, 2, ..., the principal part is zero and the Laurent series reduces to a Taylor series.
- The annular domain  $r < |z z_0| < R$  need not have a "ring" shape. Some other possible annular domains are:
  - (i) r = 0, R finite; In this case, the series converges in  $0 < |z z_0| < R$ , i.e., the domain is a punctured open disk.
  - (ii)  $r \neq 0$ ,  $R = \infty$ ; In this case, the annular domain is  $r < |z z_0|$  and consists of all points exterior to the circle  $|z z_0| = r$ .
  - (iii) r = 0,  $R = \infty$ ; In this case, the domain is defined by  $0 < |z z_0|$ . This represents the entire complex plane except the point  $z_0$ .
- Finding the Laurent series of a function in a specified annular domain is generally difficult, but in many instances we can obtain a desired Laurent series by either
  - employing a known power series expansion of a function; or by
  - creative manipulation of a suitably chosen geometric series.

# Finding Laurent Expansions I

• Expand  $f(z) = \frac{1}{z(z-1)}$  in a Laurent series valid for the following annular domains.

(a) 0 < |z| < 1 (b) 1 < |z| (c) 0 < |z - 1| < 1 (d) 1 < |z - 1|.

- In parts (a) and (b) we want only powers of z, whereas in parts (c) and (d) we want powers of z 1.
- (a) f(z) = -1/z 1/(1-z) = -1/z (1 + z + z<sup>2</sup> + z<sup>3</sup> + ···). The infinite series in the brackets converges for |z| < 1, but after we multiply this expression by 1/z, the resulting series f(z) = -1/z 1 z z<sup>2</sup> z<sup>3</sup> ··· converges for 0 < |z| < 1.</li>
  (b) To obtain a series that converges for 1 < |z|, we start by constructing a series that converges for |1/z| < 1. We write the given function f(z) = 1/(z<sup>2</sup> 1 1/z) = 1/(z<sup>2</sup> (1 + 1/z) + 1/(z<sup>2</sup> + 1/z<sup>3</sup> + ···)). The series in the brackets converges for |1/z| < 1 or equivalently for 1 < |z|. Thus, the required Laurent series is f(z) = 1/(z<sup>2</sup> + 1/z<sup>3</sup> + 1/z<sup>4</sup> + 1/z<sup>5</sup> + ···.

# Finding Laurent Expansions I

(c) We add and subtract 1 in the denominator: 
$$f(z) = \frac{1}{(1-1+z)(z-1)} = \frac{1}{z-1} \frac{1}{1+(z-1)} = \frac{1}{z-1} (1-(z-1)+(z-1)^2-(z-1)^3+\cdots) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$$
. The requirement that  $z \neq 1$  is equivalent to  $0 < |z-1|$ , and the geometric series in brackets converges for  $|z-1| < 1$ . Thus, the last series converges for  $z$  satisfying  $0 < |z-1| < 1$ .  
(d) As in part (b),  $f(z) = \frac{1}{z-1} \frac{1}{1+(z-1)} = \frac{1}{(z-1)^2} \frac{1}{1+\frac{1}{z-1}} = \frac{1}{(z-1)^2} \left(1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^5} + \cdots\right) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + \cdots$ . Because the series within the brackets converges for  $|\frac{1}{z-1}| < 1$ , the final series converges for  $1 < |z-1|$ .

### More Laurent Series Expansions I

• Expand 
$$f(z) = \frac{1}{(z-1)^2(z-3)}$$
 in a Laurent series valid for  
(a)  $0 < |z-1| < 2$  (b)  $0 < |z-3| < 2$ .

(a) We need to express z - 3 in terms of z - 1. This can be done by writing  $f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2} \frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2} \frac{1}{1-\frac{z-1}{2}} =$  $\frac{-1}{2(z-1)^2}\left(1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right)=$  $-\frac{1}{2(z-1)^2}-\frac{1}{4(z-1)}-\frac{1}{8}-\frac{1}{16}(z-1)-\cdots$ (b) To obtain powers of z - 3, we write z - 1 = 2 + (z - 3) and  $f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{z-3}[2+(z-3)]^{-2} = \frac{1}{4(z-3)}[1+\frac{z-3}{2}]^{-2} =$  $\frac{1}{4(z-3)}\left(1+\frac{(-2)}{1!}\left(\frac{z-3}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{z-3}{2}\right)^3+\cdots\right).$ The series in the brackets is valid for  $\left|\frac{z-3}{2}\right| < 1$  or |z-3| < 2. Multiplying by  $\frac{1}{4(z-3)}$  gives a series that is valid for 0 < |z-3| < 2:  $f(z) = \frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \cdots$ 

George Voutsadakis (LSSU)

#### More Laurent Series Expansions II

• Expand  $f(z) = \frac{8z+1}{z(1-z)}$  in a Laurent series valid for 0 < |z| < 1. By partial fractions we can rewrite f as  $f(z) = \frac{8z+1}{z(1-z)} = \frac{1}{z} + \frac{9}{1-z}$ . Now we have

$$\frac{9}{1-z}=9+9z+9z^2+\cdots.$$

The foregoing geometric series converges for |z| < 1, but after we add the term  $\frac{1}{z}$  to it, the resulting Laurent series

$$f(z) = \frac{1}{z} + 9 + 9z + 9z^2 + \cdots$$

is valid for 0 < |z| < 1.

## More Laurent Series Expansions III

• Expand  $f(z) = \frac{1}{z(z-1)}$  in a Laurent series valid for 1 < |z-2| < 2. The center z = 2 is a point of analyticity of the function f. Our goal now is to find two series involving integer powers of z - 2, one converging for 1 < |z - 2| and the other converging for |z - 2| < 2. Decompose f into partial fractions:  $f(z) = \frac{-1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$ . •  $f_1(z) = \frac{-1}{z} = \frac{-1}{2+z-2} = \frac{-1}{2} \frac{1}{1+\frac{z-2}{2}} = \frac{-1}{2} \left( 1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \cdots \right) =$  $\frac{-1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$ . This series converges for  $|\frac{z-2}{2}| < 1$ or |z - 2| < 2. •  $f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2} \frac{1}{1+\frac{1}{z-2}} = \frac{1}{z-2} \left( 1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \cdots \right) =$  $\frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots$  It converges for  $\left|\frac{1}{z-2}\right| < 1$  or 1 < |z - 2|. Thus, we get  $f(z) = \cdots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2}$  $-\frac{1}{2}+\frac{z-2}{2^2}-\frac{(z-2)^2}{2^3}+\frac{(z-2)^3}{2^4}-\cdots$  This representation is valid for z satisfying 1 < |z - 2| < 2.

### More Laurent Series Expansions IV

• Expand  $f(z) = \frac{e^3}{z}$  in a Laurent series valid for  $0 < |z| < \infty$ . We know that for  $|z| < \infty$ ,

$$e^{z} = 1 + z + \frac{z^{2}}{2!} + \frac{z^{3}}{3!} + \cdots$$

We obtain the Laurent series for f by simply replacing z by  $\frac{3}{z}$ , when  $z \neq 0$ :

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots$$

This series is valid for  $z \neq 0$ , that is, for  $0 < |z| < \infty$ .

### Remarks

(i) Replacing the complex variable s with the usual symbol z, we see that when k = -1, the formula for the Laurent series coefficients yields

$$a_{-1}=\frac{1}{2\pi i}\oint_C f(z)dz,$$

or more important,

$$\oint_C f(z)dz = 2\pi ia_{-1}.$$

 (ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; i.e., the series we obtain is unique.

#### Subsection 4

Zeros and Poles

## Review of Laurent Series

• Suppose  $z = z_0$  is an isolated singularity of a complex function f, and that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=1}^{\infty} a_{-k} (z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

is the Laurent series representation of f valid for the punctured open disk  $0 < |z - z_0| < R$ .

• The part of the series with the negative powers of  $z - z_0$ , i.e.,

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}$$

is the principal part of the series.

• We will classify the isolated singularity  $z = z_0$  according to the number of terms in the principal part.

# Classification of Isolated Singular Points

• An isolated singular point  $z = z_0$  of a complex function f is given a classification depending on whether the principal part of its Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=1}^{\infty} a_{-k} (z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

contains zero, a finite number, or an infinite number of terms:

- (i) If the principal part is zero, that is, all the coefficients  $a_{-k}$  are zero, then  $z = z_0$  is called a **removable singularity**.
- (ii) If the principal part contains a finite number of nonzero terms, then z = z<sub>0</sub> is called a **pole**. If, in this case, the last nonzero coefficient in ∑<sub>k=1</sub><sup>∞</sup> (z-z<sub>0</sub>)<sup>k</sup> is a<sub>-n</sub>, n ≥ 1, then z = z<sub>0</sub> is called a **pole of order** n. If z = z<sub>0</sub> is a pole of order 1, then the principal part contains exactly one term with coefficient a<sub>-1</sub> and the pole is called a **simple pole**.
  (iii) If the principal part contains infinitely many nonzero terms, then z = z<sub>0</sub> is called an **essential singularity**.

### Form of Laurent Series Based on Classification

• The form of a Laurent series for a function *f*, when *z* = *z*<sub>0</sub> is one of the various types of isolated singularities is summarized below:

$z = z_0$	Laurent Series for $0 <  z - z_0  < R$
Removable Singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of Order <i>n</i>	$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \dots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$
Simple Pole	$\frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
Essential Singularity	$ \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots $

# A Removable Singularity

Recall the Maclaurin series for sin z: sin z = z - <sup>z<sup>3</sup></sup>/<sub>3!</sub> + <sup>z<sup>5</sup></sup>/<sub>5!</sub> - ···. Divide by z to get

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

Thus, all the coefficients in the principal part of the Laurent series are zero. Hence, z = 0 is a removable singularity of the function  $f(z) = \frac{\sin z}{z}$ .

• If a function f has a removable singularity at  $z = z_0$ , then we can supply an appropriate definition for the value of  $f(z_0)$  so that f becomes analytic at  $z = z_0$ .

Example: Since the right-hand side of the series above is 1 when we set z = 0, it makes sense to define f(0) = 1. Hence the function  $f(z) = \frac{\sin z}{z}$  is now defined and continuous at every complex number z. Indeed, f is also analytic at z = 0 because it is represented by the Taylor series  $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$  centered at 0 (a Maclaurin series).

# Poles and Essential Singularities

(a) Dividing sin  $z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$  by  $z^2$  shows that, for  $0 < |z| < \infty$ ,



Since  $a_{-1} \neq 0$ , z = 0 is a simple pole of the function  $f(z) = \frac{\sin z}{z^2}$ . Similarly, z = 0 is a pole of order 3 of the function  $f(z) = \frac{\sin z}{z^4}$ . (b) The Laurent series of  $f(z) = \frac{1}{(z-1)^2(z-3)}$  for 0 < |z-1| < 2:



Since a<sub>-2</sub> = -<sup>1</sup>/<sub>2</sub> ≠ 0, we conclude that z = 1 is a pole of order 2.
(c) The principal part of the Laurent expansion of f(z) = e<sup>3/z</sup> valid for 0 < |z| < ∞ contains an infinite number of nonzero terms. This shows that z = 0 is an essential singularity of f.</li>

# Zeros and Multiplicities

- A number  $z_0$  is a **zero** of a function f if  $f(z_0) = 0$ .
- We say that an analytic function f has a **zero of order** n at  $z = z_0$  if  $z_0$  is a zero of f and of its first n 1 derivatives, but not of its n-th derivative, i.e.,  $f(z_0) = 0$ ,  $f'(z_0) = 0$ ,  $f''(z_0) = 0$ , ...,  $f^{(n-1)}(z_0) = 0$ , but  $f^{(n)}(z_0) \neq 0$ .
- A zero of order n is also referred to as a zero of multiplicity n. Example: Consider  $f(z) = (z - 5)^3$ .

$$f(5) = 0, f'(5) = 0, f''(5) = 0, \text{ but } f'''(5) = 6 \neq 0.$$

Thus, f has a zero of order (or multiplicity) 3 at  $z_0 = 5$ . • A zero of order 1 is called a **simple zero**.

# Order of Zeros

#### Theorem (Zero of Order *n*)

A function f that is analytic in some disk  $|z - z_0| < R$  has a zero of order n at  $z = z_0$  if and only if f can be written  $f(z) = (z - z_0)^n \phi(z)$ , where  $\phi$  is analytic at  $z = z_0$  and  $\phi(z_0) \neq 0$ .

• Partial Proof ("only if" Part): Given that f is analytic at  $z_0$ , it can be expanded in a Taylor series that is centered at  $z_0$  and is convergent for  $|z - z_0| < R$ . Since, in a Taylor series  $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ ,  $a_k = \frac{f^{(k)}(z_0)}{k!}$ , k = 0, 1, ..., it follows that the first n terms are zero. So  $f(z) = a_n (z - z_0)^n + a_{n+1} (z - z_0)^{n+1} + a_{n+2} (z - z_0)^{n+2} + \cdots =$  $(z - z_0)^n (a_n + a_{n+1} (z - z_0) + a_{n+2} (z - z_0)^2 + \cdots)$ . Letting  $\phi(z) = a_n + a_{n+1} (z - z_0) + a_{n+2} (z - z_0)^2 + \cdots$ , we conclude  $f(z) = (z - z_0)^n \phi(z)$ , where  $\phi$  is an analytic function, such that  $\phi(z_0) = a_n \neq 0$  because  $a_n = \frac{f^{(n)}(z_0)}{n!} \neq 0$ .

### Computing the Order of a Zero Using a Power Series

• The analytic function  $f(z) = z \sin z^2$  has a zero at z = 0. If we replace z by  $z^2$  in the Maclaurin series for  $\sin z$ , we obtain

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots$$

Then, by factoring  $z^2$  out, we can rewrite f as

$$f(z) = z \sin z^2 = z^3 \phi(z),$$

where  $\phi(z) = 1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots$  and  $\phi(0) = 1$ . This shows that z = 0 is a zero of order 3 of f.

#### Poles of Order n

• A pole of order *n* may be characterized analogously to the characterization of zeros:

#### Theorem (Pole of Order *n*)

A function f analytic in a punctured disk  $0 < |z - z_0| < R$  has a pole of order n at  $z = z_0$  if and only if f can be written  $f(z) = \frac{\phi(z)}{(z-z_0)^n}$ , where  $\phi$  is analytic at  $z = z_0$  and  $\phi(z_0) \neq 0$ .

• Partial Proof ("only if" Part): Since f is assumed to have a pole of order n at  $z_0$ , it can be expanded in a Laurent series  $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots$ , valid in some punctured disk  $0 < |z - z_0| < R$ . By factoring out  $\frac{1}{(z-z_0)^n}$ ,  $f(z) = \frac{\phi(z)}{(z-z_0)^n}$ , where  $\phi(z) = a_{-n} + \dots + a_{-2}(z-z_0)^{n-2} + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \dots$ . This is a power series valid for the open disk  $|z - z_0| < R$ . Since  $z = z_0$  is a pole of order n of f,  $a_{-n} \neq 0$ .

#### Zeros and Poles

- A zero  $z = z_0$  of an analytic function f is *isolated* in the sense that there exists some neighborhood of  $z_0$  for which  $f(z) \neq 0$  at every point z in that neighborhood except at  $z = z_0$ .
- As a consequence, if z<sub>0</sub> is a zero of a nontrivial analytic function f, then the function <sup>1</sup>/<sub>f(z)</sub> has an isolated singularity at the point z = z<sub>0</sub>.

#### Theorem (Pole of Order *n*)

If the functions g and h are analytic at  $z = z_0$  and h has a zero of order n at  $z = z_0$  and  $g(z_0) \neq 0$ , then the function  $f(z) = \frac{g(z)}{h(z)}$  has a pole of order n at  $z = z_0$ .

• Because *h* has a zero of order *n*,  $h(z) = (z - z_0)^n \phi(z)$ , where  $\phi$  is analytic at  $z = z_0$  and  $\phi(z_0) \neq 0$ . Thus, *f* can be written  $f(z) = \frac{g(z)/\phi(z)}{(z-z_0)^n}$ . Since *g* and  $\phi$  are analytic at  $z = z_0$  and  $\phi(z_0) \neq 0$ , it follows that the function  $g/\phi$  is analytic at  $z_0$  and  $g(z_0)/\phi(z_0) \neq 0$ . We conclude that the function *f* has a pole of order *n* at  $z_0$ .

### Examples

(a) Inspection of the rational function

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

shows that the denominator has zeros of order 1 at z = 1 and z = -5, and a zero of order 4 at z = 2. Since the numerator is not zero at any of these points, it follows from the theorem that f has simple poles at z = 1 and z = -5, and a pole of order 4 at z = 2.

(b) z = 0 is a zero of order 3 of  $z \sin z^2$ . The reciprocal function

$$f(z) = \frac{1}{z \sin z^2}$$

has a pole of order 3 at z = 0.

## Remarks

- (i) If a function f has a pole at  $z = z_0$ , then  $|f(z)| \to \infty$  as  $z \to z_0$  from any direction. Thus, we can write  $\lim_{z \to z_0} f(z) = \infty$ .
- (ii) A function f is **meromorphic** if it is analytic throughout a domain D, except possibly for poles in D. It can be proved that a meromorphic function can have at most a finite number of poles in D.

E.g., the rational function

$$f(z) = \frac{1}{z^2 + 1}$$

is meromorphic in the complex plane.

#### Subsection 5

#### Residues and Residue Theorem

#### Residue

- If a complex function f has an isolated singularity at a point  $z_0$ , then f has a Laurent series representation  $f(z) = \sum_{k=-\infty}^{\infty} a_k (z z_0)^k = \cdots + \frac{a_{-2}}{(z z_0)^2} + \frac{a_{-1}}{z z_0} + a_0 + a_1 (z z_0) + \cdots$ , which converges for all z in some deleted neighborhood  $0 < |z z_0| < R$  of  $z_0$ .
- We now focus on the coefficient *a*<sub>-1</sub> and its importance in the evaluation of contour integrals.
- The coefficient  $a_{-1}$  is called the **residue** of the function f at the isolated singularity  $z_0$  and denoted

$$a_{-1} = \operatorname{Res}(f(z), z_0).$$

• Recall, if the principal part of the series valid for  $0 < |z - z_0| < R$  contains a finite number of terms with  $a_{-n}$  the last nonzero coefficient, then  $z_0$  is a pole of order n; if the principal part contains an infinite number of terms with nonzero coefficients, then  $z_0$  is an essential singularity.

George Voutsadakis (LSSU)

### Examples of Residues

(a) We have seen that z = 1 is a pole of order two of the function  $f(z) = \frac{1}{(z-1)^2(z-3)}$ . The Laurent series valid for the deleted neighborhood 0 < |z - 1| < 2 is

$$f(z) = -\frac{1/2}{(z-1)^2} + \frac{-1/4}{z-1} - \frac{1}{8} - \frac{z-1}{16} - \cdots$$

Thus, the coefficient of  $\frac{1}{z-1}$  is  $a_{-1} = \operatorname{Res}(f(z), 1) = -\frac{1}{4}$ .

(b) We also saw that z = 0 is an essential singularity of  $f(z) = e^{3/z}$ . Its Laurent series is

$$e^{3/z} = 1 + rac{3}{z} + rac{3^2}{2!z^2} + rac{3^3}{3!z^3} + \cdots, \ 0 < |z| < \infty.$$

Hence, the coefficient of  $\frac{1}{z}$  is  $a_{-1} = \text{Res}(f(z), 0) = 3$ .

# Residue at a Simple Pole

- We examine ways of obtaining a<sub>-1</sub> when z<sub>0</sub> is a pole of a function f, without the necessity of expanding f in a Laurent series at z<sub>0</sub>.
- We begin with the residue at a simple pole:

#### Theorem (Residue at a Simple Pole)

If f has a simple pole at  $z = z_0$ , then

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

• Since f has a simple pole at  $z = z_0$ , its Laurent expansion convergent on a punctured disk  $0 < |z - z_0| < R$  has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

where  $a_{-1} \neq 0$ . By multiplying both sides of this series by  $z - z_0$  and then taking the limit as  $z \to z_0$  we obtain  $\lim_{z\to z_0} (z - z_0)f(z) =$  $\lim_{z\to z_0} [a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots] = a_{-1} = \operatorname{Res}(f(z), z_0).$ 

# Residue at a Pole of Order n

Theorem (Residue at a Pole of Order n)

If f has a pole of order n at  $z = z_0$ , then

$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z).$$

• Since f has a pole of order n at  $z = z_0$ , its Laurent expansion, convergent on a punctured disk  $0 < |z - z_0| < R$ , has the form  $f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$ , where  $a_{-n} \neq 0$ . We multiply by  $(z - z_0)^n$ ,  $(z - z_0)^n f(z) = a_{-n} + \cdots + a_{-2}(z - z_0)^{n-2} + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \cdots$ and then differentiate n - 1 times:

$$\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z) = (n-1)!a_{-1} + n!a_0(z-z_0) + \cdots$$

Therefore, as  $z \to z_0$ ,  $\lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) = (n-1)! a_{-1}$ .

# Finding Residue at a Pole

• The function  $f(z) = \frac{1}{(z-1)^2(z-3)}$  has a simple pole at z = 3 and a pole of order 2 at z = 1. Use the theorems to find the residues. Since z = 3 is a simple pole,

$$\operatorname{Res}(f(z),3) = \lim_{z \to 3} (z-3)f(z) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4}$$

At the pole of order 2,

$$\operatorname{Res}(f(z),1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz} (z-1)^2 f(z) = \lim_{z \to 1} \frac{d}{dz} \frac{1}{z-3}$$
$$= \lim_{z \to 1} \frac{-1}{(z-3)^2} = -\frac{1}{4}.$$

### Second Method for Computing a Residue at a Simple Pole

• Suppose a function f can be written as a quotient  $f(z) = \frac{g(z)}{h(z)}$ , where g and h are analytic at  $z = z_0$ . If  $g(z_0) \neq 0$  and if the function h has a zero of order 1 at  $z_0$ , then f has a simple pole at  $z = z_0$  and

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

• Since h has a zero of order 1 at  $z_0$ , we must have  $h(z_0) = 0$  and  $h'(z_0) \neq 0$ . By definition of the derivative,  $h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0}$ . Therefore,  $\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\frac{h(z)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}$ .

#### Applying the Second Method

• The polynomial  $z^4 + 1$  can be factored as

$$(z-z_1)(z-z_2)(z-z_3)(z-z_4),$$

where  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  are the four distinct roots of the equation  $z^4 + 1 = 0$  (or, equivalently, the four fourth roots of -1). It follows that the function  $f(z) = \frac{1}{z^4+1}$  has four simple poles. By the root formula  $z_1 = e^{\pi i/4}$ ,  $z_2 = e^{3\pi i/4}$ ,  $z_3 = e^{5\pi i/4}$ , and  $z_4 = e^{7\pi i/4}$ . We compute the residues:

$$\begin{aligned} &\operatorname{Res}(f(z), z_1) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i\\ &\operatorname{Res}(f(z), z_2) = \frac{1}{4z_2^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i\\ &\operatorname{Res}(f(z), z_3) = \frac{1}{4z_3^3} = \frac{1}{4}e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i\\ &\operatorname{Res}(f(z), z_4) = \frac{1}{4z_4^3} = \frac{1}{4}e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i. \end{aligned}$$

# Using the Original Formula

• We could have calculated each of the residues of  $f(z) = \frac{1}{z^4+1}$  using  $\operatorname{Res}(f(z), z_i) = \lim_{z \to z_i} (z - z_i) f(z)$ .

• E.g., at *z*<sub>1</sub>,

$$\begin{split} \mathsf{Res}(f(z),z_1) &= \lim_{z \to z_1} (z-z_1) \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\ &= \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} \\ &= \frac{1}{(e^{\pi i/4}-e^{3\pi i/4})(e^{\pi i/4}-e^{5\pi i/4})(e^{\pi i/4}-e^{7\pi i/4})}. \end{split}$$

In simplifying the denominator of the last expression considerably more algebra is involved than using the second method.

# Cauchy's Residue Theorem

 Complex integrals ∮<sub>C</sub> f(z)dz can sometimes be evaluated by summing the residues at the isolated singularities of f within C:

#### Theorem (Cauchy's Residue Theorem)

Let *D* be a simply connected domain and *C* a simple closed contour lying entirely within *D*. If a function *f* is analytic on and within *C*, except at a finite number of isolated singular points  $z_1, z_2, \ldots, z_n$  within *C*, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k).$$

• Suppose  $C_1, C_2, \ldots, C_n$  are circles centered at  $z_1, z_2, \ldots, z_n$ , respectively, such that  $C_k$  has a radius  $r_k$  small enough so that  $C_1, C_2, \ldots, C_n$  are mutually disjoint and are interior to the simple closed curve C. We saw that  $\oint_{C_k} f(z)dz = 2\pi i \operatorname{Res}(f(z), z_k)$ , whence, we have  $\oint_C f(z)dz = \sum_{k=1}^n \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k)$ .
# Evaluation by the Residue Theorem I

• Evaluate 
$$\oint_C \frac{1}{(z-1)^2(z-3)} dz$$
, where  
(a) the contour *C* is the rectangle defined by  $x = 0, x = 4, y = -1, y = 1$ ;  
(b) the contour *C* is the circle  $|z| = 2$ .  
(a) Since both  $z = 1$  and  $z = 3$  are poles within the rectangle, we have  
 $\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i [\operatorname{Res}(f(z), 1) + \operatorname{Res}(f(z), 3)]$ . We found  
these residues already:  $\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i (-\frac{1}{4} + \frac{1}{4}) = 0$ .  
(b) Since only the pole  $z = 1$  lies within the circle  $|z| = 2$ , we have  
 $\oint_C \frac{1}{(z-1)^2(z-3)} dz = 2\pi i \operatorname{Res}(f(z), 1) = 2\pi i (-\frac{1}{4}) = -\frac{\pi}{2} i$ .

## Evaluation by the Residue Theorem II

• Evaluate 
$$\oint_C \frac{2z+6}{z^2+4} dz$$
, where the contour C is the circle  $|z-i| = 2$ .  
By factoring the denominator  $z^2 + 4 = (z-2i)(z+2i)$ , we see that  
the integrand has simple poles at  $-2i$  and  $2i$ . Only  $2i$  lies within the  
contour C. Thus,  $\oint_C \frac{2z+6}{z^2+4} dz = 2\pi i \operatorname{Res}(f(z), 2i)$ . But  
 $\operatorname{Res}(f(z), 2i) = \lim_{z \to 2i} (z-2i) \frac{2z+6}{(z-2i)(z+2i)} = \frac{6+4i}{4i} = \frac{3+2i}{2i}$ . Hence,  
 $\oint_C \frac{2z+6}{z^2+4} dz = 2\pi i \left(\frac{3+2i}{2i}\right) = \pi(3+2i)$ .

## Evaluation by the Residue Theorem III

• Evaluate 
$$\oint_C \frac{e^z}{z^4 + 5z^3} dz$$
, where the contour *C* is the circle  $|z| = 2$ .  
Writing the denominator as  $z^4 + 5z^3 = z^3(z+5)$  reveals that the integrand  $f(z)$  has a pole of order 3 at  $z = 0$  and a simple pole at  $z = -5$ . Only the pole  $z = 0$  lies within the given contour. Thus, we have  
 $\oint_C \frac{e^z}{z^4 + 5z^3} dz = 2\pi i \operatorname{Res}(f(z), 0) = 2\pi i \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} z^3 \cdot \frac{e^z}{z^3(z+5)} = \pi i \lim_{z \to 0} \frac{d}{dz} \frac{e^z(z+4)}{(z+5)^2} = \pi i \lim_{z \to 0} \frac{(z^2 + 8z + 17)e^z}{(z+5)^3} = \frac{17\pi}{125}i$ .

## Evaluation by the Residue Theorem IV

• Evaluate 
$$\oint_C \tan z dz$$
, where the contour *C* is the circle  $|z| = 2$ .  
The integrand  $f(z) = \tan z = \frac{\sin z}{\cos z}$  has simple poles at the points  
where  $\cos z = 0$ . We saw that the only zeros of  $\cos z$  are the real  
numbers  $z = \frac{(2n+1)\pi}{2}$ ,  $n = 0, \pm 1, \pm 2, \ldots$  Only  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$  are within  
the circle  $|z| = 2$ . Thus, we have  
 $\oint_C \tan z dz = 2\pi i [\operatorname{Res}(f(z), -\frac{\pi}{2}) + \operatorname{Res}(f(z), \frac{\pi}{2})]$ . With  $f(z) = \frac{g(z)}{h(z)}$ ,  
where  $g(z) = \sin z$ ,  $h(z) = \cos z$ , and  $h'(z) = -\sin z$ , we get  
 $\operatorname{Res}(f(z), -\frac{\pi}{2}) = \frac{\sin(-\frac{\pi}{2})}{-\sin(-\frac{\pi}{2})} = -1$ .  $\operatorname{Res}(f(z), \frac{\pi}{2}) = \frac{\sin(\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = -1$ .  
Therefore,  $\oint_C \tan z dz = 2\pi i [-1 - 1] = -4\pi i$ .

#### Evaluation by the Residue Theorem V

• Evaluate  $\oint_C e^{3/z} dz$ , where the contour C is the circle |z| = 1. We saw that z = 0 is an essential singularity of the integrand  $f(z) = e^{3/z}$ . So we cannot use the formulas

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$
$$\operatorname{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

or

to find the residue of f at that point. Nevertheless, the Laurent series of f at z = 0 gives

$$\mathsf{Res}(f(z),0)=3.$$

Hence, we have

$$\oint_C e^{3/z} dz = 2\pi i \operatorname{Res}(f(z), 0) = 2\pi i (3) = 6\pi i.$$