# Introduction to Complex Analysis 

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(1) Series and Residues

- Sequences and Series
- Taylor Series
- Laurent Series
- Zeros and Poles
- Residues and Residue Theorem


## Subsection 1

## Sequences and Series

## Sequences

- A sequence $\left\{z_{n}\right\}$ is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers $\mathbb{C}$.
- Example: The sequence $\left\{1+i^{n}\right\}$ is $\underset{n=1}{1+i,} \underset{n=2}{0}, 1_{n=3}^{1-i}, \underset{n=4}{2}, \underset{n=5}{1+i}, \ldots$.
- If $\lim _{n \rightarrow \infty} z_{n}=L$, we say the sequence $\left\{z_{n}\right\}$ is convergent, i.e., $\left\{z_{n}\right\}$ converges to the number $L$ if, for each positive real number $\varepsilon$, an $N$ can be found, such that $\left|z_{n}-L\right|<\varepsilon$, whenever $n>N$.
- Since $\left|z_{n}-L\right|$ is distance, the terms $z_{n}$ of a sequence that converges to $L$ can be made arbitrarily close to $L$. In a different way, when a sequence $\left\{z_{n}\right\}$ converges to $L$, then all but a finite number of the terms of the sequence are within every $\varepsilon$-neighborhood of $L$.

- A sequence that is not convergent is said to be divergent. Example: The sequence $\left\{1+i^{n}\right\}$ is divergent since the general term $z_{n}=1+i^{n}$ does not approach a fixed complex number as $n \rightarrow \infty$.


## An Example of a Convergent Sequence

- The sequence $\left\{\frac{i^{n+1}}{n}\right\}$ converges since $\lim _{n \rightarrow \infty} \frac{i^{n+1}}{n}=0$. As we see from

$$
-1,-\frac{i}{2}, \frac{1}{3}, \frac{i}{4},-\frac{1}{5}, \ldots
$$

the terms of the sequence spiral in toward the point $z=0$ as $n$ increases.


## Criterion for Convergence

## Theorem (Criterion for Convergence)

A sequence $\left\{z_{n}\right\}$ converges to a complex number $L=a+i b$ if and only if $\operatorname{Re}\left(z_{n}\right)$ converges to $\operatorname{Re}(L)=a$ and $\operatorname{Im}\left(z_{n}\right)$ converges to $\operatorname{Im}(L)=b$.

- Example: Consider the sequence $\left\{\frac{3+n i}{n+2 n i}\right\}$.

$$
z_{n}=\frac{3+n i}{n+2 n i}=\frac{(3+n i)(n-2 n i)}{n^{2}+4 n^{2}}=\frac{2 n^{2}+3 n}{5 n^{2}}+i \frac{n^{2}-6 n}{5 n^{2}} .
$$

Thus, we get

$$
\begin{aligned}
& \operatorname{Re}\left(z_{n}\right)=\frac{2 n^{2}+3 n}{5 n^{2}}=\frac{2}{5}+\frac{3}{5 n} \rightarrow \frac{2}{5} \\
& \operatorname{Im}\left(z_{n}\right)=\frac{n^{2}-6 n}{5 n^{2}}=\frac{1}{5}-\frac{6}{5 n} \rightarrow \frac{1}{5}
\end{aligned}
$$

By the theorem, the given sequence converges to $a+i b=\frac{2}{5}+\frac{1}{5} i$.

## Series and Geometric Series

- An infinite series or series of complex numbers $\sum_{k=1}^{\infty} z_{k}=z_{1}+z_{2}$ $+z_{3}+\cdots+z_{n}+\cdots$ is convergent if the sequence of partial sums $\left\{S_{n}\right\}$, where $S_{n}=z_{1}+z_{2}+z_{3}+\cdots+z_{n}$ converges. If $S_{n} \rightarrow L$ as $n \rightarrow \infty$, we say that the series converges to $L$ or that the sum of the series is $L$.
- Geometric Series: A geometric series is any series of the form $\sum_{k=1}^{\infty} a z^{k-1}=a+a z+a z^{2}+\cdots+a z^{n-1}+\cdots$. The $n$-th term of the sequence of partial sums is $S_{n}=a+a z+a z^{2}+\cdots+a z^{n-1}$. To get a formula for $S_{n}$, multiply by $z: ~ z S_{n}=a z+a z^{2}+a z^{3}+\cdots+a z^{n}$. Subtract this from $S_{n}: S_{n}-z S_{n}=\left(a+a z+a z^{2}+\cdots+a z^{n-1}\right)-$ $\left(a z+a z^{2}+a z^{3}+\cdots+a z^{n-1}+a z^{n}\right)=a-a z^{n}$. Thus,
$(1-z) S_{n}=a\left(1-z^{n}\right)$, and, hence, $S_{n}=\frac{a\left(1-z^{n}\right)}{1-z}$.
- If $|z|<1, z^{n} \rightarrow 0$ as $n \rightarrow \infty$. So $S_{n} \rightarrow \frac{a}{1-z}$. I.e., for $|z|<1$, $\frac{a}{1-z}=a+a z+a z^{2}+\cdots+a z^{n-1}+\cdots$.
- If $|z| \geq 1$, a geometric series diverges.


## Special Geometric Series

- Recall the sum formulas

$$
S_{n}=\frac{a\left(1-z^{n}\right)}{1-z}, \quad \frac{a}{1-z}=a+a z+a z^{2}+\cdots+a z^{n-1}+\cdots
$$

- If we set $a=1$, we get

$$
\frac{1}{1-z}=1+z+z^{2}+z^{3}+\cdots
$$

- If we then replace $z$ by $-z$ :

$$
\frac{1}{1+z}=1-z+z^{2}-z^{3}+\cdots
$$

- For the finite sum, we have $\frac{1-z^{n}}{1-z}=1+z+z^{2}+z^{3}+\cdots+z^{n-1}$. Rewriting the left side of the above equation as $\frac{1-z^{n}}{1-z}=\frac{1}{1-z}+\frac{-z^{n}}{1-z}$, we get

$$
\frac{1}{1-z}=1+z+z^{2}+z^{3}+\cdots+z^{n-1}+\frac{z^{n}}{1-z}
$$

## A Convergent Geometric Series

- The infinite series

$$
\sum_{k=1}^{\infty} \frac{(1+2 i)^{k}}{5^{k}}=\frac{1+2 i}{5}+\frac{(1+2 i)^{2}}{5^{2}}+\frac{(1+2 i)^{3}}{5^{3}}+\cdots
$$

is a geometric series.
It has the standard form, with $a=\frac{1}{5}(1+2 i)$ and $z=\frac{1}{5}(1+2 i)$. Since $|z|=\frac{\sqrt{5}}{5}<1$, the series is convergent and its sum is given by:

$$
\sum_{k=1}^{\infty} \frac{(1+2 i)^{k}}{5^{k}}=\frac{\frac{1+2 i}{5}}{1-\frac{1+2 i}{5}}=\frac{1+2 i}{4-2 i}=\frac{1}{2} i
$$

## Necessary Condition for Convergence

- We turn to some important theorems about convergence and divergence of an infinite series:


## Theorem (A Necessary Condition for Convergence)

If $\sum_{k=1}^{\infty} z_{k}$ converges, then $\lim _{n \rightarrow \infty} z_{n}=0$.

- Let $L$ denote the sum of the series. Then $S_{n} \rightarrow L$ and $S_{n-1} \rightarrow L$ as $n \rightarrow \infty$. By taking the limit of both sides of $S_{n}-S_{n-1}=z_{n}$ as $n \rightarrow \infty$, we obtain the desired conclusion.


## Theorem (The $n$-th Term Test for Divergence)

If $\lim _{n \rightarrow \infty} z_{n} \neq 0$, then $\sum_{k=1}^{\infty} z_{k}$ diverges.

- Example: The series $\sum_{k=1}^{\infty} \frac{i k+5}{k}$ diverges, since $z_{n}=\frac{i n+5}{n} \rightarrow i \neq 0$ as $n \rightarrow \infty$.
The geometric series $\sum_{k=1}^{\infty} a z^{k}$ diverges if $|z| \geq 1$ because even in the case when $\lim _{n \rightarrow \infty}\left|z^{n}\right|$ exists, the limit is not zero.


## Absolute and Conditional Convergence

## Definition (Absolute and Conditional Convergence)

An infinite series $\sum_{k=1}^{\infty} z_{k}$ is said to be absolutely convergent if $\sum_{k=1}^{\infty}\left|z_{k}\right|$ converges. An infinite series $\sum_{k=1}^{\infty} z_{k}$ is said to be conditionally convergent if it converges but $\sum_{k=1}^{\infty}\left|z_{k}\right|$ diverges.

- In elementary calculus a real series of the form $\sum_{k=1}^{\infty} \frac{1}{k^{p}}$ is called a $p$-series and
- converges for $p>1$;
- diverges for $p \leq 1$.
- Example: The series $\sum_{k=1}^{\infty} \frac{i^{k}}{k^{2}}$ is absolutely convergent: The series $\sum_{k=1}^{\infty}\left|\frac{i^{k}}{k^{2}}\right|$ is the same as the real convergent $p$-series $\sum_{k=1}^{\infty} \frac{1}{k^{2}}$.
- As in real calculus, absolute convergence implies convergence.
- Example: The series $\sum_{k=1}^{\infty} \frac{i^{k}}{k^{2}}=i-\frac{1}{2^{2}}-\frac{i}{3^{2}}+\cdots$ converges, because it is was shown to be absolutely convergent.


## Tests for Convergence

## Theorem (The Ratio Test)

Let $\sum_{k=1}^{\infty} z_{k}$ be a series of nonzero terms, with $\lim _{n \rightarrow \infty}\left|\frac{z_{n+1}}{z_{n}}\right|=L$.
(i) If $L<1$, then the series converges absolutely.
(ii) If $L>1$ or $L=\infty$, then the series diverges.
(iii) If $L=1$, the test is inconclusive.

## Theorem (The Root Test)

Let $\sum_{k=1}^{\infty} z_{k}$ be a series of complex terms, with $\lim _{n \rightarrow \infty} \sqrt[n]{\left|z_{n}\right|}=L$.
(i) If $L<1$, then the series converges absolutely.
(ii) If $L>1$ or $L=\infty$, then the series diverges.
(iii) If $L=1$, the test is inconclusive.

- We are interested primarily in applying these tests to power series.


## Power Series and Circle of Convergence

- An infinite series of the form $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}=a_{0}+a_{1}\left(z-z_{0}\right)+$ $a_{2}\left(z-z_{0}\right)^{2}+\cdots$, where the coefficients $a_{k}$ are complex constants, is called a power series in $z-z_{0}$.
- The power series is said to be centered at $z_{0}$ and the complex point $z_{0}$ is referred to as the center of the series.
- It is also convenient to define $\left(z-z_{0}\right)^{0}=1$ even when $z=z_{0}$.
- Every complex power series has a radius of convergence and a circle of convergence: It is the circle centered at $z_{0}$ of largest radius $R>0$ for which the series converges at every point within the circle $\left|z-z_{0}\right|=R$.


A power series converges absolutely at all points $z$ satisfying $\left|z-z_{0}\right|<R$, and diverges at all points $z$, with $\left|z-z_{0}\right|>R$.

## Possibilities for Radius of Convergence

- The radius of convergence can be:
(i) $R=0$ (series converges only at its center $z=z_{0}$ );
(ii) $R$ a finite positive number (series converges in interior of $\left|z-z_{0}\right|=R$ );
(iii) $R=\infty$ (series converges for all $z$ ).

A power series may converge at some, all, or at none of the points on the actual circle of convergence.

- Example: Consider $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test, $\lim _{n \rightarrow \infty}\left|\frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}}\right|=$ $\lim _{n \rightarrow \infty} \frac{n}{n+1}|z|=|z|$. Thus, the series converges absolutely for $|z|<1$. The circle of convergence is $|z|=1$ and the radius of convergence is $R=1$. On the circle $|z|=1$, the series does not converge absolutely since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the well-known divergent harmonic series. This does not mean that the series diverges on the circle of convergence. In fact, at $z=-1, \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is the convergent alternating harmonic series. It can be shown that the series converges at all points on the circle $|z|=1$ except at $z=1$.


## Dependence of the Radius on the Coefficients

- For a power series

$$
\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}
$$

the limit depends only on the coefficients $a_{k}$. Thus:
(i) if $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=L \neq 0$, the radius of convergence is $R=\frac{1}{L}$;
(ii) if $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=0$, the radius of convergence is $R=\infty$;
(iii) if $\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|=\infty$, the radius of convergence is $R=0$.

- Similar conclusions can be made for the root test by utilizing $\lim _{n \rightarrow \infty} \sqrt[n]{\left|a_{n}\right|}$. E.g., if $\lim _{n \rightarrow \infty} \sqrt[n]{\left|a_{n}\right|}=L \neq 0$, then $R=\frac{1}{L}$.


## Finding Radius of Convergence Using Ratio Test

- Consider the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z-1-i)^{k}$.

With the identification $a_{n}=\frac{(-1)^{n+1}}{n!}$, we have

$$
\lim _{n \rightarrow \infty}\left|\frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}}\right|=\lim _{n \rightarrow \infty} \frac{1}{n+1}=0
$$

Hence, the radius of convergence is $\infty$. The power series with center $z_{0}=1+i$ converges absolutely for all $z$, i.e., for $|z-1-i|<\infty$.

## Finding Radius of Convergence Using Root Test

- Consider the power series $\sum_{k=1}^{\infty}\left(\frac{6 k+1}{2 k+5}\right)^{k}(z-2 i)^{k}$.
- With $a_{n}=\left(\frac{6 n+1}{2 n+5}\right)^{n}$, the root test gives

$$
\lim _{n \rightarrow \infty} \sqrt[n]{\left|a_{n}\right|}=\lim _{n \rightarrow \infty}\left(\frac{6 n+1}{2 n+5}\right)=3
$$

We conclude that the radius of convergence of the series is $R=\frac{1}{3}$. The circle of convergence is $|z-2 i|=\frac{1}{3}$; the power series converges absolutely for $|z-2 i|<\frac{1}{3}$.

## The Arithmetic of Power Series

- Some facts concerning power-series stated informally:
- A power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ can be multiplied by a nonzero complex constant $c$ without affecting its convergence or divergence.
- A power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ converges absolutely within its circle of convergence. As a consequence, within the circle of convergence the terms of the series can be rearranged and the rearranged series has the same sum $L$ as the original series.
- Two power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ and $\sum_{k=0}^{\infty} b_{k}\left(z-z_{0}\right)^{k}$ can be added and subtracted by adding or subtracting like terms:

$$
\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k} \pm \sum_{k=0}^{\infty} b_{k}\left(z-z_{0}\right)^{k}=\sum_{k=0}^{\infty}\left(a_{k} \pm b_{k}\right)\left(z-z_{0}\right)^{k}
$$

- If both series have the same nonzero radius $R$ of convergence, the radius of convergence of $\sum_{k=0}^{\infty}\left(a_{k} \pm b_{k}\right)\left(z-z_{0}\right)^{k}$ is $R$.
- If one series has radius of convergence $r>0$ and the other $R>0$, where $r \neq R$, then $\sum_{k=0}^{\infty}\left(a_{k} \pm b_{k}\right)\left(z-z_{0}\right)^{k}$ has radius of convergence the smaller of $r$ and $R$.
- Two power series can (with care) be multiplied and divided.


## Final Remarks on Series and Power Series

- If $z_{n}=a_{n}+i b_{n}$ then the $n$-th term of the sequence of partial sums for $\sum_{k=1}^{\infty} z_{k}$ is $S_{n}=\sum_{k=1}^{n}\left(a_{k}+i b_{k}\right)=\sum_{k=1}^{n} a_{k}+i \sum_{k=1}^{n} b_{k}$. Thus, $\sum_{k=1}^{\infty} z_{k}$ converges to $L=a+i b$ if and only if $\operatorname{Re}\left(S_{n}\right)=\sum_{k=1}^{n} a_{k}$ converges to $a$ and $\operatorname{Im}\left(S_{n}\right)=\sum_{k=1}^{n} b_{k}$ converges to $b$.
- In summation notation a geometric series need not start at $k=1$ nor does the general term have to appear precisely as $a z^{k-1}$.
- Example: Consider $\sum_{k=3}^{\infty} 40 \frac{i^{k+2}}{2^{k-1}}$. It does not appear to match the form $\sum_{k=1}^{\infty} a z^{k-1}$ of a geometric series. By writing out three terms, $\sum_{k=3}^{\infty} 40 \frac{i^{k+2}}{2^{k-1}}=40 \frac{i^{5}}{2^{2}}+40 \frac{i^{6}}{2^{3}}+40 \frac{i^{7}}{2^{4}}+\cdots$ we see $a=40 \frac{i^{5}}{2^{2}}$ and $z=\frac{i}{2}$. Since $|z|=\frac{1}{2}<1$, the sum is $\sum_{k=3}^{\infty} 40 \frac{i^{k+2}}{2^{k-1}}=\frac{40 \frac{i^{5}}{2^{2}}}{1-\frac{i}{2}}=-4+8 i$.
- A power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ always possesses a radius of convergence $R$. The ratio and root tests lead to $\frac{1}{R}=\lim _{n \rightarrow \infty}\left|\frac{a_{n+1}}{a_{n}}\right|$ and $\frac{1}{R}=\lim _{n \rightarrow \infty} \sqrt[n]{\left|a_{n}\right|}$ assuming the appropriate limit exists.


## Subsection 2

## Taylor Series

## Differentiation of Power Series

## Theorem (Continuity)

A power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ represents a continuous function $f$ within its circle of convergence $\left|z-z_{0}\right|=R$.

## Theorem (Term-by-Term Differentiation)

A power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ can be differentiated term by term within its circle of convergence $\left|z-z_{0}\right|=R$.

- Differentiating a power series term-by-term gives,

$$
\frac{d}{d z} \sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}=\sum_{k=0}^{\infty} a_{k} \frac{d}{d z}\left(z-z_{0}\right)^{k}=\sum_{k=1}^{\infty} a_{k} k\left(z-z_{0}\right)^{k-1}
$$

- Using the ratio test, it can be shown that the original series and the differentiated series have the same circle of convergence.
- Since the derivative of a power series is another power series, the first series can be differentiated as many times as we wish.


## Integration of Power Series

## Theorem (Term-by-Term Integration)

A power series $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ can be integrated term-by-term within its circle of convergence $\left|z-z_{0}\right|=R$, for every contour $C$ lying entirely within the circle of convergence.

- The theorem states that

$$
\int_{C} \sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k} d z=\sum_{k=0}^{\infty} a_{k} \int_{C}\left(z-z_{0}\right)^{k} d z
$$

whenever $C$ lies in the interior of $\left|z-z_{0}\right|=R$.

- Indefinite integration can also be carried out term by term:
$\int \sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k} d z=\sum_{k=0}^{\infty} a_{k} \int\left(z-z_{0}\right)^{k} d z=\sum_{k=0}^{\infty} \frac{a_{k}}{k+1}\left(z-z_{0}\right)^{k+1}+K$.
- The ratio test can be used to prove that both series have the same circle of convergence.


## Analyticity

- Suppose a power series represents a function $f$ within $\left|z-z_{0}\right|=R$,

$$
\begin{aligned}
& \text { i.e., } f(z)=\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}= \\
& a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+a_{3}\left(z-z_{0}\right)^{3}+\cdots .
\end{aligned}
$$

- Then, the derivatives of $f$ are the series

$$
\begin{aligned}
f^{\prime}(z) & =\sum_{k=1}^{\infty} a_{k} k\left(z-z_{0}\right)^{k-1}=a_{1}+2 a_{2}\left(z-z_{0}\right)+3 a_{3}\left(z-z_{0}\right)^{2}+\cdots \\
f^{\prime \prime}(z) & =\sum_{k=2}^{\infty} a_{k} k(k-1)\left(z-z_{0}\right)^{k-2}=2 \cdot 1 a_{2}+3 \cdot 2 a_{3}\left(z-z_{0}\right)+\cdots \\
f^{\prime \prime \prime}(z) & =\sum_{k=3}^{\infty} a_{k} k(k-1)(k-2)\left(z-z_{0}\right)^{k-3}=3 \cdot 2 \cdot 1 a_{3}+\cdots
\end{aligned}
$$

- Since the power series represents a differentiable function $f$ within its circle of convergence $\left|z-z_{0}\right|=R$, it represents an analytic function within its circle of convergence.


## Taylor Series and Maclaurin Series

- Evaluating the derivatives at $z=z_{0}$ gives

$$
f\left(z_{0}\right)=a_{0}, f^{\prime}\left(z_{0}\right)=1!a_{1}, f^{\prime \prime}\left(z_{0}\right)=2!a_{2}, f^{\prime \prime \prime}\left(z_{0}\right)=3!a_{3} .
$$

- In general, $f^{(n)}\left(z_{0}\right)=n!a_{n}$, or $a_{n}=\frac{f^{(n)}\left(z_{0}\right)}{n!}, n \geq 0$.
- When $n=0$, we interpret the zero-order derivative as $f\left(z_{0}\right)$ and $0!=1$, so that the formula gives $a_{0}=f\left(z_{0}\right)$.
- Substituting into the series yields

$$
f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}\left(z_{0}\right)}{k!}\left(z-z_{0}\right)^{k}
$$

This series is called the Taylor series for $f$ centered at $z_{0}$.

- A Taylor series with center $z_{0}=0$,

$$
f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^{k}
$$

is referred to as a Maclaurin series.

## Taylor's Theorem

- Since a power series converges in a circular domain, and a domain $D$ is generally not circular, the following question arises:

Can we expand $f$ in one or more power series that are valid, i.e., a power series that converges at $z$ and the number to which the series converges is $f(z)$, in circular domains that are all contained in $D$ ?

## Theorem (Taylor's Theorem)

Let $f$ be analytic within a domain $D$ and let $z_{0}$ be a point in $D$. Then $f$ has the series representation $f(z)=\sum_{k=0}^{\infty} \frac{f^{(k)}\left(z_{0}\right)}{k!}\left(z-z_{0}\right)^{k}$ valid for the largest circle $C$ with center at $z_{0}$ and radius $R$ that lies entirely within $D$.

- Let $z$ be a fixed point within the circle $C$ and let $s$ denote the variable of integration. The circle $C$ is then described by $\left|s-z_{0}\right|=R$. We use the Cauchy integral formula to obtain the value of $f$ at $z$ :


D

## Proof of Taylor's Theorem I

- $f(z)=\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s-z} d s=\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{\left(s-z_{0}\right)-\left(z-z_{0}\right)} d s=$
$\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s-z_{0}}\left(\frac{1}{1-\frac{z-z_{0}}{s-z_{0}}}\right) d s$. By the power series for $\frac{1}{1-z}$, we get
$\frac{1}{1-\frac{z-z_{0}}{s-z_{0}}}=1+\frac{z-z_{0}}{s-z_{0}}+\left(\frac{z-z_{0}}{s-z_{0}}\right)^{2}+\cdots+\left(\frac{z-z_{0}}{s-z_{0}}\right)^{n-1}+\frac{\left(z-z_{0}\right)^{n}}{(s-z)\left(s-z_{0}\right)^{n-1}}$,
whence, we get
$f(z)=\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{s-z_{0}} d s+\frac{z-z_{0}}{2 \pi i} \oint_{C} \frac{f(s)}{\left(s-z_{0}\right)^{2}} d s+\frac{\left(z-z_{0}\right)^{2}}{2 \pi i} \oint_{C} \frac{f(s)}{\left(s-z_{0}\right)^{3}} d s+$
$\cdots+\frac{\left(z-z_{0}\right)^{n-1}}{2 \pi i} \oint_{C} \frac{f(s)}{\left(s-z_{0}\right)^{n}} d s+\frac{\left(z-z_{0}\right)^{n}}{2 \pi i} \oint_{C} \frac{f(s)}{(s-z)\left(s-z_{0}\right)^{n}} d s$. By Cauchy's
integral formula for derivatives, $f(z)=$
$f\left(z_{0}\right)+\frac{f^{\prime}\left(z_{0}\right)}{1!}\left(z-z_{0}\right)+\frac{f^{\prime \prime}\left(z_{0}\right)}{2!}\left(z-z_{0}\right)^{2}+\cdots+\frac{f^{(n-1)}\left(z_{0}\right)}{(n-1)!}\left(z-z_{0}\right)^{n-1}+R_{n}(z)$,
where $R_{n}(z)=\frac{\left(z-z_{0}\right)^{n}}{2 \pi i} \oint_{C} \frac{f(s)}{(s-z)\left(s-z_{0}\right)^{n}} d s$. This is called Taylor's formula with remainder $R_{n}$. The goal now is to show that $R_{n}(z) \rightarrow 0$ as $n \rightarrow \infty$.


## Proof of Taylor's Theorem II

- To see that $R_{n}(z)=\frac{\left(z-z_{0}\right)^{n}}{2 \pi i} \oint_{C} \frac{f(s)}{(s-z)\left(s-z_{0}\right)^{n}} d s \rightarrow 0$, it suffices to show that $\left|R_{n}(z)\right| \rightarrow 0$ as $n \rightarrow \infty$. Since $f$ is analytic in $D$, we know that $|f(z)|$ has a maximum value $M$ on the contour $C$. In addition, since $z$ is inside $C,\left|z-z_{0}\right|<R$ and, consequently,
$|s-z|=\left|s-z_{0}-\left(z-z_{0}\right)\right| \geq\left|s-z_{0}\right|-\left|z-z_{0}\right|=R-d$, where $d=\left|z-z_{0}\right|$ is the distance from $z$ to $z_{0}$. The $M L$-inequality then gives
$\left|R_{n}(z)\right|=\left|\frac{\left(z-z_{0}\right)^{n}}{2 \pi i} \oint_{C} \frac{f(s)}{(s-z)\left(s-z_{0}\right)^{n}} d s\right| \leq \frac{d^{n}}{2 \pi} \cdot \frac{M}{(R-d) R^{n}} \cdot 2 \pi R=\frac{M R}{R-d}\left(\frac{d}{R}\right)^{n}$. Because $d<R,\left(\frac{d}{R}\right)^{n} \rightarrow 0$ as $n \rightarrow \infty$, we conclude that $\left|R_{n}(z)\right| \rightarrow 0$ as $n \rightarrow \infty$. It follows that the infinite series $f\left(z_{0}\right)+\frac{f^{\prime}\left(z_{0}\right)}{1!}\left(z-z_{0}\right)+\frac{f^{\prime \prime}\left(z_{0}\right)}{2!}\left(z-z_{0}\right)^{2}+\cdots$ converges to $f(z)$.


## Isolated Singularities and Important Maclaurin Series

- An isolated singularity of a function $f$ is a point at which $f$ fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.
Example: $f(z)=\frac{1}{z-5 i}$ has an isolated singularity at $z=5 i$.
- The radius of convergence $R$ of a Taylor series for $f$ is the distance from the center $z_{0}$ of the series to the nearest isolated singularity of $f$.
- Thus, if the function $f$ is entire, then the radius of convergence of a Taylor series centered at any point $z_{0}$ is necessarily $R=\infty$.
- We summarize some Important Maclaurin Series:

$$
\begin{aligned}
e^{z} & =1+\frac{z}{1!}+\frac{z^{2}}{2!}+\cdots=\sum_{k=0}^{\infty} \frac{z^{k}}{k!} \\
\sin z & =z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}-\cdots=\sum_{k=0}^{\infty}(-1)^{k} \frac{z^{2 k+1}}{(2 k+1)!} \\
\cos z & =1-\frac{z^{2}}{2!}+\frac{z^{4}}{4!}-\cdots=\sum_{k=0}^{\infty}(-1)^{k} \frac{z^{2 k}}{(2 k)!}
\end{aligned}
$$

## Finding Radius of Convergence

- Suppose the function $f(z)=\frac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_{0}=4-2 i$. What is its radius of convergence $R$ ?
Observe that the function is analytic at every point except at $z=-1+i$, which is an isolated singularity of $f$. The distance from $z=-1+i$ to $z_{0}=4-2 i$ is

$$
\left|z-z_{0}\right|=\sqrt{(-1-4)^{2}+(1-(-2))^{2}}=\sqrt{34}
$$

Thus, the radius of convergence for the Taylor series centered at $4-2 i$ is $R=\sqrt{34}$.

## Uniqueness of the Series Expansion

- If two power series with center $z_{0}$,

$$
\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k} \quad \text { and } \quad \sum_{k=0}^{\infty} b_{k}\left(z-z_{0}\right)^{k}
$$

represent the same function $f$ and have the same nonzero radius $R$ of convergence, then $a_{k}=b_{k}=\frac{f^{(k)}\left(z_{0}\right)}{k!}, k=0,1,2, \ldots$

- Stated in another way, the power series expansion of a function, with center $z_{0}$, is unique.
- Thus, a power series expansion of an analytic function $f$ centered at $z_{0}$, irrespective of the method used to obtain it, is the Taylor series expansion of the function.


## Finding a Maclaurin Series

- Find the Maclaurin expansion of $f(z)=\frac{1}{(1-z)^{2}}$.

Recall that for $|z|<1$,

$$
\frac{1}{1-z}=1+z+z^{2}+z^{3}+\cdots
$$

If we differentiate both sides of the last result with respect to $z$,

$$
\frac{d}{d z} \frac{1}{1-z}=\frac{d}{d z} 1+\frac{d}{d z} z+\frac{d}{d z} z^{2}+\frac{d}{d z} z^{3}+\cdots
$$

or

$$
\frac{1}{(1-z)^{2}}=0+1+2 z+3 z^{2}+\cdots=\sum_{k=1}^{\infty} k z^{k-1}
$$

The radius of convergence of the last power series is the same as the original series $R=1$.

## Finding a Taylor Series

- Expand $f(z)=\frac{1}{1-z}$ in a Taylor series with center $z_{0}=2 i$.

We use again $\frac{1}{1-z}=1+z+z^{2}+\cdots$. By adding and subtracting $2 i$ in the denominator, $\frac{1}{1-z}=\frac{1}{1-z+2 i-2 i}=\frac{1}{1-2 i-(z-2 i)}=\frac{1}{1-2 i} \cdot \frac{1}{1-\frac{z-2 i}{1-2 i}}$. We now write $\frac{1}{1-\frac{z-2 i}{1-2 i}}$ as a power series:

$$
\begin{aligned}
& \frac{1}{1-z}=\frac{1}{1-2 i}\left[1+\frac{z-2 i}{1-2 i}+\left(\frac{z-2 i}{1-2 i}\right)^{2}+\left(\frac{z-2 i}{1-2 i}\right)^{3}+\cdots\right] \text { or } \\
& \frac{1}{1-z}=\frac{1}{1-2 i}+\frac{1}{(1-2 i)^{2}}(z-2 i)+\frac{1}{(1-2 i)^{3}}(z-2 i)^{2}+\frac{1}{(1-2 i)^{4}}(z-2 i)^{3}+\cdots .
\end{aligned}
$$

Because the distance from the center $z_{0}=2 i$ to the nearest singularity $z=1$ is $\sqrt{5}$, we conclude that the circle of convergence is $|z-2 i|=\sqrt{5}$.

## Power Series for the Same Function

- We have represented the same function $f(z)=\frac{1}{1-z}$ by two different power series; one with center $z_{0}=0$ and radius of convergence $R=1$; another with center $z_{0}=2 i$ and radius of convergence $R=\sqrt{5}$.


The interior of the intersection of the two circles is the region where both series converge, i.e., at a specified point $z^{*}$ in this region, both series converge to same value $f\left(z^{*}\right)=\frac{1}{1-z^{*}}$. Outside the colored region at least one of the two series must diverge.

## Subsection 3

## Laurent Series

## Isolated Singularities

- Suppose that $z=z_{0}$ is a singularity of a complex function $f$, i.e., a point at which $f$ fails to be analytic.
- The point $z=z_{0}$ is said to be an isolated singularity of the function $f$ if there exists some deleted neighborhood, or punctured open disk, $0<\left|z-z_{0}\right|<R$ of $z_{0}$ throughout which $f$ is analytic.
Example: The points $z=2 i$ and $z=-2 i$ are singularities of $f(z)=\frac{z}{z^{2}+4}$. Both $2 i$ and $-2 i$ are isolated singularities since $f$ is analytic at every point in the neighborhood defined by $|z-2 i|<1$, except at $z=2 i$, and at every point in the neighborhood defined by $|z-(-2 i)|<1$, except at $z=-2 i$. In other words, $f$ is analytic in the deleted neighborhoods $0<|z-2 i|<1$ and $0<|z+2 i|<1$.
- A singular point $z=z_{0}$ of a function $f$ is nonisolated if every neighborhood of $z_{0}$ contains at least one singularity of $f$ other than $z_{0}$. Example: The branch point $z=0$ is a nonisolated singularity of Lnz since every neighborhood of $z=0$ contains points on the negative real axis.


## A New Kind of Series

- If $z=z_{0}$ is a singularity of a function $f$, then certainly $f$ cannot be expanded in a power series with $z_{0}$ as its center.
- About an isolated singularity $z=z_{0}$, it is still possible to represent $f$ by a series involving both negative and nonnegative integer powers of $z-z_{0}$, i.e.,

$$
f(z)=\cdots+\frac{a_{-2}}{\left(z-z_{0}\right)^{2}}+\frac{a_{-1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots .
$$

- Example: Consider the function $f(z)=\frac{1}{z-1}$. The point $z=1$ is an isolated singularity of $f$ and, consequently, the function cannot be expanded in a Taylor series centered at that point. Nevertheless, $f$ can expanded in a series of the previous form that is valid for all $z$ near 1: $f(z)=\cdots+\frac{0}{(z-1)^{2}}+\frac{1}{z-1}+0+0 \cdot(z-1)+0 \cdot(z-1)^{2}+\cdots$. This series representation is valid for $0<|z-1|<\infty$.


## Principal Part and Analytic Part

- Using summation notation, we can rewrite

$$
f(z)=\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{-k}+\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k} .
$$

- The part with negative powers $\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{-k}=\sum_{k=1}^{\infty} \frac{a_{-k}}{\left(z-z_{0}\right)^{k}}$ is called the principal part of the series. It converges for $\left|\frac{1}{z-z_{0}}\right|<r^{*}$ or, equivalently, for $\left|z-z_{0}\right|>\frac{1}{r^{*}}=r$.
- The part consisting of the nonnegative powers $\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$, is called the analytic part of the series. It converges for $\left|z-z_{0}\right|<R$.
- Thus, the sum converges when $z$ satisfies both $\left|z-z_{0}\right|>r$ and $\left|z-z_{0}\right|<R$, i.e., when $z$ is a point in an annular domain defined by $r<\left|z-z_{0}\right|<R$.
- By summing over negative and nonnegative integers, we can rewrite $f(z)=\sum_{k=-\infty}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$.


## An Example

- The function $f(z)=\frac{\sin z}{z^{4}}$ is not analytic at the isolated singularity $z=0$ and hence cannot be expanded in a Maclaurin series.
- However, $\sin z$ is an entire function having Maclaurin series

$$
\sin z=z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}-\frac{z^{7}}{7!}+\frac{z^{9}}{9!}-\cdots
$$

which converges for $|z|<\infty$.

- By dividing this power series by $z^{4}$ we obtain a series for $f$ with negative and positive integer powers of $z$ :

$$
f(z)=\frac{\sin z}{z^{4}}=\overbrace{\frac{1}{z^{3}}-\frac{1}{3!z}}^{\text {principal part }} \overbrace{+\frac{z}{5!}-\frac{z^{3}}{7!}+\frac{z^{5}}{9!}-\cdots}^{\text {analytic part }} .
$$

- The analytic part converges for $|z|<\infty$.
- The principal part is valid for $|z|>0$.
- The series converges for all $z$, but $z=0$, i.e., is valid for $0<|z|<\infty$.


## Laurent Series and Laurent's Theorem

- A series representation of a function $f$ consisting of both negative and nonnegative powers of $z-z_{0}$ is called a Laurent series or a Laurent expansion of $f$ about $z_{0}$ on the annulus $r<\left|z-z_{0}\right|<R$.


## Theorem (Laurent's Theorem)

Let $f$ be analytic within the annulus $D$ defined by $r<\left|z-z_{0}\right|<R$. Then $f$ has the series representation $f(z)=\sum_{k=-\infty}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$ valid for $r<\left|z-z_{0}\right|<R$.
The coefficients $a_{k}$ are given by

$$
a_{k}=\frac{1}{2 \pi i} \oint_{C} \frac{f(s)}{\left(s-z_{0}\right)^{k+1}} d s
$$

$k=0, \pm 1, \pm 2, \ldots$, where $C$ is a simple closed curve that lies entirely within $D$ and has $z_{0}$ in its interior.

## Proof of Laurent's Theorem I

- Let $C_{1}$ and $C_{2}$ be concentric circles with center $z_{0}$ and radii $r_{1}$ and $R_{2}$, where $r<r_{1}<R_{2}<$ $R$. Let $z$ be a fixed point in $D$ that satisfies $r_{1}<\left|z-z_{0}\right|<R_{2}$. By introducing a crosscut between $C_{2}$ and $C_{1}$, Cauchy's formula gives $f(z)=\frac{1}{2 \pi i} \oint_{C_{2}} \frac{f(s)}{s-z} d s-\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{s-z} d s$.


We can write $\frac{1}{2 \pi i} \oint_{C_{2}} \frac{f(s)}{s-z} d s=\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$, where $a_{k}=\frac{1}{2 \pi i} \oint_{C_{2}} \frac{f(s)}{\left(s-z_{0}\right)^{k+1}} d s, k=0,1,2, \ldots$ We have $-\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{s-z} d s=$ $\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{\left(z-z_{0}\right)-\left(s-z_{0}\right)} d s=\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{z-z_{0}}\left(\frac{1}{1-\frac{s-z_{0}}{z-z_{0}}}\right) d s=$
$\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{z-z_{0}}\left(1+\frac{s-z_{0}}{z-z_{0}}+\cdots+\left(\frac{s-z_{0}}{z-z_{0}}\right)^{n-1}+\frac{\left(s-z_{0}\right)^{n}}{(z-s)\left(z-z_{0}\right)^{n-1}}\right) d s=$
$\sum_{k=1}^{n} \frac{a_{-k}}{\left(z-z_{0}\right)^{k}}+R_{n}(z), a_{-k}=\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{\left(s-z_{0}\right)^{-k+1}} d s$,
$R_{n}(z)=\frac{1}{2 \pi i\left(z-z_{0}\right)^{n}} \oint_{C_{1}} \frac{f(s)\left(s-z_{0}\right)^{n}}{z-s} d s$.

## Proof of Laurent's Theorem II

- Now let $d=\left|z-z_{0}\right|$ and let $M$ denote the maximum value of $|f(z)|$ on $C_{1}$. Using $\left|s-z_{0}\right|=r_{1}$ and $|z-s|=\left|z-z_{0}-\left(s-z_{0}\right)\right|$
$\geq\left|z-z_{0}\right|-\left|s-z_{0}\right|=d-r_{1}$, the ML-inequality gives:
$\left|R_{n}(z)\right|=\left|\frac{1}{2 \pi i\left(z-z_{0}\right)^{n}} \oint_{C_{1}} \frac{f(s)\left(s-z_{0}\right)^{n}}{z-s} d s\right| \leq \frac{1}{2 \pi d^{n}} \frac{M r_{1}^{n}}{d-r_{1}} 2 \pi r_{1}=\frac{M r_{1}}{d-r_{1}}\left(\frac{r_{1}}{d}\right)^{n}$.
Because $r_{1}<d,\left(\frac{r_{1}}{d}\right)^{n} \rightarrow 0$ as $n \rightarrow \infty$, and so $\left|R_{n}(z)\right| \rightarrow 0$ as $n \rightarrow \infty$.
Thus we have shown that $-\frac{1}{2 \pi i} \oint_{C_{1}} \frac{f(s)}{s-z} d s=\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{k}$.
- Therefore, overall we have

$$
f(z)=\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{k}+\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}
$$

By summing over all integer powers,

$$
f(z)=\sum_{k=-\infty}^{\infty} a_{k}\left(z-z_{0}\right)^{k}, a_{k}=\oint_{C} \frac{f(z)}{\left(z-z_{0}\right)^{k+1}} d z, k=0, \pm 1, \pm 2, \ldots
$$

## Remarks

- In the case when $a_{-k}=0$ for $k=1,2, \ldots$, the principal part is zero and the Laurent series reduces to a Taylor series.
- The annular domain $r<\left|z-z_{0}\right|<R$ need not have a "ring" shape. Some other possible annular domains are:
(i) $r=0, R$ finite; In this case, the series converges in $0<\left|z-z_{0}\right|<R$, i.e., the domain is a punctured open disk.
(ii) $r \neq 0, R=\infty$; In this case, the annular domain is $r<\left|z-z_{0}\right|$ and consists of all points exterior to the circle $\left|z-z_{0}\right|=r$.
(iii) $r=0, R=\infty$; In this case, the domain is defined by $0<\left|z-z_{0}\right|$. This represents the entire complex plane except the point $z_{0}$.
- Finding the Laurent series of a function in a specified annular domain is generally difficult, but in many instances we can obtain a desired Laurent series by either
- employing a known power series expansion of a function; or by
- creative manipulation of a suitably chosen geometric series.


## Finding Laurent Expansions I

- Expand $f(z)=\frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains.
(a) $0<|z|<1$
(b) $1<|z|$
(c) $0<|z-1|<1$
(d) $1<|z-1|$.
- In parts (a) and (b) we want only powers of $z$, whereas in parts (c) and (d) we want powers of $z-1$.
(a) $f(z)=-\frac{1}{z} \frac{1}{1-z}=-\frac{1}{z}\left(1+z+z^{2}+z^{3}+\cdots\right)$. The infinite series in the brackets converges for $|z|<1$, but after we multiply this expression by $\frac{1}{z}$, the resulting series $f(z)=-\frac{1}{z}-1-z-z^{2}-z^{3}-\cdots$ converges for $0<|z|<1$.
(b) To obtain a series that converges for $1<|z|$, we start by constructing a series that converges for $|1 / z|<1$. We write the given function $f(z)=\frac{1}{z^{2}} \frac{1}{1-\frac{1}{z}}=\frac{1}{z^{2}}\left(1+\frac{1}{z}+\frac{1}{z^{2}}+\frac{1}{z^{3}}+\cdots\right)$. The series in the brackets converges for $\left|\frac{1}{z}\right|<1$ or equivalently for $1<|z|$. Thus, the required Laurent series is $f(z)=\frac{1}{z^{2}}+\frac{1}{z^{3}}+\frac{1}{z^{4}}+\frac{1}{z^{5}}+\cdots$.


## Finding Laurent Expansions I

(c) We add and subtract 1 in the denominator: $f(z)=\frac{1}{(1-1+z)(z-1)}=$ $\frac{1}{z-1} \frac{1}{1+(z-1)}=\frac{1}{z-1}\left(1-(z-1)+(z-1)^{2}-(z-1)^{3}+\cdots\right)=$ $\frac{1}{z-1}-1+(z-1)-(z-1)^{2}+\cdots$. The requirement that $z \neq 1$ is equivalent to $0<|z-1|$, and the geometric series in brackets converges for $|z-1|<1$. Thus, the last series converges for $z$ satisfying $0<|z-1|<1$.
(d) As in part (b), $f(z)=\frac{1}{z-1} \frac{1}{1+(z-1)}=\frac{1}{(z-1)^{2}} \frac{1}{1+\frac{1}{z-1}}=$
$\frac{1}{(z-1)^{2}}\left(1-\frac{1}{z-1}+\frac{1}{(z-1)^{2}}-\frac{1}{(z-1)^{3}}+\cdots\right)=$ $\frac{1}{(z-1)^{2}}-\frac{1}{(z-1)^{3}}+\frac{1}{(z-1)^{4}}-\frac{1}{(z-1)^{5}}+\cdots$. Because the series within the brackets converges for $\left|\frac{1}{z-1}\right|<1$, the final series converges for $1<|z-1|$.

## More Laurent Series Expansions I

- Expand $f(z)=\frac{1}{(z-1)^{2}(z-3)}$ in a Laurent series valid for

$$
\begin{array}{ll}
\text { (a) } 0<|z-1|<2 & \text { (b) } 0<|z-3|<2 .
\end{array}
$$

(a) We need to express $z-3$ in terms of $z-1$. This can be done by writing $f(z)=\frac{1}{(z-1)^{2}(z-3)}=\frac{1}{(z-1)^{2}} \frac{1}{-2+(z-1)}=\frac{-1}{2(z-1)^{2}} \frac{1}{1-\frac{z-1}{2}}=$

$$
\begin{aligned}
& \frac{-1}{2(z-1)^{2}}\left(1+\frac{z-1}{2}+\frac{(z-1)^{2}}{2^{2}}+\frac{(z-1)^{3}}{2^{3}}+\cdots\right)= \\
& -\frac{1}{2(z-1)^{2}}-\frac{1}{4(z-1)}-\frac{1}{8}-\frac{1}{16}(z-1)-\cdots
\end{aligned}
$$

(b) To obtain powers of $z-3$, we write $z-1=2+(z-3)$ and

$$
\begin{gathered}
f(z)=\frac{1}{(z-1)^{2}(z-3)}=\frac{1}{z-3}[2+(z-3)]^{-2}=\frac{1}{4(z-3)}\left[1+\frac{z-3}{2}\right]^{-2}= \\
\frac{1}{4(z-3)}\left(1+\frac{(-2)}{1!}\left(\frac{z-3}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^{2}+\frac{(-2)(-3)(-4)}{3!}\left(\frac{z-3}{2}\right)^{3}+\cdots\right)
\end{gathered}
$$

The series in the brackets is valid for $\left|\frac{z-3}{2}\right|<1$ or $|z-3|<2$.
Multiplying by $\frac{1}{4(z-3)}$ gives a series that is valid for $0<|z-3|<2$ :

$$
f(z)=\frac{1}{4(z-3)}-\frac{1}{4}+\frac{3}{16}(z-3)-\frac{1}{8}(z-3)^{2}+\cdots .
$$

## More Laurent Series Expansions II

- Expand $f(z)=\frac{8 z+1}{z(1-z)}$ in a Laurent series valid for $0<|z|<1$. By partial fractions we can rewrite $f$ as $f(z)=\frac{8 z+1}{z(1-z)}=\frac{1}{z}+\frac{9}{1-z}$. Now we have

$$
\frac{9}{1-z}=9+9 z+9 z^{2}+\cdots .
$$

The foregoing geometric series converges for $|z|<1$, but after we add the term $\frac{1}{z}$ to it, the resulting Laurent series

$$
f(z)=\frac{1}{z}+9+9 z+9 z^{2}+\cdots
$$

is valid for $0<|z|<1$.

## More Laurent Series Expansions III

- Expand $f(z)=\frac{1}{z(z-1)}$ in a Laurent series valid for $1<|z-2|<2$. The center $z=2$ is a point of analyticity of the function $f$. Our goal now is to find two series involving integer powers of $z-2$, one converging for $1<|z-2|$ and the other converging for $|z-2|<2$. Decompose $f$ into partial fractions: $f(z)=\frac{-1}{z}+\frac{1}{z-1}=f_{1}(z)+f_{2}(z)$.

$$
\begin{aligned}
& f_{1}(z)=\frac{-1}{z}=\frac{-1}{2+z-2}=\frac{-1}{2} \frac{1}{1+\frac{z-2}{2}}=\frac{-1}{2}\left(1-\frac{z-2}{2}+\frac{(z-2)^{2}}{2^{2}}-\cdots\right)= \\
& \frac{-1}{2}+\frac{z-2}{2^{2}}-\frac{(z-2)^{2}}{2^{3}}+\frac{(z-2)^{3}}{2^{4}}-\cdots \text {. This series converges for }\left|\frac{z-2}{2}\right|<1 \\
& \text { or }|z-2|<2
\end{aligned}
$$

$$
f_{2}(z)=\frac{1}{z-1}=\frac{1}{1+z-2}=\frac{1}{z-2} \frac{1}{1+\frac{1}{z-2}}=\frac{1}{z-2}\left(1-\frac{1}{z-2}+\frac{1}{(z-2)^{2}}-\cdots\right)=
$$

$$
\frac{1}{z-2}-\frac{1}{(z-2)^{2}}+\frac{1}{(z-2)^{3}}-\frac{1}{(z-2)^{4}}+\cdots . \text { It converges for }\left|\frac{1}{z-2}\right|<1 \text { or }
$$

$$
1<|z-2|
$$

Thus, we get $f(z)=\cdots-\frac{1}{(z-2)^{4}}+\frac{1}{(z-2)^{3}}-\frac{1}{(z-2)^{2}}+\frac{1}{z-2}$
$-\frac{1}{2}+\frac{z-2}{2^{2}}-\frac{(z-2)^{2}}{2^{3}}+\frac{(z-2)^{3}}{2^{4}}-\cdots$. This representation is valid for $z$ satisfying $1<|z-2|<2$.

## More Laurent Series Expansions IV

- Expand $f(z)=\frac{e^{3}}{z}$ in a Laurent series valid for $0<|z|<\infty$. We know that for $|z|<\infty$,

$$
e^{z}=1+z+\frac{z^{2}}{2!}+\frac{z^{3}}{3!}+\cdots
$$

We obtain the Laurent series for $f$ by simply replacing $z$ by $\frac{3}{z}$, when $z \neq 0$ :

$$
e^{3 / z}=1+\frac{3}{z}+\frac{3^{2}}{2!z^{2}}+\frac{3^{3}}{3!z^{3}}+\cdots
$$

This series is valid for $z \neq 0$, that is, for $0<|z|<\infty$.

## Remarks

(i) Replacing the complex variable $s$ with the usual symbol $z$, we see that when $k=-1$, the formula for the Laurent series coefficients yields

$$
a_{-1}=\frac{1}{2 \pi i} \oint_{C} f(z) d z
$$

or more important,

$$
\oint_{C} f(z) d z=2 \pi i a_{-1}
$$

(ii) Regardless how a Laurent expansion of a function $f$ is obtained in a specified annular domain it is the Laurent series; i.e., the series we obtain is unique.

## Subsection 4

## Zeros and Poles

## Review of Laurent Series

- Suppose $z=z_{0}$ is an isolated singularity of a complex function $f$, and that

$$
f(z)=\sum_{k=-\infty}^{\infty} a_{k}\left(z-z_{0}\right)^{k}=\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{-k}+\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}
$$

is the Laurent series representation of $f$ valid for the punctured open disk $0<\left|z-z_{0}\right|<R$.

- The part of the series with the negative powers of $z-z_{0}$, i.e.,

$$
\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{-k}=\sum_{k=1}^{\infty} \frac{a_{-k}}{\left(z-z_{0}\right)^{k}}
$$

is the principal part of the series.

- We will classify the isolated singularity $z=z_{0}$ according to the number of terms in the principal part.


## Classification of Isolated Singular Points

- An isolated singular point $z=z_{0}$ of a complex function $f$ is given a classification depending on whether the principal part of its Laurent expansion

$$
f(z)=\sum_{k=-\infty}^{\infty} a_{k}\left(z-z_{0}\right)^{k}=\sum_{k=1}^{\infty} a_{-k}\left(z-z_{0}\right)^{-k}+\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}
$$

contains zero, a finite number, or an infinite number of terms:
(i) If the principal part is zero, that is, all the coefficients $a_{-k}$ are zero, then $z=z_{0}$ is called a removable singularity.
(ii) If the principal part contains a finite number of nonzero terms, then $z=z_{0}$ is called a pole. If, in this case, the last nonzero coefficient in $\sum_{k=1}^{\infty} \frac{a-k}{\left(z-z_{0}\right)^{k}}$ is $a_{-n}, n \geq 1$, then $z=z_{0}$ is called a pole of order $n$. If $z=z_{0}$ is a pole of order 1 , then the principal part contains exactly one term with coefficient $a_{-1}$ and the pole is called a simple pole.
(iii) If the principal part contains infinitely many nonzero terms, then $z=z_{0}$ is called an essential singularity.

## Form of Laurent Series Based on Classification

- The form of a Laurent series for a function $f$, when $z=z_{0}$ is one of the various types of isolated singularities is summarized below:

| $z=z_{0}$ | Laurent Series for $0<\left\|z-z_{0}\right\|<R$ |
| :--- | :--- |
| Removable Singularity | $a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots$ |
| Pole of Order $n$ | $\frac{a_{-n}}{\left(z-z_{0}\right)^{n}}+\frac{a_{-(n-1)}}{\left(z-z_{0}\right)^{n-1}}+\cdots+\frac{a_{-1}}{z-z_{0}}$ |
| $+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots$ |  |$]$| $\frac{a_{-1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots$ |  |
| :--- | :--- |
| Simple Pole | $\cdots+\frac{a-2}{\left(z-z_{0}\right)^{2}}+\frac{a_{-1}}{z-z_{0}}$ <br> Essential Singularity$+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots$ |

## A Removable Singularity

- Recall the Maclaurin series for $\sin z: \sin z=z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}-\cdots$. Divide by $z$ to get

$$
\frac{\sin z}{z}=1-\frac{z^{2}}{3!}+\frac{z^{4}}{5!}-\cdots
$$

Thus, all the coefficients in the principal part of the Laurent series are zero. Hence, $z=0$ is a removable singularity of the function $f(z)=\frac{\sin z}{z}$.

- If a function $f$ has a removable singularity at $z=z_{0}$, then we can supply an appropriate definition for the value of $f\left(z_{0}\right)$ so that $f$ becomes analytic at $z=z_{0}$.
Example: Since the right-hand side of the series above is 1 when we set $z=0$, it makes sense to define $f(0)=1$. Hence the function $f(z)=\frac{\sin z}{z}$ is now defined and continuous at every complex number $z$. Indeed, $f$ is also analytic at $z=0$ because it is represented by the Taylor series $1-\frac{z^{2}}{3!}+\frac{z^{4}}{5!}-\cdots$ centered at 0 (a Maclaurin series).


## Poles and Essential Singularities

(a) Dividing $\sin z=z-\frac{z^{3}}{3!}+\frac{z^{5}}{5!}-\cdots$ by $z^{2}$ shows that, for $0<|z|<\infty$,

$$
\frac{\sin z}{z^{2}}=\overbrace{\frac{1}{z}}^{\text {principal part }}-\frac{z}{3!}+\frac{z^{3}}{5!}-\cdots .
$$

Since $a_{-1} \neq 0, z=0$ is a simple pole of the function $f(z)=\frac{\sin z}{z^{2}}$.
Similarly, $z=0$ is a pole of order 3 of the function $f(z)=\frac{\sin z}{z^{4}}$.
(b) The Laurent series of $f(z)=\frac{1}{(z-1)^{2}(z-3)}$ for $0<|z-1|<2$ :

$$
f(z)=\overbrace{-\frac{1}{2(z-1)^{2}}-\frac{1}{4(z-1)}}^{\text {principal part }}-\frac{1}{8}-\frac{z-1}{16}-\cdots .
$$

Since $a_{-2}=-\frac{1}{2} \neq 0$, we conclude that $z=1$ is a pole of order 2 .
(c) The principal part of the Laurent expansion of $f(z)=e^{3 / z}$ valid for $0<|z|<\infty$ contains an infinite number of nonzero terms. This shows that $z=0$ is an essential singularity of $f$.

## Zeros and Multiplicities

- A number $z_{0}$ is a zero of a function $f$ if $f\left(z_{0}\right)=0$.
- We say that an analytic function $f$ has a zero of order $n$ at $z=z_{0}$ if $z_{0}$ is a zero of $f$ and of its first $n-1$ derivatives, but not of its $n$-th derivative, i.e., $f\left(z_{0}\right)=0, f^{\prime}\left(z_{0}\right)=0, f^{\prime \prime}\left(z_{0}\right)=0, \ldots, f^{(n-1)}\left(z_{0}\right)=0$, but $f^{(n)}\left(z_{0}\right) \neq 0$.
- A zero of order n is also referred to as a zero of multiplicity $n$.

Example: Consider $f(z)=(z-5)^{3}$.

$$
f(5)=0, f^{\prime}(5)=0, f^{\prime \prime}(5)=0, \text { but } f^{\prime \prime \prime}(5)=6 \neq 0
$$

Thus, $f$ has a zero of order (or multiplicity) 3 at $z_{0}=5$.

- A zero of order 1 is called a simple zero.


## Order of Zeros

## Theorem (Zero of Order n)

A function $f$ that is analytic in some disk $\left|z-z_{0}\right|<R$ has a zero of order $n$ at $z=z_{0}$ if and only if $f$ can be written $f(z)=\left(z-z_{0}\right)^{n} \phi(z)$, where $\phi$ is analytic at $z=z_{0}$ and $\phi\left(z_{0}\right) \neq 0$.

- Partial Proof ("only if" Part): Given that $f$ is analytic at $z_{0}$, it can be expanded in a Taylor series that is centered at $z_{0}$ and is convergent for $\left|z-z_{0}\right|<R$. Since, in a Taylor series $f(z)=\sum_{k=0}^{\infty} a_{k}\left(z-z_{0}\right)^{k}$, $a_{k}=\frac{f^{(k)}\left(z_{0}\right)}{k!}, k=0,1, \ldots$, it follows that the first $n$ terms are zero.
So $f(z)=a_{n}\left(z-z_{0}\right)^{n}+a_{n+1}\left(z-z_{0}\right)^{n+1}+a_{n+2}\left(z-z_{0}\right)^{n+2}+\cdots=$ $\left(z-z_{0}\right)^{n}\left(a_{n}+a_{n+1}\left(z-z_{0}\right)+a_{n+2}\left(z-z_{0}\right)^{2}+\cdots\right)$. Letting $\phi(z)=a_{n}+a_{n+1}\left(z-z_{0}\right)+a_{n+2}\left(z-z_{0}\right)^{2}+\cdots$, we conclude $f(z)=\left(z-z_{0}\right)^{n} \phi(z)$, where $\phi$ is an analytic function, such that $\phi\left(z_{0}\right)=a_{n} \neq 0$ because $a_{n}=\frac{f^{(n)}\left(z_{0}\right)}{n!} \neq 0$.


## Computing the Order of a Zero Using a Power Series

- The analytic function $f(z)=z \sin z^{2}$ has a zero at $z=0$.

If we replace $z$ by $z^{2}$ in the Maclaurin series for $\sin z$, we obtain

$$
\sin z^{2}=z^{2}-\frac{z^{6}}{3!}+\frac{z^{10}}{5!}-\cdots
$$

Then, by factoring $z^{2}$ out, we can rewrite $f$ as

$$
f(z)=z \sin z^{2}=z^{3} \phi(z)
$$

where $\phi(z)=1-\frac{z^{4}}{3!}+\frac{z^{8}}{5!}-\cdots$ and $\phi(0)=1$.
This shows that $z=0$ is a zero of order 3 of $f$.

## Poles of Order $n$

- A pole of order $n$ may be characterized analogously to the characterization of zeros:


## Theorem (Pole of Order $n$ )

A function $f$ analytic in a punctured disk $0<\left|z-z_{0}\right|<R$ has a pole of order $n$ at $z=z_{0}$ if and only if f can be written $f(z)=\frac{\phi(z)}{\left(z-z_{0}\right)^{n}}$, where $\phi$ is analytic at $z=z_{0}$ and $\phi\left(z_{0}\right) \neq 0$.

- Partial Proof ("only if" Part): Since $f$ is assumed to have a pole of order $n$ at $z_{0}$, it can be expanded in a Laurent series $f(z)=\frac{a-n}{\left(z-z_{0}\right)^{n}}$ $+\cdots+\frac{a_{-2}}{\left(z-z_{0}\right)^{2}}+\frac{a_{-1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+\cdots$, valid in some punctured disk $0<\left|z-z_{0}\right|<R$. By factoring out $\frac{1}{\left(z-z_{0}\right)^{n}}, f(z)=\frac{\phi(z)}{\left(z-z_{0}\right)^{n}}$, where $\phi(z)=a_{-n}+\cdots+a_{-2}\left(z-z_{0}\right)^{n-2}+a_{-1}\left(z-z_{0}\right)^{n-1}+a_{0}\left(z-z_{0}\right)^{n}+$ $a_{1}\left(z-z_{0}\right)^{n+1}+\cdots$. This is a power series valid for the open disk $\left|z-z_{0}\right|<R$. Since $z=z_{0}$ is a pole of order $n$ of $f, a_{-n} \neq 0$.


## Zeros and Poles

- A zero $z=z_{0}$ of an analytic function $f$ is isolated in the sense that there exists some neighborhood of $z_{0}$ for which $f(z) \neq 0$ at every point $z$ in that neighborhood except at $z=z_{0}$.
- As a consequence, if $z_{0}$ is a zero of a nontrivial analytic function $f$, then the function $\frac{1}{f(z)}$ has an isolated singularity at the point $z=z_{0}$.


## Theorem (Pole of Order $n$ )

If the functions $g$ and $h$ are analytic at $z=z_{0}$ and $h$ has a zero of order $n$ at $z=z_{0}$ and $g\left(z_{0}\right) \neq 0$, then the function $f(z)=\frac{g(z)}{h(z)}$ has a pole of order $n$ at $z=z_{0}$.

- Because $h$ has a zero of order $n, h(z)=\left(z-z_{0}\right)^{n} \phi(z)$, where $\phi$ is analytic at $z=z_{0}$ and $\phi\left(z_{0}\right) \neq 0$. Thus, $f$ can be written $f(z)=\frac{g(z) / \phi(z)}{\left(z-z_{0}\right)^{n}}$. Since $g$ and $\phi$ are analytic at $z=z_{0}$ and $\phi\left(z_{0}\right) \neq 0$, it follows that the function $g / \phi$ is analytic at $z_{0}$ and $g\left(z_{0}\right) / \phi\left(z_{0}\right) \neq 0$. We conclude that the function $f$ has a pole of order $n$ at $z_{0}$.


## Examples

(a) Inspection of the rational function

$$
f(z)=\frac{2 z+5}{(z-1)(z+5)(z-2)^{4}}
$$

shows that the denominator has zeros of order 1 at $z=1$ and $z=-5$, and a zero of order 4 at $z=2$. Since the numerator is not zero at any of these points, it follows from the theorem that $f$ has simple poles at $z=1$ and $z=-5$, and a pole of order 4 at $z=2$.
(b) $z=0$ is a zero of order 3 of $z \sin z^{2}$. The reciprocal function

$$
f(z)=\frac{1}{z \sin z^{2}}
$$

has a pole of order 3 at $z=0$.

## Remarks

(i) If a function $f$ has a pole at $z=z_{0}$, then $|f(z)| \rightarrow \infty$ as $z \rightarrow z_{0}$ from any direction. Thus, we can write $\lim _{z \rightarrow z_{0}} f(z)=\infty$.
(ii) A function $f$ is meromorphic if it is analytic throughout a domain $D$, except possibly for poles in $D$. It can be proved that a meromorphic function can have at most a finite number of poles in $D$.
E.g., the rational function

$$
f(z)=\frac{1}{z^{2}+1}
$$

is meromorphic in the complex plane.

## Subsection 5

## Residues and Residue Theorem

## Residue

- If a complex function $f$ has an isolated singularity at a point $z_{0}$, then $f$ has a Laurent series representation $f(z)=\sum_{k=-\infty}^{\infty} a_{k}\left(z-z_{0}\right)^{k}=$ $\cdots+\frac{a-2}{\left(z-z_{0}\right)^{2}}+\frac{a-1}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+\cdots$, which converges for all $z$ in some deleted neighborhood $0<\left|z-z_{0}\right|<R$ of $z_{0}$.
- We now focus on the coefficient $a_{-1}$ and its importance in the evaluation of contour integrals.
- The coefficient $a_{-1}$ is called the residue of the function $f$ at the isolated singularity $z_{0}$ and denoted

$$
a_{-1}=\operatorname{Res}\left(f(z), z_{0}\right)
$$

- Recall, if the principal part of the series valid for $0<\left|z-z_{0}\right|<R$ contains a finite number of terms with $a_{-n}$ the last nonzero coefficient, then $z_{0}$ is a pole of order $n$; if the principal part contains an infinite number of terms with nonzero coefficients, then $z_{0}$ is an essential singularity.


## Examples of Residues

(a) We have seen that $z=1$ is a pole of order two of the function $f(z)=\frac{1}{(z-1)^{2}(z-3)}$. The Laurent series valid for the deleted neighborhood $0<|z-1|<2$ is

$$
f(z)=-\frac{1 / 2}{(z-1)^{2}}+\frac{-1 / 4}{z-1}-\frac{1}{8}-\frac{z-1}{16}-\cdots
$$

Thus, the coefficient of $\frac{1}{z-1}$ is $a_{-1}=\operatorname{Res}(f(z), 1)=-\frac{1}{4}$.
(b) We also saw that $z=0$ is an essential singularity of $f(z)=e^{3 / z}$. Its Laurent series is

$$
e^{3 / z}=1+\frac{3}{z}+\frac{3^{2}}{2!z^{2}}+\frac{3^{3}}{3!z^{3}}+\cdots, 0<|z|<\infty
$$

Hence, the coefficient of $\frac{1}{z}$ is $a_{-1}=\operatorname{Res}(f(z), 0)=3$.

## Residue at a Simple Pole

- We examine ways of obtaining $a_{-1}$ when $z_{0}$ is a pole of a function $f$, without the necessity of expanding $f$ in a Laurent series at $z_{0}$.
- We begin with the residue at a simple pole:


## Theorem (Residue at a Simple Pole)

If $f$ has a simple pole at $z=z_{0}$, then

$$
\operatorname{Res}\left(f(z), z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right) f(z)
$$

- Since $f$ has a simple pole at $z=z_{0}$, its Laurent expansion convergent on a punctured disk $0<\left|z-z_{0}\right|<R$ has the form

$$
f(z)=\frac{a_{-1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\cdots
$$

where $a_{-1} \neq 0$. By multiplying both sides of this series by $z-z_{0}$ and then taking the limit as $z \rightarrow z_{0}$ we obtain $\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right) f(z)=$

$$
\lim _{z \rightarrow z_{0}}\left[a_{-1}+a_{0}\left(z-z_{0}\right)+a_{1}\left(z-z_{0}\right)^{2}+\cdots\right]=a_{-1}=\operatorname{Res}\left(f(z), z_{0}\right)
$$

## Residue at a Pole of Order $n$

## Theorem (Residue at a Pole of Order $n$ )

If $f$ has a pole of order $n$ at $z=z_{0}$, then

$$
\operatorname{Res}\left(f(z), z_{0}\right)=\frac{1}{(n-1)!} \lim _{z \rightarrow z_{0}} \frac{d^{n-1}}{d z^{n-1}}\left(z-z_{0}\right)^{n} f(z)
$$

- Since $f$ has a pole of order $n$ at $z=z_{0}$, its Laurent expansion, convergent on a punctured disk $0<\left|z-z_{0}\right|<R$, has the form $f(z)=\frac{a-n}{\left(z-z_{0}\right)^{n}}+\cdots+\frac{a-2}{\left(z-z_{0}\right)^{2}}+\frac{a_{-1}}{z-z_{0}}+a_{0}+a_{1}\left(z-z_{0}\right)+\cdots$, where $a_{-n} \neq 0$. We multiply by $\left(z-z_{0}\right)^{n},\left(z-z_{0}\right)^{n} f(z)=a_{-n}+\cdots+$ $a_{-2}\left(z-z_{0}\right)^{n-2}+a_{-1}\left(z-z_{0}\right)^{n-1}+a_{0}\left(z-z_{0}\right)^{n}+a_{1}\left(z-z_{0}\right)^{n+1}+\cdots$ and then differentiate $n-1$ times:

$$
\frac{d^{n-1}}{d z^{n-1}}\left(z-z_{0}\right)^{n} f(z)=(n-1)!a_{-1}+n!a_{0}\left(z-z_{0}\right)+\cdots
$$

Therefore, as $z \rightarrow z_{0}, \lim _{z \rightarrow z_{0}} \frac{d^{n-1}}{d z^{n-1}}\left(z-z_{0}\right)^{n} f(z)=(n-1)!a_{-1}$.

## Finding Residue at a Pole

- The function $f(z)=\frac{1}{(z-1)^{2}(z-3)}$ has a simple pole at $z=3$ and a pole of order 2 at $z=1$. Use the theorems to find the residues.
Since $z=3$ is a simple pole,

$$
\operatorname{Res}(f(z), 3)=\lim _{z \rightarrow 3}(z-3) f(z)=\lim _{z \rightarrow 3} \frac{1}{(z-1)^{2}}=\frac{1}{4}
$$

At the pole of order 2,

$$
\begin{aligned}
\operatorname{Res}(f(z), 1) & =\frac{1}{1!} \lim _{z \rightarrow 1} \frac{d}{d z}(z-1)^{2} f(z)=\lim _{z \rightarrow 1} \frac{d}{d z} \frac{1}{z-3} \\
& =\lim _{z \rightarrow 1} \frac{-1}{(z-3)^{2}}=-\frac{1}{4} .
\end{aligned}
$$

## Second Method for Computing a Residue at a Simple Pole

- Suppose a function $f$ can be written as a quotient $f(z)=\frac{g(z)}{h(z)}$, where $g$ and $h$ are analytic at $z=z_{0}$. If $g\left(z_{0}\right) \neq 0$ and if the function $h$ has a zero of order 1 at $z_{0}$, then $f$ has a simple pole at $z=z_{0}$ and

$$
\operatorname{Res}\left(f(z), z_{0}\right)=\frac{g\left(z_{0}\right)}{h^{\prime}\left(z_{0}\right)}
$$

- Since $h$ has a zero of order 1 at $z_{0}$, we must have $h\left(z_{0}\right)=0$ and $h^{\prime}\left(z_{0}\right) \neq 0$. By definition of the derivative,

$$
\begin{aligned}
& h^{\prime}\left(z_{0}\right)=\lim _{z \rightarrow z_{0}} \frac{h(z)-h\left(z_{0}\right)}{z-z_{0}}=\lim _{z \rightarrow z_{0}} \frac{h(z)}{z-z_{0}} . \text { Therefore, } \\
& \operatorname{Res}\left(f(z), z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right) \frac{g(z)}{h(z)}=\lim _{z \rightarrow z_{0}} \frac{g(z)}{\frac{h(z)}{z-z_{0}}}=\frac{g\left(z_{0}\right)}{h^{\prime}\left(z_{0}\right)} .
\end{aligned}
$$

## Applying the Second Method

- The polynomial $z^{4}+1$ can be factored as

$$
\left(z-z_{1}\right)\left(z-z_{2}\right)\left(z-z_{3}\right)\left(z-z_{4}\right),
$$

where $z_{1}, z_{2}, z_{3}$, and $z_{4}$ are the four distinct roots of the equation $z^{4}+1=0$ (or, equivalently, the four fourth roots of -1 ). It follows that the function $f(z)=\frac{1}{z^{4}+1}$ has four simple poles. By the root formula $z_{1}=e^{\pi i / 4}, z_{2}=e^{3 \pi i / 4}, z_{3}=e^{5 \pi i / 4}$, and $z_{4}=e^{7 \pi i / 4}$. We compute the residues:

$$
\begin{aligned}
& \operatorname{Res}\left(f(z), z_{1}\right)=\frac{1}{4 z_{1}^{3}}=\frac{1}{4} e^{-3 \pi i / 4}=-\frac{1}{4 \sqrt{2}}-\frac{1}{4 \sqrt{2}} i \\
& \operatorname{Res}\left(f(z), z_{2}\right)=\frac{1}{4 z_{2}^{3}}=\frac{1}{4} e^{-9 \pi i / 4}=\frac{1}{4 \sqrt{2}}-\frac{1}{4 \sqrt{2}} i \\
& \operatorname{Res}\left(f(z), z_{3}\right)=\frac{1}{4 z_{3}^{3}}=\frac{1}{4} e^{-15 \pi i / 4}=\frac{1}{4 \sqrt{2}}+\frac{1}{4 \sqrt{2}} i \\
& \operatorname{Res}\left(f(z), z_{4}\right)=\frac{1}{4 z_{4}^{3}}=\frac{1}{4} e^{-21 \pi i / 4}=-\frac{1}{4 \sqrt{2}}+\frac{1}{4 \sqrt{2}} i .
\end{aligned}
$$

## Using the Original Formula

- We could have calculated each of the residues of $f(z)=\frac{1}{z^{4}+1}$ using $\operatorname{Res}\left(f(z), z_{i}\right)=\lim _{z \rightarrow z_{i}}\left(z-z_{i}\right) f(z)$.
- E.g., at $z_{1}$,

$$
\begin{aligned}
\operatorname{Res}\left(f(z), z_{1}\right) & =\lim _{z \rightarrow z_{1}}\left(z-z_{1}\right) \frac{1}{\left(z-z_{1}\right)\left(z-z_{2}\right)\left(z-z_{3}\right)\left(z-z_{4}\right)} \\
& =\frac{1}{\left(z_{1}-z_{2}\right)\left(z_{1}-z_{3}\right)\left(z_{1}-z_{4}\right)} \\
& =\frac{1}{\left(e^{\pi i / 4}-e^{3 \pi i / 4}\right)\left(e^{\pi i / 4}-e^{5 \pi i / 4}\right)\left(e^{\pi i / 4}-e^{7 \pi i / 4}\right)} .
\end{aligned}
$$

In simplifying the denominator of the last expression considerably more algebra is involved than using the second method.

## Cauchy's Residue Theorem

- Complex integrals $\oint_{C} f(z) d z$ can sometimes be evaluated by summing the residues at the isolated singularities of $f$ within $C$ :


## Theorem (Cauchy's Residue Theorem)

Let $D$ be a simply connected domain and $C$ a simple closed contour lying entirely within $D$. If a function $f$ is analytic on and within $C$, except at a finite number of isolated singular points $z_{1}, z_{2}, \ldots, z_{n}$ within $C$, then

$$
\oint_{C} f(z) d z=2 \pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_{k}\right)
$$

- Suppose $C_{1}, C_{2}, \ldots, C_{n}$ are circles centered at $z_{1}, z_{2}, \ldots, z_{n}$, respectively, such that $C_{k}$ has a radius $r_{k}$ small enough so that $C_{1}, C_{2}, \ldots, C_{n}$ are mutually disjoint and are interior to the simple closed curve $C$. We saw that $\oint_{C_{k}} f(z) d z=2 \pi i \operatorname{Res}\left(f(z), z_{k}\right)$, whence, we have $\oint_{C} f(z) d z=\sum_{k=1}^{n} \oint_{C_{k}} f(z) d z=2 \pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_{k}\right)$.


## Evaluation by the Residue Theorem I

- Evaluate $\oint_{C} \frac{1}{(z-1)^{2}(z-3)} d z$, where
(a) the contour $C$ is the rectangle defined by $x=0, x=4, y=-1, y=1$;
(b) the contour $C$ is the circle $|z|=2$.
(a) Since both $z=1$ and $z=3$ are poles within the rectangle, we have $\oint_{C} \frac{1}{(z-1)^{2}(z-3)} d z=2 \pi i[\operatorname{Res}(f(z), 1)+\operatorname{Res}(f(z), 3)]$. We found these residues already: $\oint_{C} \frac{1}{(z-1)^{2}(z-3)} d z=2 \pi i\left(-\frac{1}{4}+\frac{1}{4}\right)=0$.
(b) Since only the pole $z=1$ lies within the circle $|z|=2$, we have $\oint_{C} \frac{1}{(z-1)^{2}(z-3)} d z=2 \pi i \operatorname{Res}(f(z), 1)=2 \pi i\left(-\frac{1}{4}\right)=-\frac{\pi}{2} i$.


## Evaluation by the Residue Theorem II

- Evaluate $\oint_{C} \frac{2 z+6}{z^{2}+4} d z$, where the contour $C$ is the circle $|z-i|=2$. By factoring the denominator $z^{2}+4=(z-2 i)(z+2 i)$, we see that the integrand has simple poles at $-2 i$ and $2 i$. Only $2 i$ lies within the contour $C$. Thus, $\oint_{C} \frac{2 z+6}{z^{2}+4} d z=2 \pi i \operatorname{Res}(f(z), 2 i)$. But $\operatorname{Res}(f(z), 2 i)=\lim _{z \rightarrow 2 i}(z-2 i) \frac{2 z+6}{(z-2 i)(z+2 i)}=\frac{6+4 i}{4 i}=\frac{3+2 i}{2 i}$. Hence, $\oint_{C} \frac{2 z+6}{z^{2}+4} d z=2 \pi i\left(\frac{3+2 i}{2 i}\right)=\pi(3+2 i)$.


## Evaluation by the Residue Theorem III

- Evaluate $\oint_{C} \frac{e^{z}}{z^{4}+5 z^{3}} d z$, where the contour $C$ is the circle $|z|=2$. Writing the denominator as $z^{4}+5 z^{3}=z^{3}(z+5)$ reveals that the integrand $f(z)$ has a pole of order 3 at $z=0$ and a simple pole at $z=-5$. Only the pole $z=0$ lies within the given contour. Thus, we have

$$
\pi i \lim _{z \rightarrow 0} \frac{d}{d z} \frac{e^{z}(z+4)}{(z+5)^{2}}=\pi i \lim _{z \rightarrow 0} \frac{\left(z^{2}+8 z+17\right) e^{z}}{(z+5)^{3}}=\frac{17 \pi}{125} i
$$

## Evaluation by the Residue Theorem IV

- Evaluate $\oint_{C} \tan z d z$, where the contour $C$ is the circle $|z|=2$. The integrand $f(z)=\tan z=\frac{\sin z}{\cos z}$ has simple poles at the points where $\cos z=0$. We saw that the only zeros of $\cos z$ are the real numbers $z=\frac{(2 n+1) \pi}{2}, n=0, \pm 1, \pm 2, \ldots$ Only $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are within the circle $|z|=2$. Thus, we have
$\oint_{C} \tan z d z=2 \pi i\left[\operatorname{Res}\left(f(z),-\frac{\pi}{2}\right)+\operatorname{Res}\left(f(z), \frac{\pi}{2}\right)\right]$. With $f(z)=\frac{g(z)}{h(z)}$, where $g(z)=\sin z, h(z)=\cos z$, and $h^{\prime}(z)=-\sin z$, we get

$$
\operatorname{Res}\left(f(z),-\frac{\pi}{2}\right)=\frac{\sin \left(-\frac{\pi}{2}\right)}{-\sin \left(-\frac{\pi}{2}\right)}=-1 . \operatorname{Res}\left(f(z), \frac{\pi}{2}\right)=\frac{\sin \left(\frac{\pi}{2}\right)}{-\sin \left(\frac{\pi}{2}\right)}=-1
$$

Therefore, $\oint_{C} \tan z d z=2 \pi i[-1-1]=-4 \pi i$.

## Evaluation by the Residue Theorem V

- Evaluate $\oint_{C} e^{3 / z} d z$, where the contour $C$ is the circle $|z|=1$. We saw that $z=0$ is an essential singularity of the integrand $f(z)=e^{3 / z}$. So we cannot use the formulas

$$
\operatorname{Res}\left(f(z), z_{0}\right)=\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right) f(z)
$$

or

$$
\operatorname{Res}\left(f(z), z_{0}\right)=\frac{1}{(n-1)!} \lim _{z \rightarrow z_{0}} \frac{d^{n-1}}{d z^{n-1}}\left(z-z_{0}\right)^{n} f(z)
$$

to find the residue of $f$ at that point. Nevertheless, the Laurent series of $f$ at $z=0$ gives

$$
\operatorname{Res}(f(z), 0)=3
$$

Hence, we have

$$
\oint_{C} e^{3 / z} d z=2 \pi i \operatorname{Res}(f(z), 0)=2 \pi i(3)=6 \pi i
$$

