# Introduction to Complex Analysis 

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(1) Consequences and Applications of the Residue Theorem

- Evaluation of Real Trigonometric Integrals
- Evaluation of Real Improper Integrals
- Integration along a Branch Cut
- The Argument Principle and Rouché's Theorem
- Summing Infinite Series
- Laplace and Fourier Transforms


## Overview of Consequences of the Residue Theorem

- The residue theory can be used to evaluate real integrals of the forms
- $\int_{0}^{2 \pi} F(\cos \theta, \sin \theta) d \theta$;
- $\int_{-\infty}^{\infty} f(x) d x$;
- $\int_{-\infty}^{\infty} f(x) \cos \alpha x d x$;
- $\int_{-\infty}^{\infty} f(x) \sin \alpha x d x$.

Here $F$ and $f$ are rational functions of the form $f(x)=\frac{p(x)}{q(x)}$ in which the polynomials $p$ and $q$ are assumed to have no common factors.

- Residues can be used to evaluate real improper integrals that require integration along a branch cut.
- A relationship exists between the residue theory and the zeros of an analytic function.
- Residues can, in certain cases, be used to find the sum of an infinite series.


## Subsection 1

## Evaluation of Real Trigonometric Integrals

## Integrals of the Form $\int_{0}^{2 \pi} F(\cos \theta, \sin \theta) d \theta$

- The basic idea is to convert those into a complex integral, where the contour $C$ is the unit circle $|z|=1$ centered at the origin.
- To do this we parametrize this contour by $z=e^{i \theta}, 0 \leq \theta \leq 2 \pi$. We write $d z=i e^{i \theta} d \theta, \cos \theta=\frac{e^{i \theta}+e^{-i \theta}}{2}, \sin \theta=\frac{e^{i \theta}-e^{-i \theta}}{2 i}$. Since $d z=i e^{i \theta} d \theta=i z d \theta$ and $z^{-1}=\frac{1}{z}=e^{-i \theta}$, these three quantities are equivalent to $d \theta=\frac{d z}{i z}, \cos \theta=\frac{1}{2}\left(z+z^{-1}\right), \sin \theta=\frac{1}{2 i}\left(z-z^{-1}\right)$. The conversion of the given integral into a contour integral is

$$
\oint_{C} F\left(\frac{1}{2}\left(z+z^{-1}\right), \frac{1}{2 i}\left(z-z^{-1}\right)\right) \frac{d z}{i z},
$$

where $C$ is the unit circle $|z|=1$.

## A Real Trigonometric Integral

- Evaluate $\int_{0}^{2 \pi} \frac{1}{(2+\cos \theta)^{2}} d \theta$.
- We use the substitutions: $\oint_{C} \frac{1}{\left(2+\frac{1}{2}\left(z+z^{-1}\right)\right)^{2}} \frac{d z}{i z}=\oint_{C} \frac{1}{\left(2+\frac{z^{2}+1}{2 z}\right)^{2}} \frac{d z}{i z}$. Simplifying, $\frac{4}{i} \oint_{C} \frac{z}{\left(z^{2}+4 z+1\right)^{2}} d z$. Factoring the denominator $z^{2}+4 z+1=\left(z-z_{1}\right)\left(z-z_{2}\right)$, where $z_{1}=-2-\sqrt{3}$ and $z_{2}=-2+\sqrt{3}$. Thus, $\frac{z}{\left(z^{2}+4 z+1\right)^{2}}=\frac{z}{\left(z-z_{1}\right)^{2}\left(z-z_{2}\right)^{2}}$. Only $z_{2}$ is inside the unit circle $C$. Thus, we have $\oint_{C} \frac{z}{\left(z^{2}+4 z+1\right)^{2}} d z=2 \pi i \operatorname{Res}\left(f(z), z_{2}\right)$. To calculate the residue, note that $z_{2}$ is a pole of order 2 : $\operatorname{Res}\left(f(z), z_{2}\right)=\lim _{z \rightarrow z_{2}} \frac{d}{d z}\left(z-z_{2}\right)^{2} f(z)=\lim _{z \rightarrow z_{2}} \frac{d}{d z} \frac{z}{\left(z-z_{1}\right)^{2}}=$ $\lim _{z \rightarrow z_{2}} \frac{-z-z_{1}}{\left(z-z_{1}\right)^{3}}=\frac{1}{6 \sqrt{3}}$.
Hence, $\frac{4}{i} \oint_{C} \frac{z}{\left(z^{2}+4 z+1\right)^{2}} d z=\frac{4}{i} \cdot 2 \pi i \operatorname{Res}\left(f(z), z_{1}\right)=\frac{4}{i} \cdot 2 \pi i \cdot \frac{1}{6 \sqrt{3}}$
and, finally, $\int_{0}^{2 \pi} \frac{1}{(2+\cos \theta)^{2}} d \theta=\frac{4 \pi}{3 \sqrt{3}}$.


## Subsection 2

## Evaluation of Real Improper Integrals

## Integrals of the Form $\int_{-\infty}^{\infty} f(x) d x$

- Suppose $y=f(x)$ is a real function that is defined and continuous on the interval $[0, \infty)$.
- In elementary calculus the improper integral $I_{1}=\int_{0}^{\infty} f(x) d x$ is defined as the limit $I_{1}=\int_{0}^{\infty} f(x) d x=\lim _{R \rightarrow \infty} \int_{0}^{R} f(x) d x$. If the limit exists, the integral $I_{1}$ is said to be convergent; otherwise, it is divergent.
- The improper integral $I_{2}=\int_{-\infty}^{0} f(x) d x$ is defined similarly: $I_{2}=\int_{-\infty}^{0} f(x) d x=\lim _{R \rightarrow \infty} \int_{-R}^{0} f(x) d x$.
- Finally, if $f$ is continuous on $(-\infty, \infty)$, then

$$
\int_{-\infty}^{\infty} f(x) d x=\int_{-\infty}^{0} f(x) d x+\int_{0}^{\infty} f(x) d x=I_{1}+l_{2}
$$

provided both integrals $I_{1}$ and $I_{2}$ are convergent. If either one, $I_{1}$ or $I_{2}$, is divergent, then $\int_{-\infty}^{\infty} f(x) d x$ is divergent.

## Cauchy Principal Value of $\int_{-\infty}^{\infty} f(x) d x$

- It is important to remember that $\lim _{R \rightarrow \infty} \int_{-R}^{0} f(x) d x+\lim _{R \rightarrow \infty} \int_{0}^{R} f(x) d x$ is not the same as $\lim _{R \rightarrow \infty}\left(\int_{-R}^{0} f(x) d x+\int_{0}^{R} f(x) d x\right)=\lim _{R \rightarrow \infty} \int_{-R}^{R} f(x) d x$.
- For the integral $\int_{-\infty}^{\infty} f(x) d x$ to be convergent, the limits $\lim _{R \rightarrow \infty} \int_{-R}^{0} f(x) d x$ and $\lim _{R \rightarrow \infty} \int_{0}^{R} f(x) d x$ must exist independently of one another.
- In the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x) d x$ converges, we can then evaluate it by means of the single limiting process $\int_{-\infty}^{\infty} f(x) d x=\lim _{R \rightarrow \infty} \int_{-R}^{R} f(x) d x$.
- On the other hand, the symmetric limit may exist even though the improper integral $\int_{-\infty}^{\infty} f(x) d x$ is divergent.
- The limit $\lim _{R \rightarrow \infty} \int_{-R}^{R} f(x) d x$, if it exists, is called the Cauchy principal value (P.V.) of the integral and is written P.V. $\int_{-\infty}^{\infty} f(x) d x=\lim _{R \rightarrow \infty} \int_{-R}^{R} f(x) d x$.


## Principal Value and Integrals of Even Functions

- Suppose $f(x)$ is continuous on $(-\infty, \infty)$ and is an even function, i.e., $f(-x)=f(x)$. Then its graph is symmetric with respect to the $y$-axis. As a consequence, $\int_{-R}^{0} f(x) d x=\int_{0}^{R} f(x) d x$. Therefore, $\int_{-R}^{R} f(x) d x=\int_{-R}^{0} f(x) d x+\int_{0}^{R} f(x) d x=2 \int_{0}^{R} f(x) d x$. If the Cauchy principal value exists, $\int_{0}^{\infty} f(x) d x$ and $\int_{-\infty}^{\infty} f(x) d x$ converge. The values of the integrals are

$$
\int_{0}^{\infty} f(x) d x=\frac{1}{2} \text { P.V. } \int_{-\infty}^{\infty} f(x) d x
$$

and

$$
\int_{-\infty}^{\infty} f(x) d x=\text { P.V. } \int_{-\infty}^{\infty} f(x) d x
$$

## Evaluation of Integral $\int_{-\infty}^{\infty} f(x) d x$

- To evaluate $\int_{-\infty}^{\infty} f(x) d x$, where the rational function $f(x)=\frac{p(x)}{q(x)}$ is continuous on $(-\infty, \infty)$, we replace $x$ by the complex variable $z$ and integrate the complex function $f$ over a closed contour $C$ that consists of the interval $[-R, R]$ on the real axis and a semicircle $C_{R}$ of radius large enough to enclose all the poles of $f(z)=\frac{p(z)}{q(z)}$ in the upper half-plane
 $\operatorname{Im}(z)>0$.
Then,
$\oint_{C} f(z) d z=\int_{C_{R}} f(z) d z+\int_{-R}^{R} f(x) d x=2 \pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_{k}\right)$, where $z_{k}, k=1,2, \ldots, n$ denotes poles in the upper half-plane. If we can show that the $\int_{C_{R}} f(z) d z \rightarrow 0$ as $R \rightarrow \infty$, then we have P.V. $\int_{-\infty}^{\infty} f(x) d x=\lim _{R \rightarrow \infty} \int_{-R}^{R} f(x) d x=2 \pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_{k}\right)$.


## Cauchy P.V. of an Improper Integral

- Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{\left(x^{2}+1\right)\left(x^{2}+9\right)} d x$. Let $f(z)=\frac{1}{\left(z^{2}+1\right)\left(z^{2}+9\right)}$.
Since $\left(z^{2}+1\right)\left(z^{2}+9\right)=(z-i)(z+i)(z-$ $3 i)(z+3 i)$, we take $C$ be the closed contour consisting of the interval $[-R, R]$ on the $x-$ axis and the semicircle $C_{R}$ of radius $R>3$.
Then,
 $\oint_{C} \frac{1}{\left(z^{2}+1\right)\left(z^{2}+9\right)} d z=\int_{-R}^{R} \frac{1}{\left(x^{2}+1\right)\left(x^{2}+9\right)} d x+\int_{C_{R}} \frac{1}{\left(z^{2}+1\right)\left(z^{2}+9\right)} d z=I_{1}+I_{2}$ and $I_{1}+I_{2}=2 \pi i[\operatorname{Res}(f(z), i)+\operatorname{Res}(f(z), 3 i)]$. At the simple poles $z=i$ and $z=3 i$ we find $\operatorname{Res}(f(z), i)=\frac{1}{16 i}$ and $\operatorname{Res}(f(z), 3 i)=-\frac{1}{48 i}$, whence $I_{1}+I_{2}=2 \pi i\left[\frac{1}{16 i}+\left(-\frac{1}{48 i}\right)\right]=\frac{\pi}{12}$.


## Letting $R \rightarrow \infty$

- $\oint_{C} \frac{1}{\left(z^{2}+1\right)\left(z^{2}+9\right)} d z=\int_{-R}^{R} \frac{1}{\left(x^{2}+1\right)\left(x^{2}+9\right)} d x+\int_{C_{R}} \frac{1}{\left(z^{2}+1\right)\left(z^{2}+9\right)} d z=\frac{\pi}{12}$.

Before letting $R \rightarrow \infty$, note that $\left|\left(z^{2}+1\right)\left(z^{2}+9\right)\right|=$ $\left|z^{2}+1\right| \cdot\left|z^{2}+9\right| \geq\left|\left|z^{2}\right|-1\right| \cdot| | z^{2}|-9|=\left(R^{2}-1\right)\left(R^{2}-9\right)$. Since the length $L$ of the semicircle is $\pi R$, it follows, by the $M L$-inequality, $\left|I_{2}\right|=\left|\int_{C_{R}} \frac{1}{\left(z^{2}+1\right)\left(z^{2}+9\right)} d z\right| \leq \frac{\pi R}{\left(R^{2}-1\right)\left(R^{2}-9\right)}$. Hence, $\left|I_{2}\right| \rightarrow 0$ as $R \rightarrow \infty$, and we conclude that $\lim _{R \rightarrow \infty} I_{2}=0$. It follows that $\lim _{R \rightarrow \infty} \iota_{1}=\frac{\pi}{12}$. I.e., $\lim _{R \rightarrow \infty} \int_{-R}^{R} \frac{1}{\left(x^{2}+1\right)\left(x^{2}+9\right)} d x=\frac{\pi}{12}$ or
P.V. $\int_{-\infty}^{\infty} \frac{1}{\left(x^{2}+1\right)\left(x^{2}+9\right)} d x=\frac{\pi}{12}$.

## Behavior of Integral as $R \rightarrow \infty$

- To show that the contour integral along $C_{R}$ approaches zero as $R \rightarrow \infty$ the following sufficient conditions are useful:


## Theorem (Behavior of Integral as $R \rightarrow \infty$ )

Suppose $f(z)=\frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is $n$ and the degree of $q(z)$ is $m \geq n+2$. If $C_{R}$ is a semicircular contour $z=R e^{i \theta}, 0 \leq \theta \leq \pi$, then $\int_{C_{R}} f(z) d z \rightarrow 0$ as $R \rightarrow \infty$.

- In other words, the integral along $C_{R}$ approaches zero as $R \rightarrow \infty$ when the denominator of $f$ is of a power at least 2 more than its numerator.
- The proof of this fact follows as in the preceding example, in which degree of $p(z)=1$ is 0 and degree of $q(z)=\left(z^{2}+1\right)\left(z^{2}+9\right)$ is 4 .


## Another Cauchy P.V. of an Improper Integral

- Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^{4}+1} d x$.

The conditions given in the preceding theorem are satisfied. Moreover, $f(z)=\frac{1}{z^{4}+1}$ has simple poles in the upper half-plane at $z_{1}=e^{\pi i / 4}$ and $z_{2}=e^{3 \pi i / 4}$. We have seen the residues at these poles are

$$
\operatorname{Res}\left(f(z), z_{1}\right)=-\frac{1}{4 \sqrt{2}}-\frac{1}{4 \sqrt{2}} i \quad \text { and } \quad \operatorname{Res}\left(f(z), z_{2}\right)=\frac{1}{4 \sqrt{2}}-\frac{1}{4 \sqrt{2}} i
$$

Thus,

$$
\text { P.V. } \int_{-\infty}^{\infty} \frac{1}{x^{4}+1} d x=2 \pi i\left[\operatorname{Res}\left(f(z), z_{1}\right)+\operatorname{Res}\left(f(z), z_{2}\right)\right]=\frac{\pi}{\sqrt{2}}
$$

Since the integrand is an even function, the original integral converges to $\frac{\pi}{\sqrt{2}}$.

## Integrals $\int_{-\infty}^{\infty} f(x) \sin \alpha x d x$ and $\int_{-\infty}^{\infty} f(x) \cos \alpha x d x$

- Integrals of Form $\int_{-\infty}^{\infty} f(x) \sin \alpha x d x$ and $\int_{-\infty}^{\infty} f(x) \cos \alpha x d x$ are referred to as Fourier integrals.
- They appear as the real and imaginary parts of $\int_{-\infty}^{\infty} f(x) e^{i \alpha x} d x$.
- Suppose $f(x)=\frac{p(x)}{q(x)}$ is a rational function continuous on $(-\infty, \infty)$. Then both Fourier integrals can be evaluated by considering the complex integral $\oint_{C} f(z) e^{i \alpha z} d z$, where $\alpha>0$, and the contour $C$ consists of $[-R, R]$ and a semicircular contour $C_{R}$ with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.
- Sufficient conditions under which the contour integral along $C_{R}$ approaches zero as $R \rightarrow \infty$ are given by


## Theorem (Behavior of Integral as $R \rightarrow \infty$ )

Suppose $f(z)=\frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is $n$ and the degree of $q(z)$ is $m \geq n+2$. If $C_{R}$ is a semicircular contour $z=R e^{i \theta}, 0 \leq \theta \leq \pi$, and $\alpha>0$, then $\int_{C_{R}} f(z) e^{i \alpha z} d z \rightarrow 0$ as $R \rightarrow \infty$.

## Evaluating a Fourier Integral

- Evaluate the Cauchy principal value of $\int_{0}^{\infty} \frac{x \sin x}{x^{2}+9} d x$.

First note that the limits of integration in the given integral are not from $-\infty$ to $\infty$ as required by the method just described. Since the integrand is an even function of $x, \int_{0}^{\infty} \frac{x \sin x}{x^{2}+9} d x=\frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^{2}+9} d x$. We now form the contour integral $\oint_{C} \frac{z}{z^{2}+9} e^{i z} d z$, where $C$ is the contour described before, with $R>3$. We have $\int_{C_{R}} \frac{z}{z^{2}+9} e^{i z} d z+\int_{-R}^{R} \frac{x}{x^{2}+9} e^{i x} d x=2 \pi i \operatorname{Res}\left(f(z) e^{i z}, 3 i\right)$, where $f(z)=\frac{z}{z^{2}+9}$, and $\operatorname{Res}\left(f(z) e^{i z}, 3 i\right)=\left.\frac{z e^{i z}}{z+3 i}\right|_{z=3 i}=\frac{e^{-3}}{2}$. Since, by the theorem, $\int_{C_{R}} f(z) e^{i z} d z \rightarrow 0$ as $R \rightarrow \infty$, we get
P.V. $\int_{-\infty}^{\infty} \frac{x}{x^{2}+9} e^{i x} d x=2 \pi i\left(\frac{e^{-3}}{2}\right)=\frac{\pi}{e^{3}} i$. Note that $\int_{-\infty}^{\infty} \frac{x}{x^{2}+9} e^{i x} d x=$ $\int_{-\infty}^{\infty} \frac{x \cos x}{x^{2}+9} d x+i \int_{-\infty}^{\infty} \frac{x \sin x}{x^{2}+9} d x=\frac{\pi}{e^{3}} i$. Equating real and imaginary parts: P.V. $\int_{-\infty}^{\infty} \frac{x \cos x}{x^{2}+9} d x=0$ and P.V. $\int_{-\infty}^{\infty} \frac{x \sin x}{x^{2}+9} d x=\frac{\pi}{e^{3}}$. This implies that $\int_{0}^{\infty} \frac{x \sin x}{x^{2}+9} d x=\frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^{2}+9} d x=\frac{\pi}{2 e^{3}}$.

## Indented Contours

- Up to this point we considered improper integrals of functions continuous on the interval $(-\infty, \infty)$, i.e., the complex function $f(z)=\frac{p(z)}{q(z)}$ did not have poles on the real axis.
- Suppose we want to evaluate $\int_{-\infty}^{\infty} f(x) d x$ by residues when $f(z)$ has a pole at $z=c$, where $c$ is a real number. Then we use an indented contour: The symbol $C_{r}$ denotes a semicircular contour centered at $z=c$ and oriented in the positive direction.



## Theorem (Behavior of Integral as $r \rightarrow 0$ )

Suppose $f$ has a simple pole $z=c$ on the real axis. If $C_{r}$ is the contour defined by $z=c+r e^{i \theta}, 0 \leq \theta \leq \pi$, then

$$
\lim _{r \rightarrow 0} \int_{C_{r}} f(z) d z=\pi i \operatorname{Res}(f(z), c)
$$

## Proof of the Theorem

- Since $f$ has a simple pole at $z=c$, its Laurent series is

$$
f(z)=\frac{a_{-1}}{z-c}+g(z)
$$

where $a_{-1}=\operatorname{Res}(f(z), c)$ and $g$ is analytic at the point $c$. Using the Laurent series and the parametrization of $C_{r}$, we have

$$
\int_{C_{r}} f(z) d z=a_{-1} \int_{0}^{\pi} \frac{i r e^{i \theta}}{r e^{i \theta}} d \theta+i r \int_{0}^{\pi} g\left(c+r e^{i \theta}\right) e^{i \theta} d \theta=I_{1}+I_{2} .
$$

- $I_{1}=a_{-1} \int_{0}^{\pi} \frac{i r e^{i \theta}}{r e e^{i \theta}} d \theta=a_{-1} \int_{0}^{\pi} i d \theta=\pi i a_{-1}=\pi i \operatorname{Res}(f(z), c)$.
- Since $g$ is analytic at $c$, it is continuous at this point and bounded in a neighborhood of the point. I.e., there exists an $M>0$ for which $\left|g\left(c+r e^{i \theta}\right)\right| \leq M$. Hence,

$$
\left|I_{2}\right|=\left|i r \int_{0}^{\pi} g\left(c+r e^{i \theta}\right) d \theta\right| \leq r \int_{0}^{\pi} M d \theta=\pi r M .
$$

It follows that $\lim _{r \rightarrow 0}\left|I_{2}\right|=0$ and, consequently, $\lim _{r \rightarrow 0} I_{2}=0$.
By taking the limit of the sum as $r \rightarrow 0$, we get the conclusion.

## Using an Indented Contour

- Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\sin x}{x\left(x^{2}-2 x+2\right)} d x$.

We consider $\oint_{C} \frac{e^{i z}}{z\left(z^{2}-2 z+2\right)} d z$.
$f(z)=\frac{1}{z\left(z^{2}-2 z+2\right)}$ has a pole at $z=0$ and at $z=1+i$ in the upper half-plane. The contour $C$, is indented at the origin. We have $\oint_{C}=\int_{C_{R}}+\int_{-R}^{-r}+\int_{-C_{r}}+\int_{r}^{R}=$ $2 \pi i \operatorname{Res}\left(f(z) e^{i z}, 1+i\right), \int_{-C_{r}}=-\int_{C_{r}}$.
 If we take the limits as $R \rightarrow \infty$ and as $r \rightarrow 0$,
P.V. $\int_{-\infty}^{\infty} \frac{e^{i x}}{x\left(x^{2}-2 x+2\right)} d x-\pi i \operatorname{Res}\left(f(z) e^{i z}, 0\right)=2 \pi i \operatorname{Res}\left(f(z) e^{i z}, 1+i\right)$.

Now, $\operatorname{Res}\left(f(z) e^{i z}, 0\right)=\frac{1}{2}$ and $\operatorname{Res}\left(f(z) e^{i z}, 1+i\right)=-\frac{e^{-1+i}}{4}(1+i)$.
Therefore, P.V. $\int_{-\infty}^{\infty} \frac{e^{i x}}{x\left(x^{2}-2 x+2\right)} d x=\pi i \frac{1}{2}+2 \pi i\left(-\frac{e^{-1+i}}{4}(1+i)\right)$. Using $e^{-1+i}=e^{-1}(\cos 1+i \sin 1)$ and equating real and imaginary parts:
P.V. $\int_{-\infty}^{\infty} \frac{\cos x}{x\left(x^{2}-2 x+2\right)} d x=\frac{\pi}{2} e^{-1}(\sin 1+\cos 1)$,
P.V. $\int_{-\infty}^{\infty} \frac{\sin x}{x\left(x^{2}-2 x+2\right)} d x=\frac{\pi}{2}\left[1+e^{-1}(\sin 1-\cos 1)\right]$.

## Subsection 3

## Integration along a Branch Cut

## Branch Point at $z=0$

- Suppose that, if $f(x)$ is converted to a complex function, $f(z)$ has, in addition to poles, a nonisolated singularity at $z=0$.
- In that case, computing $\int_{0}^{\infty} f(x) d x$ requires a special type of contour.
- Example: Consider the real integral $\int_{0}^{\infty} \frac{x^{\alpha-1}}{x+1} d x$, (21) where $\alpha$ is a real constant restricted to the interval $0<\alpha<1$. When $\alpha=\frac{1}{2}$ and $x$ is replaced by $z$, the integrand becomes the multiple-valued function $\frac{1}{z^{1 / 2}(z+1)}$. The origin is a branch point because $z^{1 / 2}$ has two values for any $z \neq 0$. Traveling in a complete circle around the origin $z=0$, starting from a point $z=r e^{i \theta}, r>0$, we return to the same starting point $z$, but $\theta$ has increased by $2 \pi$. Thus, the value of $z^{1 / 2}$ changes from $z^{1 / 2}=\sqrt{r} e^{i \theta / 2}$ to a different value or different branch: $z^{1 / 2}=\sqrt{r} e^{i(\theta+2 \pi) / 2}=\sqrt{r} e^{i \theta / 2} e^{i \pi}=-\sqrt{r} e^{i \pi / 2}$.
Recall, we can force $z^{1 / 2}$ to be single valued by restricting $\theta$ to some interval of length $2 \pi$. E.g., by restricting $\theta$ to $0<\theta<2 \pi$, we guarantee that $z^{1 / 2}=\sqrt{r} e^{i \theta / 2}$ is single valued.


## Integration along a Branch Cut

- Evaluate $\int_{0}^{\infty} \frac{1}{\sqrt{x}(x+1)} d x$.

The real integral is improper for two reasons:

- There is an infinite discontinuity at $x=0$;
- The limit of integration is infinite.

We form the integral $\int_{C} \frac{1}{z^{1 / 2}(z+1)} d z$, where $C$ is the contour shown, which consists of

- $C_{r}$ and $C_{R}$, which are portions of circles;
- $A B$ and $E D$, which are parallel horizontal line segments running along opposite sides of the branch cut.


The integrand $f(z)$ of the contour integral is single valued and analytic on and within $C$, except for the simple pole at $z=-1=e^{\pi i}$. Hence, we can write $\oint_{C} \frac{1}{z^{1 / 2}(z+1)} d z=2 \pi i \operatorname{Res}(f(z),-1)$ or

$$
\int_{C_{R}}+\int_{E D}+\int_{C_{r}}+\int_{A B}=2 \pi i \operatorname{Res}(f(z),-1) .
$$

## Integration along a Branch Cut (Cont'd)

- We think of $A B$ as coinciding with the upper side of the positive real axis for which $\theta=0$ and of $E D$ with the lower side of the positive real axis for which $\theta=2 \pi$.
On $A B, z=x e^{0 i}$;
On $E D, z=x e^{(0+2 \pi) i}=x e^{2 \pi i}$; Thus,
$\int_{E D}=\int_{R}^{r} \frac{\left(x e^{2 \pi i}\right)^{-1 / 2}}{x e^{2 \pi i}+1}\left(e^{2 \pi i} d x\right)=-\int_{R}^{r} \frac{x^{-1 / 2}}{x+1} d x=\int_{r}^{R} \frac{x^{-1 / 2}}{x+1} d x$ and
$\int_{A B}=\int_{r}^{R} \frac{\left(x e^{0 i}\right)^{-1 / 2}}{x e^{0 i}+1}\left(e^{0 i} d x\right)=\int_{r}^{R} \frac{x^{-1 / 2}}{x+1} d x$.
Now with $z=r e^{i \theta}$ and $z=R e^{i \theta}$ on $C_{r}$ and $C_{R}$, respectively, it can be shown that $\int_{C_{r}} \rightarrow 0$ as $r \rightarrow 0$ and $\int_{C_{R}} \rightarrow 0$ as $R \rightarrow \infty$. Thus, $\lim _{\substack{r \rightarrow 0 \\ R \rightarrow \infty}}\left[\int_{C_{R}}+\int_{E D}+\int_{C_{r}}+\int_{A B}=2 \pi i \operatorname{Res}(f(z),-1)\right]$ is the same as $2 \int_{0}^{\infty} \frac{1}{\sqrt{x}(x+1)} d x=2 \pi i \operatorname{Res}(f(z),-1)$. Since $\operatorname{Res}(f(z),-1)=\left.z^{-1 / 2}\right|_{z=e^{\pi i}}=e^{-\pi i / 2}=-i, \int_{0}^{\infty} \frac{1}{\sqrt{x}(x+1)} d x=\pi$.


## Subsection 4

## The Argument Principle and Rouché's Theorem

## Number of Zeros and Poles

- We apply residue theory to the location of zeros of an analytic function.
- In the first theorem we need to count the number of zeros and poles of a function $f$ that are located within a simple closed contour $C$, taking into account the order or multiplicity of each zero and pole.
- Example: If $f(z)=\frac{(z-1)(z-9)^{4}(z+i)^{2}}{\left(z^{2}-2 z+2\right)^{2}(z-i)^{6}(z+6 i)^{7}}$ and $C$ is taken to be the circle $|z|=2$, then:
- Inspection of the numerator of $f$ reveals that the zeros inside $C$ are $z=1$ (a simple zero) and $z=-i$ (a zero of order or multiplicity 2 ). Therefore, the number $N_{0}$ of zeros inside $C$ is taken to be $N_{0}=1+2=3$.
- Similarly, inspection of the denominator of $f$ shows, after factoring $z^{2}-2 z+2=(z-1-i)(z-1+i)$, that the poles inside $C$ are $z=1-i$ (pole of order 2), $z=1+i$ (pole of order 2), and $z=i$ (pole of order 6). The number $N_{p}$ of poles inside $C$ is taken to be $N_{p}=2+2+6=10$.


## Argument Principle

## Theorem (Argument Principle)

Let $C$ be a simple closed contour lying entirely within a domain $D$. Suppose $f$ is analytic in $D$ except at a finite number of poles inside $C$, and that $f(z) \neq 0$ on $C$. Then $\frac{1}{2 \pi i} \oint_{C} \frac{f^{\prime}(z)}{f(z)} d z=N_{0}-N_{p}$, where $N_{0}$ is the total number of zeros of $f$ inside $C$ and $N_{p}$ is the total number of poles of $f$ inside $C$, counting their order or multiplicities.

- The integrand $\frac{f^{\prime}(z)}{f(z)}$ is analytic in and on the contour $C$ except at the points in the interior of $C$ where $f$ has a zero or a pole. If $z_{0}$ is a zero of order $n$ of $f$ inside $C$, then we can write $f(z)=\left(z-z_{0}\right)^{n} \phi(z)$, where $\phi$ is analytic at $z_{0}$ and $\phi\left(z_{0}\right) \neq 0$. We differentiate $f$ by the product rule, $f^{\prime}(z)=\left(z-z_{0}\right)^{n} \phi^{\prime}(z)+n\left(z-z_{0}\right)^{n-1} \phi(z)$, and divide this expression by $f$. In some punctured disk centered at $z_{0}$, we have $\frac{f^{\prime}(z)}{f(z)}=\frac{\left(z-z_{0}\right)^{n} \phi^{\prime}(z)+n\left(z-z_{0}\right)^{n-1} \phi(z)}{\left(z-z_{0}\right)^{n} \phi(z)}=\frac{\phi^{\prime}(z)}{\phi(z)}+\frac{n}{z-z_{0}}$. Thus, the integrand $\frac{f^{\prime}(z)}{f(z)}$ has a simple pole at $z_{0}$.


## Proof of the Argument Principle

- We found $\frac{f^{\prime}(z)}{f(z)}=\frac{\phi^{\prime}(z)}{\phi(z)}+\frac{n}{z-z_{0}}$. The residue at $z_{0}$ is $\operatorname{Res}\left(\frac{f^{\prime}(z)}{f(z)}, z_{0}\right)=$ $\lim _{z \rightarrow z_{0}}\left(z-z_{0}\right)\left(\frac{\phi^{\prime}(z)}{\phi(z)}+\frac{n}{z-z_{0}}\right)=\lim _{z \rightarrow z_{0}}\left(\frac{\left(z-z_{0}\right) \phi^{\prime}(z)}{\phi(z)}+n\right)=0+n=n$, which is the order of the zero $z_{0}$.
Now if $z_{p}$ is a pole of order $m$ of $f$ within $C$, then $f(z)=\frac{g(z)}{\left(z-z_{p}\right)^{m}}$, where $g$ is analytic at $z_{p}$ and $g\left(z_{p}\right) \neq 0$. By differentiating, $f^{\prime}(z)=\left(z-z_{p}\right)^{-m} g^{\prime}(z)-m\left(z-z_{p}\right)^{-m-1} g(z)$. Therefore, in some punctured disk centered at $z_{p}$,
$\frac{f^{\prime}(z)}{f(z)}=\frac{\left(z-z_{p}\right)^{-m} g^{\prime}(z)-m\left(z-z_{p}\right)^{-m-1} g(z)}{\left(z-z_{p}\right)^{-m} g(z)}=\frac{g^{\prime}(z)}{g(z)}+\frac{-m}{z-z_{p}}$. Thus, $\frac{f^{\prime}(z)}{f(z)}$ has
a simple pole at $z_{p}$. We also see that the residue at $z_{p}$ is equal to $-m$, which is the negative of the order of the pole of $f$.


## Proof of the Argument Principle (Cont'd)

- Finally, suppose that $z_{0_{1}}, z_{0_{2}}, \ldots, z_{0_{r}}$ and $z_{p_{1}}, z_{p_{2}}, \ldots, z_{p_{s}}$ are the zeros and poles of $f$ within $C$ and that the order of the zeros are $n_{1}, n_{2}, \ldots, n_{r}$ and that order of the poles are $m_{1}, m_{2}, \ldots, m_{s}$. Then each of these points is a simple pole of the integrand $\frac{f^{\prime}(z)}{f(z)}$ with corresponding residues $n_{1}, n_{2}, \ldots, n_{r}$ and $-m_{1},-m_{2}, \ldots,-m_{s}$. It follows from the residue theorem that $\oint_{C} \frac{f^{\prime}(z)}{f(z)} d z$ is equal to $2 \pi i$ times the sum of the residues at the poles:
$\oint_{C} \frac{f^{\prime}(z)}{f(z)} d z=2 \pi i\left[\sum_{k=1}^{r} \operatorname{Res}\left(\frac{f^{\prime}(z)}{f(z)}, z_{0_{k}}\right)+\sum_{k=1}^{s} \operatorname{Res}\left(\frac{f^{\prime}(z)}{f(z)}, z_{p_{k}}\right)\right]=$ $2 \pi i\left(\sum_{k=1}^{r} n_{k}+\sum_{k=1}^{s}\left(-m_{k}\right)\right)=2 \pi i\left[N_{0}-N_{p}\right]$.


## Illustrating the Argument Principle

- Suppose the simple closed contour is $|z|=2$ and the function

$$
f(z)=\frac{(z-1)(z-9)^{4}(z+i)^{2}}{\left(z^{2}-2 z+2\right)^{2}(z-i)^{6}(z+6 i)^{7}}
$$

In the evaluation of $\oint_{C} \frac{f^{\prime}(z)}{f(z)} d z$, each zero of $f$ within $C$ contributes
$2 \pi i$ times the order of multiplicity of the zero and each pole contributes $2 \pi i$ times the negative of the order of the pole:
$\oint_{C} \frac{f^{\prime}(z)}{f(z)} d z$
$=[2 \pi i(1)+2 \pi i(2)]+[2 \pi i(-2)+2 \pi i(-2)+2 \pi i(-6)]=-14 \pi i$.

- The name "argument principle" originates from a relation between the number $N_{0}-N_{p}$ and $\arg (f(z))$ : We have

$$
\begin{aligned}
& N_{0}-N_{p}=\frac{1}{2 \pi} \text { [change in } \arg (f(z)) \text { as } z \text { traverses } C \\
& \text { once in the positive direction]. }
\end{aligned}
$$

## Rouché's Theorem

- The following theorem is helpful in determining the number of zeros of an analytic function.


## Theorem (Rouché's Theorem)

Let $C$ be a simple closed contour lying entirely within a domain $D$. Suppose $f$ and $g$ are analytic in $D$. If the strict inequality $|f(z)-g(z)|<|f(z)|$ holds for all $z$ on $C$, then $f$ and $g$ have the same number of zeros, counting their order or multiplicities, inside $C$.

- The hypothesis that $|f(z)-g(z)|<|f(z)|$ holds, for all $z$ on $C$, indicates that both $f$ and $g$ have no zeros on the contour $C$. From $|f(z)-g(z)|=|g(z)-f(z)|$, we see that, by dividing the inequality by $|f(z)|$. we have, for all $z$ on $C,|F(z)-1|<1$, where $F(z)=\frac{g(z)}{f(z)}$.


## Proof of Rouché's Theorem

- We have $|F(z)-1|<1$, where $F(z)=\frac{g(z)}{f(z)}$. This inequality shows that the image $C^{\prime}$ in the $w$-plane of the curve $C$ under the mapping $w=F(z)$ is a closed path and must lie within the unit open disk $|w-1|<1$ centered at $w=1$.


As a consequence, the curve $C^{\prime}$ does not enclose $w=0$, and therefore $\frac{1}{w}$ is analytic in and on $C^{\prime}$. By the Cauchy-Goursat Theorem, $\oint_{C^{\prime}} \frac{1}{w} d w=0$. Since $w=F(z)$ and $d w=F^{\prime}(z) d z$, $\oint_{C} \frac{F^{\prime}(z)}{F(z)} d z=0$. From the quotient rule, $F^{\prime}(z)=\frac{f(z) g^{\prime}(z)-g(z) f^{\prime}(z)}{[f(z)]^{2}}$, we get $\frac{F^{\prime}(z)}{F(z)}=\frac{g^{\prime}(z)}{g(z)}-\frac{f^{\prime}(z)}{f(z)}$. Therefore, $\oint_{C}\left(\frac{g^{\prime}(z)}{g(z)}-\frac{f^{\prime}(z)}{f(z)}\right) d z=0$ or $\oint_{C} \frac{g^{\prime}(z)}{g(z)} d z=\oint_{C} \frac{f^{\prime}(z)}{f(z)} d z$. By the argument principle, the number of zeros of $g$ inside $C$ is the same as the number of zeros of $f$ inside $C$.

## Location of Zeros

- Locate the zeros of the polynomial function $g(z)=z^{9}-8 z^{2}+5$.

We begin by choosing $f(z)=z^{9}$ because it has the same number of zeros as $g$. Since $f$ has a zero of order 9 at $z=0$, we search for the zeros of $g$ by examining circles centered at $z=0$. If we can establish that $|f(z)-g(z)|<|f(z)|$, for all $z$ on some circle $|z|=R$, then Rouché's Theorem asserts that $f$ and $g$ have the same number of zeros within $|z|<R$.
By the triangle inequality, $|f(z)-g(z)|=\left|z^{9}-\left(z^{9}-8 z^{2}+5\right)\right|=$ $\left|8 z^{2}-5\right| \leq 8|z|^{2}+5$. Also, $|f(z)|=|z|^{9}$.
Since $|f(z)-g(z)|<|f(z)|$ or $8|z|^{2}+5<|z|^{9}$ is not true for all $z$ on $|z|=1$, we can draw no conclusion.
By expanding the search to the larger circle $|z|=\frac{3}{2}$, we see $|f(z)-g(z)| \leq 8|z|^{2}+5=8 \cdot\left(\frac{3}{2}\right)^{2}+5=23<\left(\frac{3}{2}\right)^{9}=|f(z)|$. Thus, since $f$ has a zero of order 9 within $|z|<\frac{3}{2}$, all nine zeros of $g$ lie within the same disk.

## Revisiting the Zeros of $g l$

- By more refined reasoning, we can show that $g(z)=z^{9}-8 z^{2}+5$ has some zeros inside $|z|<1$.
To see this suppose we choose $f(z)=-8 z^{2}+5$. Then, for all $z$ on $|z|=1$,
$|f(z)-g(z)|=\left|\left(-8 z^{2}+5\right)-\left(z^{9}-8 z^{2}+5\right)\right|=\left|-z^{9}\right|=|z|^{9}=$ $(1)^{9}=1$.
For all $z$ on $|z|=1$,
$|f(z)|=|-f(z)|=\left|8 z^{2}-5\right| \geq\left.|8| z\right|^{2}-|-5||=|8-5|=3$.
Therefore, for all $z$ on $|z|=1,|f(z)-g(z)|<|f(z)|$.
Because $f$ has two zeros within $|z|<1$ (namely, $\pm \sqrt{\frac{5}{8}}$ ), we can conclude, by Rouché's Theorem, that two zeros of $g$ also lie within this disk.


## Revisiting the Zeros of $g$ II

- Continuing to reason about the zeros of $g(z)=z^{9}-8 z^{2}+5$, suppose we choose $f(z)=5$ and $|z|=\frac{1}{2}$. Then, for all $z$ on $|z|=\frac{1}{2}$, $|f(z)-g(z)|=\left|5-\left(z^{9}-8 z^{2}+5\right)\right|=\left|-z^{9}+8 z^{2}\right| \leq|z|^{9}+8|z|^{2}=$ $\left(\frac{1}{2}\right)^{9}+2 \approx 2.002$.
We now have $|f(z)-g(z)|<|f(z)|=5$, for all $z$ on $|z|=\frac{1}{2}$. Since $f$ has no zeros within the disk $|z|<\frac{1}{2}$, neither does $g$.
At this point we are able to conclude that all nine zeros of $g(z)=z^{9}-8 z^{2}+5$ lie within the annular region $\frac{1}{2}<|z|<\frac{3}{2}$. Moreover, two of these zeros lie within $\frac{1}{2}<|z|<1$.


## Subsection 5

## Summing Infinite Series

## Using cot $\pi z$

- The residues at the simple poles of cot $\pi z$ can help find the sum of an infinite series.
- The zeros of $\sin z$ are the reals $z=k \pi, k=0, \pm 1, \pm 2, \ldots$. Thus, $\cot \pi z$ has simple poles at $\pi z=k \pi$ or $z=k, k=0, \pm 1, \pm 2, \ldots$.
- If a polynomial function $p(z)$ has (i) real coefficients; (ii) degree $n \geq 2$, and (iii) no integer zeros, then the function $f(z)=\frac{\pi \cot \pi z}{p(z)}$ has an infinite number of simple poles $z=0, \pm 1, \pm 2, \ldots$ from $\cot \pi z$ and a finite number of poles $z_{p_{1}}, z_{p_{2}}, \ldots, z_{p_{r}}$ from the zeros of $p(z)$.
- The closed rectangular contour is $C$, where $n$ is taken large enough so that $C$ encloses the simple poles $z=0$, $\pm 1, \pm 2, \ldots, \pm n$ and all of the poles $z_{p_{1}}, z_{p_{2}}, \ldots, z_{p_{r}}$. By the residue theorem,

$\oint_{C} \frac{\pi \cot \pi z}{p(z)} d z=2 \pi i\left(\sum_{k=-n}^{n} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right)+\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_{j}}\right)\right)$.


## Using $\cot \pi z$ (Cont'd)

- Since it can be shown that $\oint_{C} \frac{\pi \cot \pi z}{p(z)} d z \rightarrow 0$ as $n \rightarrow \infty$, we get $0=\sum_{k}$ residues $+\sum_{j}$ residues. That is,

$$
\sum_{k=-\infty}^{\infty} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right)=-\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_{j}}\right)
$$

- If a function $f$ can be written as a quotient $f(z)=\frac{g(z)}{h(z)}$, where $g$ and $h$ are analytic at $z=z_{0}, g\left(z_{0}\right) \neq 0$ and $h$ has a zero of order 1 at $z_{0}$, then $f$ has a simple pole at $z=z_{0}$ and $\operatorname{Res}\left(f(z), z_{0}\right)=\frac{g\left(z_{0}\right)}{h^{\prime}\left(z_{0}\right)}$.
- Hence, with $g(z)=\frac{\pi \cos \pi z}{p(z)}$ and $h(z)=\sin \pi z$, we get

$$
\operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k\right)=\frac{\frac{\pi \cos k \pi}{p(k)}}{\pi \cos k \pi}=\frac{1}{p(k)} .
$$

- Therefore, we arrive at

$$
\sum_{k=-\infty}^{\infty} \frac{1}{p(k)}=-\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_{j}}\right)
$$

## Using $\csc \pi z$

- If $p(z)$ is a polynomial function satisfying the same assumptions, i.e.,
(i) has real coefficients;
(ii) has degree $n \geq 2$, and
(iii) no integer zeros,
then the function $f(z)=\frac{\pi \csc \pi z}{p(z)}$ has an infinite number of simple poles $z=0, \pm 1, \pm 2, \ldots$ from $\csc \pi z$ and a finite number of poles $z_{p_{1}}, z_{p_{2}}, \ldots, z_{p_{r}}$ from the zeros of $p(z)$.
- In this case it can be shown that

$$
\sum_{k=-\infty}^{\infty} \frac{(-1)^{k}}{p(k)}=-\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \csc \pi z}{p(z)}, z_{p_{j}}\right)
$$

## Summing an Infinite Series

- Find the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k^{2}+4}$.

If we identify $p(z)=z^{2}+4$, then the three assumptions (i)-(iii) are satisfied. The zeros of $p(z)$ are $\pm 2 i$ and correspond to simple poles of $f(z)=\frac{\pi \cot \pi z}{z^{2}+4}$. According to the formula
$\sum_{k=-\infty}^{\infty} \frac{1}{k^{2}+4}=-\left(\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2}+4},-2 i\right)+\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2}+4}, 2 i\right)\right)$. Since $\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2}+4},-2 i\right)=\frac{\pi \cot 2 \pi i}{4 i}$ and $\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2}+4}, 2 i\right)=\frac{\pi \cot 2 \pi i}{4 i}$, the sum of the residues is $\frac{\pi}{2 i} \cot 2 \pi i$. This sum is a real quantity because $\frac{\pi}{2 i} \cot 2 \pi i=\frac{\pi}{2 i} \frac{\cosh (-2 \pi)}{(-i \sinh (-2 \pi))}=-\frac{\pi}{2} \operatorname{coth} 2 \pi$. Hence,

$$
\sum_{k=-\infty}^{\infty} \frac{1}{k^{2}+4}=\frac{\pi}{2} \operatorname{coth} 2 \pi
$$

## Summing an Infinite Series (Cont'd)

- To get the desired sum, we must manipulate the summation $\sum_{-\infty}^{\infty}$ in order to put it in the form $\sum_{k=0}^{\infty}$.
We have

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{k^{2}+4} & =\sum_{k=-\infty}^{-1} \frac{1}{k^{2}+4}+\frac{1}{4}+\sum_{k=1}^{\infty} \frac{1}{k^{2}+4} \\
& =\sum_{k=1}^{\infty} \frac{1}{(-k)^{2}+4}+\frac{1}{4}+\sum_{k=1}^{\infty} \frac{1}{k^{2}+4} \\
& =2 \sum_{k=1}^{\infty} \frac{1}{k^{2}+4}+\frac{1}{4}=2 \sum_{k=0}^{\infty} \frac{1}{k^{2}+4}-\frac{1}{4}
\end{aligned}
$$

Finally, since $\sum_{k=-\infty}^{\infty} \frac{1}{k^{2}+4}=2 \sum_{k=0}^{\infty} \frac{1}{k^{2}+4}-\frac{1}{4}=\frac{\pi}{2} \operatorname{coth} 2 \pi$, we obtain

$$
\sum_{k=0}^{\infty} \frac{1}{k^{2}+4}=\frac{1}{8}+\frac{\pi}{4} \operatorname{coth} 2 \pi
$$

## Subsection 6

## Laplace and Fourier Transforms

## Laplace and Inverse Laplace Transforms

- The Laplace transform of a real function $f$ is defined, for $t \geq 0$, by $\mathcal{L}\{f(t)\}=\int_{0}^{\infty} e^{-s t} f(t) d t$.
(i) The direct problem: Given a function $f(t)$ satisfying certain conditions, find its Laplace transform. When the integral converges, the result is a function of $s$. The relationship between a function and its transform is exhibited by using a lowercase letter to denote the function and the corresponding uppercase letter to denote its Laplace transform, e.g., $\mathcal{L}\{f(t)\}=F(s), \mathcal{L}\{y(t)\}=Y(s)$, and so on.
(ii) The inverse problem: Find the function $f(t)$ that has a given transform $F(s)$. The function $f(t)$ is called the inverse Laplace transform and is denoted by $\mathcal{L}^{-1}\{F(s)\}$.
- We will see that the inverse Laplace transform is not merely a symbol but actually another integral transform, actually a special type of complex contour integral.


## Integral Transforms

- Suppose $f(x, y)$ is a real-valued function of two real variables.
- A definite integral of $f$ with respect to one of the variables leads to a function of the other variable.
Example: If we hold $y$ constant, integration with respect to the real variable $x$ gives $\int_{1}^{2} 4 x y^{2} d x=\left.2 x^{2} y^{2}\right|_{1} ^{2}=8 y^{2}-2 y^{2}=6 y^{2}$.
- Thus, a definite integral such as $F(\alpha)=\int_{a}^{b} f(x) K(\alpha, x) d x$ transforms a function $f$ of the variable $x$ into a function $F$ of the variable $\alpha$.
- We say that $F(\alpha)=\int_{a}^{b} f(x) K(\alpha, x) d x$ is an integral transform of the function $f$.
- Integral transforms appear in transform pairs, meaning that the original function $f$ can be recovered by another integral transform $f(x)=\int_{c}^{d} F(\alpha) H(\alpha, x) d \alpha$, called the inverse transform.
- The functions $K(\alpha, x)$ and $H(\alpha, x)$ are the kernels of the transforms.
- If $\alpha$ represents a complex variable, then the second definite integral is replaced by a contour integral.


## The Laplace Transform

- Suppose that, in $F(\alpha)=\int_{a}^{b} f(x) K(\alpha, x) d x, \alpha$ is replaced by the symbol $s$, and that $f$ represents a real function that is defined on the unbounded interval $[0, \infty)$.
- Then $F(s)=\int_{0}^{\infty} f(t) K(s, t) d t$ is an improper integral, defined by

$$
\int_{0}^{\infty} K(s, t) f(t) d t=\lim _{b \rightarrow \infty} \int_{0}^{b} K(s, t) f(t) d t
$$

- If the limit exists, we say that the integral exists or is convergent; otherwise, the integral does not exist and is said to be divergent.
- The choice $K(s, t)=e^{-s t}$, where $s$ is a complex variable, gives the Laplace transform $\mathcal{L}\{f(t)\}$ defined previously.
- The integral that defines the Laplace transform may not converge for certain kinds of functions $f$.
Example: Neither $\mathcal{L}\left\{e^{t^{2}}\right\}$ nor $\mathcal{L}\left\{\frac{1}{t}\right\}$ exists.
- Also, the limit may exist for only certain values of the variable $s$.


## Existence of a Laplace Transform

- The Laplace transform of $f(t)=1, t \geq 0$, is

$$
\begin{aligned}
\mathcal{L}\{1\} & =\int_{0}^{\infty} e^{-s t}(1) d t \\
& =\lim _{b \rightarrow \infty} \int_{0}^{b} e^{-s t} d t \\
& =\lim _{b \rightarrow \infty}-\left.\frac{e^{-s t}}{s}\right|_{0} ^{b} \\
& =\lim _{b \rightarrow \infty}\left[\frac{1-e^{-s b}}{s}\right] .
\end{aligned}
$$

If $s=x+i y$, then $e^{-s b}=e^{-b x}(\cos b y-i \sin b y)$. Thus, $e^{-s b} \rightarrow 0$ as $b \rightarrow \infty$, if $x>0$. In other words,

$$
\mathcal{L}\{1\}=\frac{1}{s}, \text { provided } \operatorname{Re}(s)>0
$$

## Existence of $\mathcal{L}\{f(t)\}$

- Conditions that are sufficient to guarantee the existence of $\mathcal{L}\{f(t)\}$ are that $f$ be piecewise continuous on $[0, \infty)$ and that $f$ be of exponential order.
- Piecewise continuity on $[0, \infty)$ means that, on any interval, there are at most a finite number of points $t_{k}, k=1,2, \ldots, n, t_{k-1}<t_{k}$, at which $f$ has finite discontinuities and is continuous on each open interval $t_{k-1}<t<t_{k}$.
- A function $f$ is said to be of exponential order $c$ if there exist constants $c, M>0$, and $T>0$, so that $|f(t)| \leq M e^{c t}$, for $t>T$. The condition $|f(t)| \leq M e^{c t}$, for $t>T$, states that the graph of $f$ on the interval $(T, \infty)$ does not grow faster than the graph of the exponential function $M e^{c t}$.
Alternatively, $e^{-c t}|f(t)|$ is bounded, i.e., $e^{-c t}|f(t)| \leq M$, for $t>T$.
- All bounded functions are necessarily of exponential order $c=0$.


## Existence Theorem for $\mathcal{L}\{f(t)\}$

## Theorem (Sufficient Conditions for Existence)

Suppose $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for $t>T$. Then $\mathcal{L}\{f(t)\}$ exists for $\operatorname{Re}(s)>c$.

- We have $\mathcal{L}\{f(t)\}=\int_{0}^{T} e^{-s t} f(t) d t+\int_{T}^{\infty} e^{-s t} f(t) d t=I_{1}+I_{2}$.
- The integral $I_{1}$ exists since it can be written as a sum of integrals over intervals on which $e^{-s t} f(t)$ is continuous.
- To prove the existence of $I_{2}$, let $s=x+i y$. Then $\left|e^{-s t}\right|=\left|e^{-x t}(\cos y t-i \sin y t)\right|=e^{-x t}$. Further, by the definition of exponential order, $|f(t)| \leq M e^{c t}, t>T$. Hence, $\left|l_{2}\right| \leq \int_{T}^{\infty}\left|e^{-s t} f(t)\right| d t$ $\leq M \int_{T}^{\infty} e^{-x t} e^{c t} d t=M \int_{T}^{\infty} e^{-(x-c) t} d t=-\left.M \frac{e^{-(x-c) t}}{x-c}\right|_{T} ^{\infty}=$ $M \frac{e^{-(x-c) T}}{x-c}$, for $x=\operatorname{Re}(s)>c$. Since $\int_{T}^{\infty} M e^{-(x-c) t} d t$ converges, $\int_{T}^{\infty}\left|e^{-s t} f(t)\right| d t$ converges by the comparison test. This, in turn, implies that $I_{2}$ exists for $\operatorname{Re}(s)>c$.
The existence of $I_{1}$ and $I_{2}$ implies that $\mathcal{L}\{f(t)\}=\int_{0}^{\infty} e^{-s t} f(t) d t$ exists for $\operatorname{Re}(s)>c$.


## Analyticity of the Laplace Transform

- The following theorem is stated without proof:


## Theorem (Analyticity of the Laplace Transform)

Suppose $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for $t \geq 0$. Then the Laplace transform of $f$,

$$
F(s)=\int_{0}^{\infty} e^{-s t} f(t) d t
$$

is an analytic function in the right half-plane defined by $\operatorname{Re}(s)>c$.

- Although the complex function $F(s)$ is analytic to the right of the line $x=c$ in the complex plane, $F(s)$ will, in general, have singularities to the left of that line.


## The Inverse Laplace Transform

## Theorem (Inverse Laplace Transform)

If $f$ and $f^{\prime}$ are piecewise continuous on $[0, \infty)$ and $f$ is of exponential order $c$ for $t \geq 0$, and $F(s)$ is a Laplace transform, then the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ is

$$
f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2 \pi i} \lim _{R \rightarrow \infty} \int_{\gamma-i R}^{\gamma+i R} e^{s t} F(s) d s,
$$

where $\gamma>c$.

- We write $f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2 \pi i} \int_{\gamma-i \infty}^{\gamma+i \infty} e^{s t} F(s) d s$, where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\operatorname{Re}(s)=x=\gamma$.
- $\gamma$ is a positive real constant greater than $c$ and greater than all the real parts of the singularities in the left half-plane.
- This integral is called a Bromwich contour integral.
- The kernel of the inverse transform is $H(s, t)=\frac{e^{s t}}{2 \pi i}$.


## Evaluating the Inverse Laplace Transform

- The Bromwich contour integral

$$
f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2 \pi i} \int_{\gamma-i \infty}^{\gamma+i \infty} e^{s t} F(s) d s
$$

- The fact that $F(s)$ has singularities $s_{1}, s_{2}, \ldots, s_{n}$ to the left of the line $x=\gamma$ makes it possible to evaluate the integral by using an appropriate closed contour encircling the singularities. A closed contour $C$ that is commonly used consists of a semicircle $C_{R}$ of radius $R$ centered at $(\gamma, 0)$ and a vertical line segment $L_{R}$ parallel to the $y$-axis passing through the point $(\gamma, 0)$ and extending from $y=$ $\gamma-i R$ to $y=\gamma+i R . \quad R$ is larger than the largest number in $\left\{\left|s_{1}\right|,\left|s_{2}\right|, \ldots,\left|s_{n}\right|\right\}$.


With the contour $C$ chosen in this manner, the integral can often be evaluated using Cauchy's residue theorem. If we allow the radius $R$ of the semicircle to approach $\infty$, the vertical part of the contour approaches the infinite vertical line of the Bromwich integral.

## Inverse Laplace Transform Theorem

## Theorem (Inverse Laplace Transform)

Suppose $F(s)$ is a Laplace transform that has a finite number of poles $s_{1}, s_{2}, \ldots, s_{n}$ to the left of the vertical line $\operatorname{Re}(s)=\gamma$ and that $C$ is the contour on the preceding slide. If $s F(s)$ is bounded as $R \rightarrow \infty$, then $\mathcal{L}^{-1}\{F(s)\}=\sum_{k=1}^{n} \operatorname{Res}\left(e^{s t} F(s), s_{k}\right)$.

- By Cauchy's residue theorem, we have $\int_{C_{R}} e^{s t} F(s) d s+\int_{L_{R}} e^{s t} F(s) d s=2 \pi i \sum_{k=1}^{n} \operatorname{Res}\left(e^{s t} F(s), s_{k}\right)$ or $\frac{1}{2 \pi i} \int_{\gamma-i R}^{\gamma+i R} e^{s t} F(s) d s=\sum_{k=1}^{n} \operatorname{Res}\left(e^{s t} F(s), s_{k}\right)-\frac{1}{2 \pi i} \int_{C_{R}} e^{s t} F(s) d s$. We let $R \rightarrow \infty$ and show that $\lim _{R \rightarrow \infty} \int_{C_{R}} e^{s t} F(s) d s=0$. If the semicircle $C_{R}$ is parametrized by $s=\gamma+\operatorname{Re}^{i \theta}, \frac{\pi}{2} \leq \theta \leq \frac{3 \pi}{2}$, then $d s=R i e^{i \theta} d \theta=(s-\gamma) i d \theta$, and so,

$$
\begin{aligned}
& \frac{1}{2 \pi i} \int_{C_{R}} e^{s t} F(s) d s=\frac{1}{2 \pi i} \int_{\pi / 2}^{3 \pi / 2} e^{\gamma t+R t e^{i \theta}} F\left(\gamma+R^{i \theta}\right) R e^{i \theta} d \theta \text {, whence } \\
& \frac{1}{2 \pi}\left|\int_{C_{R}} e^{s t} F(s) d s\right| \leq \frac{1}{2 \pi} \int_{\pi / 2}^{3 \pi / 2}\left|e^{\gamma t+R t e^{i \theta}}\right|\left|F\left(\gamma+R e^{i \theta}\right)\right|\left|R i e^{i \theta}\right| d \theta
\end{aligned}
$$

## Proof of the Inverse Laplace Transform Theorem

- We examine the three moduli involved:
- $\left|e^{\gamma t+R t e^{i \theta}}\right|=\left|e^{\gamma t} e^{R t(\cos \theta+i \sin \theta)}\right|=e^{\gamma t} e^{R t \cos \theta}$.
- For $|s|$ sufficiently large, we can write

$$
\left|R i e^{i \theta}\right|=|s-\gamma||i| \leq|s|+|\gamma|<|s|+|s|=2|s|
$$

- Finally, by hypothesis, $|s F(s)|<M$.

Thus, we get $\frac{1}{2 \pi}\left|\int_{C_{R}} e^{s t} F(s) d s\right| \leq$
$\frac{1}{2 \pi} \int_{\pi / 2}^{3 \pi / 2}\left|e^{\gamma t+R t e^{i \theta}}\right|\left|F\left(\gamma+R e^{i \theta}\right)\right|\left|R i e^{i \theta}\right| d \theta \leq \frac{M}{\pi} e^{\gamma t} \int_{\pi / 2}^{3 \pi / 2} e^{R t \cos \theta} d \theta$.
Let $\theta=\phi+\frac{\pi}{2}$ and notice that the integral becomes
$\int_{0}^{\pi} e^{-R t \sin \phi} d \phi=2 \int_{0}^{\pi / 2} e^{-R t \sin \phi} d \phi$. We have $\sin \phi \geq \frac{2 \phi}{\pi}$, whence
$2 \int_{0}^{\pi / 2} e^{-R t \sin \phi} d \phi \leq 2 \int_{0}^{\pi / 2} e^{-2 R t \phi / \pi} d \phi=-\left.\frac{\pi}{R t} e^{-2 R t \phi / \pi}\right|_{0} ^{\pi / 2}=$
$\frac{\pi}{R t}\left[1-e^{-R t}\right]$. We conclude that $\frac{1}{2 \pi}\left|\int_{C_{R}} e^{s t} F(s) d s\right| \leq \frac{M e^{\gamma t}}{R t}\left[1-e^{-R t}\right]$.
The right-hand side approaches zero as $R \rightarrow \infty$ for $t>0$, whence $\lim _{R \rightarrow \infty} \int_{C_{R}} e^{s t} F(s) d s=0$.

## An Inverse Laplace Transform

- Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^{3}}\right\}, \operatorname{Re}(s)>0$.

The function $F(s)=\frac{1}{s^{3}}$ has a pole of order 3 at $s=0$. Thus, by the theorem,

$$
\begin{aligned}
f(t) & =\mathcal{L}^{-1}\left\{\frac{1}{s^{3}}\right\} \\
& =\operatorname{Res}\left(e^{s t} \frac{1}{s^{3}}, 0\right) \\
& =\frac{1}{2} \lim _{s \rightarrow 0} \frac{d^{2}}{d s^{2}}(s-0)^{3} \frac{e^{s t}}{s^{3}} \\
& =\frac{1}{2} \lim _{s \rightarrow 0} \frac{d^{2}}{d s^{2}} e^{s t} \\
& =\frac{1}{2} \lim _{s \rightarrow 0} t^{2} e^{s t} \\
& =\frac{1}{2} t^{2}
\end{aligned}
$$

## Fourier Transform

- Suppose now that $f(x)$ is a real function defined on the interval $(-\infty, \infty)$.
- Another important transform pair consists of
- the Fourier transform

$$
\mathfrak{F}\{f(x)\}=\int_{-\infty}^{\infty} f(x) e^{i \alpha x} d x=F(\alpha)
$$

- the inverse Fourier transform

$$
\mathfrak{F}^{-1}\{F(\alpha)\}=\frac{1}{2 \pi} \int_{-\infty}^{\infty} F(\alpha) e^{-i \alpha x} d \alpha=f(x) .
$$

- The kernel of the Fourier transform is $K(\alpha, x)=e^{i \alpha x}$, whereas the kernel of the inverse transform is $H(\alpha, x)=\frac{e^{-i \alpha x}}{2 \pi}$.
- We assume that $\alpha$ is a real variable.
- In contrast to the Laplace case, the inverse Fourier transform is not a contour integral.


## Computing a Fourier Transform

- Find the Fourier transform of $f(x)=e^{-|x|}$.

We have $f(x)=\left\{\begin{array}{ll}e^{x}, & \text { if } x<0 \\ e^{-x}, & \text { if } x \geq 0\end{array}\right.$. The Fourier transform of $f$ is $\mathfrak{F}\{f(x)\}=\int_{-\infty}^{0} e^{x} e^{i \alpha x} d x+\int_{0}^{\infty} e^{-x} e^{i \alpha x} d x=I_{1}+I_{2}$.

- For $I_{2}$, we have $I_{2}=\lim _{b \rightarrow \infty} \int_{0}^{b} e^{-x} e^{i \alpha x} d x=$

$$
\begin{aligned}
& \lim _{b \rightarrow \infty} \int_{0}^{b} e^{-x(1-\alpha i)} d x=\left.\lim _{b \rightarrow \infty} \frac{e^{-x(1-\alpha i)}}{\alpha i-1}\right|_{0} ^{b}=\lim _{b \rightarrow \infty} \frac{e^{-b(1-\alpha i)}-1}{\alpha i-1}= \\
& \frac{1}{\alpha i-1} \lim _{b \rightarrow \infty}\left(e^{-b} \cos b \alpha+i e^{-b} \sin b \alpha-1\right)=\frac{1}{1-\alpha i} .
\end{aligned}
$$

- The integral $I_{1}$ can be evaluate similarly to obtain $I_{1}=\frac{1}{1+\alpha i}$.

Adding $I_{1}$ and $I_{2}$ gives the value of the Fourier transform:

$$
\mathcal{F}\{f(x)\}=\frac{1}{1-\alpha i}+\frac{1}{1+\alpha i}=\frac{2}{1+\alpha^{2}} .
$$

## Computing an Inverse Fourier Transform

- Find the inverse Fourier transform of $F(\alpha)=\frac{2}{1+\alpha^{2}}$.

The idea here is to recover the function $f$ of the preceding example.
We have $\mathfrak{F}^{-1}\{F(\alpha)\}=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^{2}} e^{-i \alpha x} d \alpha=f(x)$.


Let $z$ be a complex variable and introduce the contour integral $\oint_{C} \frac{1}{\pi\left(1+z^{2}\right)} e^{-i z x} d z$. The integrand has simple poles at $z= \pm i$. The contour $C$ is shown in the figure.

We get $\oint_{C} \frac{1}{\pi\left(1+z^{2}\right)} e^{-i z x} d z=2 \pi i \operatorname{Res}\left(\frac{1}{\pi\left(1+z^{2}\right)} e^{-i z x}, i\right)=e^{x}$. The contour integral along $C_{R}$ approaches zero as $R \rightarrow \infty$ only if we assume that $x<0$. Thus, the answer is $e^{x}, x<0$.

## Computing an Inverse Fourier Transform (Cont'd)

0


If we consider $\oint_{C} \frac{1}{\pi\left(1+z^{2}\right)} e^{-i z x} d z$, where $C$ is the contour on the left, it can be shown that the integral along $C_{R}$ now approaches zero as $R \rightarrow \infty$ when $x$ is assumed to be positive. Hence, $\oint_{C} \frac{1}{\pi\left(1+z^{2}\right)} e^{-i z x} d z=$
$-2 \pi i \operatorname{Res}\left(\frac{1}{\pi\left(1+z^{2}\right)} e^{-i z x},-i\right)=e^{-x}, x>0$. The extra minus sign appearing in front of the factor $2 \pi i$ comes from the fact that on $C$, $\int_{C}=\int_{C_{R}}+\int_{R}^{-R}=\int_{C_{R}}-\int_{-R}^{R}=2 \pi i \operatorname{Res}(z=-i)$. As $R \rightarrow \infty$, $\int_{C_{R}} \rightarrow 0$, for $x>0$, whence $-\lim _{R \rightarrow \infty} \int_{-R}^{R}=2 \pi i \operatorname{Res}(z=-i)$ or $\lim _{R \rightarrow \infty} \int_{-R}^{R}=-2 \pi i \operatorname{Res}(z=-i)$.

- By combining the findings, we get

$$
\mathfrak{F}^{-1}\{F(\alpha)\}=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^{2}} e^{-i \alpha x} d \alpha=\left\{\begin{array}{ll}
e^{x}, & \text { if } x<0 \\
e^{-x}, & \text { if } x>0
\end{array} .\right.
$$

