Fundamental Concepts of Mathematics

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1. Collections
   - Lists
   - Factorial
   - Sets I: Introduction, Subsets
   - Quantifiers
   - Sets II: Operations

Subsection 1

Lists

Definition of List

- A **list** is an ordered sequence of objects.
- Lists are presented by placing their elements separated by commas inside parentheses.
- Example: $(1, 2, \mathbb{Z})$ is a list whose first element is the number 1, second element is the number 2, and third element is the set of integers $\mathbb{Z}$.
- Some important features of lists:
  - The order in which elements appear in a list is significant; so the list $(1, 2, 3)$ is not the same as the list $(3, 2, 1)$.
  - Elements in a list might be repeated, as in $(3, 3, 2)$.
- The number of elements in a list is called its **length**.
- Example: The list $(1, 1, 2, 1)$ is a list of length four.
- A list of length two is called an **ordered pair**.
- A list of length zero is called the **empty list** and is denoted $()$.

Equality of Lists and Counting Lists

- Two lists are **equal** provided they have the same length, and elements in the corresponding positions in the two lists are equal.
- Example: Lists (a, b, c) and (x, y, z) are equal iff $a = x$, $b = y$ and $c = z$.
- Suppose we wish to make a two-element list where the entries in the list may be any of the digits $1, 2, 3$, and 4. How many such lists are possible?
  The most direct approach to answering this question is to write out all the possibilities:

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{array}$$

  So, there are 16 such lists.

Counting Number of Lists I

- Suppose we wish to know the number of two-element lists where there are n possible choices for each entry in the list.
- We may assume the possible elements are the integers 1 through n.
- Again, we organize all the possible lists:

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{array}$$

The first row contains all the lists that begin with 1, the second those that begin with 2, and so forth. There are n rows in all. Each row has exactly n lists. Therefore, there are $n \times n = n^2$ possible lists.

Counting Number of Lists II

- How many two-element lists are possible in which there are n choices for the first element and m choices for the second element?
- Suppose that the possible entries in the first position of the list are the integers 1 through n, and the possible entries in the second position are 1 through m.
- We construct a chart of all the possibilities:

$$\begin{array}{cccc} (1,1) & (1,2) & \cdots & (1,m) \\ (2,1) & (2,2) & \cdots & (2,m) \\ \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & \cdots & (n,m) \end{array}$$

There are n rows (for each possible first choice), and each row contains m entries. Thus, the number of possible such lists is $\underbrace{m+m+\cdots+m}_{n \text{ times}} = n \times m$.

More Specialized Counting

- Sometimes the elements of a list satisfy special properties.
- In particular, the choice of the second element might depend on what the first element is.
- Example: Suppose we want to count the number of two-element lists we can form from the integers 1 through 5, in which the two numbers on the list must be different.
  We again list the possibilities:

$$\begin{array}{ccccc} & (1,2) & (1,3) & (1,4) & (1,5) \\ (2,1) & & (2,3) & (2,4) & (2,5) \\ (3,1) & (3,2) & & (3,4) & (3,5) \\ (4,1) & (4,2) & (4,3) & & (4,5) \\ (5,1) & (5,2) & (5,3) & (5,4) & \end{array}$$

  As before, the first row contains all the possible lists that begin with 1, the second row those lists that start with 2, and so on, so there are 5 rows. Notice that each row contains exactly $5-1=4$ lists, so the number of lists is $5 \times 4 = 20$.

The Multiplication Principle

Theorem (Multiplication Principle)

Consider two-element lists for which there are n choices for the first element, and for each choice of the first element there are m choices for the second element. Then the number of such lists is nm.

- Construct a chart of all the possible lists.
  Each row of this chart contains all the two-element lists that begin with a particular element. Since there are n choices for the first element, there are n rows in the chart.
  Since, for each choice of the first element, there are m choices for the second element, we know that every row of the chart has m entries.
  Therefore the number of lists is $\underbrace{m + m + \cdots + m}_{n \text{ times}} = n \times m$.

Example I

- A person's initials form the two-element list consisting of the initial letters of their first and last names. In how many ways can we form a person's initials? In how many ways can we form initials where the two letters are different?
- Method I: The first question asks for the number of two-element lists where there are 26 choices for each element. There are 26^2 such lists. The second question asks for the number of two-element lists where there are 26 choices for the first element and, for each choice of first element, 25 choices for the second element. Thus there are 26×25 such lists.
- Method II: There are 26^2 ways to form initials (repetitions allowed). However, 26 of these initials have a repetition, namely, AA, BB, CC, $\ldots$, ZZ. The remaining lists are the ones we want to count, so there are $26^2 - 26$ possibilities. The two answers are the same: $26 \times 25 = 26 \times (26 - 1) = 26^2 - 26$.

Example II

- A club has ten members. The members wish to elect a president and to elect someone else as a vice president. In how many ways can these posts be filled?
- We recast this question as a list-counting problem. How many two-element lists of people can be formed in which the two people in the list are selected from a collection of ten candidates and the same person may not be selected twice?
- There are ten choices for the first element of the list. For each choice of the first element (for each president), there are nine possible choices for the second element of the list (vice president). By the Multiplication Principle, there are 10×9 possibilities.

Counting Longer Lists

- How many lists of three elements can we make using the numbers $1, 2, 3, 4$ and 5?
  We organize all the possibilities:

$$\begin{array}{ccccc} (1,1,1) & (1,1,2) & (1,1,3) & (1,1,4) & (1,1,5) \\ (1,2,1) & (1,2,2) & (1,2,3) & (1,2,4) & (1,2,5) \\ & & \cdots & & \\ (5,5,1) & (5,5,2) & (5,5,3) & (5,5,4) & (5,5,5) \end{array}$$

  The first line of this chart contains all lists that begin $(1, 1, \ldots)$. The second line all lists that begin $(1, 2, \ldots)$ and so forth. Clearly, each line has five lists. So, the question becomes: How many lines are there in this chart? Since each line of the chart begins with a different two-element list and the number of two-element lists where each element is one of five possible values is 5×5, this chart has 5×5 lines. Therefore, since each line of the chart has five entries, the number of three-element lists is $(5 \times 5) \times 5 = 5^3$.

Special Counting With Longer Lists

- Count three-element lists whose elements are the integers 1 through 5 in which no number is repeated.
  The following chart organizes the possible choices:

$$\begin{array}{ccc} (1,2,3) & (1,2,4) & (1,2,5) \\ (1,3,2) & (1,3,4) & (1,3,5) \\ & \cdots & \\ (5,4,1) & (5,4,2) & (5,4,3) \end{array}$$

  The first line of the chart contains all the lists that begin $(1,2,\ldots)$. The second line contains all lists that begin $(1,3,\ldots)$, and so on. Each line of the chart contains just three lists.
  So, as before, the question becomes: How many lines are on this chart? The first two elements of the list form a two-element list with each element chosen from a list of five possible objects and without repetition. So, by the Multiplication Principle, there are 5×4 lines. So, there are a total of $5 \times 4 \times 3$ possible lists in all.

Multiplication Principle for Longer Lists

- Consider length three lists. Suppose that we have:
  - a choices for the first element of the list,
  - for each choice of first element, b choices for the second element, and
  - for each choice of first and second elements, c choices for the third element.

  Then, in all, there are abc possible lists.
- To see why, imagine that the three-element list consists of two parts: the initial two elements and the final element. There are ab ways to fill in the first two elements (by the Multiplication Principle) and there are c ways to complete the last element once the first two are specified. So, by the Multiplication Principle again, there are $(ab)c$ ways to make the lists.
- The extension of these ideas to lists of length-four or more is analogous.

An Example of the Extended Principle

- Suppose the members of a club with ten members want to elect an executive board consisting of a president, a vice president, a secretary, and a treasurer. In how many ways can they do this (assuming no member of the club can fill two offices)?
  The following diagram gives the number of available choices:

President	V.P.	Secretary	Treasurer
10	9	8	7

  - There are ten choices for president.
  - Once the president is selected, there are are nine choices for vice president, so there are 10×9 ways to fill in the first two elements of the list.
  - Once these are filled, there are eight ways to fill in the next element of the list (secretary), so there are $(10 \times 9) \times 8$ ways to complete the first three slots.
  - Finally, once the first three offices are filled, there are seven ways to select a treasurer, so there are $(10 \times 9 \times 8) \times 7$ ways to select the entire slate of officers.

Lists With/Without Repetitions

- Make a list of length k in which each element of the list is selected from among n possibilities in two different ways:
  - In the first case, we count all such lists (with possible repetitions). When repetitions are allowed, we have n choices for the first element of the list, n choices for the second element of the list, and so on, and n choices for the last element of the list.
    All told, there are $\underbrace{n \times n \times \cdots \times n}_{k \text{ times}} = n^k$ possible lists.
  - In the second problem, we count those without repeated elements. There are n ways to select the first element of the list. Once this is done, there are $n-1$ choices for the second element of the list. There are $n-2$ ways to fill in position three, $n-3$ ways to fill in position four, and so on, and finally, there are $n-(k-1) = n-k+1$ ways to fill in position k.
    Therefore, the number of ways to make a list of length k where the elements are chosen from n possibilities and no repetitions are allowed is $n \times [n-1] \times [n-2] \times \cdots \times [n-(k-1)]$.

Falling Factorial and Summary of the Results

- Recall that the number of ways to make a list of length k where the elements are chosen from n possibilities and no repetitions are allowed is $n \times [n-1] \times [n-2] \times \cdots \times [n-(k-1)]$.
- Since the expression $n(n-1)(n-2)\cdots(n-k+1)$ occurs fairly often, there is a special notation for it:

$$(n)_k = n(n-1)(n-2)(n-k+1).$$

This notation is called **falling factorial**.

Theorem

The number of lists of length k whose elements are chosen from a pool of n possible elements

$$= \begin{cases} n^k, & \text{if repetitions are permitted} \\ (n)_k, & \text{if repetitions are forbidden} \end{cases}$$

Subsection 2

Factorial

The Factorial

- Count the number of length n lists chosen from a pool of n objects in which repetition is forbidden. Equivalently, arrange n objects into a list, using each object exactly once.
- We saw the number of such lists is $(n)_n = n(n-1)(n-2)\cdots(n-n+1) = n(n-1)(n-2)\cdots(1)$.
- The quantity $(n)_n$ is called n **factorial** and is written $n!$.
- Example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.
- Consider the special case $1!$. This is the result of multiplying all the numbers starting from 1 all the way down to 1. The answer is 1. Think, also, in how many ways we can make a length 1 list where there is only one possible element to fill the only position. There is only one possible list, which also justifies $1! = 1$.

Why should $0! = 1$?

- In how many ways can we make a length 0 list whose elements come from a pool of 0 elements in which there is no repetition?
  It is tempting to say that no such list is possible, but this is not correct. There is a list whose length is zero: the empty list (). The empty list has zero length, and its elements satisfy the conditions of the problem. So the answer to the problem is $0! = 1$.
- Here is another explanation why $0! = 1$.
  Consider the equation $n! = n \cdot (n-1)!$. For example,
  $5! = 5 \cdot (4 \cdot 3 \cdot 2 \cdot 1) = 5 \cdot 4!$.
  This equation makes sense for $n = 2$ since $2! = 2 \cdot 1! = 2 \cdot 1$. Does it make sense for $n = 1$? If we want it to also work when $n = 1$, we need $1! = 1 \cdot 0!$. This forces us to choose $0! = 1$.

More Reasons for $0! = 1$

- Here is another explanation why $0! = 1$.
  We can think of $n!$ as the result of multiplying n numbers together.
  For example, 5! is the result of multiplying the numbers on the list $(5, 4, 3, 2, 1)$.
  What should it mean to multiply together the numbers on the empty list? We'll see that the sensible answer is 1.
  But, first, we'll consider what it means to add the numbers on the empty list.

Adding the Elements in the Empty List

- Alice and Bob work in a number factory and are given a list of numbers to add.
- They decide to break the list in two. Alice will add her numbers, Bob will add his numbers, and then they will add their results to get the final answer.
- They ask Carlos to break the list in two for them.
- Carlos decides to give Alice all of the numbers and Bob none of the numbers.
- Alice receives the full list and Bob receives the empty list.
- Alice adds her numbers as usual, but what is Bob to report as the sum of the numbers on his list? If he gives any answer other than 0, the final answer to the problem will be incorrect. The only sensible thing for Bob to say is that his list - the empty list - sums to 0. The sum of the numbers in the empty list is 0.

Multiplying the Elements in the Empty List

- Alice, Bob and Carlos are now working on multiplication and their multiplication procedure is the same as their addition procedure.
- They are asked to multiply a list of numbers.
- They again ask Carlos to break the list into two parts. Alice multiplies the numbers on her list, and Bob multiplies the numbers on his. They then multiply together their individual results to get the final answer.
- Carlos, now, gives all the numbers to Bob; to Alice, he gives the empty list.
- Bob reports the product of his numbers as usual.
- What should Alice say? What is the product of the numbers in ()? If she says 0, then when her answer is multiplied by Bob's answer, the final result will be 0, and this is likely to be the wrong answer. Indeed, the only sensible reply that Alice can give is 1.
  The product of the numbers in the empty list is 1.

Product Notation I

- Another way to write $n!$ is $n! = \prod_{k=1}^{n} k$.
- The symbol $\prod$ is the uppercase form of the Greek letter π and it stands for product (i.e., multiply).
- This notation is similar to using $\sum$ for summation.
- The letter k is called a **dummy variable** and is a place holder that ranges from the lower value (written below the $\prod$ symbol) to the upper value (written at the top).
- The variable k takes on the values $1, 2, \ldots, n$.
- To the right of the $\prod$ symbol are the quantities we multiply. In this case, we just multiply the values of k as k goes from 1 to n; that is, we multiply $1 \cdot 2 \cdot \cdots \cdot n$.

Product Notation II

- The expression on the right of the $\prod$ symbol can be more complex.
- Example: Consider the product $\prod_{k=1}^{5}(2k+3)$;

  This specifies that we multiply together the various values of $(2k+3)$ for $k = 1, 2, 3, 4, 5$. In other words, $\prod_{k=1}^{5}(2k+3) = 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13$.
- Example: The expression $\prod_{k=1}^{n} 2$ is a fancy way to write 2^n.
- Example: Consider $\prod_{k=1}^{0} k$;

  This means that k starts at 1 and goes up to 0. Since there is no possible value of k with $1 \leq k \leq 0$, there are no terms to multiply. Therefore the product is empty and evaluates to 1.

Subsection 3

Sets I: Introduction, Subsets

Introducing Sets and Membership

- A **set** is an unordered collection of objects without repetitions.
- The simplest way to specify a set is to list its elements between curly braces. For example, $\{2, 3, \frac{1}{2}\}$ is a set with exactly three members: the integers 2 and 3, and the rational number $\frac{1}{2}$.
- $\{3, \frac{1}{2}, 2\}$ and $\{2, 2, 3, \frac{1}{2}\}$ denote the same set.
- The notation for three special sets of numbers is worth remembering: $\mathbb{Z}$ (the **integers**), $\mathbb{N}$ (the **natural numbers**), and $\mathbb{Q}$ (the **rational numbers**).
- An object that belongs to a set is called an **element** of the set.
- **Membership** in a set is denoted with the symbol $\in$. So $x \in A$ means that the object x is a member of the set A. For example, $2 \in \{2, 3, \frac{1}{2}\}$ is true, but $5 \in \{2, 3, \frac{1}{2}\}$ is false. In the latter case, we write $5 \notin \{2, 3, \frac{1}{2}\}$.
- When read aloud, $\in$ is pronounced "is a member of" or "is an element of" or "is in."

Cardinality of a Set

- Often mathematicians write, "If $x \in \mathbb{Z}$, then ..." This means exactly the same thing as "If x is an integer, then ..."
- If we write "Let $x \in \mathbb{Z}$," we mean "Let x be an integer."
- The number of elements in a set A is denoted $|A|$. The **cardinality** of A is simply the number of objects in the set. The cardinality of the set $\{2, 3, \frac{1}{2}\}$ is 3. The cardinality of $\mathbb{Z}$ is infinite.
- We also call $|A|$ the **size** of the set A.
- A set is called **finite** if its cardinality is an integer (i.e., is finite). Otherwise, it is called **infinite**.
- The **empty set** is the set with no members. The empty set may be denoted $\{\}$, but it is better to use the special symbol $\emptyset$.
- The statement "$x \in \emptyset$" is false regardless of what object x represents.
- The cardinality of the empty set is zero (i.e., $|\emptyset| = 0$).
- Be careful to distinguish between $\emptyset$ and ϕ (Greek phi).

Presentation of a Set

- There are two principal ways of specifying a set.
  - The most direct way is to list the elements of the set between curly braces, as in $\{3, 4, 9\}$. This notation is appropriate for small sets.
  - More often, set-builder notation is used. The form of this notation is $\{\text{dummy variable} : \text{conditions}\}$.
    - For example, consider $\{x : x \in \mathbb{Z}, x \geq 0\}$. This is the set of all objects x that satisfy the two conditions $x \in \mathbb{Z}$ and $x \geq 0$. Therefore, this set is $\mathbb{N}$, the set of the natural numbers.

  An alternative way of writing set-builder notation is $\{\text{dummy variable} \in \text{set} : \text{conditions}\}$. This is the set of all objects drawn from the set and subject to the conditions.
    - $\{x \in \mathbb{Z} : 2 \mid x\}$ is the set of all integers that are divisible by 2 (i.e., the set of even integers).
    - $\{x \in \mathbb{Z} : 1 \leq x \leq 100\}$ denotes the set of integers from 1 to 100 inclusive.
- Care is needed with the list notation! Ambiguity, such as $\{3, 5, 7, \ldots\}$ (odd integers greater than 1 or odd prime numbers?) should be avoided.

Equality of Sets

- Two sets are **equal** when they have exactly the same elements.
- To prove that sets A and B are equal, one shows that every element of A is also an element of B, and vice versa.
- Example: Show that the sets $E = \{x \in \mathbb{Z} : x \text{ is even}\}$ and $F = \{z \in \mathbb{Z} : z = a + b \text{ where } a \text{ and } b \text{ are both odd}\}$ are equal.
  - We show, first, that, if $x \in E$, then $x \in F$.
    Suppose $x \in E$. Therefore x is even, and hence divisible by 2, so $x = 2y$ for some integer y. Now note that $x = 2y = (2y + 1) + (-1)$ and $(2y + 1)$ and -1 are both odd. Therefore, x is the sum of two odd numbers. This shows that $x \in F$.
  - We show, next, that, if $x \in F$, then $x \in E$.
    Suppose $x \in F$. Therefore x is the sum of two odd numbers. We showed before that, then, x must be even. Therefore, $x \in E$.

Subsets

- Suppose A and B are sets. We say that A is a **subset** of B, written $A \subseteq B$, provided every element of A is also an element of B.
- Example: $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$.
- For any set A, we have $A \subseteq A$ because every element of A is in A.
- For any set A, we also have $\emptyset \subseteq A$. (Every element of $\emptyset$ is in A.)
- It is important to distinguish between $\in$ and $\subseteq$:
  - The notation $x \in A$ means that x is an element (or member) of A.
  - The notation $A \subseteq B$ means that every element of A is also in B.
- Example: $\emptyset \subseteq \{1, 2, 3\}$ is true, but $\emptyset \in \{1, 2, 3\}$ is false.
- Note also the difference between x and $\{x\}$.
  - The symbol x refers to some object (a number or whatever).
  - The notation $\{x\}$ means the set whose one and only element is the object x.
  - It is always correct to write $x \in \{x\}$, but, in general, it is incorrect to write $x = \{x\}$ or $x \subseteq \{x\}$.
- To prove that one set is a subset of another, we need to show that every element of the first set is also a member of the second set.

Membership and Subsets

Proposition

Let x be anything and let A be a set. Then $x \in A$ if and only if $\{x\} \subseteq A$.

- We show, first, that, if $x \in A$, then $\{x\} \subseteq A$.
  Suppose that $x \in A$. We want to show $\{x\} \subseteq A$. To do this, we need to show that every element of $\{x\}$ is also an element of A. But the only element of $\{x\}$ is x, and we are given that $x \in A$. Therefore $\{x\} \subseteq A$.
- We show, next, that, if $\{x\} \subseteq A$, then $x \in A$.
  Suppose that $\{x\} \subseteq A$. This means that every element of the set $\{x\}$ is also a member of A. But x is an (the only) element of $\{x\}$ and so $x \in A$.

Pythagorean Triples

- A list of integers (a, b, c) is a **Pythagorean triple** if $a^2 + b^2 = c^2$.

Proposition

Let P be the set of Pythagorean triples, i.e., $P = \{(a, b, c) : a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2\}$ and let T be the set $T = \{(p, q, r) : p = x^2 - y^2,$ $q = 2xy$ and $r = x^2 + y^2$ where $x, y \in \mathbb{Z}\}$. Then $T \subseteq P$.

- Suppose $(p, q, r) \in T$. Therefore, there are integers x and y such that $p = x^2 - y^2$, $q = 2xy$ and $r = x^2 + y^2$. Note that p, q and r are integers because x and y are integers. We calculate

$$\begin{aligned} p^2 + q^2 &= (x^2 - y^2)^2 + (2xy)^2 \\ &= (x^4 - 2x^2y^2 + y^4) + 4x^2y^2 \\ &= x^4 + 2x^2y^2 + y^4 \\ &= (x^2 + y^2)^2 = r^2. \end{aligned}$$

Therefore (p, q, r) is a Pythagorean triple, i.e., $(p, q, r) \in P$.

Contains and Superset

- The symbols $\in$ and $\subseteq$ may be written backward: $\ni$ and $\supseteq$.
- The notation $A \ni x$ means exactly the same thing as $x \in A$.
- The symbol $\ni$ can be read, "contains the element."
- The notation $B \supseteq A$ means exactly the same thing as $A \subseteq B$.
- We say that "B is a superset of A."

Counting Number of Subsets

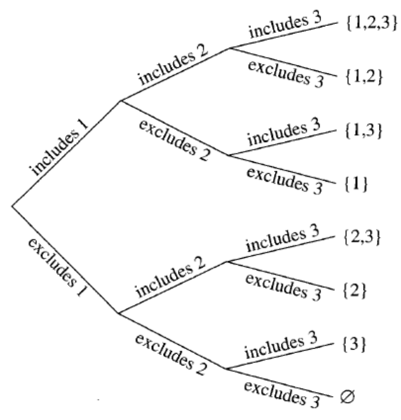
- How many subsets does $A = \{1, 2, 3\}$ have?
  The easiest way to find out is to list all the possibilities. Since $|A| = 3$, a subset of A can have anywhere from zero to three elements.

Number of elements	Subsets	Number
0	$\emptyset$	1
1	$\{1\}, \{2\}, \{3\}$	3
2	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	3
3	$\{1, 2, 3\}$	1

  Therefore, there are eight subsets of $\{1, 2, 3\}$.

An Alternative Way to Count Number of Subsets

- Each element of the set $\{1, 2, 3\}$ either is a member of or is not a member of a subset.



For each element, we have two choices: to include or not to include that element in the subset. For example, if for the elements 1, 2, and 3 the answers are (yes, yes, no), then the subset is $\{1, 2\}$. The problem of counting subsets of $\{1, 2, 3\}$ reduces to the problem of counting lists, and we know how to count lists! The number of lists of length three where each entry on the list is either "yes" or "no" is $2 \times 2 \times 2 = 8$.

Number of Subsets of a Finite Set

Theorem

Let A be a finite set. The number of subsets of A is $2^{|A|}$.

- Let A be a finite set and let $n = |A|$. Suppose $A = \{a_1, a_2, \ldots, a_n\}$. To each subset B of A we can associate a list of length n; each element of the list is one of the words "yes" or "no." The kth element of the list is "yes" precisely when $a_k \in B$. This establishes a correspondence between length n yes-no lists and subsets of A. Each subset of A gives a yes-no list, and each yes-no list determines a different subset of A. Therefore, the number of subsets of A is exactly the same as the number of length n yes-no lists, which is 2^n, i.e., the number of subsets of A is 2^n, where $n = |A|$.
- This style of proof is called a **bijective proof**. To show that two counting problems have the same answer, we establish a one-to-one correspondence between the two sets we want to count. Knowing the answer to one provides the answer to the other.

Powerset

- The **power set** of a set A is the set of all subsets of A.
- Example: We saw that the power set of $\{1, 2, 3\}$ is the set $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
- We also proved that if a set $|A| = n$, then its power set contains 2^n elements.
- As a mnemonic, the notation for the power set of A is 2^A. This notation was created so that we would have $|2^A| = 2^{|A|}$.
- The left side of this equation is the cardinality of the power set of A; the right side is 2 raised to the cardinality of A. On the left, the superscript on 2 is a set, so the notation means power set; on the right, the superscript on 2 is a number, so the notation means ordinary exponentiation.

Subsection 4

Quantifiers

Existential Statements

- Consider “There is a natural number that is prime and even”.
- The general form of this sentence is “There is an object x, a member of set A, that has the following properties.”
- Fot the example sentence: “There is an x, a member of $\mathbb{N}$, such that x is prime and even.”
- It says “at least one element in $\mathbb{N}$ has the required properties”.
- The phrase “there exists” is synonymous with “there is”.
- The formal notation $\exists x \in A$, assertions about x means “There is an x in set A such that ... ”.
- Example: The sentence “There is a natural number that is prime and even” would be written $\exists x \in \mathbb{N}$, x is prime and even.
- $\exists$ is called the **existential quantifier**.
- To prove a statement of the form “$\exists x \in A$, assertions about x,” we have to show that some element in A satisfies the assertions.

Proving Existential Statements

- To prove $\exists x \in A$, assertions about x:
  - Provide an explicit example for x;
  - Show that the x provided satisfies the assertions.
- Proving an existential statement is akin to finding a counterexample. We simply have to find one object with the required properties.
- Example: Prove that there is an integer that is even and prime.

  Consider the integer 2. Clearly 2 is even and 2 is prime.

Universal Statements

- Consider now "Every integer is even or odd."
- Alternative phrases for "every" include "all", "each", and "any".
- Thus, all of the following sentences mean the same thing:
  - Every integer is either even or odd.
  - All integers are either even or odd.
  - Each integer is either even or odd.
  - Let x be any integer. Then x is even or odd.
- The meaning is that the condition applies to all integers without exception.
- We use $\forall$ as a notation for "all". The general form for this notation is $\forall x \in A$, assertions about x.
- This means that all elements of the set A satisfy the assertions, as in $\forall x \in \mathbb{Z}$, x is odd or x is even.
- $\forall$ is called the **universal quantifier**.
- To prove an "all" theorem, we need to show that every element of the set satisfies the required assertions.

Proving Universal Statements

- To prove $\forall x \in A$, assertions about x:
  - Let x be any element of A;
  - Show that x satisfies the assertions using only the fact that $x \in A$ and no further assumptions on x.
- Example: Every integer that is divisible by 6 is even. More formally, let $A = \{x \in \mathbb{Z} : 6 \mid x\}$. Then the statement is $\forall x \in A$, x is even.

  Let $x \in A$; that is, x is an integer that is divisible by 6. This means there is an integer y such that $x = 6y$; We rewrite this as $x = (2 \cdot 3)y = 2(3y)$. Since $3y$ is an integer, x is divisible by 2; This proves that x is even.

  Notice that the structure of this proof is identical with the proof of: If $x \in A$, then x is even.

Negation of Quantified Statements I

- Consider the statements
  - There is no integer that is both even and odd.
  - Not all integers are prime.
- Symbolically, these can be written
  - $\neg(\exists x \in \mathbb{Z},\ x \text{ is even and } x \text{ is odd})$.
  - $\neg(\forall x \in \mathbb{Z},\ x \text{ is prime})$.
- What do these negated quantified statements mean?
- Consider first $\neg(\exists x \in A,\ \text{assertions about } x)$.
  This means that none of the elements of A satisfy the assertions. This is equivalent to saying that all of the elements of A fail to satisfy the assertions. In other words, the following two sentences are equivalent:
  - $\neg(\exists x \in A,\ \text{assertions about } x)$
  - $\forall x \in A, \neg(\text{assertions about } x)$
- Example: The statement "There is no integer that is both even and odd" says the same as "Every integer is not both even and odd."

Negation of Quantified Statements II

- Consider now the negation of universal statements.
- Such a statement is of the form

$$\neg(\forall x \in A, \text{ assertions about } x).$$

  This means that not all of the elements of x have the requisite assertions (i.e., some do not). Thus, the following two statements are equivalent:
  - $\neg(\forall x \in A, \text{ assertions about } x)$
  - $\exists x \in A, \neg(\text{assertions about } x)$
- Example: The statement "Not all integers are prime" is equivalent to "There is an integer that is not prime."
- To remember these equivalences note that

$$\neg\forall \ldots = \exists\neg \ldots \quad \text{and} \quad \neg\exists \ldots = \forall\neg \ldots$$

  i.e., when the $\neg$ "moves" inside the quantifier, it toggles the quantifier between $\forall$ and $\exists$.

Combining Quantifiers I

- Quantified statements can become difficult and confusing when there are two or more quantifiers in the same statement.
- Consider the following statements about integers:
  - For every x, there is a y, such that $x + y = 0$.
  - There is a y, such that for every x, we have $x + y = 0$.
- In symbols, these statements are written
  - $\forall x, \exists y, x + y = 0$
  - $\exists y, \forall x, x + y = 0$
- The first sentence says that for any integer x we can find an integer y such that $x + y = 0$.
  Let's say $x = 12$. Can we find a y such that $x + y = 0$? Of course! We just need $y = -12$. Say $x = -53$. Can we find a y such that $x + y = 0$? Yes! Take $y = 53$.
  The statement just requires that no matter how we pick x, we can find a y such that $x + y = 0$. This is a true statement.

Combining Quantifiers II

- Proof of $\forall x, \exists y, x + y = 0$.
  Let x be any integer. Pick $y = -x$. Then $x + y = x + (-x) = 0$.
- Now let us examine the similar statement $\exists y, \forall x, x + y = 0$.
  This sentence alleges that there is an integer y with the property that no matter what number x we add to it we get 0. There is no such integer y! No matter what integer y you might think of, we can always find an integer x, such that $x + y$ is not zero.
- The statements $\forall x, \exists y, x + y = 0$ and $\exists y, \forall x, x + y = 0$ are made a bit clearer through the use of parentheses: They may be rewritten as: $\forall x, (\exists y, x + y = 0)$ and $\exists y, (\forall x, x + y = 0)$. These additional parentheses are not necessary. We may use them, if they make the statements clearer.

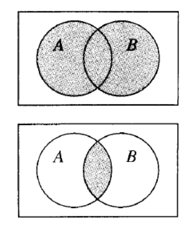
Subsection 5

Sets II: Operations

Union and Intersection of Sets

- Let A and B be sets.
  - The **union** of A and B is the set of all elements that are in A or B. The union of A and B is denoted $A \cup B$.
  - The **intersection** of A and B is the set of all elements that are in both A and B. The intersection of A and B is denoted $A \cap B$.
- In symbols, we can write:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}, \quad A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- Example: Suppose $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \{3, 4\}$.
- 

A **Venn diagram** depicts sets as circles or other shapes. In the figure, the shaded region in the upper diagram is $A \cup B$, and the shaded region in the lower diagram is $A \cap B$.

Some Algebraic Properties of Union and Intersection

Theorem

Let A, B and C denote sets. The following are true:

- $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (**Commutative properties**)
- $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$ (**Associative properties**)
- $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ (**Identity** and **Absorption**)
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (**Distributive properties**)

- We prove the associative property for union. Let A, B and C be sets.

$$
\begin{aligned}
A \cup (B \cup C) &= \{x : (x \in A) \vee (x \in B \cup C)\} \text{ (definition of union)} \\
&= \{x : (x \in A) \vee ((x \in B) \vee (x \in C))\} \text{ (union)} \\
&= \{x : ((x \in A) \vee (x \in B)) \vee (x \in C)\} \text{ (assoc. of } \vee) \\
&= \{x : (x \in A \cup B) \vee (x \in C)\} \text{ (definition of union)} \\
&= (A \cup B) \cup C \text{ (definition of union)}
\end{aligned}
$$

The Size of the Union

Proposition

Let A and B be finite sets. Then $|A| + |B| = |A \cup B| + |A \cap B|$.

- Imagine attaching a label A to objects in the set A, and attaching a label B to objects in B. How many labels (in total) have we assigned?
  - On the one hand, the answer to this question is $|A| + |B|$ because we assign $|A|$ labels to the objects in A and $|B|$ labels to the objects in B.
  - On the other hand, we have assigned at least one label to the elements in $|A \cup B|$. So $|A \cup B|$ counts the number of objects that get at least one label. Elements in $A \cap B$ receive two labels. Thus, $|A \cup B| + |A \cap B|$ counts all elements that receive a label and double counts those elements that receive two labels. This gives the number of labels.

Since these two quantities, $|A| + |B|$ and $|A \cup B| + |A \cap B|$, answer the same question, they must be equal.

Combinatorial Proofs: The Basic Idea

- A **combinatorial proof** is used to demonstrate that an equation is true.
    - This is accomplished by creating a question and then arguing that both sides of the equation give a correct answer to the question.
    - It then follows, since both sides are correct answers to the *same question*, that the two sides of the alleged equation must be equal.
- Thus to **prove combinatorially** an equation of the form LHS = RHS:
    - Pose a question of the form, "In how many ways ... ?"
    - On the one hand, argue why LHS is a correct answer to the question.
    - On the other hand, argue why RHS is a correct answer.
    - Therefore, conclude that LHS = RHS.

Inclusion-Exclusion Principle

- A useful way to rewrite $|A| + |B| = |A \cup B| + |A \cap B|$ is
$$|A \cup B| = |A| + |B| - |A \cap B|;$$
- This is a special case of a counting method called the **inclusion-exclusion principle**.
- It can be interpreted as follows: Suppose we want to count the number of things that have one property or another. If the set A contains those things that have the one property and the set B contains those that have the other, then the set $A \cup B$ contains those things that have one property or the other. We can count those things by calculating $|A| + |B| - |A \cap B|$.
- This is useful when calculating $|A|$, $|B|$ and $|A \cap B|$ is easier than calculating $|A \cup B|$.

Inclusion-Exclusion: An Example

- Example: How many integers in the range 1 to 1000 (inclusive) are divisible by 2 or by 5?
  Set
  $$\begin{array}{rcl} A & = & \{x \in \mathbb{Z} : 1 \leq x \leq 1000 \text{ and } 2 \mid x\} \\ B & = & \{x \in \mathbb{Z} : 1 \leq x \leq 1000 \text{ and } 5 \mid x\} \end{array}$$
  The problem asks for $|A \cup B|$.
  It is not hard to see that $|A| = 500$ and $|B| = 200$. Now $A \cap B$ are those numbers (in the range from 1 to 1000) that are divisible by both 2 and 5, i.e., that are divisible by 10, so
  $A \cap B = \{x \in \mathbb{Z} : 1 \leq x \leq 1000 \text{ and } 10 \mid x\}$. It follows that
  $|A \cap B| = 100$. Therefore,
  $|A \cup B| = |A| + |B| - |A \cap B| = 500 + 200 - 100 = 600$.

Disjoint Sets

- Let A and B be sets. We call A and B **disjoint** provided $A \cap B = \emptyset$.
- Let $A_1, A_2, \ldots, A_n$ be a collection of sets. These sets are called **pairwise disjoint** provided $A_i \cap A_j = \emptyset$ whenever $i \neq j$. In other words, they are pairwise disjoint provided no two of them have an element in common.
- Example: Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ and $C = \{7, 8, 9\}$. These sets are pairwise disjoint because $A \cap B = A \cap C = B \cap C = \emptyset$. However, let $X = \{1, 2, 3\}$, $Y = \{4, 5, 6, 7\}$ and $Z = \{7, 8, 9, 10\}$. This collection of sets is not pairwise disjoint because $Y \cap Z \neq \emptyset$.

The Addition Principle

Corollary (Addition Principle)

Let A and B be finite sets. If A and B are disjoint, then $|A \cup B| = |A| + |B|$.

- There is an extension of the Addition Principle to more than two sets.
- If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

- A fancy way to write this is

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n} |A_k|.$$

Difference and Symmetric Difference

- Let A and B be sets. The **set difference**, $A - B$, is the set of all elements of A that are not in B:

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

- The **symmetric difference** of A and B, denoted $A \triangle B$, is the set of all elements in A but not B or in B but not A. That is, $A \triangle B = (A - B) \cup (B - A)$.
- Example: Suppose $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then $A - B = \{1, 2\}$, $B - A = \{5, 6\}$, and $A \triangle B = \{1, 2, 5, 6\}$.

Symmetric Difference in terms of Union and Intersection

Proposition

Let A and B be sets. Then $A \triangle B = (A \cup B) - (A \cap B)$.

- Suppose $x \in A \triangle B$. Thus $x \in (A - B) \cup (B - A)$. This means either $x \in A - B$ or $x \in B - A$.
  - Suppose $x \in A - B$. So $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B$. Since $x \notin B$, we have $x \notin A \cap B$. Thus, $x \in A \cup B$, but $x \notin A \cap B$. Therefore, $x \in (A \cup B) - (A \cap B)$.
  - Suppose $x \in B - A$. By the same argument as above, we have $x \in (A \cup B) - (A \cap B)$. Therefore, $x \in (A \cup B) - (A \cap B)$.
- Suppose $x \in (A \cup B) - (A \cap B)$. Thus, $x \in A \cup B$ and $x \notin A \cap B$. This means that x is in A or B but not both. Thus, either x is in A but not B or x is in B but not A. That is, $x \in (A - B)$ or $x \in (B - A)$. So $x \in (A - B) \cup (B - A)$. Therefore, $x \in A \triangle B$.
- The conclusion $A \triangle B = (A \cup B) - (A \cap B)$ follows.

DeMorgan's Laws

Proposition (DeMorgan's Laws)

Let A, B and C be sets. Then $A - (B \cup C) = (A - B) \cap (A - C)$ and $A - (B \cap C) = (A - B) \cup (A - C)$.

$$\begin{array}{ll} & x \in A - (B \cup C) \\ \text{iff} & x \in A \text{ and } x \notin B \cup C \\ \text{iff} & x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ \text{iff} & (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ \text{iff} & x \in A - B \text{ and } x \in A - C \\ \text{iff} & x \in (A - B) \cap (A - C). \end{array}$$

Cartesian Product

- Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (two-element lists) formed by taking an element from A together with an element from B in all possible ways. That is,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

- Example: Suppose $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Then $A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5)\}$, $B \times A = \{(3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3)\}$.
- Notice that $A \times B \neq B \times A$, so the Cartesian product of sets is not a commutative operation.

Proposition

Let A and B be finite sets. Then $|A \times B| = |A| \times |B|$.