# Fundamental Concepts of Mathematics 

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(1) Counting and Relations

- Relations
- Equivalence Relations
- Partitions
- Binomial Coefficients
- Counting Multisets
- The Inclusion-Exclusion Principle


## Subsection 1

## Relations

## Introducing Relations

- A relation is a comparison between two objects. The two objects either are or are not related according to some rule.
- Example: Less than $(<)$ is a relation defined on integers. Some pairs of numbers, such as $(2,8)$, satisfy the less-than relation, but other pairs of numbers do not, such as $(10,3)$.
- There are many other relations defined on the integers, such as divisibility, greater than, equality, and so on.
- There are relations on other sorts of objects. We can ask whether a pair of sets satisfies the $\subseteq$ relation or whether a pair of triangles satisfies the "is-congruent-to" relation.
- Our purpose is to study the relations themselves.


## Relations More Formally

- A relation is a set of ordered pairs.
- Example: $R=\{(1,2),(1,3),(3,0)\}$ is a relation.
- In truth, we think of a relation $R$ as a "test." If $x$ and $y$ are related by $R$ - if they pass the test - then we write $\times R y$. Otherwise, if they are not related by the relation $R$, we put a slash through the relation symbol, as in $x \neq y$ or $A \nsubseteq B$.
- The set of ordered pairs should be understood as a complete listing of all pairs of objects that "satisfy" the relation.
- Example: Think again of $R=\{(1,2),(1,3),(3,0)\}$. This says that, for the relation $R, 1$ is related to 2,1 is related to 3 , and 3 is related to 0 , and for any other objects $x, y$, it is not the case that $x$ is related to $y$. We can write, $(1,2) \in R,(1,3) \in R,(3,0) \in R,(5,6) \notin R$. This means that $(1,2),(1,3)$ and $(3,0)$ are related by $R$, but $(5,6)$ is not. We would usually write, $1 R 2,1 R 3,3 R 0,5 R 6$.
- $x R$ y means $(x, y) \in R$ and $x$ Ry means $(x, y) \notin R$.


## Familiar Relations

- The familiar relations of mathematics can be thought of in these terms.
- For example, the "less-than-or-equal-to" relation on the set of integers can be written as

$$
\{(x, y): x, y \in \mathbb{Z} \text { and } y-x \in \mathbb{N}\} .
$$

This says that $(x, y)$ is in the relation provided $y-x \in \mathbb{N}$, i.e., provided $y-x$ is a nonnegative integer, which is equivalent to $x \leq y$.

- We re-emphasize two important points:
- A relation $R$ is a set of ordered pairs $(x, y)$. We include an ordered pair in $R$ just when $(x, y)$ "satisfies" the relation $R$. Any set of ordered pairs constitutes a relation, and a relation does not have to be specified by a general "rule" or special principle.
- Even though relations are sets of ordered pairs, we usually do not write $(x, y) \in R$. Rather, we write $x R y$ and say, " $x$ is related to $y$ (by the relation $R$ )."


## Relation On/Between Sets

- Let $R$ be a relation and let $A$ and $B$ be sets.
- We say $R$ is a relation on $A$ provided $R \subseteq A \times A$.
- We say $R$ is a relation from $A$ to $B$ provided $R \subseteq A \times B$.
- Example: Let $A=\{1,2,3,4\}$ and $B=\{4,5,6,7\}$. Consider

$$
\begin{aligned}
R & =\{(1,1),(2,2),(3,3),(4,4)\} \\
S & =\{(1,2),(3,2)\} \\
T & =\{(1,4),(1,5),(4,7)\} \\
U & =\{(4,4),(5,2),(6,2),(7,3)\} \\
V & =\{(1,7),(7,1)\}
\end{aligned}
$$

- $R$ is a relation on $A$. Note that it is the equality relation on $A$.
- $S$ is a relation on $A$. Note that element 4 is never mentioned.
- $T$ is a relation from $A$ to $B$. Note that elements $2,3 \in A$, and $6 \in B$ are never mentioned.
- $U$ is a relation from $B$ to $A$. Note that $1 \in A$ is never mentioned.
- $V$ is a relation, but it is neither a relation from $A$ to $B$ nor a relation from $B$ to $A$.


## Set Operations and Inverse Relations

- Since a relation is a set, all the various set operations apply to relations, such as intersection, union, etc.
- Let R be a relation. The inverse of $R$, denoted $R^{-1}$, is the relation formed by reversing the order of all the ordered pairs in $R$. In symbols,

$$
R^{-1}=\{(x, y):(y, x) \in R\}
$$

- Example: Let $R=\{(1,5),(2,6),(3,7),(3,8)\}$. Then $R^{-1}=\{(5,1),(6,2),(7,3),(8,3)\}$.
- If $R$ is a relation on $A$, so is $R^{-1}$.
- If $R$ is a relation from $A$ to $B$, then $R^{-1}$ is a relation from $B$ to $A$.


## Proposition

Let $R$ be a relation. Then $\left(R^{-1}\right)^{-1}=R$.

- Suppose $(x, y) \in R$. Then $(y, x) \in R^{-1}$ and thus $(x, y) \in\left(R^{-1}\right)^{-1}$. Now suppose $(x, y) \in\left(R^{-1}\right)^{-1}$. Then $(y, x) \in R^{-1}$ and so $(x, y) \in R$.


## Properties of Relations

- Let $R$ be a relation defined on a set $A$.
- If for all $x \in A$ we have $x R x$, we call $R$ reflexive.
- If for all $x \in A$ we have $x \mathbb{R} x$, we call $R$ irrefiexive.
- If for all $x, y \in A$ we have $x R y \Rightarrow y R x$, we call $R$ symmetric.
- If for all $x, y \in A$ we have $(x R y \wedge y R x) \Rightarrow x=y$, we call $R$ antisymmetric.
- If for all $x, y, z \in A$ we have $(x R y \wedge y R z) \Rightarrow x R z$, we call $R$ transitive.


## Examples I

- Consider the relation $=$ (equality) on the integers. It is
- reflexive (any integer is equal to itself),
- symmetric (if $x=y$, then $y=x$ ),
- transitive (if $x=y$ and $y=z$, then we must have $x=z$ ).

The relation $=$ is also antisymmetric. However, it is not irreflexive (which would say that $x \neq x$ for all $x \in \mathbb{Z}$ ).

- Consider the relation $\leq$ (less than or equal to) on the integers. Note that $\leq$ is
- reflexive because for any integer $x$, it is true that $x \leq x$;
- transitive, since $x \leq y$ and $y \leq z$ imply that $x \leq z$.

The relation $\leq$ is not symmetric because that would mean that $x \leq y \Rightarrow y \leq x$, which is false. However, $\leq$ is antisymmetric: If we know $x \leq y$ and $y \leq x$, it must be the case that $x=y$. Finally, $\leq$ is not irreflexive; for instance, $5 \leq 5$.

## Examples ||

- Consider the relation $<$ (strict less than) on the integers.
- $<$ is not reflexive because, for example, $3<3$ is false.
- Further, $<$ is irreflexive because $x<x$ is never true.
- The relation $<$ is not symmetric because $x<y$ does not imply $y<x$; for example, $0<5$ but $5 \nless 0$.
- The relation $<$ is antisymmetric, but it fulfills the condition vacuously. The condition states $(x<y$ and $y<x) \Rightarrow x=y$. However, it is impossible to have both $x<y$ and $y<x$, so the hypothesis of this if-then statement can never be satisfied. Therefore it is true.
- Finally, < is transitive.
- Consider the relation $\mid$ (divides) on the natural numbers.
- | is antisymmetric because, if $x$ and $y$ are natural numbers with $x \mid y$ and $y \mid x$, then $x=y$. However, the relation $\mid$ on the integers is not antisymmetric. For example, $3 \mid-3$ and $-3 \mid 3$, but $3 \neq-3$.
- Also notice that $\mid$ is not symmetric (e.g., $3 \mid 9$, but $9 \nmid 3$ ).
- The properties of relations depend on the context of the relation. The | on the integers is different from the $\mid$ on the natural numbers.


## Subsection 2

## Equivalence Relations

## Definition of Equivalence Relation

- Certain relations bear a strong resemblance to the relation equality. A good example (from geometry) is the "is-congruent-to" relation (often denoted by $\cong$ ) on the set of triangles. Congruent triangles are not equal, but in a sense, they act like equal triangles. What is special about $\cong$ that it acts like equality?
- Of the five properties of relations, $\cong$ is reflexive, symmetric, and transitive (but it is neither irreflexive nor antisymmetric). Relations with these three properties are akin to equality.


## Definition (Equivalence Relation)

Let $R$ be a relation on a set $A$. We say $R$ is an equivalence relation provided it is reflexive, symmetric, and transitive.

## Equipotence Relation on Finite Sets

- Consider the "has-the-same-size-as" relation on finite sets. For finite sets of integers $A$ and $B$, let $A R B$ provided $|A|=|B|$. $R$ is reflexive, symmetric, and transitive and, therefore, is an equivalence relation. It is not the case that two sets with the same size are the same. For example, $\{1,2,3\} R\{2,3,4\}$, but $\{1,2,3\} \neq\{2,3,4\}$. Nonetheless, sets related by $R$ are "like" each other in that they share a common property: their size.


## Congruence Modulo n

- Let $n$ be a positive integer. We say that integers $x$ and $y$ are congruent modulo $n$, and we write

$$
x \equiv y \quad(\bmod n)
$$

provided $n \mid(x-y)$.

- In other words, $x \equiv y(\bmod n)$ if and only if $x$ and $y$ differ by a multiple of $n$.
- Example:
- $3 \equiv 13(\bmod 5)$ because $3-13=-10$ is a multiple of 5 .
- $4 \equiv 4(\bmod 5)$ because $4-4=0$ is a multiple of 5 .
- $16 \not \equiv 3(\bmod 5)$ because $16-3=13$ is not a multiple of 5 .


## Congruence Modulo $n$ is an Equivalence Relation on $\mathbb{Z}$

## Theorem

Let $n$ be a positive integer. The "is-congruent-to-mod- $n$ " relation is an equivalence relation on the set of integers.

- Let $n$ be a positive integer and let $\equiv$ denote congruence mod $n$. We need to show that $\equiv$ is reflexive, symmetric, and transitive.
- $\equiv$ is reflexive: Let $x$ be an arbitrary integer. Since $0 \cdot n=0$, we have $n \mid 0$, which we can rewrite as $n \mid(x-x)$. Therefore $x \equiv x$.
$0 \equiv$ is symmetric: Let $x$ and $y$ be integers and suppose $x \equiv y$. This means that $n \mid(x-y)$. So there is an integer $k$ such that $(x-y)=k n$. But then $(y-x)=(-k) n$. And so $n \mid(y-x)$. Therefore $y \equiv x$.
$0 \equiv$ is transitive: Let $x, y$ and $z$ be integers and suppose $x \equiv y$ and $y \equiv z$. This means that $n \mid(x-y)$ and $n \mid(y-z)$. So there are integers $k$, $l$ such that $(x-y)=k n$ and $(y-z)=I n$. But then

$$
x-z=(x-y)+(y-z)=k n+\ln =(k+I) n .
$$

And so $n \mid(x-z)$. Thus, $x \equiv z$. Therefore, $\equiv$ is transitive.

- So $\equiv$ is an equivalence relation.


## Equivalence Classes

- Let $R$ be an equivalence relation on a set $A$ and let $a \in A$. The equivalence class of $a$, denoted [a], is the set of all elements of $A$ related by $R$ to $a$; that is,

$$
[a]=\{x \in A: x R a\} .
$$

- Example: Consider the equivalence relation "congruence mod 2". What is $[1]$ ? By definition, $[1]=\{x \in \mathbb{Z}: x \equiv 1(\bmod 2)\}$. This is the set of all integers $x$ such that $2 \mid(x-1)$ i.e., $x-1=2 k$ for some $k$. So $x=2 k+1$ (i.e., $x$ is odd)! The set [1] is the set of odd numbers.
Similarly, it is not hard to see that [0] is the set of even numbers. Consider [3]. We can prove that [3] is the set of odd numbers, so $[1]=[3]$.
The equivalence relation "congruence mod 2 " has only two equivalence classes: the set of odd integers [1] and the set of even integers [0].


## Equipotency Classes

- Let $R$ be the "has-the-same-size-as" relation defined on the set of finite subsets of $\mathbb{Z}$.
What is $[\emptyset]$ ? By definition, $[\emptyset]=\{A \subseteq \mathbb{Z}:|A|=0\}=\{\emptyset\}$.
What is $[\{2,4,6,8\}]$ ? The set of all finite subsets of $\mathbb{Z}$ related to $\{2,4,6,8\}$ are exactly those of size 4 :
$[\{2,4,6,8\}]=\{A \subseteq \mathbb{Z}:|A|=4\}$.
The relation $R$ separates the set of finite subsets of $\mathbb{Z}$ into infinitely many equivalence classes (one for each element of $\mathbb{N}$ ). Every class contains sets that are related to each other but not to any elements outside that class.


## Properties of Equivalence Classes I

## Proposition (Equivalence Classes Are Not Empty)

Let $R$ be an equivalence relation on a set $A$ and let $a \in A$. Then $a \in[a]$.

- By definition [a] $=\{x \in A: x R a\}$. To show that $a \in[a]$, we just need to show that a $R$ a. This is true by definition since $R$ is reflexive.


## Proposition (Equality of Classes of Related Elements)

Let $R$ be an equivalence relation on a set $A$ and let $a, b \in A$. Then a $R b$ if and only if $[a]=[b]$.

- Suppose a $R b$. We need to show that $[a]$ and $[b]$ are the same.
- Suppose $x \in[a]$. This means that $x R$ a. Since a $R b$, transitivity yields $x R b$. Therefore, $x \in[b]$.
- Suppose $y \in[b]$. This means that y $R b$. We are given a $R b$, and we get, by symmetry, $b R$ a. By transitivity, $y R a$. Therefore $y \in[a]$.
Hence $[a]=[b]$.
- Suppose $[a]=[b]$. We know that $a \in[a]=[b]$. Therefore a $R b$.


## Properties of Equivalence Classes II

## Proposition (Elements in the Same Class are Related)

Let $R$ be an equivalence relation on a set $A$ and let $a, x, y \in A$. If $x, y \in[a]$, then $x R y$.

- Suppose $x, y \in[a]$. Then $x R$ a and $y R$ a. By symmetry, a $R$ and, by transitivity, $x R y$.


## Proposition (Different Classes are Disjoint)

Let $R$ be an equivalence relation on $A$ and $[a] \cap[b] \neq \emptyset$. Then $[a]=[b]$.

- Suppose $[a] \cap[b] \neq \emptyset$. There is an $x \in[a] \cap[b]$. So $x \in[a]$ and $x \in[b]$. Thus, $x R$ a and $x R b$. By symmetry, a $R x$. By transitivity, a $R$. By previous slide, $[a]=[b]$.


## Corollary (Equivalence Classes Partition A)

Let $R$ be an equivalence relation on a set $A$. The equivalence classes of $R$ are nonempty, pairwise disjoint subsets of $A$ whose union is $A$.

## Subsection 3

## Partitions

## Partitions

- We saw that if $R$ be an equivalence relation on a set $A$, the equivalence classes of $R$ are nonempty, pairwise disjoint subsets of $A$ whose union is $A$.



## Definition (Partition)

Let $A$ be a set. A partition of (or on) $A$ is a set of nonempty, pairwise disjoint sets whose union is $A$.

- There are four key points in this definition:
- A partition is a set of sets; each member of a partition is a subset of $A$. The members of the partition are called parts.
- The parts of a partition are nonempty. The empty set is never a part.
- The parts of a partition are pairwise disjoint. No two parts have an element in common.
- The union of the parts is the original set.


## Examples

- Let $A=\{1,2,3,4,5,6\}$ and let $\mathcal{P}=\{\{1,2\},\{3\},\{4,5,6\}\}$. This is a partition of $A$ into three parts. The three parts are $\{1,2\},\{3\}$ and $\{4,5,6\}$. They are nonempty, pairwise disjoint, and their union is $A$.
- The partition $\{\{1,2\},\{3\},\{4,5,6\}\}$ is not the only partition of $A=\{1,2,3,4,5,6\}$. Two more partitions are
- $\{\{1,2,3,4,5,6\}\}$; Just one part containing all the elements of $A$.
- $\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$. Six parts, each with just one element.


## Corollary

Let $R$ be an equivalence relation on a set $A$. The equivalence classes of $R$ form a partition of the set $A$.

- Given an equivalence relation on a set, the equivalence classes of that relation form a partition of the set. We start with an equivalence relation, and we form a partition.
- We can also go the other way: given a partition, there is a natural way to construct an equivalence relation.


## From Partitions to Equivalence Relations

- Let $\mathcal{P}$ be a partition of a set $A$. We use $\mathcal{P}$ to form a relation on $A$. We call this relation the "is-in-the-same-part-as" relation and denote it by $\stackrel{\mathcal{P}}{\equiv}$.
- If $a, b \in A$, then

$$
a \stackrel{\mathcal{P}}{\equiv} b \quad \Longleftrightarrow \quad \exists P \in \mathcal{P}, a, b \in P
$$

In words, $a$ and $b$ are related by $\stackrel{\mathcal{P}}{=}$ provided there is some part of the partition $\mathcal{P}$ that contains both $a$ and $b$.

Proposition ( $\stackrel{\mathcal{P}}{=}$ is an Equivalence Relation)
Let $A$ be a set and let $\mathcal{P}$ be a partition on $A$. The relation $\stackrel{\mathcal{P}}{\equiv}$ is an equivalence relation on $A$.

## $\stackrel{\mathcal{P}}{\underline{\mathcal{P}}}$ is an Equivalence Relation

- To show that $\stackrel{\mathcal{P}}{\equiv}$ is an equivalence relation, we must show that it is reflexive, symmetric and transitive.
- $\stackrel{\mathcal{P}}{\equiv}$ is reflexive: Let a be an arbitrary element of $A$. Since $\mathcal{P}$ is a partition, i.e., the union of the parts is $A$, there must be a part $P \in \mathcal{P}$ that contains $a$. We have $a \stackrel{\mathcal{P}}{=} a$, since $a, a \in P \in \mathcal{P}$.
- $\stackrel{\mathcal{P}}{=}$ is symmetric: Suppose $a \stackrel{\mathcal{P}}{=} b$ for $a, b \in A$. This means there is a $P \in \mathcal{P}$ such that $a, b \in \mathcal{P}$. Since $b$ and $a$ are in the same part of $\mathcal{P}$, we have $b \stackrel{\mathcal{P}}{=} a$.
- $\stackrel{\mathcal{P}}{=}$ is transitive: Let $a, b, c \in A$ and suppose $a \stackrel{\mathcal{P}}{=} b$ and $b \stackrel{\mathcal{P}}{=} c$. Since $a \stackrel{\mathcal{P}}{\equiv} b$, there is a part $P \in \mathcal{P}$ containing both $a$ and $b$. Since $b \stackrel{\mathcal{P}}{=} c$, there is a part $Q \in \mathcal{P}$ with $b, c \in Q$. Notice that $b$ is in both $P$ and $Q$. Thus, parts $P$ and $Q$ have a common element. Since parts of a partition must be pairwise disjoint, it must be the case that $P=Q$. Therefore, all three of $a, b, c$ are together in the same part of $\mathcal{P}$. Since $a, c$ are in a common part of $\mathcal{P}$, we have $a \stackrel{\mathcal{P}}{\equiv} c$.


## Equivalence Classes of $\underset{=}{\underline{P}}$

## Proposition (Classes of $\stackrel{\mathcal{P}}{=}$ and Parts of $\mathcal{P}$ )

Let $\mathcal{P}$ be a partition on a set $A$ and let $\stackrel{\mathcal{P}}{\equiv}$ be the "is-in-the-same-part-as" relation. The equivalence classes of $\underset{\equiv}{\equiv}$ are exactly the parts of $\mathcal{P}$.

- For $x \in A$, let $[x]$ denote the equivalence class of $x$ in $\stackrel{\mathcal{P}}{\equiv}$ and $P_{x}$ the part of $\mathcal{P}$ in which $x$ belongs. We want to show $\{[a]: a \in A\}=\mathcal{P}$. Indeed, we have: $[a]=\{b \in A: b \stackrel{\mathcal{P}}{\equiv} a\}=\left\{b \in A: b \in P_{a}\right\}=P_{a}$.
- Thus, equivalence relations and partitions are flip sides of the same mathematical coin.
- Given a partition, we can form the "in-the-same-part-as" equivalence relation.
- Given an equivalence relation, we can form the partition into equivalence classes.


## Counting Words With Different Letters

- In how many ways can the letters in the word WORD be rearranged?

A word is simply a list of letters. We have a list of four possible letters, and we want to count lists using each of them exactly once. This is a problem we have already solved. The answer is $4!=24$.

## Counting Words With Repeated Letters

- In how many ways can the letters in the word HELLO be rearranged? If there were no repeated letters, then the answer would be $5!=120$. Imagine for a moment that the two Ls are different letters, for instance, one larger than the other: HELLO. If we were to write down all 120 different ways to rearrange the letters in HELLO, we would have a chart that looks like this:
hello helol hello helol heoll heoll

Now we shrink the large Ls back to their proper size. When we do, we can no longer distinguish between HELLO and HELLO, or between LEHLO and LEHLO. There are 120 entries in the chart and each rearrangement of HELLO appears exactly twice on the chart. So there are exactly 60 different arrangements!

## Using Equivalence Relations to Count

- Think about this by using equivalence relations and partitions.
- The set $A$ is the set of all 120 different rearrangements of HELLO.
- Suppose $a$ and $b$ are elements of A (anagrams of HELLO). Define a relation $R$ with a $R$ brovided that $a$ and $b$ give the same rearrangement of HELLO when we shrink $L$ to $L$. For example, (HELOL) $R$ (HELOL).
- $R$ is reflexive, symmetric, and transitive and so it is an equivalence relation.
- The equivalence classes of $R$ are all the different ways of rearranging HELLO that look the same when we shrink L. For example, $[\mathrm{HLEOL}]=\{\mathrm{HLEOL}, \mathrm{HLEOL}\}$.
- The important point is that the number of ways to rearrange the letters in HELLO is exactly the same as the number of equivalence classes of $R$ !


## Finishing Up the Arithmetic

- Let's do the arithmetic:
(1) There are 120 different ways to rearrange the letters in HELLO (i.e., $|A|=120)$.
(2) The relation $R$ partitions the set $A$ into a certain number of equivalence classes. Each equivalence class has exactly two elements in it.
(3) So all told, there are $\frac{120}{2}=60$ different equivalence classes.
(9) Hence, there are 60 different ways to rearrange HELLO.


## Another Example

- In how many different ways can we rearrange the letters in the word AARDVARK?
- This 8-letter word features two Rs and three As. We use $R$ and $R$ and A, A and A, so the word is AARDVARK.
- Let $X$ be the set of all rearrangements of AARDVARK.
- We consider two spellings to be related by relation $R$ if they are the same once their letters are restored to black. $R$ is an equivalence relation on $X$.
- We want to count the number of equivalence classes. Let us consider the size of the equivalence class [RADAKRAV]. These are all the rearrangements that become RADAKRAV when their letters are all the same color. How many are there?


## Counting the Number in [ ADAKRAV]

- We want to count the number of lists wherein the entries on the list satisfy the following restrictions:
- Elements 3,5, and 8 of the list must be $\mathrm{D}, \mathrm{K}$, and V ;
- Elements 1 and 6 must be one each of two different colors of R;
- Elements 2, 4, and 7 must be one each of three different colors of A.
- In how many ways can we build this list?
- There are two choices for the first position (we can use R or R);
- There are three choices for the second position (we can use A, A or A);
- There is only one choice for position 3 (it must be D);
- Now, given the choices thus far, there are only two choices for position 4 (the first A has already been selected);
- For each of the remaining positions, there is only one choice (the K and $V$ are predetermined, and we have one choice each on $A$ and $R$ ).
- Therefore, the number of rearrangements of AARDVARK in [RADAKRAV] is $2 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1=3!\cdot 2!=12$.
- All equivalence classes have the same size! So the number of rearrangements of AARDVARK is $\frac{8!}{3!2!}=\frac{40320}{12}=3360$.


## Counting Theorem

## Theorem (Counting Using Equivalence Classes)

Let $R$ be an equivalence relation on a finite set $A$. If all the equivalence classes of $R$ have the same size $m$, then the number of equivalence classes is $|A| / m$.

- There is an important hypothesis in this result: The equivalence classes must all be the same size. This does not always happen.
- Example: Let $A=2^{\{1,2,3,4\}}$, i.e., the set of all subsets of $\{1,2,3,4\}$. Let $R$ be the "has-the-same-size-as" relation. This relation partitions $A$ into five parts (subsets of size 0 through 4 ). The sizes of these equivalence classes are not all the same!
For example, $[\emptyset]$ contains only $\emptyset$, so that class has size 1 . However, $[\{1\}]=\{\{1\},\{2\},\{3\},\{4\}\}$, so this class contains four members of $A$.


## Subsection 4

## Binomial Coefficients

## Definition of Binomial Coefficients

- How many subsets of size $k$ does an $n$-element set have?


## Definition (Binomial Coefficients)

Let $n, k \in \mathbb{N}$. The symbol $\binom{n}{k}$ denotes the number of $k$-element subsets of an $n$-element set.

- We call the number $\binom{n}{k}$ a binomial coefficient because $\binom{n}{k}$ are the coefficients in the expansion of the binomial $(x+y)^{n}$.
- Example: Evaluate ( $\left.\begin{array}{l}5 \\ 0\end{array}\right)$.

We need to count the number of subsets of a 5-element set that have zero elements. The only possible such set is $\emptyset$, so $\binom{5}{0}=1$.

- Example: Evaluate $\binom{5}{1}$.

We need to count the number of subsets of a 5 -element set that have one element. For example, if the set is $\{1,2,3,4,5\}$, then the subsets are $\{1\},\{2\},\{3\},\{4\},\{5\}$, so the answer is $\binom{5}{1}=5$.

- In general $\binom{n}{0}=1$ and $\binom{n}{1}=n$;


## Number of 2-Element Subsets

- Evaluate $\binom{5}{2}$.

The symbol $\binom{5}{2}$ stands for the number of 2-element subsets of a 5-element set.
The simplest thing to do is to list all the possibilities:

$$
\begin{array}{llll}
\{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} \\
\{2,3\} & \{2,4\} & \{2,5\} & \\
\{3,4\} & \{3,5\} & & \\
\{4,5\} & & &
\end{array}
$$

Therefore, there are 10 2-element subsets of a five-element set, which yields $\binom{5}{2}=4+3+2+1=10$.

## 2-Element Subsets of an $n$-Element Set

- Suppose that the $n$-element set is $\{1,2,3, \ldots, n\}$.
- We can make a chart as in the example.
- The first row lists the 2-element subsets whose smaller element is 1 ;
- The second row lists those 2-element subsets whose smaller element is 2, and so on;
- The last row lists the (one and only) 2-element subset whose smaller element is $n-1$ (i.e., $\{n-1, n\}$ ).
- Our chart exhausts all the possibilities and no duplication takes place.
- The number of sets
- in the first row is $n-1$;
- in the second row is $n-2$;
- in row $k$ there are $n-k$ elements.


## Proposition

Let $n$ be an integer with $n \geq 2$. Then
$\binom{n}{2}=1+2+3+\ldots+(n-1)=\sum_{k=1}^{n-1} k$.

## 2-Element and 3-Element Subsets of a 5-Element Set

- Evaluate $\binom{5}{3}$.

We simply list the three-element subsets of $\{1,2,3,4,5\}$ :

$$
\begin{array}{lllll}
\{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,3,4\} & \{1,3,5\} \\
\{1,4,5\} & \{2,3,4\} & \{2,3,5\} & \{2,4,5\} & \{3,4,5\}
\end{array}
$$

There are ten such sets, so $\binom{5}{3}=10$.

- Notice that $\binom{5}{2}=\binom{5}{3}=10$. Why are these two numbers equal?

Because there is a natural way to match up the 2-element subsets of $\{1,2,3,4,5\}$ with the 3 -element subsets. More precisely, we take the complement of a 2-element subset to form a 3-element subset, or vice versa:

| $A$ | $\bar{A}$ | $A$ | $\bar{A}$ |
| :---: | :---: | :---: | :---: |
| $\{1,2\}$ | $\{3,4,5\}$ | $\{2,4\}$ | $\{1,3,5\}$ |
| $\{1,3\}$ | $\{2,4,5\}$ | $\{2,5\}$ | $\{1,3,4\}$ |
| $\{1,4\}$ | $\{2,3,5\}$ | $\{3,4\}$ | $\{1,2,5\}$ |
| $\{1,5\}$ | $\{2,3,4\}$ | $\{3,5\}$ | $\{1,2,4\}$ |
| $\{2,3\}$ | $\{1,4,5\}$ | $\{4,5\}$ | $\{1,2,3\}$ |

The $A \leftrightarrow \bar{A}$ is a one-to-one correspondence between the 2-element and 3 -element subsets of $\{1,2,3,4,5\}$.

## Obtaining a General Combinatorial Formula

- Instead of forming the complement of the two-element subsets of $\{1,2, \ldots, n\}$, we can form the complements of subsets of another size.
- What are the complements of the $k$-element subsets of $\{1,2, \ldots, n\}$ ? They are the $(n-k)$-element subsets.
- The correspondence $A \leftrightarrow \bar{A}$ gives a one-to-one pairing of the $k$-element and ( $n-k$ )-element subsets of $\{1,2, \ldots, n\}$.
- Thus, the number of $k$ - and $(n-k)$-element subsets of an $n$-element set must be the same:


## Proposition

Let $n, k \in \mathbb{N}$ with $0 \leq k \leq n$. Then $\binom{n}{k}=\binom{n}{n-k}$.

## Alternative Interpretation of $\binom{n}{k}=\binom{n}{n-k}$

- Imagine a class with $n$ children. The teacher has $k$ identical candy bars to give to exactly $k$ of the children. In how many ways can the candy bars be distributed?
- The answer is $\binom{n}{k}$ because we are selecting a set of $k$ lucky children to get candy.
- But, alternatively, we can select the unfortunate children who will not be receiving candy.
There are $n-k$ children who do not get candy, and we can select that subset in $\binom{n}{n-k}$ ways.
- Since the two counting problems are clearly the same, we must have $\binom{n}{k}=\binom{n}{n-k}$.


## Extending the Symbol $\binom{n}{k}$ for $k>n$

- Thus far we have evaluated $\binom{5}{0},\binom{5}{1},\binom{5}{2}$ and $\binom{5}{3}$.
- By the previous Proposition $\binom{5}{4}=\binom{5}{1}=5$.
- Also by the previous Proposition $\binom{5}{5}=\binom{5}{0}=1$.
- If we attempted to use the same proposition for $\binom{5}{6}$, we get $\binom{5}{6}=\binom{5}{5-6}=\binom{5}{-1}$. Not only do we not know what $\binom{5}{-1}$ is, but, in addition, it does not make sense to ask for the number of subsets of a five-element set that have -1 elements! However, a set can have six elements, so $\binom{5}{6}$ is not nonsense; it is simply zero.
- Similarly, $\binom{5}{7}=\binom{5}{8}=\cdots=0$;
- Summary:
- The values of $\binom{5}{k}$ are $1,5,10,10,5,1,0,0, \ldots$, for $k=0,1,2, \ldots$;
- $\binom{n}{0}=1$ and $\binom{n}{1}=n$;
- $\binom{n}{2}=1+2+\cdots+(n-1)$.
- $\binom{n}{k}=\binom{n}{n-k}$;
- If $k>n,\binom{n}{k}=0$.


## The Binomial Theorem

- We found that the nonzero values of $\binom{5}{k}$ are $1,5,10,10,5,1$.
- If we expand $(x+y)^{5}$, we get

$$
\begin{aligned}
(x+y)^{5} & =1 x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 x y^{4}+1 y^{5} \\
& =\binom{5}{0} x^{5}+\binom{5}{1} x^{4} y+\binom{5}{2} x^{3} y^{2}+\binom{5}{3} x^{2} y^{3}+\binom{5}{4} x y^{4}+\binom{5}{5} y^{5}
\end{aligned}
$$

- This suggests a way to calculate $\binom{n}{k}$ : Expand $(x+y)^{n}$ and $\binom{n}{k}$ is the coefficient of $x^{n-k} y^{k}$.


## Binomial Theorem

Let $n \in \mathbb{N}$. Then

$$
(x+y)^{n}=\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k} .
$$

## Proof of the Binomial Theorem

- Think about how we multiply polynomials: When we multiply $(x+y)^{2}$, we calculate as follows: $(x+y)^{2}=(x+y)(x+y)=$ $x x+x y+y x+y y$ and then we collect like terms to get $x^{2}+2 x y+y^{2}$.
- The procedure for $(x+y)^{n}$ is similar: Write $n$ factors: $\underbrace{(x+y)}_{\text {factor } 1} \underbrace{(x+y)}_{\text {factor } 2} \cdots \underbrace{(x+y)}_{\text {factor } n}$. Form all possible terms by taking either an $x$ or a $y$ from factors $1,2, \ldots, n$. This is like forming all possible $n$-element lists where each element is either an $x$ or a $y$.
- How many terms in $(x+y)^{n}$ have precisely $k y s$ (and $n-k x s$ )? We can specify all the lists with $k y s$ (and $n-k x s$ ) by reporting the positions of the $y s$ (the $x s$ fill in the remaining positions). For example, if $n=10$ and we say that the set of $y$ positions is $\{2,3,7\}$, then we know we are speaking of the term xyyxxxyxxx. So the number of lists with $k y s$ and $n-k x$ s is exactly the same as the number of $k$-element subsets of $\{1,2, \ldots, n\}$. Therefore the number of $x^{n-k} y^{k}$ terms is $\binom{n}{k}$.


## Pascal's Identity

## Theorem (Pascal's Identity)

Let $n$ and $k$ be integers with $0<k<n$. Then

$$
\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k} .
$$

- Consider the question: How many k-element subsets does the set $\{1,2,3, \ldots, n\}$ have?
- Answer 1: $\binom{n}{k}$ by definition;
- Answer 2: Distinguish an element - from the given $n$-element set. To pick $k$ elements from the $n$-element set, we either
- include - and we pick $k-1$ more elements from the remaining $n-1$ elements, which can be done in $\binom{n-1}{k-1}$ ways, or
- we do no include - and we pick all $k$ elements from the $n-1$ other elements, which can be done in $\binom{n-1}{k}$ ways.
These mutually exclusive possibilities cover all cases. The sum of these two must equal the total number $\binom{n}{k}$ of $k$ subsets of the $n$-element set.


## An Example of Pascal's Identity

- We show that $\binom{6}{2}=\binom{5}{1}+\binom{5}{2}$ by listing all the two-element subsets of $\{1,2,3,4,5,6\}$.
- Let us consider 6 a distinguished element.
- There are $\binom{5}{1}=5$ two-element subsets that include the distinguished element 6:

$$
\{1,6\} \quad\{2,6\} \quad\{3,6\} \quad\{4,6\} \quad\{5,6\} .
$$

- There are $\binom{5}{2}=10$ two-element subsets that do not include 6 :

$$
\begin{array}{lllll}
\{1,2\} & \{1,3\} & \{1,4\} & \{1,5\} & \{2,3\} \\
\{2,4\} & \{2,5\} & \{3,4\} & \{3,5\} & \{4,5\}
\end{array}
$$

- These $15=\binom{6}{2}$ sets are all 2-element subsets of $\{1,2,3,4,5,6\}$; So, we get an illustration of why $\binom{6}{2}=\binom{5}{1}+\binom{5}{2}$.


## A Formula for $\binom{n}{k}$

## Theorem (Formula for $\binom{n}{k}$ )

Let $n$ and $k$ be integers with $0 \leq k \leq n$. Then

$$
\binom{n}{k}=\frac{n!}{k!(n-k)!} .
$$

- We want to count the number of $k$-element subsets of $\{1,2, \ldots, n\}$.
- Instead, we consider the $k$-element, repetition-free lists we can form from $\{1,2, \ldots, n\}$.
The number of $k$-element, repetition-free lists we can form from $\{1,2, \ldots, n\}$ is $(n)_{k}=\frac{n!}{(n-k)!}$.
- We declare two lists equivalent if they contain the same members. The number of elements in each equivalence class is $k$ !.
- We compute the number of equivalence classes to calculate $\binom{n}{k}$.

The number of equivalence classes is $\frac{(n)_{k}}{k!}=\frac{n!}{k!(n-k)!}$.

## Examples

- Find the number of 16 -element subsets of a 20 -element set.

$$
\binom{20}{16}=\frac{20!}{16!\cdot 4!}=\frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1}=4845 .
$$

- In how many ways can a subcommittee of 3 people be formed out of the members of a committee of 25 people?

$$
\binom{25}{3}=\frac{25!}{3!\cdot 22!}=\frac{25 \cdot 24 \cdot 23}{3 \cdot 2 \cdot 1}=2300 .
$$

## Subsection 5

## Counting Multisets

## Multisets and Equality

- A given object either is or is not in a set. An element cannot be in a set "twice". So, the following sets are all identical: $\{1,2,3\}=\{3,1,2\}=\{1,1,2,2,3,3\}=\{1,2,3,1,2,3,1,1,1,1\}$.
- A multiset is a generalization of a set.
- A multiset is, like a set, an unordered collection of elements.
- However, in a multiset, an object may be considered to be in the multiset more than once.
- We write a multiset as $\langle 1,2,3,3\rangle$. This multiset contains four elements: 1,2 , and 3 counted twice. We say that 3 has multiplicity 2 in the multiset $\langle 1,2,3,3\rangle$.
- The multiplicity of an element is the number of times it is a member of the multiset.
- Two multisets are the same or equal provided they contain the same elements with the same multiplicities. For example, $\langle 1,2,3,3\rangle=\langle 3,1,3,2\rangle$, but $\langle 1,2,3,3\rangle \neq\langle 1,2,3,3,3\rangle$.


## Cardinality and the Symbol $\left.\binom{n}{k}\right)$

- The cardinality of a multi set is the sum of the multiplicities of its elements, i.e., the number of elements in the multi set where we take into account the number of times each element is present.
- If $M$ is a multiset, then $|M|$ denotes its cardinality. $|\langle 1,2,3,3\rangle|=4$.
- How many $k$-element multisets can we form by choosing elements from an $n$-element set? I.e., how many unordered length- $k$ lists can we form using the elements $\{1,2, \ldots, n\}$ with repetition allowed?


## Definition of $\left.\binom{n}{k}\right)$

Let $n, k \in \mathbb{N}$. The symbol $\left.\binom{n}{k}\right)$ denotes the number of multisets with cardinality equal to $k$ whose elements belong to an $n$-element set such as $\{1,2, \ldots, n\}$.

- Example: Evaluate $\binom{n}{1}$ ). How many one-element multisets can be formed from $\{1,2, \ldots, n\}$ ? The multisets are $\langle 1\rangle,\langle 2\rangle, \ldots,\langle n\rangle$, whence $\left(\binom{n}{1}\right)=n$.


## More Examples I

- Let $k$ be a positive integer. Evaluate $\left.\binom{1}{k}\right)$.

This asks for the number of $k$-element multisets whose elements are selected from $\{1\}$. Since there is only one possible member and the multiset has cardinality $k$, the only possibility is $\underbrace{\langle 1,1, \ldots, 1\rangle}_{k \text { occurrences }}$. So $\left.\binom{1}{k}\right)=1$.

- Evaluate $\binom{2}{2}$ ). We need to count the number of 2-element multisets whose elements are selected from the set $\{1,2\}$. We list all the possibilities:

$$
\langle 1,1\rangle \quad\langle 1,2\rangle \quad\langle 2,2\rangle .
$$

Therefore $\left(\binom{2}{2}\right)=3$.

## More Examples II

- Evaluate $\left.\binom{2}{k}\right)$.

We need to form a $k$-element multiset using only the elements 1 and 2. We can decide how many 1 s are in the multiset $(k+1$ possibilities), and then the remaining elements of the multiset must be 2 s . Therefore $\left(\binom{2}{k}\right)=k+1$.

- Evaluate ( $\binom{3}{3}$ ).

We need to count the number of 3-element multi sets whose elements are selected from the set $\{1,2,3\}$. We list all the possibilities:

$$
\begin{array}{lllll}
\langle 1,1,1\rangle & \langle 1,1,2\rangle & \langle 1,1,3\rangle & \langle 1,2,2\rangle & \langle 1,2,3\rangle \\
\langle 1,3,3\rangle & \langle 2,2,2\rangle & \langle 2,2,3\rangle & \langle 2,3,3\rangle & \langle 3,3,3\rangle
\end{array}
$$

Therefore $\left.\binom{3}{3}\right)=10$.

## A Pascal-Like Identity

## Theorem

Let $n, k$ be positive integers. Then

$$
\left(\binom{n}{k}\right)=\left(\binom{n-1}{k}\right)+\left(\binom{n}{k-1}\right)
$$

- We use a combinatorial proof: How many multisets of size $k$ can we form using the elements $\{1,2, \ldots, n\}$ ?
- Answer 1: A simple answer to this question is $\left.\binom{n}{k}\right)$.
- Answer 2: Next, if we must use element $n$ when forming a $k$-element multiset, we may complete the multiset by picking $k-1$ more elements from $\{1,2, \ldots, n\}$. The number of ways to do that is precisely $\left(\binom{n}{k-1}\right)$. If element $n$ is never used, then we can form the multiset in $\left(\binom{n-1}{k}\right)$ ways. Clearly the multisets of size $k$ containing at least one occurrence of $n$ together with those containing no occurrences of $n$ cover all multisets of size $k$.


## Illustration of the Proof

- Consider $\left.\binom{3}{4}\right)=\left(\binom{2}{4}\right)+\left(\binom{3}{3}\right)$.

We list all the multisets of size 4 we can form using the elements $\{1,2,3\}$.

- First, we list all those that do not use element 3. There are $\left.\binom{2}{4}\right)=5$ of them. They are

$$
\langle 1,1,1,1\rangle \quad\langle 1,1,1,2\rangle \quad\langle 1,1,2,2\rangle \quad\langle 1,2,2,2\rangle \quad\langle 2,2,2,2\rangle
$$

- Second, we list all those that include the element 3 (at least once). They are

$$
\begin{array}{lllll}
\langle 1,1,1,3\rangle & \langle 1,1,2,3\rangle & \langle 1,1,3,3\rangle & \langle 1,2,2,3\rangle & \langle 1,2,3,3\rangle \\
\langle 1,3,3,3\rangle & \langle 2,2,2,3\rangle & \langle 2,2,3,3\rangle & \langle 2,3,3,3\rangle & \langle 3,3,3,3\rangle
\end{array}
$$

If we ignore the mandatory 3 , we have listed all the 3 -element multisets we can form from the elements in $\{1,2,3\}$. There are $\left(\begin{array}{l}\binom{3}{3}\end{array}\right)=10$ of them.

## Connection Between (( )) and ( )

## Theorem

Let $n, k \in \mathbb{N}$. Then

$$
\left(\binom{n}{k}\right)=\binom{n+k-1}{k} .
$$

- We work with $n>0$ and develop a way to encode multisets and then count their encodings. To find $\left.\binom{n}{k}\right)$, we list all (encodings of) the k-element multisets we can form using the integers 1 through $n$. Let $M$ be a $k$-element multiset formed using the integers 1 through $n$. We denote $M$ by a "stars-and-bars" notation that contains exactly $k$ $\star \mathrm{S}$ (one for each element of $M$ ) and exactly $n-1 \mathrm{~s}$ (to separate $n$ different compartments). Given any sequence of $k \star s$ and $n-1 \mid s$, we can recover a unique multiset of cardinality $k$ whose elements are chosen from the integers 1 through $n$.


## Connection Between (( )) and ( ) (Cont'd)

- For example, if $n=5$ and the multi set is $M=\langle 1,1,1,2,3,3,5\rangle$, we get the "stars-and-bars" code

$$
\star \star \star|\star| \star \star|\mid \star .
$$

- It is easy to count the number of such "stars-and-bars" lists. Each contains exactly $n+k-1$ symbols, of which exactly $k$ are $\star$ s. The number of such lists is $\binom{n+k-1}{k}$ because we can think of choosing exactly $k$ positions on the length- $(n+k-1)$ list to be $\star$ s.
- Therefore $\left.\binom{n}{k}\right)=\binom{n+k-1}{k}$.


## An Example

- Size three mutisets formed using the integers 1,2 and 3 .

| Multiset | Stars-and-bars | Subset |
| :---: | :---: | :---: |
| $\langle 1,1,1\rangle$ | $\star \star \star\|\mid$ | $\{1,2,3\}$ |
| $\langle 1,1,2\rangle$ | $\star \star\|\star\|$ | $\{1,2,4\}$ |
| $\langle 1,1,3\rangle$ | $\star \star \mid \\| \star$ | $\{1,2,5\}$ |
| $\langle 1,2,2\rangle$ | $\star\|\star \star\|$ | $\{1,3,4\}$ |
| $\langle 1,2,3\rangle$ | $\star\|\star\| \star$ | $\{1,3,5\}$ |
| $\langle 1,3,3\rangle$ | $\star\|\mid \star \star$ | $\{1,4,5\}$ |
| $\langle 2,2,2\rangle$ | $\|\star \star \star\|$ | $\{2,3,4\}$ |
| $\langle 2,2,3\rangle$ | $\|\star \star\| \star$ | $\{2,3,5\}$ |
| $\langle 2,3,3\rangle$ | $\|\star\| \star \star$ | $\{2,4,5\}$ |
| $\langle 3,3,3\rangle$ | $\\| \star \star \star$ | $\{3,4,5\}$ |

- We now list their "stars-and-bars" notation.
- The column labeled "Subset" lists those of the five positions in the "stars-and-bars" encoding that are occupied by $\star \mathrm{s}$.


## Summary

- We considered the problem of counting the number of $k$-element multisets that can be formed with elements selected from $\{1,2, \ldots, n\}$.
- The answer is denoted by $\left(\binom{n}{k}\right)$.
- We showed using an encoding that $\left.\binom{n}{k}\right)=\binom{n+k-1}{k}$.
- We have studied four counting problems:
- counting lists with repetitions,
- counting lists without repetitions,
- counting subsets,
- counting multisets.

|  | Repetition Allowed | Repetition Forbidden |
| :---: | :---: | :---: |
| Ordered | $n^{k}$ | $(n)_{k}$ |
| Unordered | $\left(\binom{n}{k}\right)$ | $\binom{n}{k}$ |

## Subsection 6

## The Inclusion-Exclusion Principle

## Inclusion-Exclusion for Two and Three Sets

- For finite sets $A$ and $B$, we have

$$
|A|+|B|=|A \cup B|+|A \cap B| .
$$

The identity can be rewritten as

$$
|A \cup B|=|A|+|B|-|A \cap B| .
$$

- This result can be extended to three sets $A, B$ and $C$ :

$$
|A \cup B \cup C|=|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C| .
$$

The size of the union is expressed in terms of the sizes of the individual sets and their various intersections.

- These equations are called inclusion-exclusion formulas.


## Inclusion-Exclusion Principle

## Theorem (Inclusion-Exclusion Principle)

Let $A_{1}, A_{2}, \ldots, A_{n}$ be finite sets. Then
$\left|A_{1} \cup A_{2} \cup \cdots \cup A_{n}\right|=\left|A_{1}\right|+\left|A_{2}\right|+\cdots+\left|A_{n}\right|$

$$
\begin{gathered}
-\left|A_{1} \cap A_{2}\right|-\left|A_{1} \cap A_{3}\right|-\cdots-\left|A_{n-1} \cap A_{n}\right| \\
+\left|A_{1} \cap A_{2} \cap A_{3}\right|+\left|A_{1} \cap A_{2} \cap A_{4}\right|+\cdots \\
+\left|A_{n-2} \cap A_{n-1} \cap A_{n}\right|
\end{gathered}
$$

$$
\begin{aligned}
& -\cdots+\cdots \cdots \\
& \pm\left|A_{1} \cap A_{2} \cap \cdots \cap A_{n}\right| .
\end{aligned}
$$

- The idea is that when we add up all the sizes of the individual sets, we have added too much because some elements may be in more than one set. So we subtract off the sizes of the pairwise intersections to compensate, but now we may have subtracted too much. Thus we correct back by adding in the sizes of the triple intersections, but this overcounts, so we have to subtract, and so on.
We will show that, at the end, everything is in perfect balance.


## An Example

- At an art academy, there are 43 students taking ceramics, 57 students taking painting, and 29 students taking sculpture. There are 10 students in both ceramics and painting, 5 in both painting and sculpture, 5 in both ceramics and sculpture, and 2 taking all three courses. How many students are taking at least one course at the art academy?

Let $C, P$ and $S$ denote the sets of students taking ceramics, painting, and sculpture, respectively. We want to calculate $|C \cup P \cup S|$. Applying Inclusion-Exclusion gives

$$
\begin{aligned}
|C \cup P \cup S|= & |C|+|P|+|S| \\
& -|C \cap P|-|P \cap S|-|C \cap S| \\
& +|C \cap P \cap S| \\
= & 43+57+29-10-5-5+2 \\
= & 111 .
\end{aligned}
$$

## Proof of Inclusion-Exclusion I

- Let the $n$ sets be $A_{1}, A_{2}, \ldots, A_{n}$ and let the elements in their union be named $x_{1}, x_{2}, \ldots, x_{m}$.
- We create a chart:
- The rows of the chart are labeled by the elements $x_{1}$ through $x_{m}$.
- The chart has $2^{n}-1$ columns that correspond to all the terms on the right-hand side of the inclusion-exclusion formula.
- The first $n$ columns are labeled $A_{1}$ through $A_{n}$.
- The next $\binom{n}{2}$ columns are labeled by all the pairwise intersections from $A_{1} \cap A_{2}$ through $A_{n-1} \cap A_{n}$.
- The next $\binom{n}{3}$ columns are labeled by the triple intersections, and so on.
- If the element labeling a row is not in the set labeling the column, the entry in that position is blank. If the element is a member of the set, we put a + sign when the column label is an intersection of an odd number of sets or else a - sign when the column label is an intersection of an even number of sets.


## Proof of Inclusion-Exclusion II

- Notice the following:
- The number of marks in each column is the cardinality of that column's set.
- The sign of the mark (+ or - ) corresponds to whether we are adding or subtracting that set's cardinality in the Inclusion-Exclusion formula. So, if we add 1 for every + sign in the chart and subtract 1 for every sign, we get precisely the right-hand side of the inclusion-exclusion formula.
- There is always one more + than - in each row. If we can prove this, we will be finished because, then, the net effect of all the +s and -s is to count 1 for each element in the union of the sets $A_{1} \cup A_{2} \cup \cdots \cup A_{n}$.
- Can we prove that every row has exactly one more + than - ?


## Proof that every row has exactly one more + than -

- Let $x$ be an element of $A_{1} \cup A_{2} \cup \cdots \cup A_{n}$. Suppose it is in exactly $k$ of the $A_{i} \mathrm{~s}$. How many +s and -s are in $x$ 's row?
- In the columns indexed by single sets, there will be $k=\binom{k}{1}+\mathrm{s}$.
- In the columns indexed by pairwise intersections, there will be $\binom{k}{2}$ s.
- In the columns indexed by triple intersections, there will be $\binom{k}{3}+\mathrm{s}$.
- In general, in the columns indexed by $j$-fold intersections, there will be $\binom{k}{j}$ marks. The marks are + if $j$ is odd and - if $j$ is even.
- Thus the number of +s is $\binom{k}{1}+\binom{k}{3}+\binom{k}{5}+\cdots$, and the number of $-s$ is $\binom{k}{2}+\binom{k}{4}+\binom{k}{6}+\cdots$; Note the term $\binom{k}{0}$ is absent.
- It can be seen that $\binom{k}{0}-\binom{k}{1}+\binom{k}{2}-\cdots \pm\binom{ k}{k}=0$. So $1=\binom{k}{0}=\left[\binom{k}{1}+\binom{k}{3}+\binom{k}{5}+\cdots\right]-\left[\binom{k}{2}+\binom{k}{4}+\binom{k}{6}+\cdots\right]$. This finishes the proof.


## Counting Lists I

- The number of length- $k$ lists whose elements are chosen from the set $\{1,2, \ldots, n\}$ is $n^{k}$. How many of these lists use all of the elements in $\{1,2, \ldots, n\}$ at least once?
- Let $U$ be the set of all length- $k$ lists whose elements are chosen from $\{1,2, \ldots, n\}$. Thus $|U|=n^{k}$.
We call "good" the ones that contain all the elements of $\{1,2, \ldots, n\}$ and "bad" the ones that miss one or more elements in $\{1,2, \ldots, n\}$. If we can count the number of bad lists, we'll be finished because the number of good lists is $n^{k}$ minus the number of bad lists.
- Let $B_{1}$ be the set of all lists in $U$ that do not contain the element 1 .
- Let $B_{2}$ be the set of all lists in $U$ that do not contain the element 2 , etc.
- and let $B_{n}$ be the set of all lists in $U$ that do not contain the element $n$. The set $B_{1} \cup B_{2} \cup \cdots \cup B_{n}$ contains precisely all the bad lists. We want to calculate the size of this union. We can apply inclusion-exclusion!


## Counting Lists II

- To calculate the size of $B_{1} \cup \cdots \cup B_{n}$, we need to calculate the sizes of each of the sets $B_{i}$ and all possible intersections;
- We calculate the size of $B_{1}$. This is the number of length- $k$ lists whose elements are chosen from $\{2,3, \ldots, n\}$. So $\left|B_{1}\right|=(n-1)^{k}$. The analysis is exactly the same as for $\left|B_{2}\right|,\left|B_{3}\right|$, etc. Thus, $\left|B_{j}\right|=(n-1)^{k}$, for all $j=1, \ldots, n$.
- The number $\left|B_{1} \cap B_{2}\right|$ is the number of lists that do not include the element 1 and do not include the element 2 . So, it equals the number of length- $k$ lists whose elements are chosen from the set $\{3,4, \ldots, n\}$. The number of these lists is $\left|B_{1} \cap B_{2}\right|=(n-2)^{k}$. The same analysis works for $B_{i} \cap B_{j}, i \neq j$. Note that there are $\binom{n}{2}$ pairwise intersections of $n$ sets. Therefore
$-\left|B_{1} \cap B_{2}\right|-\left|B_{1} \cap B_{3}\right|-\cdots-\left|B_{n-1} \cap B_{n}\right|=-\binom{n}{2}(n-2)^{k}$.
- The size of a $j$-fold intersection of the $B$ sets consists of the number of the length- $k$ lists that avoid all $j$ elements. So, it is equal to $(n-j)^{k}$. Moreover, there are $\binom{n}{j}$ such intersections. Thus, the $j$-th term in the inclusion-exclusion is $\pm\binom{ n}{j}(n-j)^{k}$. The sign is positive when $j$ is odd and negative when $j$ is even.


## Counting Lists III

- Applying now Inclusion-Exclusion, we get

$$
\begin{aligned}
\left|B_{1} \cup \cdots \cup B_{n}\right| & =\left(\begin{array}{c}
n \\
1 \\
n
\end{array}\right)(n-1)^{k}-\binom{n}{2}(n-2)^{k}+\cdots \pm\binom{ n}{n}(n-n)^{k} \\
& =\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}(n-j)^{k} .
\end{aligned}
$$

- Therefore,
$\#$ of good lists $=n^{k}-\#$ of bad lists

$$
\begin{aligned}
& =n^{k}-\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}(n-j)^{k} \\
& =\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}(n-j)^{k} .
\end{aligned}
$$

## Counting Derangements I

- There are $n$ ! ways to make lists of length $n$ using the elements of $\{1,2, \ldots, n\}$ without repetition. Such a list is called a derangement if the number $j$ does not occupy position $j$ for any $j=1,2, \ldots, n$. How many derangements are there?
- There are $n$ ! lists under consideration. The "good" lists are the derangements. The "bad" lists are the lists in which at least one $j$ in $\{1,2, \ldots, n\}$ appears at position $j$ of the list.
- We count the number of bad lists and subtract from $n$ ! to count the good lists. There are $n$ ways in which a list might be bad:
- 1 might be in position 1 ,
- 2 might be in position 2, and so forth, and
- $n$ might be in position $n$.

So we define the following sets:

- $B_{1}=\{$ lists with 1 in position 1$\}$
- $B_{2}=\{$ lists with 2 in position 2$\}$ etc.
- $B_{n}=\{$ lists with $n$ in position $n\}$.

Our goal is to count $\left|B_{1} \cup \cdots \cup B_{n}\right|$ and finally to subtract from $n!$.

## Counting Derangements II

- To compute the size of a union, we use Inclusion-Exclusion.
- $\left|B_{1}\right|$ is the number of lists with 1 in position 1 . The other $n-1$ elements may be anywhere. There are ( $n-1$ )! such lists. The same reasoning yields $\left|B_{i}\right|=(n-1)$ !, for all $i$. Therefore, $\left|B_{1}\right|+\left|B_{2}\right|+\cdots+\left|B_{n}\right|=n(n-1)!$.
- $\left|B_{1} \cap B_{2}\right|$ is the number of lists in which 1 must be in position 1,2 must be in position 2 , and the remaining $n-2$ elements may be anywhere. There are $(n-2)$ ! such lists. Indeed, for any $i \neq j$, we have $\left|B_{i} \cap B_{j}\right|=(n-2)$ !. Since, there are $\binom{n}{2}$ pairwise intersections, and they all have size ( $n-2$ )!, we get $-\left|B_{1} \cap B_{2}\right|-\cdots-\left|B_{n-1} \cap B_{n}\right|=-\binom{n}{2}(n-2)!$.
- For each of the $\binom{n}{k} k$-fold intersections, such as $B_{1} \cap B_{2} \cap \cdots \cap B_{k}$, we have $\left|B_{1} \cap B_{2} \cap \cdots \cap B_{k}\right|=(n-k)$ !. Therefore, their contribution to the Inclusion-Exclusion sum is $\pm\binom{ n}{k}(n-k)$ !.


## Counting Derangements III

- The Inclusion-Exclusion Formula gives

$$
\left|B_{1} \cup B_{2} \cup \cdots \cup B_{n}\right|=\binom{n}{1}(n-1)!-\binom{n}{2}(n-2)!+\cdots \pm\binom{ n}{n}(n-n)!
$$

- Recall that to compute the number of derangements, we must subtract $\left|B_{1} \cup B_{2} \cup \cdots \cup B_{n}\right|$ from $n!$ :

$$
\begin{aligned}
& n!-\binom{n}{1}(n-1)!+\binom{n}{2}(n-2)!-\binom{n}{3}(n-3)!+\cdots \mp\binom{n}{n}(n-n)! \\
= & \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(n-k)!=\sum_{k=0}^{n}(-1)^{k} \frac{n!}{k!(n-k)!}(n-k)! \\
= & \sum_{k=0}^{n}(-1)^{k} \frac{n!}{k!}=n!\sum_{k=0}^{n} \frac{(-1)^{k}}{k!} .
\end{aligned}
$$

