Fundamental Concepts of Mathematics

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1. More on Proofs
   - Contradiction
   - Smallest Counterexample
   - Induction
   - Recurrence Relations

Subsection 1

Contradiction

The Contrapositive

- The statement "If A, then B" is logically equivalent to the statement "If (not B), then (not A)".
- The statement "If (not B), then (not A)" is called the **contrapositive** of "If A, then B."
- Why are a statement and its contrapositive logically equivalent?
  - For "If A, then B" to be true, it must be the case that whenever A is true, B must also be true. If it ever should happen that B is false, then it must have been the case that A was false. In other words, if B is false, then A must be false. Thus we have "If (not B), then (not A)."
  - Alternatively, "If A, then B" is logically equivalent to "(not A) or B". By the same reasoning, "If (not B), then (not A)" is equivalent to "(not (not B)) or (not A)", but "not (not B)" is the same as B, so this becomes "B or (not A)", which is equivalent to "(not A) or B".
- In symbols,

$$a \to b = (\neg a) \vee b = (\neg(\neg b)) \vee (\neg a) = (\neg b) \to (\neg a).$$

Equivalence of $a \to b$ and $(\neg b) \to (\neg a)$

- We can also verify the equivalence of $a \to b$ and $(\neg b) \to (\neg a)$ mechanically by looking at truth tables:

a	b	$a \to b$	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Therefore, $a \to b$ and $(\neg b) \to (\neg a)$ are logically equivalent.

Proof by Contraposition

Proposition

Let R be an equivalence relation on a set A and let $a, b \in A$. If $a \not R\ b$, then $[a] \cap [b] = \emptyset$.

- Let R be an equivalence relation on a set A and let $a, b \in A$.
- We prove the contrapositive of the statement:
  - Suppose $[a] \cap [b] \neq \emptyset$.
  - Thus, there is an $x \in [a] \cap [b]$.
  - This means that $x \in [a]$ and $x \in [b]$.
  - Hence $x \; R \; a$ and $x \; R \; b$.
  - By symmetry, $a \; R \; x$.
  - Since $x \; R \; b$, by transitivity, $a \; R \; b$.

Reductio ad Absurdum (Proof by Contradiction)

- Proof by contrapositive is an alternative to direct proof. If we cannot find a direct proof, we try to prove the contrapositive.
- A proof technique that combines direct proof and proof by contrapositive is called **proof by contradiction** or, in Latin, **reductio ad absurdum**.
  - We want to prove "If A, then B".
  - We show that it is impossible for A to be true while B is false. In other words, we show that "A and (not B)" is impossible.
  - How do we prove that something is impossible? We suppose the impossible thing is true and prove that this supposition leads to an absurd conclusion. If a statement implies something clearly wrong, then that statement must have been false!
- To prove "If A, then B", we assume the hypothesis A and we assume the opposite of the conclusion, (not B). From these two assumptions, we try to reach a clearly false statement.

Equivalence of $a \to b$ and $(a \wedge \neg b) \to F$

- We can also verify the equivalence of $a \to b$ and $(a \wedge \neg b) \to F$ mechanically by looking at truth tables:

a	b	$a \to b$	$\neg b$	$a \wedge \neg b$	$(a \wedge \neg b) \to F$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	F	T

Therefore, $a \to b$ and $(a \wedge \neg b) \to F$ are logically equivalent.

Proof by Contradiction: An Example

Proposition

No integer is both even and odd.

- Let x be an integer.
  - Suppose, for the sake of contradiction, that x is both even and odd.
  - Since x is even, we know $2 \mid x$.
  - Thus, there is an integer a such that $x = 2a$.
  - Since x is odd, we know that there is an integer b such that $x = 2b + 1$.
  - Therefore, $2a = 2b + 1$.
  - Dividing both sides by 2 gives $a = b + \frac{1}{2}$, i.e., $a - b = \frac{1}{2}$.
  - But $a - b$ is an integer (since a and b are integers) whereas $\frac{1}{2}$ is not an integer, which is a contradiction!
  - Therefore, x is not both even and odd, and the proposition is proved.

Special Cases: Emptiness of a Set and Uniqueness

- To prove a set is empty:
  - Assume the set is nonempty and argue to a contradiction.
- To prove there is at most one object that satisfies conditions:
  - Suppose there are two different objects, x and y, that satisfy conditions. Argue to a contradiction.

Proposition

Let a and b be numbers with $a \neq 0$. There is at most one number x with $ax + b = 0$.

- Suppose there are two different numbers x and y such that $ax + b = 0$ and $ay + b = 0$. This gives $ax + b = ay + b$. Subtracting b from both sides gives $ax = ay$. Since $a \neq 0$, we can divide both sides by a to give $x = y$. This is a contradiction!

Subsection 2

Smallest Counterexample

Proof by Contradiction: Counterexample View

- The method of proof by contradiction:
  - We want to prove a result of the form "If A, then B".
  - Suppose this result were false. Then, there would be a counterexample to the statement. That is, there would be an instance where A is true and B is false.
  - We analyze the alleged counterexample and produce a contradiction. Since the supposition that there is a counterexample leads to an absurd conclusion (a contradiction), that supposition must be wrong; there is no counterexample.
  - Since there is no counterexample, the result must be true.
- Example: No integer could be both even and odd.
  - Suppose the statement "No integer is both even and odd" were false.
  - Then there would be an integer x that is both even and odd.
  - Since x is even, there is an integer a such that $x = 2a$.
  - Since x is odd, there is an integer b such that $x = 2b + 1$.
  - Thus $2a = 2b + 1$, which implies $a - b = \frac{1}{2}$.
  - Since a and b are integers, so is $a - b = \frac{1}{2}$, a contradiction!

Smallest Counterexample: Main Idea

Proposition

Every natural number is either even or odd.

- If not all natural numbers are even or odd, there is a smallest natural number, x, that is neither even nor odd. We know $x \neq 0$ because 0 is even. Therefore $x \geq 1$. Since $0 \leq x - 1 < x$, we see that $x - 1$ is smaller than x, so $x - 1$ is either even or odd.
  - If $x - 1$ is odd, then $x - 1 = 2a + 1$ for some integer a. Thus, $x = 2a + 2 = 2(a + 1)$, so x is even, a contradiction!
  - If $x - 1$ is even, then $x - 1 = 2b$ for some integer b. Thus, $x = 2b + 1$, so x is odd, a contradiction!

  In every case, we have a contradiction, so the supposition is false!
- We observe that in the Smallest Counterexample Proof:
  - We use proof by contradiction;
  - We consider a smallest counterexample to the result.
  - We need to treat the very smallest possibility as a special case.
  - We descend to a smaller case for which the theorem is true and work back.

Extending the Proposition to Integers

Proposition

Every integer is either even or odd.

- The key idea is that either $x \geq 0$ (in which case we are finished by the previous Proposition) or else $x < 0$ (in which case $-x \in \mathbb{N}$, and again we can use the previous Proposition).
- Let x be any integer.
  - If $x \geq 0$, then $x \in \mathbb{N}$, so by the previous proposition, x is either even or odd.
  - Otherwise, $x < 0$. In this case $-x > 0$, so $-x$ is either even or odd.
    - If $-x$ is even, then $-x = 2a$ for some integer a. But then $x = -2a = 2(-a)$, so x is even.
    - If $-x$ is odd, then $-x = 2b + 1$ for some integer b. From this we have $x = -2b - 1 = 2(-b - 1) + 1$, so x is odd.

  In every case, x is either even or odd.

Another Proof by Smallest Counterexample

Proposition

Let n be a positive integer. The sum of first n odd natural numbers is n^2.

- The first n odd natural numbers are $1, 3, 5, \ldots, 2n-1$. The proposition claims that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ or, $\sum_{k=1}^{n}(2k-1) = n^2$.
- Suppose the statement is false. There is a smallest positive integer x for which the statement is false (i.e., the sum of the first x odd numbers is not x^2). So $1 + 3 + 5 + \cdots + (2x-1) \neq x^2$.
  - Note that $x \neq 1$ because the sum of the first 1 odd number is $1 = 1^2$. So $x > 1$.
  - Since x is the smallest counterexample and $x > 1$, the sum of the first $x-1$ odd numbers must equal $(x-1)^2$. So $1+3+5+\cdots+[2(x-1)-1] = (x-1)^2$. We add $2x-1$ to both sides: $1+3+5+\cdots+[2(x-1)-1]+(2x-1) = (x-1)^2+(2x-1) = (x^2-2x+1)+(2x-1) = x^2$, which is a contradiction!

Never Omit the Basis Step!!

A False Proposition

Every natural number is both even and odd.

- We give a bogus proof using the Smallest Counterexample method, but omitting the basis step.
- Suppose the statement is false. Then there is a smallest natural number x that is not both even and odd. Consider $x-1$. Since $x-1<x$, $x-1$ is not a counterexample. Therefore $x-1$ is both even and odd.
  - Since $x-1$ is even, $x-1=2a$ for some integer a, whence $x=2a+1$, and x is odd.
  - Since $x-1$ is odd, $x-1=2b+1$ for some integer b, and so $x=2b+2=2(b+1)$, so x is even.

  Thus x is both even and odd, a contradiction!
- Where is the mistake? The error is in the sentence "Therefore $x-1$ is both even and odd." It is correct that $x-1$ is not a counterexample, but we do not know that $x-1$ is a natural number!

Not Applicable for Integers or Rational Numbers

Another False Proposition

Every nonnegative rational number is an integer.

- Recall that a **rational number** is any number that can be expressed as a fraction a/b, where $a, b \in \mathbb{Z}$ and $b \neq 0$.
- This statement is asserting that numbers such as $\frac{1}{4}$ are integers. Obviously false!
- The following is a bogus proof:
  - Suppose the statement is false.
  - Let x be a smallest counterexample.
  - Notice that $x = 0$ is not a counterexample because 0 is an integer.
  - Since x is a nonnegative rational, so is $x/2$. Furthermore, since $x \neq 0$, we know that $x/2 < x$, so $x/2$ being smaller than x, is not a counterexample, and $x/2$ must be an integer.
  - Now $x = 2(x/2)$, and 2 times an integer is an integer;
  - Therefore x is an integer, a contradiction!

The Well-Ordering Principle

- What is wrong with this "proof"?
  It looks like we followed the right steps, and even remembered to do a basis step.
- The problem is in the sentence "Let x be a smallest counterexample."
  There are infinitely many counterexamples to this statement, including $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ These form an **infinite descent of counterexamples**, and so there can be no smallest counterexample!
- When can we be certain to find a smallest counterexample?

The Well-Ordering Principle

Every nonempty set of natural numbers contains a least element.

Examples

- Let $P = \{x \in \mathbb{N} : x \text{ is prime}\}$.
  This set is a non empty subset of the natural numbers. By the Well-Ordering Principle, P contains a least element. Of course, the least element in P is 2.
- Consider the set $X = \{x \in \mathbb{N} : x \text{ is even and odd}\}$.
  We know that this set is empty because we have shown that no natural number is both even and odd. But for the sake of contradiction, we suppose that $X \neq \emptyset$; then, by the Well-Ordering Principle, X would contain a smallest element. This is the central idea in the proof that showed that $X = \emptyset$.
- In contradistinction, consider the set $Y = \{y \in \mathbb{Q} : y \geq 0, y \notin \mathbb{Z}\}$.
  The bogus proof we studied sought a least element of Y. We subsequently realized that Y has no least element, and that was the error in our "proof." The Well-Ordering Principle applies to $\mathbb{N}$, but not to $\mathbb{Q}$.

Sum of a Geometric Series

Proposition

Let $n \in \mathbb{N}$. If $a \neq 0$ and $a \neq 1$, then

$$\sum_{k=0}^{n} a^k = a^0 + a^1 + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}.$$

- Suppose, for the sake of contradiction, that the statement is false.
- Let X be the set of counterexamples: $X = \{n \in \mathbb{N} : \sum_{k=0}^{n} a^k \neq \frac{a^{n+1}-1}{a-1}\} \neq \emptyset$. Since $X \neq \emptyset$ subset of $\mathbb{N}$, by the Well-Ordering Principle, it contains a least element x.
  - Note that for $n = 0$, the equation gives $a^0 = \frac{a^1-1}{a-1}$ and this is true. This means that $n = 0$ is not a counterexample. Thus $x \neq 0$.
  - Since $x > 0$, $x - 1 \in \mathbb{N}$ and $x - 1 \notin X$ because $x - 1$ is smaller than the least element of X. Therefore, $a^0 + a^1 + \cdots + a^{x-1} + a^x = \frac{a^x-1}{a-1} + a^x = \frac{a^x-1+a^x(a-1)}{a-1} = \frac{a^x-1+a^{x+1}-a^x}{a-1} = \frac{a^{x+1}-1}{a-1}$, This shows that x satisfies the proposition contradicting $x \in X$.

Proving an Inequality

Proposition

For all integers $n \geq 5$, we have $2^n > n^2$.

- Suppose that the statement is false.
- Let X be the set of counterexamples:
  $X = \{n \in \mathbb{Z} : n \geq 5, 2^n \not> n^2\} \neq \emptyset$. By the Well-Ordering Principle, X contains a least element x.
  - We claim that $x \neq 5$. Note that $2^5 = 32 > 25 = 5^2$, so 5 is not a counterexample to the proposition. Thus $x \geq 6$.
  - Since $x \geq 6$, we have $x - 1 \geq 5$, and, since x is the least element of X, we know that the proposition is true for $n = x - 1$. Thus,
    $2^{x-1} > (x-1)^2 \Rightarrow \frac{1}{2} \cdot 2^x > x^2 - 2x + 1 \Rightarrow 2^x > 2x^2 - 4x + 2$.
    Now it suffices to show $2x^2 - 4x + 2 \geq x^2$. This is equivalent to $x^2 - 4x + 4 \geq 2$, i.e., to $(x-2)^2 \geq 2$, which clearly holds for all $x \geq 6$!
    But this contradicts $x \in X$.

The Fibonacci Sequence

- The **Fibonacci numbers** are the list of integers $(1, 1, 2, 3, 5, 8, \ldots) = (F_0, F_1, F_2, \ldots)$, where $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$.

Proposition

For all $n \in \mathbb{N}$, we have $F_n \leq 1.7^n$.

- Suppose that the statement is false.
- Let X be the set of counterexamples $X = \{n \in \mathbb{N} : F_n \nleq 1.7^n\} \neq \emptyset$. By the Well-Ordering Principle, X contains a least element x.
  - Observe that $x \neq 0$ because $F_0 = 1 = 1.7^0$ and $x \neq 1$ because $F_1 = 1 \leq 1.7^1$. Thus, $x \geq 2$.
  - Now we know that $F_x = F_{x-1} + F_{x-2}$ and, since $x - 1$ and $x - 2$ are natural numbers less than x, $F_{x-1} \leq 1.7^{x-1}$ and $F_{x-2} \leq 1.7^{x-2}$. We work as follows: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$ $= 1.7^{x-2}(1.7 + 1) = 1.7^{x-2}(2.7) < 1.7^{x-2}(2.89) = 1.7^{x-2}(1.7^2)$ $= 1.7^x$. Therefore the statement is true for $n = x$, contradicting $x \in X$.

Subsection 3

Induction

Illustrating with a Proof

Proposition

Let n be a positive integer. The sum of first n odd natural numbers is n^2.

- We would like to prove

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

- We prove that the statement is true for $n = 1$: $1 = 1^2$.
- We assume that it is true for $n = k$:

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

- We now try to show it is true for $n = k + 1$:

$$\begin{aligned} &1 + 3 + 5 + \cdots + (2k - 1) = k^2 \\ &1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) \\ &1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2. \end{aligned}$$

Now we conclude that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all natural numbers $n \geq 1$!

The Principle of Mathematical Induction

Theorem (Principle of Mathematical Induction)

Let A be a set of natural numbers. If

- $0 \in A$;
- For all $k \in \mathbb{N}$, $k \in A \Rightarrow k+1 \in A$,

then $A = \mathbb{N}$.

- We use the Well-Ordering and argue by Least Counterexample;
- Suppose that $A \neq \mathbb{N}$.
- Let $X = \mathbb{N} - A \neq \emptyset$. By the Well-Ordering Principle, since X is a nonempty set of natural numbers, it contains a least element x. Thus, x is the smallest natural number not in A.
  - Note $x \neq 0$ because, by hypothesis, $0 \in A$, so $0 \notin X$. Therefore, $x \geq 1$.
  - Thus $x - 1 \geq 0$, so $x - 1 \in \mathbb{N}$. and, since x is the smallest element not in A, we have $x - 1 \in A$. Now the second condition of the theorem says that whenever a natural number is in A, so is the next larger natural number. Since $x - 1 \in A$, we know that $(x-1)+1 = x$ is in A. This contradicts $x \notin A$.

Proof by Induction I

Proposition

Let n be a natural number. Then $0^2+1^2+2^2+\cdots+n^2=\frac{(2n+1)(n+1)(n)}{6}$.

- Let A be the set of natural numbers for which the statement is true.
  - $0 \in A$ because $0^2=\frac{(2\cdot 0+1)(0+1)(0)}{6}$.
  - Suppose the result is true for $n=k$, i.e., assume
    $0^2+1^2+\cdots+k^2=\frac{(2k+1)(k+1)(k)}{6}$;
    We must show that the equation is true for $n=k+1$, i.e., that
    $0^2+1^2+\cdots+k^2+(k+1)^2=\frac{(2k+3)(k+2)(k+1)}{6}$;

$$\begin{aligned} 0^2+1^2+\cdots+k^2+(k+1)^2 &= \frac{(2k+1)(k+1)(k)}{6}+(k+1)^2 \\ &= [\frac{(2k+1)(k)}{6}+k+1](k+1) \\ &= [\frac{2k^2+k+6k+6}{6}](k+1) \\ &= \frac{(2k+3)(k+2)(k+1)}{6}. \end{aligned}$$

- We have shown $0 \in A$ and $k \in A \Rightarrow (k+1) \in A$. Therefore, by induction, $A=\mathbb{N}$; so the proposition is true for all natural numbers.

Proof by Induction II

Proposition

Let n be a positive integer. Then $2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1$.

- We prove this by induction on n.
  - For $n = 1$, $2^0 = 2^1 - 1$ holds.
  - Suppose the result is true for $n = k$, i.e., assume $2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$;
    We must show that the equation is true for $n = k + 1$, i.e., that $2^0 + 2^1 + \cdots + 2^{k-1} + 2^k = 2^{k+1} - 1$;

$$\begin{aligned} 2^0 + 2^1 + \cdots + 2^{k-1} + 2^k &= 2^k - 1 + 2^k \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

- Thus, the proposition is true for all positive integers.

Proof by Induction III

Proposition

Let n be a positive integer. Then $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.

- We prove this by induction on n.
  - For $n = 1$, $1 \cdot 1! = (1+1)! - 1$ holds.
  - Suppose the result is true for $n = k$, i.e., assume
    $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$;
    We must show that the equation is true for $n = k+1$, i.e., that
    $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$;

$$\begin{aligned}
&1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! \\
&= (k+1)! - 1 + (k+1) \cdot (k+1)! \\
&= (1 + (k+1)) \cdot (k+1)! - 1 \\
&= (k+2) \cdot (k+1)! - 1 \\
&= (k+2)! - 1.
\end{aligned}$$

- Thus, the proposition is true for all positive integers.

Proof of an Inequality by Induction

Proposition

Let n be a natural number. Then $10^0 + 10^1 + \cdots + 10^n < 10^{n+1}$.

- We prove this by induction on n.
  - For $n = 0$, $10^0 < 10^1$ holds.
  - Suppose the result is true for $n = k$, i.e., assume $10^0 + 10^1 + \cdots + 10^k < 10^{k+1}$;
    We must show that the equation is true for $n = k + 1$, i.e., that $10^0 + 10^1 + \cdots + 10^k + 10^{k+1} < 10^{k+2}$;

$$\begin{aligned} 10^0 + 10^1 + \cdots + 10^k + 10^{k+1} &< 10^{k+1} + 10^{k+1} \\ &= 2 \cdot 10^{k+1} \\ &< 10 \cdot 10^{k+1} \\ &= 10^{k+2}. \end{aligned}$$

- Thus, the proposition is true for all positive integers.

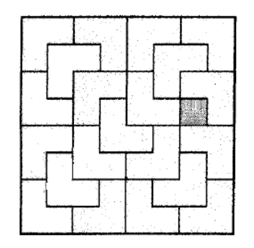
Proof of a Divisibility Relation by Induction

Proposition

Let n be a natural number. Then $4^n - 1$ is divisible by 3.

- We prove this by induction on n.
  - For $n = 0$, $4^0 - 1$ is divisible by 3.
  - Suppose the result is true for $n = k$, i.e., $3 \mid (4^k - 1)$. This means that $4^k - 1 = 3a$ for some integer a.
    We must show that the statement is true for $n = k + 1$, i.e., that $3 \mid (4^{k+1} - 1)$;
    $$\begin{aligned} 4^{k+1} - 1 &= 4 \cdot 4^k - 1 \\ &= 4(4^k - 1) + 3 \\ &= 4 \cdot 3a + 3 \\ &= 3(4a + 1). \end{aligned}$$
- Thus, the proposition is true for all natural numbers.

L-Shaped Triominoes

- 

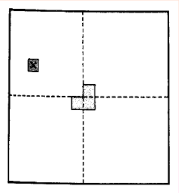
  We wish to cover a chess board with special tiles called L-**shaped triominoes**, or L-**triominoes**. These are tiles formed from three 1×1 squares joined at their edges to form an L shape. It is not possible to tile a standard 8×8 chess board with L-triominoes because there are 64 squares on the chess board and 64 is not divisible by 3. However, it is possible to cover all but one square of the chess board, and such a tiling is shown in the figure.
- Is it possible to tile larger chess boards?
  A $2^n \times 2^n$ chess board has 4^n squares, and, since we proved $4^n - 1$ is divisible by 3, there is a hope that we may be able to cover all but one of the squares.

Proof of a Tiling by Induction

Proposition

Let n be a positive integer. For every square on a $2^n \times 2^n$ chess board, there is a tiling by L-triominoes of the remaining $4^n - 1$ squares.

- We prove this by induction on n.
  - The basis case, $n = 1$, is obvious since placing an L-triomino on a 2×2 chess board covers all but one of the squares, and by rotating the triomino we can select which square is missed.
  - Suppose that the Proposition has been proved for $n = k$. We are given a $2^{k+1} \times 2^{k+1}$ chess board with one square selected. Divide the board into four $2^k x 2^k$ subboards. The selected square is in one of these.



Place an L-triomino overlapping three corners from the remaining subboards as shown. We now have four $2^k \times 2^k$ subboards each with one square that does not need to be covered. By induction, the remaining squares in the subboards can be tiled by L-triominoes.

- Thus, the proposition is true for all positive integers.

Strong Version of Mathematical Induction

Theorem (Mathematical Induction - Strong Version)

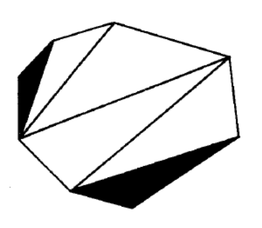
Let A be a set of natural numbers. If

- $0 \in A$;
- for all $k \in \mathbb{N}$, if $0, 1, 2, \ldots, k \in A$, then $k + 1 \in A$

then $A = \mathbb{N}$.

- Why is this called strong induction?
- Suppose we use induction to prove a proposition.
  - In both standard and strong induction, we begin by showing the basis case ($0 \in A$).
  - In standard induction, we assume the induction hypothesis ($k \in A$; i.e., the proposition is true for $n = k$) and then use that to prove $k + 1 \in A$ (i.e., the proposition is true for $n = k + 1$).
  - Strong induction gives a stronger induction hypothesis: We may assume $0, 1, 2, \ldots, k \in A$ (the proposition is true for all $n = 0, \ldots, k$) and use that to prove $k + 1 \in A$ (the proposition is true for $n = k + 1$).

Triangulating a Polygon

- 

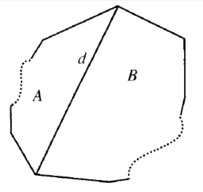
  Let P be a polygon in the plane. To **triangulate** a polygon is to draw diagonals through the interior of the polygon so that
  - the diagonals do not cross each other;
  - every region created is a triangle.
- The shaded triangles are called **exterior triangles** because two of their three sides are on the exterior of the polygon.

Proposition

If a polygon with four or more sides is triangulated, then at least two of the triangles formed are exterior.

Proof of the Proposition

- Let n denote the number of sides of the polygon. We apply strong induction on n.
  - Basis case: Since this result makes sense only for $n \geq 4$, the basis case is $n = 4$. The only way to triangulate a quadrilateral is to draw in one of the two possible diagonals. Either way, the two triangles formed must be exterior.
  - Strong induction hypothesis: Suppose the statement is true for all polygons on $n = 4, 5, \ldots, k$ sides.
  - Induction Step: Let P be any triangulated polygon with $k + 1$ sides. We must prove that at least two of its triangles are exterior.



Let d be one of the diagonals. This diagonal separates P into two polygons A and B. A and B are triangulated polygons with fewer sides than P.

Proof of the Proposition (Induction Step Continued)

- We continue the analysis of the Induction Step:
  - We drew d that separates P into two polygons A and B that are triangulated polygons with fewer sides than P.
    - If A is not a triangle, then, since A has at least four, but at most k sides, by strong induction we know that two or more of A's triangles are exterior. Are those exterior triangles of P? Not necessarily. If one of A's exterior triangles uses the diagonal d, then it is not an exterior triangle of P. Nonetheless, the other exterior triangle of A cannot also use the diagonal d; So at least one exterior triangle of A is also an exterior triangle of P.
    - If B is not a triangle, as in the previous case, B contributes at least one exterior triangle to P.
    - If A is a triangle, then A is an exterior triangle of P.
    - If B is a triangle, then B is an exterior triangle of P.
  - In every case, both A and B contribute at least one exterior triangle to P. So P has at least two exterior triangles.

Identity Involving Binomials and Fibonacci Numbers

Proposition

Let $n \in \mathbb{Z}$ and let F_n denote the nth Fibonacci number. Then $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{0}{n} = F_n$.

- We use strong induction.
  - Basis case: The result is true for $n = 0$: We get $\binom{0}{0} = 1 = F_1$; The result is also true for $n = 1$: Indeed, $\binom{1}{0} + \binom{0}{1} = 1 + 0 = 1 = F_1$.
  - Strong induction hypothesis: Suppose the statement is true for all values of n from 0 to k. We may also assume $k \geq 1$ since we have already proved the result for $n = 0$ and $n = 1$.
  - Induction Step: We want to prove the statement for $n = k + 1$, i.e., $\binom{k+1}{0} + \binom{k}{1} + \cdots + \binom{0}{k+1} = F_{k+1}$.
    By the strong induction hypothesis:

$$\begin{array}{rcl} F_{k-1} & = & \binom{k-1}{0} + \binom{k-2}{1} + \binom{k-3}{2} + \cdots + \binom{0}{k-1} \\ F_k & = & \binom{k}{0} + \binom{k-1}{1} + \binom{k-2}{2} + \cdots + \binom{1}{k-1} + \binom{0}{k}; \end{array}$$

Binomials and Fibonacci Numbers (Cont'd)

- We obtained by the Strong Induction hypothesis:

$$\begin{array}{rcl} F_{k-1} & = & \binom{k-1}{0} + \binom{k-2}{1} + \binom{k-3}{2} + \cdots + \binom{1}{k-2} + \binom{0}{k-1} \\ F_k & = & \binom{k}{0} + \binom{k-1}{1} + \binom{k-2}{2} + \binom{k-3}{3} + \cdots + \binom{1}{k-1} + \binom{0}{k}; \end{array}$$

We add these two lines to get

$$F_{k+1} = F_k + F_{k-1} =$$

$$\begin{array}{l} = \binom{k}{0} + \binom{k-1}{0} + \binom{k-1}{1} + \binom{k-2}{1} + \binom{k-2}{2} \cdots + \binom{0}{k-1} + \binom{0}{k} \\ = \binom{k}{0} + \binom{k}{1} + \binom{k-1}{2} + \cdots + \binom{2}{k-1} + \binom{1}{k} + 0 \\ = \binom{k+1}{0} + \binom{k}{1} + \binom{k-1}{2} + \cdots + \binom{2}{k-1} + \binom{1}{k} + \binom{0}{k+1}. \end{array}$$

This concludes the induction step.

Subsection 4

Recurrence Relations

Recurrence Relations and Solutions

- A **recurrence relation** is a formula that specifies how each term of a sequence is produced from earlier terms.
- Example: $a_n = 3a_{n-1} + 4a_{n-2}$, with $a_0 = 3$, $a_1 = 2$.
- **Solving a recurrence relation** means obtaining an explicit formula for the n-th term of the sequence.
- Example: The solution of $a_n = 3a_{n-1} + 4a_{n-2}$, with $a_0 = 3$, $a_1 = 2$, is $a_n = 4^n + 2 \cdot (-1)^n$.
- Example: Consider the recurrence $a_n = sa_{n-1}$, with given a_0; We obtain
$$a_n = sa_{n-1} = s^2 a_{n-2} = s^3 a_{n-3} = \cdots = s^n a_0;$$
Thus, the solution is $a_n = a_0 s^n$;

Solving a Recurrence Relation

Proposition

All solutions to the recurrence relation $a_n = sa_{n-1} + t$, where $s \neq 1$, have the form $a_n = c_1 s^n + c_2$, where c_1 and c_2 are specific numbers.

$$\begin{array}{lcl} a_0 & = & a_0 \\ a_1 & = & sa_0 + t \\ a_2 & = & sa_1 + t = s(sa_0 + t) + t = s^2 a_0 + (s+1)t \\ a_3 & = & sa_2 + t = s(s^2 a_0 + (s+1)t) + t = s^3 a_0 + (s^2 + s + 1)t \\ a_4 & = & sa_3 + t = s(s^3 a_0 + (s^2 + s + 1)t) = s^4 a_0 + (s^3 + s^2 + s + 1)t. \end{array}$$

Continuing with this pattern, we see that

$$a_n = s^n a_0 + (s^{n-1} + s^{n-2} + \cdots + s + 1)t.$$

Therefore, we obtain:

$$a_n = s^n a_0 + \frac{s^n - 1}{s - 1} t = \left(a_0 + \frac{t}{s-1} \right) s^n - \frac{t}{s-1}.$$

An Example

- Solve the recurrence $a_n = 5a_{n-1} + 3$ where $a_0 = 1$.
- The given recurrence is of the form $a_n = sa_{n-1} + t$, with $s = 5 \neq 1$. Therefore, the form of its solution is $a_n = c_1 s^n + c_2$. To find c_1 and c_2, note that

$$\begin{array}{rcccl} 1 & = & a_0 & = & c_1 + c_2 \\ 8 & = & a_1 & = & 5c_1 + c_2. \end{array}$$

  Solving these equations, we find $c_1 = \frac{7}{4}$ and $c_2 = -\frac{3}{4}$ and so

$$a_n = \frac{7}{4} \cdot 5^n - \frac{3}{4}.$$

$a_n = sa_{n-1} + t$: The Case $s = 1$

Proposition

The solution to the recurrence relation $a_n = a_{n-1} + t$ is $a_n = a_0 + nt$.

- We obtain

$$\begin{aligned}
a_n &= a_{n-1} + t \\
&= (a_{n-2} + t) + t = a_{n-2} + 2t \\
&= (a_{n-3} + t) + 2t = a_{n-3} + 3t \\
&= \cdots \\
&= a_1 + (n-1)t \\
&= (a_0 + t) + (n-1)t \\
&= a_0 + nt.
\end{aligned}$$

Second-Order Recurrence Relations

- A **second-order recurrence relation** gives each term of a sequence in terms of the previous two terms.
- Example: $a_n = 5a_{n-1} - 6a_{n-2}$.

Proposition

Let s_1, s_2 be given numbers and suppose r is a root of the quadratic equation $x^2 - s_1 x - s_2 = 0$. Then $a_n = r^n$ is a solution to the recurrence relation $a_n = s_1 a_{n-1} + s_2 a_{n-2}$.

- Let r be a root of $x^2 - s_1 x - s_2 = 0$. Observe that
  $s_1 r^{n-1} + s_2 r^{n-2} = r^{n-2}(s_1 r + s_2) = r^{n-2} r^2 = r^n$.
  Therefore r^n satisfies the recurrence $a_n = s_1 a_{n-1} + s_2 a_{n-2}$.

Example

- Find a solution to the second-order recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$.
- According to the Proposition, if r is a solution of $x^2 - 5x + 6 = 0$, then $a_n = r^n$ will be a solution to the given recurrence.
  We get $x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x = 2$ or $x = 3$.
  Therefore, one solution is $a_n = 2^n$ and another solution is $a_n = 3^n$.

The General Solution of $a_n = s_1 a_{n-1} + s_2 a_{n-2}$

Theorem

Let s_1, s_2 be numbers and let r_1, r_2 be roots of the equation $x^2 - s_1 x - s_2 = 0$. If $r_1 \neq r_2$, then every solution to the recurrence $a_n = s_1 a_{n-1} + s_2 a_{n-2}$ is of the form $a_n = c_1 r_1^n + c_2 r_2^n$.

- We know that, if r is a solution of $x^2 - s_1 x - s_2 = 0$, then r^n is a solution of $a_n = s_1 a_{n-1} + s_2 a_{n-2}$. But, if a_n is a solution, then so is any constant multiple of a_n, i.e., ca_n is also a solution. Moreover, if a_n and a'_n are two solutions, then so is their sum $a_n + a'_n$. Therefore, if r_1 and r_2 are roots of the polynomial $x^2 - s_1 x - s_2 = 0$, then $a_n = c_1 r_1^n + c_2 r_2^n$ is a solution of the recurrence relation.
- The expression $c_1 r_1^n + c_2 r_2^n$ gives all solutions provided it can produce a_0 and a_1. If we can choose c_1 and c_2 so that $a_0 = c_1 + c_2$ and $a_1 = r_1 c_1 + r_2 c_2$, then every possible sequence that satisfies the recurrence is of the form $c_1 r_1^n + c_2 r_2^n$. Solving for c_1 and c_2 gives $c_1 = \frac{a_1 - a_0 r_2}{r_1 - r_2}$, $c_2 = \frac{-a_1 + a_0 r_1}{r_1 - r_2}$. So this is possible when $r_1 \neq r_2$.

Example: Two Real Roots

- Find the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$, with $a_0 = 3$ and $a_1 = 2$.
- First, find the roots of the quadratic equation $x^2 - 3x - 4 = 0$.

$$x^2 - 3x - 4 = 0 \;\Rightarrow\; (x-4)(x+1) = 0 \;\Rightarrow\; x = 4 \text{ or } x = -1.$$

Therefore, $a_n = c_1 4^n + c_2(-1)^n$.
To determine c_1 and c_2, we note

$$\begin{array}{rcccl} 3 & = & a_0 & = & c_1 + c_2 \\ 2 & = & a_1 & = & c_1 \cdot 4 + c_2 \cdot (-1) \end{array}$$

Solving the system, we get $c_1 = 1$ and $c_2 = 2$. Thus, $a_n = 4^n + 2(-1)^n$.

The Fibonacci Numbers

- The Fibonacci numbers were defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. We find a closed form formula for F_n.
- First, solve the quadratic equation $x^2 - x - 1 = 0$. Its roots are $x = \frac{1 \pm \sqrt{5}}{2}$. Therefore, there is a formula for F_n of the form
$$F_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$
To determine c_1 and c_2, we note
$$\begin{array}{rcccl} 1 & = & F_0 & = & c_1 + c_2 \\ 1 & = & F_1 & = & c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} \end{array}$$
Solving the system, we get $c_1 = \frac{5+\sqrt{5}}{10}$ and $c_2 = \frac{5-\sqrt{5}}{10}$. Thus,
$$F_n = \frac{5+\sqrt{5}}{10} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{5-\sqrt{5}}{10} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Example: Two Complex Conjugate Roots

- Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$, where $a_0 = 1$ and $a_1 = 3$.
- The associated quadratic equation is $x^2 - 2x + 2 = 0$. This has two complex roots: $x = 1 \pm i$. So we seek a formula of the form $a_n = c_1(1+i)^n + c_2(1-i)^n$. To determine c_1 and c_2, we note

$$\begin{array}{rcccl} 1 & = & a_0 & = & c_1 + c_2 \\ 3 & = & a_1 & = & c_1(1+i) + c_2(1-i) \end{array}$$

Solving the system, we get $c_1 = \frac{1}{2} - i$ and $c_2 = \frac{1}{2} + i$. Thus,

$$a_n = (\frac{1}{2} - i)(1+i)^n + (\frac{1}{2} + i)(1-i)^n.$$

The Case of Repeated Roots

Theorem

Let s_1, s_2 be numbers so that the quadratic equation $x^2 - s_1 x - s_2 = 0$ has exactly one root $r \neq 0$. Then every solution to the recurrence relation $a_n = s_1 a_{n-1} + s_2 a_{n-2}$ is of the form $a_n = c_1 r^n + c_2 n r^n$.

- Since the quadratic equation has a single root, it must be of the form $(x-r)(x-r) = x^2 - 2rx + r^2$. Thus the recurrence must be $a_n = 2ra_{n-1} - r^2 a_{n-2}$. We show that a_n satisfies the recurrence and that c_1, c_2 can be chosen so as to produce all possible a_0, a_1.
  - To see that a_n satisfies the recurrence, note $2ra_{n-1} - r^2 a_{n-2}$
  $$\begin{aligned} &= 2r(c_1 r^{n-1} + c_2(n-1)r^{n-1}) - r^2(c_1 r^{n-2} + c_2(n-2)r^{n-2}) \\ &= (2c_l r^n - c_1 r^n) + (2c_2(n-1)r^n - c_2(n-2)r^n) \\ &= c_1 r^n + c_2 n r^n = a_n. \end{aligned}$$
  - To see that we can choose c_1, c_2 to produce all possible a_0, a_1, we solve
  $$\begin{aligned} a_0 &= c_1 r^0 + c_2 \cdot 0 \cdot r_0 &= c_1 \\ a_1 &= c_1 r^1 + c_2 \cdot 1 \cdot r &= r(c_1 + c_2). \end{aligned}$$
  So long as $r \neq 0$, we can solve these: $c_1 = a_0$ and $c_2 = \frac{a_0 r - a_1}{r}$.

Example: A Repeated Root

- Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$, where $a_0 = 1$ and $a_1 = 3$.
- The associated quadratic equation is $x^2 - 4x + 4 = 0$. We get

$$x^2 - 4x + 4 = 0 \;\Rightarrow\; (x-2)^2 = 0 \;\Rightarrow\; x = 2.$$

So we seek a formula of the form $a_n = c_1 2^n + c_2 n 2^n$. To determine c_1 and c_2, we note

$$\begin{array}{rcccl} 1 & = & a_0 & = & c_1 \\ 3 & = & a_1 & = & c_1 \cdot 2 + c_2 \cdot 1 \cdot 2 \end{array}$$

Solving the system, we get $c_1 = 1$ and $c_2 = \frac{1}{2}$. Thus,

$$a_n = 2^n + \frac{1}{2} \cdot n \cdot 2^n.$$

The Difference Operator

- Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. Let Δa denote a new sequence in which each term is the difference of two consecutive terms of the original sequence. That is, Δa is the sequence whose n-th term is
$$\Delta a_n = a_{n+1} - a_n.$$
We call Δ the **difference operator**.
- Example: Let a be the sequence
$$0, 2, 7, 15, 26, 40, 57, \ldots.$$
The sequence Δa is
$$2, 5, 8, 11, 14, 17, \ldots.$$
We may write the sequence a on one row and Δa on a second row with Δa_n written between a_n and a_{n+1}:
$$\begin{array}{r|ccccccccccccc} a & 0 & & 2 & & 7 & & 15 & & 26 & & 40 & & 57 \\ \Delta a & & 2 & & 5 & & 8 & & 11 & & 14 & & 17 & \end{array}$$

Reduction of Degree

Proposition

Let a be a sequence of numbers in which a_n is given by a degree d polynomial in n where $d \geq 1$. Then Δa is a sequence given by a polynomial of degree $d-1$.

- Suppose $a_n = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$, $c_d \neq 0$ and $d \geq 1$. We calculate Δa_n:

$$\begin{array}{rcl}
\Delta a_n & = & a_{n+1} - a_n \\
& = & [c_d (n+1)^d + c_{d-1}(n+1)^{d-1} + \cdots + c_1(n+1) + c_0] \\
& & -[c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0] \\
& = & [c_d(n+1)^d - c_d n^d] + [c_{d-1}(n+1)^{d-1} - c_{d-1} n^{d-1}] \\
& & + \cdots + [c_1(n+1) - c_1 n] + [c_0 - c_0].
\end{array}$$

Each term on the last line is of the form $c_j(n+1)^j - c_j n^j$. We expand the $(n+1)^j$ using the Binomial Theorem.

Reduction of Degree (Cont'd)

- We have $\Delta a_n = [c_d(n+1)^d - c_d n^d] + [c_{d-1}(n+1)^{d-1} - c_{d-1}n^{d-1}] + \cdots + [c_1(n+1) - c_1 n] + [c_0 - c_0]$.
- We look at $c_j(n+1)^j - c_j n^j$:

$$\begin{array}{rl} & c_j(n+1)^j - c_j n^j \\ = & c_j[n^j + \binom{j}{1}n^{j-1} + \binom{j}{2}n^{j-2} + \cdots + \binom{j}{j}n^0] - c_j n^j \\ = & c_j[\binom{j}{1}n^{j-1} + \binom{j}{2}n^{j-2} + \cdots + \binom{j}{j}]. \end{array}$$

  So $c_j(n+1)^j - c_j n^j$ is a polynomial of degree $j-1$. Therefore, $c_d(n+1)^d - c_d n^d$ is a polynomial of degree $d-1$. Moreover, none of the subsequent terms in Δa_n can cancel the n^{d-1} term because they all have degree less than $d-1$. Therefore, Δa_n is given by a polynomial of degree $d-1$.

Repeated Application of the Difference Operator

- If a is given by a polynomial of degree d, then Δa is given by a polynomial of degree $d-1$. Therefore, $\Delta^2 a = \Delta(\Delta a)$ is given by a polynomial of degree $d-2$, etc.
- Since each subsequent sequence is a polynomial of degree one lower, we eventually reach a polynomial of degree zero, i.e., a constant. One more application yields the all-zero sequence!

Corollary

If a sequence a is generated by a polynomial of degree d, then $\Delta^{d+1} a$ is the all-zeros sequence.

- Example: The sequence $0, 2, 7, 15, 26, 40, 57, \ldots$ is generated by a polynomial. We get:

$$\begin{array}{r|ccccccccccccc}
a & 0 & & 2 & & 7 & & 15 & & 26 & & 40 & & 57 \\
\Delta a & & 2 & & 5 & & 8 & & 11 & & 14 & & 17 & \\
\Delta^2 a & & & 3 & & 3 & & 3 & & 3 & & 3 & & \\
\Delta^3 a & & & & 0 & & 0 & & 0 & & 0 & & &
\end{array}$$

Linearity of Difference

- If there is a positive integer k such that $\Delta^k a_n$ is the all-zeros sequence, then a_n is given by a polynomial formula. In addition, there is a simple method for deducing the polynomial that generates a_n.

Proposition (Linearity of Difference)

Let a, b and c be sequences of numbers and s a number.

1. If, for all n, $c_n = a_n + b_n$, then $\Delta c_n = \Delta a_n + \Delta b_n$.
2. If, for all n, $b_n = sa_n$, then $\Delta b_n = s\Delta a_n$.

More succinctly, $\Delta(a_n + b_n) = \Delta a_n + \Delta b_n$ and $\Delta(sa_n) = s\Delta a_n$.

- If $c_n = a_n + b_n$, then $\Delta c_n = c_{n+1} - c_n = (a_{n+1} + b_{n+1}) - (a_n + b_n) = (a_{n+1} - a_n) + (b_{n+1} - b_n) = \Delta a_n + \Delta b_n$.
- Similarly, if $b_n = sa_n$, then
  $\Delta b_n = b_{n+1} - b_n = sa_{n+1} - sa_n = s(a_{n+1} - a_n) = s\Delta a_n$.

Binomial Coefficients and Difference

Proposition

Let k be a positive integer and let $a_n = \binom{n}{k}$, for all $n \geq 0$. Then $\Delta a_n = \binom{n}{k-1}$.

- We need to show that $\Delta \binom{n}{k} = \binom{n}{k-1}$, for all $n \geq 0$. This is equivalent to $\binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$ which is the same as $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, which holds by Pascal's Identity whenever $0 < k < n+1$.
  So we only need to prove it when $n+1 \leq k$ (i.e., $n \leq k-1$).
  - If $n < k-1$, all three terms equal zero.
  - If $n = k-1$, we have $\binom{n+1}{k} = \binom{k}{k} = 1$, $\binom{n}{k} = \binom{k-1}{k} = 0$ and $\binom{n}{k-1} = \binom{k-1}{k-1} = 1$.

Determinacy Based on First Term and Differences

Proposition

Let a and b be sequences of numbers and let k be a positive integer. Suppose that

- $\Delta^k a_n = \Delta^k b_n = 0$, for all n;
- $a_0 = b_0$;
- $\Delta^j a_0 = \Delta^j b_0$, for all $1 \leq j < k$.

Then $a_n = b_n$, for all n.

- The proof is by induction on k.
  - Basis: $k = 1$. In this case, $\Delta a_n = \Delta b_n = 0$ for all n. This means that $a_{n+1} - a_n = 0$ for all n. So $a_{n+1} = a_n$ for all n, i.e., all terms in a_n are identical. Likewise for b_n. Since, by hypothesis, $a_0 = b_0$, the two sequences are the same.
  - Induction Hypothesis: The Proposition has been proved for $k = l$.
  - Induction Step: We must prove the result in the case $k = l + 1$.

Determinacy Based on First Term and Differences (Cont'd)

- We are continuing the Induction;
  - Induction Step: We are assuming that the Proposition has been proved for $k = l$ and are working to prove it for $k = l + 1$. Let a and b be sequences such that
    - $\Delta^{l+1} a_n = \Delta^{l+1} b_n = 0$, for all n,
    - $a_0 = b_0$ and
    - $\Delta^j a_0 = \Delta^j b_0$, for all $1 \leq j < l + 1$.

    Consider the sequences $a'_n = \Delta a_n$ and $b'_n = \Delta b_n$. By our hypotheses we see that $\Delta^l a'_n = \Delta^l b'_n = 0$, for all n, $a'_0 = b'_0$, and $\Delta^j a'_0 = \Delta^j b'_0$, for all $1 \leq j < l$. Therefore, by induction, a' and b' are identical. Now use smallest counterexample proof to show that $a_n = b_n$, for all n.
    Let m be the smallest subscript so that $a_m \neq b_m$.
    - $m \neq 0$ because we are given $a_0 = b_0$; Thus $m > 0$.
    - Now, we know $a_{m-1} = b_{m-1}$ and $a'_{m-1} = b'_{m-1}$. But then $a_m = (a_m - a_{m-1}) + a_{m-1} = a'_{m-1} + a_{m-1} = b'_{m-1} + b_{m-1} = (b_m - b_{m-1}) + b_{m-1} = b_m$, a contradiction!

Deriving a Sequence from its Differences I

Theorem

Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers. The terms a_n can be expressed as polynomial expressions in n if and only if there is a nonnegative integer k such that for all $n \geq 0$ we have $\Delta^{k+1} a_n = 0$. Then,

$$a_n = a_0 \binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} + \cdots + (\Delta^k a_0)\binom{n}{k}.$$

- If a_n is given by a polynomial of degree d, then $\Delta^{d+1} a_n = 0$, for all n.
- Suppose now that, for some k and for all n, $\Delta^{k+1} a_n = 0$. We prove that a_n is given by a polynomial expression by showing that a_n is equal to $b_n = a_0\binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} + \cdots + (\Delta^k a_0)\binom{n}{k}$. By the previous proposition, we need
  - $\Delta^{k+1} a_n = \Delta^{k+1} b_n = 0$, for all n;
  - $a_0 = b_0$ and
  - $\Delta^j a_0 = \Delta^j b_0$, for all $1 \leq j \leq k$.

  We show all three in the next slide.

Deriving a Sequence from its Differences II

- Recall $b_n = a_0\binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} + \cdots + (\Delta^k a_0)\binom{n}{k}$.
  - Showing $\Delta^{k+1} a_n = \Delta^{k+1} b_n = 0$, for all n:
    $\Delta^{k+1} a_n = 0$ holds by hypothesis. Since b_n is a polynomial of degree k, $\Delta^{k+1} b_n = 0$ also.
  - Showing $a_0 = b_0$:
    $b_0 = a_0\binom{0}{0} + (\Delta a_0)\binom{0}{1} + (\Delta^2 a_0)\binom{0}{2} + \cdots + (\Delta^k a_0)\binom{0}{k} = a_0$.
  - Showing $\Delta^j a_0 = \Delta^j b_0$, for all $1 \le j \le k$:
    Set $c_j = \Delta^j a_0$, $1 \le j \le k$. Then, we get
    $b_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_k\binom{n}{k}$. We have

$$\begin{array}{rcl}
\Delta^j b_n & = & \Delta^j[c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_k\binom{n}{k}] \\
 & \stackrel{\text{Linearity}}{=} & c_0\Delta^j\binom{n}{0} + c_1\Delta^j\binom{n}{1} + \cdots + c_k\Delta^j\binom{n}{k} \\
 & \stackrel{\text{Binomials}}{=} & 0 + \cdots + 0 + c_j\Delta^j\binom{n}{j} + c_{j+1}\Delta^j\binom{n}{j+1} + \cdots + c_k\Delta^j\binom{n}{k} \\
 & = & c_j\binom{n}{0} + c_{j+1}\binom{n}{1} + \cdots + c_k\binom{n}{k-j}.
\end{array}$$

So, setting $n = 0$ yields $\Delta^j b_0 = c_j = \Delta^j a_0$.

Revisiting an Example

- Recall the sequence $0, 2, 7, 15, 26, 40, 57, \ldots$ whose differences we have computed before:

$$\begin{array}{c|ccccccccccccc}
a & 0 & & 2 & & 7 & & 15 & & 26 & & 40 & & 57 \\
\Delta a & & 2 & & 5 & & 8 & & 11 & & 14 & & 17 & \\
\Delta^2 a & & & 3 & & 3 & & 3 & & 3 & & 3 & & \\
\Delta^3 a & & & & 0 & & 0 & & 0 & & 0 & & &
\end{array}$$

By the Theorem

$$\begin{array}{lll}
a_n & = & a_0\binom{n}{0} + (\Delta a_0)\binom{n}{1} + (\Delta^2 a_0)\binom{n}{2} \\
& = & 0\binom{n}{0} + 2\binom{n}{1} + 3\binom{n}{2} \\
& = & 2\frac{n!}{1!(n-1)!} + 3\frac{n!}{2!(n-2)!} \\
& = & 2n + 3\frac{n(n-1)}{2} = \frac{3n^2+n}{2} = \frac{n(3n+1)}{2}.
\end{array}$$

Deriving a Formula for the Sum of Squares

- Our goal is to use the theorem to show:

$$0^2 + 1^2 + 2^2 + \cdots + n^2 = \frac{(2n+1)(n+1)(n)}{6}.$$

Let $a_n = 0^2 + 1^2 + \cdots + n^2$. Then, we have

$$\begin{array}{c|ccccccccccccccc}
a_n & 0 & & 1 & & 5 & & 14 & & 30 & & 55 & & 91 & & 140 \\
\Delta a_n & & 1 & & 4 & & 9 & & 16 & & 25 & & 36 & & 49 & \\
\Delta^2 a_n & & & 3 & & 5 & & 7 & & 9 & & 11 & & 13 & & \\
\Delta^3 a_n & & & & 2 & & 2 & & 2 & & 2 & & 2 & & & \\
\Delta^4 a_n & & & & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}$$

Therefore,

$$\begin{aligned}
a_n &= 0\binom{n}{0} + 1\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3} \\
&= 0 + n + \tfrac{3}{2}n(n-1) + \tfrac{2}{6}n(n-1)(n-2) \\
&= \tfrac{2n^3+3n^2+n}{6} = \tfrac{(2n+1)(n+1)(n)}{6}.
\end{aligned}$$