Introduction to Complex Analysis

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Consequences and Applications of the Residue Theorem

- Evaluation of Real Trigonometric Integrals
- Evaluation of Real Improper Integrals
- Integration along a Branch Cut
- The Argument Principle and Rouché's Theorem
- Summing Infinite Series
- Laplace and Fourier Transforms

Overview of Consequences of the Residue Theorem

- ${\ensuremath{\, \bullet }}$ The residue theory can be used to evaluate real integrals of the forms
 - $\int_{0}^{2\pi} F(\cos\theta, \sin\theta) d\theta;$ • $\int_{-\infty}^{\infty} f(x) dx;$ • $\int_{-\infty}^{\infty} f(x) \cos \alpha x dx;$
 - $\int_{-\infty}^{\infty} f(x) \sin \alpha x dx$.

Here F and f are rational functions of the form $f(x) = \frac{p(x)}{q(x)}$ in which the polynomials p and q are assumed to have no common factors.

- Residues can be used to evaluate real improper integrals that require integration along a branch cut.
- A relationship exists between the residue theory and the zeros of an analytic function.
- Residues can, in certain cases, be used to find the sum of an infinite series.

Subsection 1

Evaluation of Real Trigonometric Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$

- The basic idea is to convert those into a complex integral, where the contour C is the unit circle |z| = 1 centered at the origin.
- To do this we parametrize this contour by $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. We write $dz = ie^{i\theta}d\theta$, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} e^{-i\theta}}{2i}$. Since $dz = ie^{i\theta}d\theta = izd\theta$ and $z^{-1} = \frac{1}{z} = e^{-i\theta}$, these three quantities are equivalent to $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z + z^{-1})$, $\sin\theta = \frac{1}{2i}(z z^{-1})$. The conversion of the given integral into a contour integral is

$$\oint_C F\left(\frac{1}{2}(z+z^{-1}),\frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz},$$

where C is the unit circle |z| = 1.

A Real Trigonometric Integral

• Evaluate
$$\int_{0}^{2\pi} \frac{1}{(2+\cos\theta)^{2}} d\theta$$
.
• We use the substitutions: $\oint_{C} \frac{1}{(2+\frac{1}{2}(z+z^{-1}))^{2}} \frac{dz}{iz} = \oint_{C} \frac{1}{(2+\frac{z^{2}+1}{2z})^{2}} \frac{dz}{iz}$.
Simplifying, $\frac{4}{i} \oint_{C} \frac{z}{(z^{2}+4z+1)^{2}} dz$. Factoring the denominator
 $z^{2} + 4z + 1 = (z - z_{1})(z - z_{2})$, where $z_{1} = -2 - \sqrt{3}$ and
 $z_{2} = -2 + \sqrt{3}$. Thus, $\frac{z}{(z^{2}+4z+1)^{2}} = \frac{z}{(z-z_{1})^{2}(z-z_{2})^{2}}$. Only z_{2} is inside
the unit circle C. Thus, we have $\oint_{C} \frac{z}{(z^{2}+4z+1)^{2}} dz = 2\pi i \operatorname{Res}(f(z), z_{2})$.
To calculate the residue, note that z_{2} is a pole of order 2:
 $\operatorname{Res}(f(z), z_{2}) = \lim_{z \to z_{2}} \frac{d}{dz}(z - z_{2})^{2}f(z) = \lim_{z \to z_{2}} \frac{d}{dz}\frac{z}{(z-z_{1})^{2}} = \lim_{z \to z_{2}} \frac{-z-z_{1}}{(z-z_{1})^{3}} = \frac{1}{6\sqrt{3}}$.
Hence, $\frac{4}{i} \oint_{C} \frac{z}{(z^{2}+4z+1)^{2}} dz = \frac{4}{i} \cdot 2\pi i \operatorname{Res}(f(z), z_{1}) = \frac{4}{i} \cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}$
and, finally, $\int_{0}^{2\pi} \frac{1}{(2+\cos\theta)^{2}} d\theta = \frac{4\pi}{3\sqrt{3}}$.

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Subsection 2

Evaluation of Real Improper Integrals

Integrals of the Form $\int_{-\infty}^{\infty} f(x) dx$

- Suppose y = f(x) is a real function that is defined and continuous on the interval [0,∞).
- In elementary calculus the improper integral $I_1 = \int_0^\infty f(x) dx$ is defined as the limit $I_1 = \int_0^\infty f(x) dx = \lim_{R \to \infty} \int_0^R f(x) dx$. If the limit exists, the integral I_1 is said to be **convergent**; otherwise, it is **divergent**.
- The improper integral $I_2 = \int_{-\infty}^0 f(x) dx$ is defined similarly: $I_2 = \int_{-\infty}^0 f(x) dx = \lim_{R \to \infty} \int_{-R}^0 f(x) dx.$

• Finally, if f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = I_1 + I_2,$$

provided both integrals I_1 and I_2 are convergent. If either one, I_1 or I_2 , is divergent, then $\int_{-\infty}^{\infty} f(x) dx$ is divergent.

Cauchy Principal Value of $\int_{-\infty}^{\infty} f(x) dx$

- It is important to remember that $\lim_{R\to\infty} \int_{-R}^{0} f(x)dx + \lim_{R\to\infty} \int_{0}^{R} f(x)dx \text{ is not the same as}$ $\lim_{R\to\infty} \left(\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx\right) = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx.$
- For the integral $\int_{-\infty}^{\infty} f(x) dx$ to be convergent, the limits $\lim_{R\to\infty} \int_{-R}^{0} f(x) dx$ and $\lim_{R\to\infty} \int_{0}^{R} f(x) dx$ must exist independently of one another.
- In the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by means of the single limiting process $\int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$.
- On the other hand, the symmetric limit may exist even though the improper integral $\int_{-\infty}^{\infty} f(x) dx$ is divergent.

• The limit $\lim_{R\to\infty} \int_{-R}^{R} f(x) dx$, if it exists, is called the **Cauchy** principal value (P.V.) of the integral and is written P.V. $\int_{-\infty}^{\infty} f(x) dx = \lim_{R\to\infty} \int_{-R}^{R} f(x) dx$.

Principal Value and Integrals of Even Functions

• Suppose f(x) is continuous on $(-\infty, \infty)$ and is an even function, i.e., f(-x) = f(x). Then its graph is symmetric with respect to the y-axis. As a consequence, $\int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx$. Therefore, $\int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx$. If the Cauchy principal value exists, $\int_{0}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ converge. The values of the integrals are

$$\int_0^\infty f(x)dx = \frac{1}{2}\mathsf{P}.\mathsf{V}.\int_{-\infty}^\infty f(x)dx$$

and

$$\int_{-\infty}^{\infty} f(x) dx = \mathsf{P.V.} \int_{-\infty}^{\infty} f(x) dx.$$

Evaluation of Integral $\int_{-\infty}^{\infty} f(x) dx$

• To evaluate $\int_{-\infty}^{\infty} f(x) dx$, where the rational function $f(x) = \frac{p(x)}{q(x)}$ is continuous on $(-\infty, \infty)$,

we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval [-R, R] on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = \frac{p(z)}{q(z)}$ in the upper half-plane $\operatorname{Im}(z) > 0$.



Then,

 $\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^R f(x)dx = 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k),$ where z_k , k = 1, 2, ..., n denotes poles in the upper half-plane. If we can show that the $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have $P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx = 2\pi i \sum_{k=1}^n \operatorname{Res}(f(z), z_k).$

Cauchy P.V. of an Improper Integral

• Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx$.

Let
$$f(z) = \frac{1}{(z^2+1)(z^2+9)}$$
.

Since $(z^2 + 1)(z^2 + 9) = (z - i)(z + i)(z - 3i)(z+3i)$, we take *C* be the closed contour consisting of the interval [-R, R] on the *x*-axis and the semicircle C_R of radius R > 3. Then



$$\oint_C \frac{1}{(z^2+1)(z^2+9)} dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)} dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz = l_1 + l_2$$

and $l_1 + l_2 = 2\pi i [\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)]$. At the simple poles $z = i$ and $z = 3i$ we find $\operatorname{Res}(f(z), i) = \frac{1}{16i}$ and $\operatorname{Res}(f(z), 3i) = -\frac{1}{48i}$, whence $l_1 + l_2 = 2\pi i [\frac{1}{16i} + (-\frac{1}{48i})] = \frac{\pi}{12}$.

Letting $R o \infty$

•
$$\oint_C \frac{1}{(z^2+1)(z^2+9)} dz = \int_{-R}^R \frac{1}{(x^2+1)(x^2+9)} dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz = \frac{\pi}{12}$$
.
Before letting $R \to \infty$, note that $|(z^2+1)(z^2+9)| = |z^2+1| \cdot |z^2+9| \ge ||z^2| - 1| \cdot ||z^2| - 9| = (R^2 - 1)(R^2 - 9)$. Since the length L of the semicircle is πR , it follows, by the *ML*-inequality, $|I_2| = \left| \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} dz \right| \le \frac{\pi R}{(R^2-1)(R^2-9)}$. Hence, $|I_2| \to 0$ as $R \to \infty$, and we conclude that $\lim_{R\to\infty} I_2 = 0$. It follows that $\lim_{R\to\infty} I_1 = \frac{\pi}{12}$. I.e., $\lim_{R\to\infty} \int_{-R}^R \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}$ or P.V. $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12}$.

Behavior of Integral as $R \to \infty$

• To show that the contour integral along C_R approaches zero as $R \to \infty$ the following sufficient conditions are useful:

Theorem (Behavior of Integral as $R \to \infty$)

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of p(z) is n and the degree of q(z) is $m \ge n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

- In other words, the integral along C_R approaches zero as $R \to \infty$ when the denominator of f is of a power at least 2 more than its numerator.
- The proof of this fact follows as in the preceding example, in which degree of p(z) = 1 is 0 and degree of $q(z) = (z^2 + 1)(z^2 + 9)$ is 4.

Another Cauchy P.V. of an Improper Integral

• Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$.

The conditions given in the preceding theorem are satisfied. Moreover, $f(z) = \frac{1}{z^4+1}$ has simple poles in the upper half-plane at $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$. We have seen the residues at these poles are

$$\operatorname{Res}(f(z), z_1) = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$
 and $\operatorname{Res}(f(z), z_2) = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$.

Thus,

$$\mathsf{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx = 2\pi i [\mathsf{Res}(f(z), z_1) + \mathsf{Res}(f(z), z_2)] = \frac{\pi}{\sqrt{2}}.$$

Since the integrand is an even function, the original integral converges to $\frac{\pi}{\sqrt{2}}$.

Integrals $\int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx$ and $\int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx$

- Integrals of Form $\int_{-\infty}^{\infty} f(x) \sin \alpha x dx$ and $\int_{-\infty}^{\infty} f(x) \cos \alpha x dx$ are referred to as **Fourier integrals**.
- They appear as the real and imaginary parts of $\int_{-\infty}^{\infty} f(x)e^{i\alpha x} dx$.
- Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function continuous on $(-\infty, \infty)$. Then both Fourier integrals can be evaluated by considering the complex integral $\oint_C f(z)e^{i\alpha z} dz$, where $\alpha > 0$, and the contour C consists of [-R, R] and a semicircular contour C_R with radius large enough to enclose the poles of f(z) in the upper-half plane.
- Sufficient conditions under which the contour integral along C_R approaches zero as $R \to \infty$ are given by

Theorem (Behavior of Integral as $R o \infty$)

Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of p(z) is n and the degree of q(z) is $m \ge n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$, and $\alpha > 0$, then $\int_{C_R} f(z)e^{i\alpha z}dz \to 0$ as $R \to \infty$.

Evaluating a Fourier Integral

• Evaluate the Cauchy principal value of $\int_0^\infty \frac{x \sin x}{x^2 \pm a} dx$. First note that the limits of integration in the given integral are not from $-\infty$ to ∞ as required by the method just described. Since the integrand is an even function of x, $\int_0^\infty \frac{x \sin x}{x^{2+0}} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^{2+0}} dx$. We now form the contour integral $\oint_C \frac{z}{z^2+q} e^{iz} dz$, where C is the contour described before, with R > 3. We have $\int_{C_{R}} \frac{z}{z^{2}+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^{2}+9} e^{ix} dx = 2\pi i \operatorname{Res}(f(z)e^{iz}, 3i), \text{ where }$ $f(z) = \frac{z}{z^2+9}$, and $\text{Res}(f(z)e^{iz}, 3i) = \frac{ze^{iz}}{z+3i}\Big|_{z=3i} = \frac{e^{-3}}{2}$. Since, by the theorem, $\int_{C_n} f(z)e^{iz} dz \to 0$ as $R \to \infty$, we get P.V. $\int_{-\infty}^{\infty} \frac{x}{x^{2+0}} e^{ix} dx = 2\pi i \left(\frac{e^{-3}}{2}\right) = \frac{\pi}{a^3} i$. Note that $\int_{-\infty}^{\infty} \frac{x}{x^{2+0}} e^{ix} dx = \frac{\pi}{a^3} i e^{ix} dx$ $\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + 0} dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 0} dx = \frac{\pi}{a^3} i.$ Equating real and imaginary parts: P.V. $\int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx = 0$ and P.V. $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{x^3}$. This implies that $\int_0^\infty \frac{x \sin x}{x^2 + 0} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2 + 0} dx = \frac{\pi}{2\sigma^3}$.

Indented Contours

- Up to this point we considered improper integrals of functions continuous on the interval $(-\infty, \infty)$, i.e., the complex function $f(z) = \frac{p(z)}{q(z)}$ did not have poles on the real axis.
- Suppose we want to evaluate $\int_{-\infty}^{\infty} f(x)dx$ by residues when f(z) has a pole at z = c, where c is a real number. Then we use an indented contour: The symbol C_r denotes a semicircular contour centered at z = cand oriented in the positive direction.



Theorem (Behavior of Integral as $r \rightarrow 0$)

Suppose f has a simple pole z = c on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \le \theta \le \pi$, then

$$\lim_{r\to 0}\int_{C_r}f(z)dz=\pi i\mathrm{Res}(f(z),c).$$

Proof of the Theorem

• Since f has a simple pole at z = c, its Laurent series is

$$f(z)=\frac{a_{-1}}{z-c}+g(z),$$

where $a_{-1} = \text{Res}(f(z), c)$ and g is analytic at the point c. Using the Laurent series and the parametrization of C_r , we have

$$\int_{C_r} f(z)dz = a_{-1} \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}}d\theta + ir \int_0^\pi g(c + re^{i\theta})e^{i\theta}d\theta = l_1 + l_2.$$

•
$$I_1 = a_{-1} \int_0^\pi \frac{ire^{i\theta}}{re^{i\theta}} d\theta = a_{-1} \int_0^\pi id\theta = \pi i a_{-1} = \pi i \operatorname{Res}(f(z), c).$$

• Since g is analytic at c, it is continuous at this point and bounded in a neighborhood of the point. I.e., there exists an M > 0 for which $|g(c + re^{i\theta})| \le M$. Hence,

$$|I_2| = |ir \int_0^{\pi} g(c + re^{i\theta})d\theta| \le r \int_0^{\pi} M d\theta = \pi r M.$$

It follows that $\lim_{r\to 0} |I_2| = 0$ and, consequently, $\lim_{r\to 0} I_2 = 0$. By taking the limit of the sum as $r \to 0$, we get the conclusion.

Using an Indented Contour

• Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)} dx$. We consider $\oint_C \frac{e^{iz}}{z(z^2-2z+2)} dz$. $f(z) = \frac{1}{z(z^2-2z+2)}$ has a pole at z = 0and at z = 1 + i in the upper half-plane. The contour C, is indented at the origin. We have $\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_{r}^{R} =$ -r $2\pi i \operatorname{Res}(f(z)e^{iz}, 1+i), \ \int_{-C} = -\int_{C}$ If we take the limits as $R \to \infty$ and as $r \to 0$. P.V. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)} dx - \pi i \operatorname{Res}(f(z)e^{iz}, 0) = 2\pi i \operatorname{Res}(f(z)e^{iz}, 1+i).$ Now, $\operatorname{Res}(f(z)e^{iz}, 0) = \frac{1}{2}$ and $\operatorname{Res}(f(z)e^{iz}, 1+i) = -\frac{e^{-1+i}}{4}(1+i)$. Therefore, P.V. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)} dx = \pi i \frac{1}{2} + 2\pi i (-\frac{e^{-1+i}}{4}(1+i))$. Using $e^{-1+i} = e^{-1}(\cos 1 + i \sin 1)$ and equating real and imaginary parts: P.V. $\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)} dx = \frac{\pi}{2} e^{-1} (\sin 1 + \cos 1),$ P.V. $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - 2x + 2)} dx = \frac{\pi}{2} [1 + e^{-1} (\sin 1 - \cos 1)].$

Subsection 3

Integration along a Branch Cut

Branch Point at z = 0

- Suppose that, if f(x) is converted to a complex function, f(z) has, in addition to poles, a nonisolated singularity at z = 0.
- In that case, computing $\int_0^\infty f(x) dx$ requires a special type of contour.
- Example: Consider the real integral $\int_0^\infty \frac{x^{\alpha-1}}{x+1} dx$, (21) where α is a real constant restricted to the interval $0 < \alpha < 1$. When $\alpha = \frac{1}{2}$ and x is replaced by z, the integrand becomes the multiple-valued function $\frac{1}{z^{1/2}(z+1)}$. The origin is a branch point because $z^{1/2}$ has two values for any $z \neq 0$. Traveling in a complete circle around the origin z = 0, starting from a point $z = re^{i\theta}$, r > 0, we return to the same starting point z, but θ has increased by 2π . Thus, the value of $z^{1/2}$ changes from $z^{1/2} = \sqrt{r}e^{i\theta/2}$ to a different value or different branch: $z^{1/2} = \sqrt{r}e^{i(\theta+2\pi)/2} = \sqrt{r}e^{i\theta/2}e^{i\pi} = -\sqrt{r}e^{i\pi/2}.$ Recall, we can force $z^{1/2}$ to be single valued by restricting θ to some interval of length 2π . E.g., by restricting θ to $0 < \theta < 2\pi$, we guarantee that $z^{1/2} = \sqrt{r}e^{i\theta/2}$ is single valued.

Integration along a Branch Cut

• Evaluate
$$\int_0^\infty \frac{1}{\sqrt{x(x+1)}} dx$$
.

The real integral is improper for two reasons:

- There is an infinite discontinuity at *x* = 0;
- The limit of integration is infinite.

We form the integral $\int_C \frac{1}{z^{1/2}(z+1)} dz$, where C is the contour shown, which consists of

- C_r and C_R , which are portions of circles;
- *AB* and *ED*, which are parallel horizontal line segments running along opposite sides of the branch cut.



The integrand f(z) of the contour integral is single valued and analytic on and within C, except for the simple pole at $z = -1 = e^{\pi i}$. Hence, we can write $\oint_C \frac{1}{z^{1/2}(z+1)} dz = 2\pi i \operatorname{Res}(f(z), -1)$ or $\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \operatorname{Res}(f(z), -1)$.

Integration along a Branch Cut (Cont'd)

- We think of *AB* as coinciding with the upper side of the positive real axis for which $\theta = 0$ and of *ED* with the lower side of the positive real axis for which $\theta = 2\pi$.
 - On AB, $z = xe^{0i}$: On *ED*. $z = xe^{(0+2\pi)i} = xe^{2\pi i}$: Thus. $\int_{FD} = \int_{R}^{r} \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i}+1} (e^{2\pi i} dx) = -\int_{R}^{r} \frac{x^{-1/2}}{x+1} dx = \int_{R}^{R} \frac{x^{-1/2}}{x+1} dx$ and $\int_{AB} = \int_{r}^{R} \frac{(xe^{0i})^{-1/2}}{xe^{0i+1}} (e^{0i} dx) = \int_{r}^{R} \frac{x^{-1/2}}{x+1} dx.$ Now with $z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R , respectively, it can be shown that $\int_{C_r} \to 0$ as $r \to 0$ and $\int_{C_R} \to 0$ as $R \to \infty$. Thus, $\lim_{z \to 0} \left[\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \operatorname{Res}(f(z), -1) \right]$ is the same as $2\int_0^{\infty} \frac{1}{\sqrt{x(x+1)}} dx = 2\pi i \text{Res}(f(z), -1)$. Since $\operatorname{Res}(f(z),-1) = z^{-1/2}\Big|_{z=e^{\pi i}} = e^{-\pi i/2} = -i, \ \int_0^\infty \frac{1}{\sqrt{x(x+1)}} dx = \pi.$

Subsection 4

The Argument Principle and Rouché's Theorem

Number of Zeros and Poles

- We apply residue theory to the location of zeros of an analytic function.
- In the first theorem we need to count the number of zeros and poles of a function *f* that are located within a simple closed contour *C*, taking into account the order or multiplicity of each zero and pole.
- Example: If $f(z) = \frac{(z-1)(z-9)^4(z+i)^2}{(z^2-2z+2)^2(z-i)^6(z+6i)^7}$ and C is taken to be the circle |z| = 2, then:
 - Inspection of the numerator of f reveals that the zeros inside C are z = 1 (a simple zero) and z = -i (a zero of order or multiplicity 2). Therefore, the number N_0 of zeros inside C is taken to be $N_0 = 1 + 2 = 3$.
 - Similarly, inspection of the denominator of f shows, after factoring $z^2 2z + 2 = (z 1 i)(z 1 + i)$, that the poles inside C are z = 1 i (pole of order 2), z = 1 + i (pole of order 2), and z = i (pole of order 6). The number N_p of poles inside C is taken to be $N_p = 2 + 2 + 6 = 10$.

Argument Principle

Theorem (Argument Principle)

Let *C* be a simple closed contour lying entirely within a domain *D*. Suppose *f* is analytic in *D* except at a finite number of poles inside *C*, and that $f(z) \neq 0$ on *C*. Then $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p$, where N_0 is the total number of zeros of *f* inside *C* and N_p is the total number of poles of *f* inside *C*, counting their order or multiplicities.

• The integrand $\frac{f'(z)}{f(z)}$ is analytic in and on the contour C except at the points in the interior of C where f has a zero or a pole. If z_0 is a zero of order n of f inside C, then we can write $f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$. We differentiate f by the product rule, $f'(z) = (z - z_0)^n \phi'(z) + n(z - z_0)^{n-1} \phi(z)$, and divide this expression by f. In some punctured disk centered at z_0 , we have $\frac{f'(z)}{f(z)} = \frac{(z - z_0)^n \phi'(z) + n(z - z_0)^{n-1} \phi(z)}{(z - z_0)^n \phi(z)} = \frac{\phi'(z)}{\phi(z)} + \frac{n}{z - z_0}$. Thus, the integrand $\frac{f'(z)}{f(z)}$ has a simple pole at z_0 .

Proof of the Argument Principle

• We found $\frac{f'(z)}{f(z)} = \frac{\phi'(z)}{\phi(z)} + \frac{n}{z-z_0}$. The residue at z_0 is $\operatorname{Res}(\frac{f'(z)}{f(z)}, z_0) =$ $\lim_{z \to z_0} (z - z_0) \left(\frac{\phi'(z)}{\phi(z)} + \frac{n}{z - z_0} \right) = \lim_{z \to z_0} \left(\frac{(z - z_0)\phi'(z)}{\phi(z)} + n \right) = 0 + n = n,$ which is the order of the zero z_0 . Now if z_p is a pole of order m of f within C, then $f(z) = \frac{g(z)}{(z-z_p)^m}$, where g is analytic at z_p and $g(z_p) \neq 0$. By differentiating, $f'(z) = (z - z_p)^{-m}g'(z) - m(z - z_p)^{-m-1}g(z)$. Therefore, in some punctured disk centered at z_p , $\frac{f'(z)}{f(z)} = \frac{(z-z_p)^{-m}g'(z)-m(z-z_p)^{-m-1}g(z)}{(z-z_p)^{-m}g(z)} = \frac{g'(z)}{g(z)} + \frac{-m}{z-z_p}$. Thus, $\frac{f'(z)}{f(z)}$ has a simple pole at z_p . We also see that the residue at z_p is equal to

-m, which is the negative of the order of the pole of f.

Proof of the Argument Principle (Cont'd)

• Finally, suppose that $z_{0_1}, z_{0_2}, \ldots, z_{0_r}$ and $z_{p_1}, z_{p_2}, \ldots, z_{p_s}$ are the zeros and poles of f within C and that the order of the zeros are n_1, n_2, \ldots, n_r and that order of the poles are m_1, m_2, \ldots, m_s . Then each of these points is a simple pole of the integrand $\frac{f'(z)}{f(z)}$ with corresponding residues n_1, n_2, \ldots, n_r and $-m_1, -m_2, \ldots, -m_s$. It follows from the residue theorem that $\oint_C \frac{f'(z)}{f(z)} dz$ is equal to $2\pi i$ times the sum of the residues at the poles: $\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i \left[\sum_{k=1}^r \operatorname{Res}(\frac{f'(z)}{f(z)}, z_{0_k}) + \sum_{k=1}^s \operatorname{Res}(\frac{f'(z)}{f(z)}, z_{p_k}) \right] = 2\pi i \left[\sum_{k=1}^r n_k + \sum_{k=1}^s (-m_k) \right] = 2\pi i [N_0 - N_p].$

Illustrating the Argument Principle

• Suppose the simple closed contour is |z| = 2 and the function

$$f(z) = \frac{(z-1)(z-9)^4(z+i)^2}{(z^2-2z+2)^2(z-i)^6(z+6i)^7}.$$

In the evaluation of $\oint_C \frac{f'(z)}{f(z)} dz$, each zero of f within C contributes $2\pi i$ times the order of multiplicity of the zero and each pole contributes $2\pi i$ times the negative of the order of the pole: $\oint_C \frac{f'(z)}{f(z)} dz$ = $[2\pi i(1) + 2\pi i(2)] + [2\pi i(-2) + 2\pi i(-2) + 2\pi i(-6)] = -14\pi i$.

• The name "argument principle" originates from a relation between the number $N_0 - N_p$ and $\arg(f(z))$: We have

$$N_0 - N_p = \frac{1}{2\pi} [\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \ \text{once in the positive direction}].$$

Rouché's Theorem

• The following theorem is helpful in determining the number of zeros of an analytic function.

Theorem (Rouché's Theorem)

Let C be a simple closed contour lying entirely within a domain D. Suppose f and g are analytic in D. If the strict inequality |f(z) - g(z)| < |f(z)| holds for all z on C, then f and g have the same number of zeros, counting their order or multiplicities, inside C.

The hypothesis that |f(z) - g(z)| < |f(z)| holds, for all z on C, indicates that both f and g have no zeros on the contour C. From |f(z) - g(z)| = |g(z) - f(z)|, we see that, by dividing the inequality by |f(z)|. we have, for all z on C, |F(z) - 1| < 1, where F(z) = g(z)/f(z).

Proof of Rouché's Theorem

• We have
$$|F(z) - 1| < 1$$
, where $F(z) = \frac{g(z)}{f(z)}$.

This inequality shows that the image C' in the *w*-plane of the curve *C* under the mapping w = F(z) is a closed path and must lie within the unit open disk |w - 1| < 1 centered at w = 1.



As a consequence, the curve C' does not enclose w = 0, and therefore $\frac{1}{w}$ is analytic in and on C'. By the Cauchy-Goursat Theorem, $\oint_{C'} \frac{1}{w} dw = 0$. Since w = F(z) and dw = F'(z)dz, $\oint_C \frac{F'(z)}{F(z)} dz = 0$. From the quotient rule, $F'(z) = \frac{f(z)g'(z) - g(z)f'(z)}{[f(z)]^2}$, we get $\frac{F'(z)}{F(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$. Therefore, $\oint_C (\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)})dz = 0$ or $\oint_C \frac{g'(z)}{g(z)} dz = \oint_C \frac{f'(z)}{f(z)} dz$. By the argument principle, the number of zeros of g inside C is the same as the number of zeros of f inside C.

Location of Zeros

Locate the zeros of the polynomial function g(z) = z⁹ - 8z² + 5. We begin by choosing f(z) = z⁹ because it has the same number of zeros as g. Since f has a zero of order 9 at z = 0, we search for the zeros of g by examining circles centered at z = 0. If we can establish that |f(z) - g(z)| < |f(z)|, for all z on some circle |z| = R, then Rouché's Theorem asserts that f and g have the same number of zeros within |z| < R.

By the triangle inequality, $|f(z) - g(z)| = |z^9 - (z^9 - 8z^2 + 5)| = |8z^2 - 5| \le 8|z|^2 + 5$. Also, $|f(z)| = |z|^9$. Since |f(z) - g(z)| < |f(z)| or $8|z|^2 + 5 < |z|^9$ is not true for all z on |z| = 1, we can draw no conclusion. By expanding the search to the larger circle $|z| = \frac{3}{2}$, we see $|f(z) - g(z)| \le 8|z|^2 + 5 = 8 \cdot (\frac{3}{2})^2 + 5 = 23 < (\frac{3}{2})^9 = |f(z)|$. Thus, since f has a zero of order 9 within $|z| < \frac{3}{2}$, all nine zeros of g lie within the same disk.

Revisiting the Zeros of $g \mid$

• By more refined reasoning, we can show that $g(z) = z^9 - 8z^2 + 5$ has some zeros inside |z| < 1.

To see this suppose we choose $f(z) = -8z^2 + 5$. Then, for all z on |z| = 1. $|f(z) - g(z)| = |(-8z^2 + 5) - (z^9 - 8z^2 + 5)| = |-z^9| = |z|^9 =$ $(1)^9 = 1.$ For all z on |z| = 1, $|f(z)| = |-f(z)| = |8z^2 - 5| > |8|z|^2 - |-5|| = |8 - 5| = 3.$ Therefore, for all z on |z| = 1, |f(z) - g(z)| < |f(z)|. Because f has two zeros within |z| < 1 (namely, $\pm \sqrt{rac{5}{8}}$), we can conclude, by Rouché's Theorem, that two zeros of g also lie within this disk.

Revisiting the Zeros of $g \, II$

• Continuing to reason about the zeros of $g(z) = z^9 - 8z^2 + 5$, suppose we choose f(z) = 5 and $|z| = \frac{1}{2}$. Then, for all z on $|z| = \frac{1}{2}$, $|f(z) - g(z)| = |5 - (z^9 - 8z^2 + 5)| = |-z^9 + 8z^2| \le |z|^9 + 8|z|^2 = (\frac{1}{2})^9 + 2 \approx 2.002$. We now have |f(z) - g(z)| < |f(z)| = 5, for all z on $|z| = \frac{1}{2}$. Since f has no zeros within the disk $|z| < \frac{1}{2}$, neither does g. At this point we are able to conclude that all nine zeros of $g(z) = z^9 - 8z^2 + 5$ lie within the annular region $\frac{1}{2} < |z| < \frac{3}{2}$. Moreover, two of these zeros lie within $\frac{1}{2} < |z| < 1$.

Subsection 5

Summing Infinite Series

Using $\cot \pi z$

- The residues at the simple poles of $\cot \pi z$ can help find the sum of an infinite series.
- The zeros of sin z are the reals $z = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ Thus, cot πz has simple poles at $\pi z = k\pi$ or z = k, $k = 0, \pm 1, \pm 2, \ldots$
- If a polynomial function p(z) has (i) real coefficients; (ii) degree $n \ge 2$, and (iii) no integer zeros, then the function $f(z) = \frac{\pi \cot \pi z}{p(z)}$ has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \ldots$ from $\cot \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ from the zeros of p(z).
- The closed rectangular contour is *C*, where *n* is taken large enough so that *C* encloses the simple poles z = 0, $\pm 1, \pm 2, \dots, \pm n$ and all of the poles $z_{p_1}, z_{p_2}, \dots, z_{p_r}$. By the residue theorem, $\oint_{\alpha} \frac{\pi \cot \pi z}{r} dz = 2\pi i (\sum_{i=1}^{n} \frac{\operatorname{Res}(\pi \cot \pi z}{r} k))$



Using $\cot \pi z$ (Cont'd)

• Since it can be shown that $\oint_C \frac{\pi \cot \pi z}{p(z)} dz \to 0$ as $n \to \infty$, we get $0 = \sum_k \text{residues} + \sum_i \text{residues}$. That is,

$$\sum_{k=-\infty}^{\infty} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right) = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_{j}}\right).$$

• If a function f can be written as a quotient $f(z) = \frac{g(z)}{h(z)}$, where g and h are analytic at $z = z_0$, $g(z_0) \neq 0$ and h has a zero of order 1 at z_0 , then f has a simple pole at $z = z_0$ and $\text{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$.

• Hence, with
$$g(z) = \frac{\pi \cos \pi z}{p(z)}$$
 and $h(z) = \sin \pi z$, we get

$$\operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right) = \frac{\frac{\pi \cos k\pi}{p(k)}}{\pi \cos k\pi} = \frac{1}{p(k)}.$$
Therefore, we arrive at

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}(\frac{\pi \cot \pi z}{p(z)}, z_{p_j}).$$

Using $\overline{\csc \pi z}$

• If p(z) is a polynomial function satisfying the same assumptions, i.e.,

- (i) has real coefficients;
- (ii) has degree $n \ge 2$, and
- (iii) no integer zeros,

then the function $f(z) = \frac{\pi \csc \pi z}{p(z)}$ has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \ldots$ from $\csc \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ from the zeros of p(z).

In this case it can be shown that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{p(k)} = -\sum_{j=1}^r \operatorname{Res}\left(\frac{\pi \csc \pi z}{p(z)}, z_{p_j}\right)$$

Summing an Infinite Series

• Find the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k^2+4}$.

If we identify $p(z) = z^2 + 4$, then the three assumptions (i)-(iii) are satisfied. The zeros of p(z) are $\pm 2i$ and correspond to simple poles of $f(z) = \frac{\pi \cot \pi z}{z^2 + 4}$. According to the formula $\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 4} = -(\operatorname{Res}(\frac{\pi \cot \pi z}{z^2 + 4}, -2i) + \operatorname{Res}(\frac{\pi \cot \pi z}{z^2 + 4}, 2i))$. Since $\operatorname{Res}(\frac{\pi \cot \pi z}{z^2 + 4}, -2i) = \frac{\pi \cot 2\pi i}{4i}$ and $\operatorname{Res}(\frac{\pi \cot \pi z}{z^2 + 4}, 2i) = \frac{\pi \cot 2\pi i}{4i}$, the sum of the residues is $\frac{\pi}{2i} \cot 2\pi i$. This sum is a real quantity because $\frac{\pi}{2i} \cot 2\pi i = \frac{\pi}{2i} \frac{\cosh(-2\pi)}{(-i\sinh(-2\pi))} = -\frac{\pi}{2} \coth 2\pi$. Hence,

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+4} = \frac{\pi}{2} \coth 2\pi.$$

Summing an Infinite Series (Cont'd)

• To get the desired sum, we must manipulate the summation $\sum_{-\infty}^{\infty}$ in order to put it in the form $\sum_{k=0}^{\infty}$.

We have

$$\begin{split} \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 4} &= \sum_{k=-\infty}^{-1} \frac{1}{k^2 + 4} + \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{k^2 + 4} \\ &= \sum_{k=1}^{\infty} \frac{1}{(-k)^2 + 4} + \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{k^2 + 4} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{k^2 + 4} + \frac{1}{4} = 2 \sum_{k=0}^{\infty} \frac{1}{k^2 + 4} - \frac{1}{4}. \end{split}$$

Finally, since $\sum_{k=-\infty}^{\infty} \frac{1}{k^2+4} = 2 \sum_{k=0}^{\infty} \frac{1}{k^2+4} - \frac{1}{4} = \frac{\pi}{2} \coth 2\pi$, we obtain

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 4} = \frac{1}{8} + \frac{\pi}{4} \coth 2\pi.$$

Subsection 6

Laplace and Fourier Transforms

Laplace and Inverse Laplace Transforms

- The Laplace transform of a real function f is defined, for $t \ge 0$, by $\mathcal{L}{f(t)} = \int_0^\infty e^{-st} f(t) dt$.
 - (i) The direct problem: Given a function f(t) satisfying certain conditions, find its Laplace transform. When the integral converges, the result is a function of s. The relationship between a function and its transform is exhibited by using a lowercase letter to denote the function and the corresponding uppercase letter to denote its Laplace transform, e.g., L{f(t)} = F(s), L{y(t)} = Y(s), and so on.
 - (ii) The inverse problem: Find the function f(t) that has a given transform F(s). The function f(t) is called the inverse Laplace transform and is denoted by L⁻¹{F(s)}.
- We will see that the inverse Laplace transform is not merely a symbol but actually another integral transform, actually a special type of complex contour integral.

Integral Transforms

- Suppose f(x, y) is a real-valued function of two real variables.
- A definite integral of *f* with respect to one of the variables leads to a function of the other variable.

Example: If we hold y constant, integration with respect to the real variable x gives $\int_{1}^{2} 4xy^{2} dx = 2x^{2}y^{2}|_{1}^{2} = 8y^{2} - 2y^{2} = 6y^{2}$.

- Thus, a definite integral such as $F(\alpha) = \int_a^b f(x) K(\alpha, x) dx$ transforms a function f of the variable x into a function F of the variable α .
- We say that $F(\alpha) = \int_a^b f(x) K(\alpha, x) dx$ is an integral transform of the function f.
- Integral transforms appear in **transform pairs**, meaning that the original function f can be recovered by another integral transform $f(x) = \int_{c}^{d} F(\alpha)H(\alpha, x)d\alpha$, called the **inverse transform**.
- The functions $K(\alpha, x)$ and $H(\alpha, x)$ are the **kernels** of the transforms.
- If α represents a complex variable, then the second definite integral is replaced by a contour integral.

The Laplace Transform

- Suppose that, in $F(\alpha) = \int_a^b f(x) \mathcal{K}(\alpha, x) dx$, α is replaced by the symbol *s*, and that *f* represents a real function that is defined on the unbounded interval $[0, \infty)$.
- Then $F(s) = \int_0^\infty f(t) K(s, t) dt$ is an improper integral, defined by

$$\int_0^\infty K(s,t) f(t) dt = \lim_{b o\infty} \int_0^b K(s,t) f(t) dt.$$

- If the limit exists, we say that the integral exists or is convergent; otherwise, the integral does not exist and is said to be divergent.
- The choice $K(s, t) = e^{-st}$, where s is a complex variable, gives the Laplace transform $\mathcal{L}{f(t)}$ defined previously.
- The integral that defines the Laplace transform may not converge for certain kinds of functions *f*.
 - Example: Neither $\mathcal{L}{e^{t^2}}$ nor $\mathcal{L}{\frac{1}{t}}$ exists.
- Also, the limit may exist for only certain values of the variable s.

Existence of a Laplace Transform

• The Laplace transform of f(t) = 1, $t \ge 0$, is

$$\mathcal{L}{1} = \int_0^\infty e^{-st}(1)dt$$

= $\lim_{b\to\infty} \int_0^b e^{-st}dt$
= $\lim_{b\to\infty} -\frac{e^{-st}}{s}\Big|_0^b$
= $\lim_{b\to\infty} \left[\frac{1-e^{-sb}}{s}\right].$

If s = x + iy, then $e^{-sb} = e^{-bx}(\cos by - i \sin by)$. Thus, $e^{-sb} \to 0$ as $b \to \infty$, if x > 0. In other words,

$$\mathcal{L}{1} = \frac{1}{s}$$
, provided $\operatorname{Re}(s) > 0$.

Existence of $\mathcal{L}{f(t)}$

- Conditions that are sufficient to guarantee the existence of L{f(t)} are that f be piecewise continuous on [0,∞) and that f be of exponential order.
 - Piecewise continuity on [0,∞) means that, on any interval, there are at most a finite number of points t_k, k = 1, 2, ..., n, t_{k-1} < t_k, at which f has finite discontinuities and is continuous on each open interval t_{k-1} < t < t_k.
 - A function f is said to be **of exponential order** c if there exist constants c, M > 0, and T > 0, so that $|f(t)| \le Me^{ct}$, for t > T. The condition $|f(t)| \le Me^{ct}$, for t > T, states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct} .

Alternatively, $e^{-ct}|f(t)|$ is bounded, i.e., $e^{-ct}|f(t)| \leq M$, for t > T.

• All bounded functions are necessarily of exponential order c = 0.

Existence Theorem for $\mathcal{L}{f(t)}$

Theorem (Sufficient Conditions for Existence)

Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for t > T. Then $\mathcal{L}{f(t)}$ exists for Re(s) > c.

- We have $\mathcal{L}{f(t)} = \int_0^T e^{-st} f(t) dt + \int_T^\infty e^{-st} f(t) dt = I_1 + I_2$.
 - The integral I₁ exists since it can be written as a sum of integrals over intervals on which e^{-st}f(t) is continuous.
 - To prove the existence of I_2 , let s = x + iy. Then $|e^{-st}| = |e^{-xt}(\cos yt - i\sin yt)| = e^{-xt}$. Further, by the definition of exponential order, $|f(t)| \le Me^{ct}$, t > T. Hence, $|I_2| \le \int_T^{\infty} |e^{-st}f(t)|dt$ $\le M \int_T^{\infty} e^{-xt}e^{ct}dt = M \int_T^{\infty} e^{-(x-c)t}dt = -M \frac{e^{-(x-c)t}}{x-c}\Big|_T^{\infty} =$ $M \frac{e^{-(x-c)T}}{x-c}$, for $x = \operatorname{Re}(s) > c$. Since $\int_T^{\infty} Me^{-(x-c)t}dt$ converges, $\int_T^{\infty} |e^{-st}f(t)|dt$ converges by the comparison test. This, in turn, implies that I_2 exists for $\operatorname{Re}(s) > c$.

The existence of I_1 and I_2 implies that $\mathcal{L}{f(t)} = \int_0^\infty e^{-st} f(t) dt$ exists for $\operatorname{Re}(s) > c$.

Analyticity of the Laplace Transform

• The following theorem is stated without proof:

Theorem (Analyticity of the Laplace Transform)

Suppose f is piecewise continuous on $[0,\infty)$ and of exponential order c for $t \ge 0$. Then the Laplace transform of f,

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

is an analytic function in the right half-plane defined by Re(s) > c.

• Although the complex function F(s) is analytic to the right of the line x = c in the complex plane, F(s) will, in general, have singularities to the left of that line.

The Inverse Laplace Transform

Theorem (Inverse Laplace Transform)

If f and f' are piecewise continuous on $[0, \infty)$ and f is of exponential order c for $t \ge 0$, and F(s) is a Laplace transform, then the inverse Laplace transform $\mathcal{L}^{-1}{F(s)}$ is

$$f(t) = \mathcal{L}^{-1}{F(s)} = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s) ds$$

where $\gamma > c$.

- We write $f(t) = \mathcal{L}^{-1}{F(s)} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds$, where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\operatorname{Re}(s) = x = \gamma$.
- γ is a positive real constant greater than *c* and greater than all the real parts of the singularities in the left half-plane.
- This integral is called a **Bromwich contour integral**.
- The kernel of the inverse transform is $H(s, t) = \frac{e^{st}}{2\pi i}$.

Evaluating the Inverse Laplace Transform

- The Bromwich contour integral $f(t) = \mathcal{L}^{-1}{F(s)} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds.$
- The fact that F(s) has singularities s₁, s₂,..., s_n to the left of the line x = γ makes it possible to evaluate the integral by using an appropriate closed contour encircling the singularities.

A closed contour C that is commonly used consists of a semicircle C_R of radius R centered at $(\gamma, 0)$ and a vertical line segment L_R parallel to the y-axis passing through the point $(\gamma, 0)$ and extending from y = $\gamma - iR$ to $y = \gamma + iR$. R is larger than the largest number in $\{|s_1|, |s_2|, \dots, |s_n|\}$.



With the contour *C* chosen in this manner, the integral can often be evaluated using Cauchy's residue theorem. If we allow the radius *R* of the semicircle to approach ∞ , the vertical part of the contour approaches the infinite vertical line of the Bromwich integral.

Inverse Laplace Transform Theorem

Theorem (Inverse Laplace Transform)

Suppose F(s) is a Laplace transform that has a finite number of poles s_1, s_2, \ldots, s_n to the left of the vertical line $\operatorname{Re}(s) = \gamma$ and that C is the contour on the preceding slide. If sF(s) is bounded as $R \to \infty$, then $\mathcal{L}^{-1}{F(s)} = \sum_{k=1}^{n} \operatorname{Res}(e^{st}F(s), s_k)$.

• By Cauchy's residue theorem, we have

$$\int_{C_R} e^{st} F(s) ds + \int_{L_R} e^{st} F(s) ds = 2\pi i \sum_{k=1}^n \operatorname{Res}(e^{st} F(s), s_k) \text{ or}$$

$$\frac{1}{2\pi i} \int_{\gamma-iR}^{\gamma+iR} e^{st} F(s) ds = \sum_{k=1}^n \operatorname{Res}(e^{st} F(s), s_k) - \frac{1}{2\pi i} \int_{C_R} e^{st} F(s) ds.$$
We let $R \to \infty$ and show that $\lim_{R\to\infty} \int_{C_R} e^{st} F(s) ds = 0.$
If the semicircle C_R is parametrized by $s = \gamma + Re^{i\theta}, \frac{\pi}{2} \le \theta \le \frac{3\pi}{2},$
then $ds = Rie^{i\theta} d\theta = (s - \gamma)id\theta$, and so,

$$\frac{1}{2\pi i} \int_{C_R} e^{st} F(s) ds = \frac{1}{2\pi i} \int_{\pi/2}^{3\pi/2} e^{\gamma t + Rte^{i\theta}} F(\gamma + Re^{i\theta}) Rie^{i\theta} d\theta, \text{ whence}$$

$$\frac{1}{2\pi} \left| \int_{C_R} e^{st} F(s) ds \right| \le \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \left| e^{\gamma t + Rte^{i\theta}} \right| \left| F(\gamma + Re^{i\theta}) \right| \left| Rie^{i\theta} \right| d\theta.$$

Proof of the Inverse Laplace Transform Theorem

• We examine the three moduli involved:

•
$$\left| e^{\gamma t + Rte^{i\theta}} \right| = \left| e^{\gamma t} e^{Rt(\cos\theta + i\sin\theta)} \right| = e^{\gamma t} e^{Rt\cos\theta}.$$

- For |s| sufficiently large, we can write $|Rie^{i\theta}| = |s - \gamma||i| \le |s| + |\gamma| < |s| + |s| = 2|s|.$
- Finally, by hypothesis, |sF(s)| < M.

Thus, we get
$$\frac{1}{2\pi} \left| \int_{C_R} e^{st} F(s) ds \right| \leq \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \left| e^{\gamma t + Rte^{i\theta}} \right| \left| F(\gamma + Re^{i\theta}) \right| \left| Rie^{i\theta} \right| d\theta \leq \frac{M}{\pi} e^{\gamma t} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta} d\theta.$$

Let $\theta = \phi + \frac{\pi}{2}$ and notice that the integral becomes
 $\int_0^{\pi} e^{-Rt\sin\phi} d\phi = 2 \int_0^{\pi/2} e^{-Rt\sin\phi} d\phi.$ We have $\sin\phi \geq \frac{2\phi}{\pi}$, whence
 $2 \int_0^{\pi/2} e^{-Rt\sin\phi} d\phi \leq 2 \int_0^{\pi/2} e^{-2Rt\phi/\pi} d\phi = -\frac{\pi}{Rt} e^{-2Rt\phi/\pi} \Big|_0^{\pi/2} = \frac{\pi}{Rt} [1 - e^{-Rt}].$ We conclude that $\frac{1}{2\pi} \left| \int_{C_R} e^{st} F(s) ds \right| \leq \frac{Me^{\gamma t}}{Rt} [1 - e^{-Rt}].$
The right-hand side approaches zero as $R \to \infty$ for $t > 0$, whence
 $\lim_{R\to\infty} \int_{C_R} e^{st} F(s) ds = 0.$

An Inverse Laplace Transform

• Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$, $\operatorname{Re}(s) > 0$.

The function $F(s) = \frac{1}{s^3}$ has a pole of order 3 at s = 0. Thus, by the theorem,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{\frac{1}{s^3}\} \\ &= \operatorname{Res}(e^{st}\frac{1}{s^3}, 0) \\ &= \frac{1}{2}\lim_{s \to 0} \frac{d^2}{ds^2}(s-0)^3 \frac{e^{st}}{s^3} \\ &= \frac{1}{2}\lim_{s \to 0} \frac{d^2}{ds^2}e^{st} \\ &= \frac{1}{2}\lim_{s \to 0} t^2 e^{st} \\ &= \frac{1}{2}t^2. \end{aligned}$$

Fourier Transform

- Suppose now that f(x) is a real function defined on the interval (-∞,∞).
- Another important transform pair consists of
 - the Fourier transform

$$\mathfrak{F}{f(x)} = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = F(\alpha).$$

• the inverse Fourier transform

$$\mathfrak{F}^{-1}{F(\alpha)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha = f(x).$$

- The kernel of the Fourier transform is K(α, x) = e^{iαx}, whereas the kernel of the inverse transform is H(α, x) = e^{-iαx}/2π.
- We assume that α is a real variable.
- In contrast to the Laplace case, the inverse Fourier transform is not a contour integral.

Computing a Fourier Transform

• Find the Fourier transform of $f(x) = e^{-|x|}$.

We have
$$f(x) = \begin{cases} e^x, & \text{if } x < 0\\ e^{-x}, & \text{if } x \ge 0 \end{cases}$$
. The Fourier transform of f is $\mathfrak{F}\{f(x)\} = \int_{-\infty}^{0} e^x e^{i\alpha x} dx + \int_{0}^{\infty} e^{-x} e^{i\alpha x} dx = l_1 + l_2.$
• For l_2 , we have $l_2 = \lim_{b \to \infty} \int_{0}^{b} e^{-x} e^{i\alpha x} dx = \lim_{b \to \infty} \int_{0}^{b} e^{-x(1-\alpha i)} dx = \lim_{b \to \infty} \frac{e^{-x(1-\alpha i)}}{\alpha i - 1} \Big|_{0}^{b} = \lim_{b \to \infty} \frac{e^{-b(1-\alpha i)} - 1}{\alpha i - 1} = \frac{1}{\alpha i - 1} \lim_{b \to \infty} (e^{-b} \cos b\alpha + ie^{-b} \sin b\alpha - 1) = \frac{1}{1 - \alpha i}.$
• The integral l_1 can be evaluate similarly to obtain $l_1 = \frac{1}{1 + \alpha i}$.
Adding l_1 and l_2 gives the value of the Fourier transform:

$$\mathcal{F}{f(x)} = \frac{1}{1-\alpha i} + \frac{1}{1+\alpha i} = \frac{2}{1+\alpha^2}.$$

Computing an Inverse Fourier Transform

• Find the inverse Fourier transform of $F(\alpha) = \frac{2}{1+\alpha^2}$.

The idea here is to recover the function f of the preceding example. We have $\mathfrak{F}^{-1}{F(\alpha)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^2} e^{-i\alpha x} d\alpha = f(x).$



Let z be a complex variable and introduce the contour integral $\oint_C \frac{1}{\pi(1+z^2)}e^{-izx}dz$. The integrand has simple poles at $z = \pm i$. The contour C is shown in the figure.

We get $\oint_C \frac{1}{\pi(1+z^2)} e^{-izx} dz = 2\pi i \operatorname{Res}(\frac{1}{\pi(1+z^2)} e^{-izx}, i) = e^x$. The contour integral along C_R approaches zero as $R \to \infty$ only if we assume that x < 0. Thus, the answer is $e^x, x < 0$.

Computing an Inverse Fourier Transform (Cont'd)



If we consider $\oint_C \frac{1}{\pi(1+z^2)}e^{-izx}dz$, where *C* is the contour on the left, it can be shown that the integral along C_R now approaches zero as $R \to \infty$ when *x* is assumed to be positive. Hence, $\oint_C \frac{1}{\pi(1+z^2)}e^{-izx}dz =$

 $-2\pi i \operatorname{Res}\left(\frac{1}{\pi(1+z^2)}e^{-izx}, -i\right) = e^{-x}, x > 0.$ The extra minus sign appearing in front of the factor $2\pi i$ comes from the fact that on C, $\int_C = \int_{C_R} + \int_R^{-R} = \int_{C_R} - \int_{-R}^{R} = 2\pi i \operatorname{Res}(z = -i).$ As $R \to \infty$, $\int_{C_R} \to 0$, for x > 0, whence $-\lim_{R\to\infty} \int_{-R}^{R} = 2\pi i \operatorname{Res}(z = -i)$ or $\lim_{R\to\infty} \int_{-R}^{R} = -2\pi i \operatorname{Res}(z = -i).$

• By combining the findings, we get

$$\mathfrak{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^2} e^{-i\alpha x} d\alpha = \begin{cases} e^x, & \text{if } x < 0\\ e^{-x}, & \text{if } x > 0 \end{cases}$$