Elementary Differential Equations

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1. Introduction
   - Basic Mathematical Models: Direction Fields
   - Solutions of Some Differential Equations
   - Classification of Differential Equations

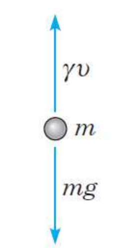
Subsection 1

Basic Mathematical Models: Direction Fields

Differential Equations and Models

- Equations containing derivatives are **differential equations**;
- A differential equation that describes some physical process is called a **mathematical model** of the process;
- Example: Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Typically, the variable t denotes time; Let v be the velocity of the falling object; We measure time t in seconds and velocity v in meters/second and assume v is positive in the downward direction; Newtons second law states: $F = ma$; Moreover, $a = \frac{dv}{dt}$; Total force acting on the falling object is



$$F = \underbrace{mg}_{\text{gravity}} - \underbrace{\gamma v}_{\text{drag}};$$

The previous two equations yield $m\frac{dv}{dt} = mg - \gamma v$;

Direction Fields

- Consider a differential equation of the form $\frac{dy}{dt} = f(t, y)$; The function $f(t, y)$ is called the **rate function**;
- A **direction field** for the differential equation is constructed by evaluating $f(t, y)$ at each point of a rectangular grid;
- At each point of the grid, a short line segment is drawn whose slope is the value of f at that point;
- Each line segment is tangent to the graph of the solution passing through that point;
- Direction fields provide a good picture of the overall behavior of solutions of a differential equation;
- In constructing a direction field, we do not have to solve the equation, but merely to evaluate the given function $f(t, y)$ many times;

Another Application: Field Mice Population

- Consider a population of field mice inhabiting a certain rural area;
- In the absence of predators we assume that the mouse population increases at a rate proportional to the current population;
- Denoting time by t and the mouse population by $p(t)$, we get

$$\frac{dp}{dt} = rp,$$

  where r is a proportionality constant called the **rate constant** or **growth rate**;
- If we assume, in addition, that owls live in the same neighborhood and that they kill field mice at a rate of k, then the new equation modeling the mouse population would be

$$\frac{dp}{dt} = \underbrace{rp}_{\text{rate of increase}} - \underbrace{k;}_{\text{rate of decrease}}$$

Guidelines to Constructing Mathematical Models

- Steps for constructing a model for a physical problem or phenomenon:
  1. Identify the independent and dependent variables and assign letters to represent them;
  2. Choose the units of measurement for each variable;
  3. Articulate the basic principle that underlies or governs the physical problem under investigation; To do this, we must often be familiar with the field in which the problem originates;
  4. Express the principle or law of the previous step in terms of the variables chosen for the modeling process;
  5. A quick check that the equation is not fundamentally inconsistent is that both terms in the equation have the same physical units;
- In more complicated problems the mathematical model may not be just a single differential equation.

Subsection 2

Solutions of Some Differential Equations

A Specific Initial Value Problem

- Consider the following:

$$\overbrace{\frac{dy}{dt} = ay - b, \qquad \underbrace{y(0) = y_0}_{\text{Initial Condition}};}^{\text{Initial Value Problem}}$$

- To solve it (i.e., find $y = y(t)$) we work as follows:

$$\frac{dy}{dt} = ay - b \quad \Rightarrow \quad \frac{dy}{dt} = a(y - \frac{b}{a}) \quad \Rightarrow \quad \frac{dy}{y - \frac{b}{a}} = adt$$
$$\Rightarrow \quad \int \frac{dy}{y - \frac{b}{a}} = \int adt \quad \Rightarrow \quad \ln|y - \frac{b}{a}| = at + C$$
$$\Rightarrow \quad y - \frac{b}{a} = e^{at+C} \quad \Rightarrow \quad y - \frac{b}{a} = e^C e^{at}$$
$$\Rightarrow \quad y = \frac{b}{a} + ce^{at} \quad \textbf{(General Solution)};$$
$$y(0) = y_0 \quad \Rightarrow \quad \frac{b}{a} + c = y_0 \quad \Rightarrow \quad c = y_0 - \frac{b}{a};$$

Thus, we get $y(t) = \frac{b}{a} + (y_0 - \frac{b}{a})e^{at}$ **(Particular Solution)**;

Free Falling Object: General Solution

- Recall the equation describing the free fall of an object of mass m:

$$m\frac{dv}{dt} = mg - \gamma v \quad \Rightarrow \quad \frac{dv}{dt} = g - \frac{\gamma}{m}v;$$

Suppose $m = 10$ Kg, and the drag coefficient $\gamma = 2$ Kg/s; Finally, recall $g = 9.8(\approx 10)$ m/s^2;

$$\begin{aligned} &\tfrac{dv}{dt} = g - \tfrac{\gamma}{m}v \quad \Rightarrow \quad \tfrac{dv}{dt} = 10 - \tfrac{2}{10}v \quad \Rightarrow \quad 5\tfrac{dv}{dt} = 50 - v \\ &\Rightarrow \quad \tfrac{dv}{50-v} = \tfrac{1}{5}dt \quad \Rightarrow \quad \int \tfrac{dv}{50-v} = \int \tfrac{1}{5}dt \\ &\Rightarrow \quad -\ln(50 - v) = \tfrac{1}{5}t + C \quad \Rightarrow \quad 50 - v = ce^{-t/5} \\ &\Rightarrow \quad v = 50 - ce^{-t/5}; \end{aligned}$$

An Initial Condition and a Particular Solution

- We found that $v = 50 - ce^{-t/5}$ is the equation describing the velocity of an object in free fall with mass $m = 10$ Kg, and drag coefficient $\gamma = 2$ Kg/s;
- If it is dropped by a height of $h_0 = 300$ meters, can we find an equation describing the distance x that the object travels in time t? At time $t = 0$, $v(0) = 0$; Therefore, $50 - c = 0 \Rightarrow c = 50$; Thus, the equation becomes: $v = 50 - 50e^{-t/5}$; Now, we get:

$$\begin{aligned} v = 50 - 50e^{-t/5} \quad &\Rightarrow \quad \frac{dx}{dt} = 50 - 50e^{-t/5} \\ \Rightarrow \quad dx = (50 - 50e^{-t/5})dt \ &\Rightarrow \ \int dx = \int (50 - 50e^{-t/5})dt \\ \Rightarrow \quad x(t) = 50t + 50 \cdot 5e^{-t/5} + C; & \end{aligned}$$

Since $x(0) = 0$ (no distance traveled yet), $0 = 50 \cdot 5 + C \Rightarrow C = -250$; So $x(t) = 50t + 250e^{-t/5} - 250$; The height of the object at time t will be $h(t) = 300 - x(t)$ (dropping by distance $x(t)$) or $h(t) = 550 - 50t - 250e^{-t/5}$.

Subsection 3

Classification of Differential Equations

Ordinary versus Partial Differential Equations

- Based on the number of independent variables on which the unknown function depends:
  - If only one independent variable is involved, only ordinary derivatives appear in the differential equation and it is said to be an **ordinary differential equation**;
  - If several independent variables appear, then the derivatives are partial derivatives, and the equation is called a **partial differential equation**;
- Some Examples:
  - The charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, and inductance L is given by the ordinary differential equation:

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t);$$

  - The heat conduction equation $\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$ is a partial differential equation, as is the wave equation: $a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$;

Systems of Differential Equations

- Based on the number of unknown functions that are involved;
  - If there is a single function to be determined, then one equation is sufficient;
  - If there are two or more unknown functions, then a system of equations is required;
- An example is the **Lotka-Volterra**, or **predator-prey**, **equations**, which are important in ecological modeling:
  - $x(t)$ and $y(t)$ are the populations of the prey and predator species;
  - a, α, c and γ are constants based on empirical observations and depend on the particular species being studied;
  - Then, the equations have the form

$$\begin{cases} \frac{dx}{dt} & = & ax - \alpha xy \\ \frac{dy}{dt} & = & -cy + \gamma xy \end{cases}$$

Order of a Differential Equation

- The **order** of a differential equation is the order of the highest derivative that appears in the equation;
- The equation $F[t, u(t), u'(t), ..., u^{(n)}(t)] = 0$ is an ordinary differential equation of the ***n*-th order**;
- It is convenient and customary in differential equations to write y for $u(t)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(t), u''(t), \ldots, u^{(n)}(t)$;
- Example: $y''' + 2e^t y'' + yy' = t^4$ is a third order differential equation for $y = u(t)$;
- We always assume that it is possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f(t, y, y', y'', ..., y^{(n-1)}).$$

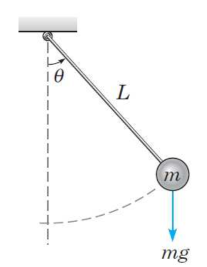
Linear and Nonlinear Equations

- The ordinary differential equation $F(t, y, y', ..., y^{(n)}) = 0$ is said to be **linear** if F is a linear function of the variables $y, y', \ldots, y^{(n)}$;
- A similar definition applies to partial differential equations;
- The general linear ordinary differential equation of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t);$$

- An equation that is not of this form is a **nonlinear equation**;
- Example: A simple physical problem that leads to a nonlinear differential equation is the oscillating pendulum. The angle θ that an oscillating pendulum of length L makes with the vertical direction satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0;$$



Advantages of Linearity and Linearization

- The mathematical theory and methods for solving linear equations are highly developed;
- For nonlinear equations the theory is more complicated, and methods of solution are less satisfactory;
- It is fortunate that many significant problems lead to linear ordinary differential equations or can be approximated by linear equations;
- Example: For the pendulum, if the angle θ is small, then $\sin\theta \cong \theta$ and the pendulum equation $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ can be approximated by the linear equation $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$;
- This process of approximating a nonlinear equation by a linear one is called **linearization** and constitutes an extremely valuable way to deal with nonlinear equations, when possible;
- Since, there are many physical phenomena that cannot be represented adequately by linear equations, to study those it is essential to deal with nonlinear equations also;

Solutions of Differential Equations

- Consider again the equation $y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$;
- A **solution** of this differential equation on the interval $\alpha < t < \beta$ is a function ϕ, such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy $\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)]$, for every t in $\alpha < t < \beta$;
- It is often not very easy to find solutions of differential equations;
- It is usually relatively easy to check whether a given function is a solution;
- Example: Check whether $y(t) = \cos t$ is a solution of $y'' + y = 0$;

$$\begin{aligned} y(t) &= \cos t; \\ y'(t) &= -\sin t; \\ y''(t) &= -\cos t; \\ y'' + y &= -\cos t + \cos t = 0; \end{aligned}$$

Existence and Uniqueness of Solutions

- Does an equation of the form $y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$ always have a solution?
- NO! Writing down an equation of this form does not necessarily mean that there is a function $y = \phi(t)$ that satisfies it;
- The question of "whether some particular equation has a solution" is the question of **existence**;
- The question of "whether a given differential equation that has a solution, has a unique solution" is the question of **uniqueness**;
- If we find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem;
- If there may be other solutions, then perhaps we should continue exploring the solution space;

Practice of Finding Solutions

- Knowledge of **existence theory** serves in avoiding pitfalls, such as using a computer to find a numerical approximation to a "solution" that does not exist;
- On the other hand, even though we may know that a solution exists, it is often the case that the solution is not expressible in terms of the usual elementary functions (polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions);
- We discuss elementary methods that can be used to obtain exact solutions of certain relatively simple problems.