Elementary Differential Equations

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Subsection 1

Homogeneous Equations with Constant Coefficients

Linear and Nonlinear Second Order Equations

- A **second order ordinary differential equation** has the form $\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt})$, where f is a given function;
- The equation is called **linear** if the function f has the form $f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y$, i.e., if f is linear in y and $\frac{dy}{dt}$;
- $g, p,$and q are specified functions of the independent variable t, but do not depend on y;
- In this case the equation can be rewritten as

$$y'' + p(t)y' + q(t)y = g(t),$$

  where the primes denote differentiation with respect to t;
- One sometimes sees the form $P(t)y'' + Q(t)y' + R(t)y = G(t)$; If $P(t) \neq 0$, we can divide by $P(t)$ to obtain the previous form;
- We operate under the hypothesis that p, q, and g are continuous functions in an interval of interest;
- Equations that are not linear are called **nonlinear**;

Homogeneous and Non-homogeneous Equations

- An initial value problem has the form
$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}),\ y(t_0) = y_0,\ y'(t_0) = y_0',$$
where y_0 and y_0' are given numbers;
- A second order linear equation is said to be **homogeneous** if the term $G(t)$ in $P(t)y'' + Q(t)y' + R(t)y = G(t)$ is zero for all t;
- Otherwise, the equation is called **nonhomogeneous**; As a result, the term $G(t)$ is sometimes called the **nonhomogeneous term**;
- We write homogeneous equations in the form $P(t)y'' + Q(t)y' + R(t)y = 0$;
- Once the homogeneous equation has been solved, it is always possible to solve the corresponding nonhomogeneous equation; Thus, solving the homogeneous equation is fundamental;

Homogeneous Equations With Constant Coefficients

- General Form $P(t)y'' + Q(t)y' + R(t)y = G(t)$;
- Homogeneous Form $P(t)y'' + Q(t)y' + R(t)y = 0$;
- We now focus on equations in which the functions P, Q, and R are constants. In this case we deal with

$$ay'' + by' + cy = 0,$$

  where a, b, and c are given constants;
- These are the **(second-order linear) homogeneous equations with constant coefficients**;
- It turns out that the equation with constant coefficients can always be solved easily in terms of the elementary functions of calculus;

Example I

- Solve the equation $y'' - y = 0$ and also find the solution that satisfies the initial conditions $y(0) = 2, y'(0) = -1$;

  This is a linear homogeneous equation with $a = 1,\ b = 0,\ c = -1$; We seek a function with the property that the second derivative of the function is the same as the function itself; We know of some such examples from calculus: $y_1(t) = e^t,\ y_2(t) = e^{-t}$; Note that constant multiples of these two solutions are also solutions, i.e., $c_1 y_1(t) = c_1 e^t$ and $c_2 y_2(t) = c_2 e^{-t}$ are solutions; Note, also, that the sum of any two solutions is also a solution; Thus,
  $y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^t + c_2 e^{-t}$ is a solution; This can be verified by calculating the second derivative;
  To pick out a particular solution satisfying our initial conditions, we first compute $y' = c_1 e^t - c_2 e^{-t}$ and then
  $$\left\{\begin{array}{c} y(0) = 2 \\ y'(0) = -1 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c_1 + c_2 = 2 \\ c_1 - c_2 = -1 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} c_1 = \frac{1}{2} \\ c_2 = \frac{3}{2} \end{array}\right\};$$
  Thus, the particular solution is $y = \frac{1}{2} e^t + \frac{3}{2} e^{-t}$;

The Characteristic Equation

- How can we solve $ay'' + by' + cy = 0$, where a, b, and c are arbitrary (real) constants?
- Seek exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined;
- Then, $y' = re^{rt}$ and $y'' = r^2 e^{rt}$;
- So, we have $(ar^2 + br + c)e^{rt} = 0$, i.e., $ar^2 + br + c = 0$;
- This equation is called the **characteristic equation**;
- Suppose that it has two real and different roots r_1 and r_2;
- Then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions and it follows $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is also a solution;
- To find the particular member of the family of these solutions that satisfy $y(t_0) = y_0$ and $y'(t_0) = y_0'$,
  - Compute the derivative;
  - Substitute $t = t_0$ in the equations for y and y';
  - Solve the resulting system for c_1 and c_2;

Example I

- Find the general solution of $y'' + 5y' + 6y = 0$;

  We assume that $y = e^{rt}$;
  Then r must be a root of $r^2 + 5r + 6 = 0$ or $(r+2)(r+3) = 0$;
  The roots are $r_1 = -2$ and $r_2 = -3$;
  The general solution is $y = c_1 e^{-2t} + c_2 e^{-3t}$;

- Find the solution of the initial value problem $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$;

  We found $y = c_1 e^{-2t} + c_2 e^{-3t}$;
  Since $y(0) = 2$, we get $c_1 + c_2 = 2$;
  Moreover, $y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$; Since $y'(0) = 3$
  $-2c_1 - 3c_2 = 3$;
  By solving those, we find that $c_1 = 9$ and $c_2 = -7$;
  Thus, the particular solution is $y = 9e^{-2t} - 7e^{-3t}$;

Example II

- Find the solution of the initial value problem

$$4y'' - 8y' + 3y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{2};$$

If $y = e^{rt}$, then the characteristic equation is $4r^2 - 8r + 3 = 0$, i.e., $(2r-3)(2r-1) = 0$;
Its roots are $r = \frac{3}{2}$ and $r = \frac{1}{2}$;
Therefore the general solution of the differential equation is $y = c_1 e^{3t/2} + c_2 e^{t/2}$;
Applying the initial conditions, we obtain the following two equations for c_1 and c_2: $c_1 + c_2 = 2, \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}$;
Thus, we get $\left\{ \begin{array}{c} c_1 + c_2 = 2 \\ 3c_1 + c_2 = 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = -\frac{1}{2} \\ c_2 = \frac{5}{2} \end{array} \right\}$
So the solution of the initial value problem is $y = -\frac{1}{2}e^{3t/2} + \frac{5}{2}e^{t/2}$;

Subsection 2

Solutions of Linear Homogeneous Equations; the Wronskian

Differential Operators

- Let p and q be continuous functions on an open interval $I = (\alpha, \beta)$; The cases $\alpha = -\infty$, or $\beta = \infty$, or both, are included;
- Then, for any function ϕ that is twice differentiable on I, we define

$$L[\phi] = \phi'' + p\phi' + q\phi;$$

- $L[\phi]$ is a function on I; The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t);$$

- The operator L is sometimes written $L = D^2 + pD + q$, where D is the **derivative operator**;
- Goal: Study second order linear homogeneous equation $L[\phi](t) = 0$;

Example

- Compute $L[\phi](t)$ for

$$p(t) = t^2, \quad q(t) = 1 + t, \quad \phi(t) = \sin 3t;$$

Since $\phi'(t) = 3\cos 3t$ and $\phi''(t) = -9\sin 3t$, we get

$$\begin{aligned} L[\phi](t) &= \phi''(t) + p(t)\phi'(t) + q(t)\phi(t) \\ &= -9\sin 3t + 3t^2\cos 3t + (1+t)\sin 3t \\ &= (t-8)\sin 3t + 3t^2\cos 3t; \end{aligned}$$

Existence and Uniqueness Theorem

Existence and Uniqueness Theorem

Consider the initial value problem $y'' + p(t)y' + q(t)y = g(t)$, with $y(t_0) = y_0, y'(t_0) = y_0'$, where p, q, and g are continuous on an open interval I that contains the point t_0; Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

- The theorem says actually three things:
  1. The initial value problem has a solution, i.e., a solution exists;
  2. The initial value problem has only one solution, i.e., the solution is unique;
  3. The solution ϕ is defined throughout the interval I where the coefficients are continuous and is at least twice differentiable there;

Example

- Find the longest interval in which the solution of the initial value problem

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0, \quad y(1) = 2, \ y'(1) = 1,$$

is guaranteed to exist;

In the standard form

$$p(t) = \frac{1}{t-3}, \quad q(t) = -\frac{t+3}{t(t-3)}, \quad g(t) = 0;$$

The only points of discontinuity of the coefficients are $t = 0$ and $t = 3$; Therefore, the longest open interval, containing the initial point $t = 1$, in which all the coefficients are continuous is $0 < t < 3$; Thus, this is the longest interval in which the theorem guarantees that the solution exists;

Example

- Find the unique solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

where p and q are continuous in an open interval I containing t_0;

The function $y = \phi(t) = 0$, for all t in I certainly satisfies the differential equation and initial conditions;

By the uniqueness part, it is the only solution of the given problem;

The Superposition Principle

- Assume that y_1 and y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0$;
- Then, we can generate more solutions by forming linear combinations of y_1 and y_2;

Theorem (Principle of Superposition)

If y_1 and y_2 are two solutions of the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$, then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

- Can the constants be chosen so as to satisfy the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$?
  This requires solving for c_1, c_2 the system

$$\left\{ \begin{array}{l} c_1y_1(t_0) + c_2y_2(t_0) = y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) = y_0' \end{array} \right\};$$

The Wronskian

- The system $\left\{ \begin{array}{c} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \end{array} \right\}$;
- By linear algebra, if
  $W = \left| \begin{array}{cc} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{array} \right| = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$, there exists a unique solution, given by
  $$c_1 = \frac{1}{W} \left| \begin{array}{cc} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{array} \right| \quad \text{and} \quad c_2 = \frac{1}{W} \left| \begin{array}{cc} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{array} \right|;$$
- The determinant W is called the **Wronskian determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2;

Theorem

Let y_1 and y_2 be two solutions of $L[y] = y'' + p(t)y' + q(t)y = 0$ and that the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$ are assigned; Then it is always possible to choose the constants c_1, c_2 so that $y = c_1 y_1(t) + c_2 y_2(t)$ satisfies the differential equation and the initial conditions if and only if the Wronskian W is not zero at t_0.

Example of Application of the Wronskian

- The functions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of the differential equation $y'' + 5y' + 6y = 0$;
- The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t};$$

- Since W is nonzero for all values of t, the functions y_1 and y_2 can be used to construct solutions of the given differential equation, together with initial conditions prescribed at any value of t;
- We already solved one of these in a previous problem;

Generality of Solutions

Theorem (Generality of Solutions for Nonzero Wronskian)

Suppose that y_1 and y_2 are two solutions of the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$; The family of solutions $y = c_1y_1(t) + c_2y_2(t)$ with arbitrary coefficients c_1 and c_2 includes every solution of the equation if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

- The theorem states that, if and only if the Wronskian of y_1 and y_2 is not everywhere zero, then the linear combination $c_1y_1 + c_2y_2$ contains all solutions of the differential equation; It is therefore natural to call the expression $y = c_1y_1(t) + c_2y_2(t)$ with arbitrary constant coefficients the **general solution** of the differential equation;
- The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of the differential equation if and only if their Wronskian is nonzero;

Example I

- Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation $y'' + p(t)y' + q(t)y = 0$; Show that they form a fundamental set of solutions if $r_1 \neq r_2$;

  Calculate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1+r_2)t};$$

  Since $e^{(r_1+r_2)t} \neq 0$, and, by hypothesis $r_1 \neq r_2$, it follows that $W \neq 0$, for all t; Consequently, y_1 and y_2 form a fundamental set of solutions;

Example II

- Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of $2t^2y'' + 3ty' - y = 0, t > 0$;

  First, verify that y_1 and y_2 are solutions of the differential equation:

$$\begin{array}{lll} y_1(t) = t^{1/2} & y_1'(t) = \frac{1}{2}t^{-1/2} & y_1''(t) = -\frac{1}{4}t^{-3/2} \\ y_2(t) = t^{-1} & y_2'(t) = -t^{-2} & y_2''(t) = 2t^{-3}; \end{array}$$

$$\begin{array}{l} 2t^2y'' + 3ty' - y = 2t^2(-\frac{1}{4}t^{-3/2}) + 3t(\frac{1}{2}t^{-1/2}) - t^{1/2} = \\ -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = 0; \\ 2t^2y'' + 3ty' - y = 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = \\ 4t^{-1} - 3t^{-1} - t^{-1} = 0; \end{array}$$

  Now, calculate the Wronskian W of y_1 and y_2:

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(y) & y_2'(t) \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2};$$

  Since $W \neq 0$ for $t > 0$, y_1 and y_2 form a fundamental set of solutions in $(0, \infty)$;

Existence of Fundamental Solutions

Theorem (Existence of Fundamental Solutions)

Consider the differential equation $L[y] = y'' + p(t)y' + q(t)y = 0$, whose coefficients p and q are continuous on some open interval I; Choose some point t_0 in I; Let y_1 be the solution that also satisfies the initial conditions $y(t_0) = 1$, $y'(t_0) = 0$, and let y_2 be the solution that satisfies the initial conditions $y(t_0) = 0$, $y'(t_0) = 1$; Then y_1 and y_2 form a fundamental set of solutions of the differential equation.

- The existence of y_1 and y_2 is ensured by the Existence Theorem;
- To see that they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1;$$

  Since the Wronskian is not zero at t_0, the functions y_1 and y_2 form a fundamental set of solutions;

Example

- Use the theorem to find the fundamental set of solutions for the differential equation $y'' - y = 0$ using the initial point $t_0 = 0$;

  The two solutions of are $y_1(t) = e^t$ and $y_2(t) = e^{-t}$; The Wronskian of these solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2 \neq 0,$$

  so they form a fundamental set of solutions;

  These are not the fundamental solutions of the Theorem because they do not satisfy the initial conditions mentioned in the theorem at $t = 0$;

Example (Cont'd)

- Let $y(t) = c_1 e^t + c_2 e^{-t}$.
  Let $y_3(t)$ be the solution that satisfies $y(0) = 1$ and $y'(0) = 0$. To find it, we solve the system:
  $$\left\{ \begin{array}{rcl} c_1 + c_2 & = & 1 \\ c_1 - c_2 & = & 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{array} \right.$$
  Let $y_4(t)$ be the solution that satisfies $y(0) = 0$ and $y'(0) = 1$; To find it, we solve the system:
  $$\left\{ \begin{array}{rcl} c_1 + c_2 & = & 0 \\ c_1 - c_2 & = & 1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{array} \right.$$
  Thus, $y_3(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$ and $y_4(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t}$;
  Since the Wronskian of y_3 and y_4 is
  $$W(y_3, y_4)(t) = \left| \begin{array}{cc} \frac{1}{2}e^t + \frac{1}{2}e^{-t} & \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} & \frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{array} \right| = 1,$$
  these functions also form a fundamental set of solutions;

Abel's Theorem

Abel's Theorem

If y_1 and y_2 are solutions of $L[y] = y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by $W(y_1, y_2)(t) = ce^{-\int p(t)dt}$, where c is a certain constant that depends on y_1 and y_2, but not on t; Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

- Note that y_1 and y_2 satisfy

$$\begin{array}{rcl} y_1'' + p(t)y_1' + q(t)y_1 & = & 0; \\ y_2'' + p(t)y_2' + q(t)y_2 & = & 0. \end{array}$$

Multiply the first by $-y_2$, the second by y_1, and add:

$$\begin{array}{rcl} -y_1''y_2 - p(t)y_1'y_2 - q(t)y_1y_2 & = & 0; \\ y_1y_2'' + p(t)y_1y_2' + q(t)y_1y_2 & = & 0; \\ (y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) & = & 0; \end{array}$$

Abel's Theorem (Cont'd)

- We got $(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0$;
  Next, we let $W(t) = W(y_1, y_2)(t)$;
  We have
  $$\begin{aligned} W' &= (y_1 y_2' - y_1' y_2)' \\ &= y_1' y_2' + y_1 y_2'' - (y_1'' y_2 + y_1' y_2') \\ &= y_1 y_2'' - y_1'' y_2; \end{aligned}$$
  Thus, we get
  $$W' + p(t)W = 0 \;\Rightarrow\; \frac{1}{W} dW = -p(t)dt \;\Rightarrow\; \ln|W| = -\int p(t)dt;$$
  Thus $W(t) = ce^{-\int p(t)dt}$, for a constant c; $W(t)$ is not zero unless $c = 0$, in which case $W(t)$ is zero for all t;

Example

- Recall that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ were shown to be solutions of $2t^2y'' + 3ty' - y = 0, t > 0$; Verify that the Wronskian of y_1 and y_2 is given by the formula in Abel's Theorem;

  We have already computed $W(y_1, y_2)(t) = -\frac{3}{2}t^{-3/2}$;
  To use Abel's Theorem, we must write the differential equation $2t^2y'' + 3ty' - y = 0$ in the standard form: $y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$;
  Thus, $p(t) = \frac{3}{2t}$; This yields

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt} = ce^{-\int \frac{3}{2t}dt} = ce^{-\frac{3}{2}\ln t} = ct^{-3/2};$$

  For the particular solutions given in the example $c = -\frac{3}{2}$, which yields the Wronskian, as computed before;

Subsection 3

Complex Roots of the Characteristic Equation

Characteristic Equations with Complex Roots

- Consider $ay'' + by' + cy = 0$, where a, b, and c are real constants;
- Solutions of the form $y = e^{rt}$ are obtained for r a root of the characteristic equation $ar^2 + br + c = 0$;
- If the roots r_1 and r_2 are real and different, which occurs when $b^2 - 4ac > 0$, then the general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$;
- If $b^2 - 4ac < 0$, then the quadratic has two complex conjugate roots, say $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, with λ, μ real;
- Then, the solutions are $y_1(t) = e^{(\lambda + i\mu)t}, y_2(t) = e^{(\lambda - i\mu)t}$;
- What is the meaning of an exponential with a complex exponent?
- For example, if $\lambda = -1, \mu = 2$, and $t = 3$, then $y_1(3) = e^{-3+6i}$;
- What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as **Eulers formula**;

Euler's Formula

- The **MacLaurin series for** e^t, $\cos t$ and $\sin t$ are (for t in $\mathbb{R}$):

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad \cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}, \quad \sin t = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!};$$

- If we can substitute it for t, then

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} \\ &= \cos t + i \sin t; \end{aligned}$$

- The equation $e^{it} = \cos t + i \sin t$ is known as **Euler's formula**;
- We adopt this equation as the **definition of** e^{it}:

$$\boxed{e^{it} = \cos t + i \sin t.}$$

Some Variations of Euler's Formula

- If we replace t by $-t$ and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have $e^{-it} = \cos t - i \sin t$;
- If t is replaced by μt, then we obtain a generalized version of Euler's formula: $e^{i\mu t} = \cos \mu t + i \sin \mu t$;
- For arbitrary complex exponents $(\lambda + i\mu)t$, we get

$$\boxed{e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos \mu t + i \sin \mu t);}$$

- We adopt this as the definition of $e^{(\lambda+i\mu)t}$;
- With these definitions, one can show that all the usual laws of exponents are valid for the complex exponential function;
- Moreover, the differentiation formula $\frac{d}{dt}(e^{rt}) = re^{rt}$ holds for complex values of r as well;

Example

- Find the general solution of $y'' + y' + \frac{37}{4}y = 0$; Also find the solution that satisfies the initial conditions $y(0) = 2$, $y'(0) = 8$;

  The characteristic equation is $r^2 + r + \frac{37}{4} = 0$; Its roots are $r_1 = -\frac{1}{2} + 3i$ and $r_2 = -\frac{1}{2} - 3i$; Therefore two solutions of the differential equation are

$$y_1(t) = e^{(-\frac{1}{2}+3i)t} = e^{-t/2}(\cos 3t + i \sin 3t)$$
$$y_2(t) = e^{(-\frac{1}{2}-3i)t} = e^{-t/2}(\cos 3t - i \sin 3t);$$

  The Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{(-\frac{1}{2}+3i)t} & e^{(-\frac{1}{2}-3i)t} \\ (-\frac{1}{2}+3i)e^{(-\frac{1}{2}+3i)t} & (-\frac{1}{2}-3i)e^{(-\frac{1}{2}-3i)t} \end{vmatrix}$$
$$= (-\frac{1}{2} - 3i)e^{-t} - (-\frac{1}{2} + 3i)e^{-t} = -6ie^{-t} \neq 0;$$

  So the general solution can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$ with arbitrary coefficients.

Example (Cont'd)

- Rather than using the complex-valued solutions

$$\begin{aligned} y_1(t) &= e^{-t/2}(\cos 3t + i \sin 3t), \\ y_2(t) &= e^{-t/2}(\cos 3t - i \sin 3t), \end{aligned}$$

  we find a fundamental set of solutions that are real-valued;
- Any linear combination of two solutions is also a solution;
- So, form the linear combinations $y_1(t) + y_2(t)$ and $y_1(t) - y_2(t)$:

$$\begin{aligned} y_1(t) + y_2(t) &= 2e^{-t/2} \cos 3t, \\ y_1(t) - y_2(t) &= 2ie^{-t/2} \sin 3t; \end{aligned}$$

- Dropping the constants 2 and $2i$, we obtain

$$u(t) = e^{-t/2} \cos 3t \quad \text{and} \quad v(t) = e^{-t/2} \sin 3t;$$

Example (Cont'd)

- We came up with the solutions
$$u(t) = e^{-t/2}\cos 3t \quad \text{and} \quad v(t) = e^{-t/2}\sin 3t;$$
- The Wronskian is
$$W(u,v)(t) = \begin{vmatrix} e^{-t/2}\cos 3t & e^{-t/2}\sin 3t \\ -\frac{1}{2}e^{-t/2}\cos 3t - 3e^{-t/2}\sin 3t & -\frac{1}{2}e^{-t/2}\sin 3t + 3e^{-t/2}\cos 3t \end{vmatrix}$$
$$= e^{-t/2}\cos 3t(-\tfrac{1}{2}e^{-t/2}\sin 3t + 3e^{-t/2}\cos 3t) - e^{-t/2}\sin 3t(-\tfrac{1}{2}e^{-t/2}\cos 3t - 3e^{-t/2}\sin 3t)$$
$$= 3e^{-t}(\cos^2 3t + \sin^2 3t) = 3e^{-t} \neq 0.$$

So $u(t)$ and $v(t)$ form a fundamental set of solutions; The general solution can be written as
$$y = c_1u(t) + c_2v(t) = e^{-t/2}(c_1\cos 3t + c_2\sin 3t);$$

Example (Cont'd)

- So we have

$$\begin{array}{rcl} y(t) & = & e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t); \\ y'(t) & = & -\frac{1}{2}c_1 e^{-t/2}\cos 3t - 3c_1 e^{-t/2}\sin 3t \\ & & \quad -\frac{1}{2}c_2 e^{-t/2}\sin 3t + 3c_2 e^{-t/2}\cos 3t \\ & = & -\frac{1}{2}e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t) \\ & & \quad + e^{-t/2}(3c_2 \cos 3t - 3c_1 \sin 3t). \end{array}$$

- To satisfy the initial conditions, we set

$$\left\{ \begin{array}{l} y(0) = 2 \\ y'(0) = 8 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = 2 \\ -\frac{1}{2}c_1 + 3c_2 = 8 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = 2 \\ c_2 = 3 \end{array} \right\};$$

- Therefore $y = e^{-t/2}(2\cos 3t + 3\sin 3t)$;

Complex Roots: The General Case

- The functions $y_1(t) = e^{(\lambda+i\mu)t}$ and $y_2(t) = e^{(\lambda-i\mu)t}$ are solutions of $ay'' + by' + cy = 0$ when the roots of the characteristic equation $ar^2 + br + c = 0$ are the complex numbers $\lambda \pm i\mu$;
- To find real-valued solutions, we proceed just as in the preceding example: We form the sum and then the difference of y_1 and y_2; We have

$$\begin{aligned} y_1(t) + y_2(t) &= e^{\lambda t}(\cos \mu t + i \sin \mu t) + e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2e^{\lambda t} \cos \mu t; \\ y_1(t) - y_2(t) &= e^{\lambda t}(\cos \mu t + i \sin \mu t) - e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2ie^{\lambda t} \sin \mu t; \end{aligned}$$

Neglecting constants, we get

$$u(t) = e^{\lambda t} \cos \mu t \quad \text{and} \quad v(t) = e^{\lambda t} \sin \mu t;$$

Complex Roots: The General Case (Cont'd)

- We found

$$u(t) = e^{\lambda t}\cos\mu t \quad \text{and} \quad v(t) = e^{\lambda t}\sin\mu t;$$

- The Wronskian of u and v is

$$\begin{aligned}
&W(u,v)(t)\\
&= \left|\begin{array}{cc} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{array}\right|\\
&= e^{2\lambda t}\cos\mu t(\lambda\sin\mu t + \mu\cos\mu t)\\
&\qquad - e^{2\lambda t}\sin\mu t(\lambda\cos\mu t - \mu\sin\mu t)\\
&= \mu e^{2\lambda t}(\cos^2\mu t + \sin^2\mu t) = \mu e^{2\lambda t}.
\end{aligned}$$

- If $\mu \neq 0$, u and v form a fundamental set of solutions;
- If the roots of the characteristic equation are $\lambda \pm i\mu$, with $\mu \neq 0$, then the general solution is

$$y = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t;$$

Example I

- Find the solution of the initial value problem

$$16y'' - 8y' + 145y = 0, \ y(0) = -2, \ y'(0) = 1;$$

The characteristic equation is $16r^2 - 8r + 145 = 0$ and its roots are $r = \frac{1}{4} \pm 3i$;
General solution of the differential equation is
$y = c_1 e^{t/4} \cos 3t + c_2 e^{t/4} \sin 3t$;
To apply the first initial condition, we set $t = 0$; this gives
$y(0) = c_1 = -2$; For the second initial condition we first differentiate and then set $t = 0$; In this way we find that $y'(0) = \frac{1}{4} c_1 + 3c_2 = 1$;
So, $c_2 = \frac{1}{2}$;
Thus, the solution of the initial value problem is
$y = -2e^{t/4} \cos 3t + \frac{1}{2} e^{t/4} \sin 3t$;

Example II

- Find the general solution of $y'' + 9y = 0$;

  The characteristic equation is $r^2 + 9 = 0$ with the roots $r = \pm 3i$;
  Thus, $\lambda = 0$ and $\mu = 3$;
  The general solution is $y = c_1 \cos 3t + c_2 \sin 3t$;
  Note that if the real part of the roots is zero, then there is no exponential factor in the solution.

Subsection 4

Repeated Roots; Reduction of Order

The Case of a Repeated Root

- We saw how to solve $ay'' + by' + cy = 0$, when the roots of $ar^2 + br + c = 0$ are
  - real and different or
  - complex conjugates;
- What if the two roots r_1 and r_2 are equal?
- Recall that this occurs when the discriminant $b^2 - 4ac = 0$ and the roots are $r_1 = r_2 = -\frac{b}{2a}$;
- In this case both roots yield the same solution: $y_1(t) = e^{-bt/2a}$;
- How do we find a second solution?

Example

- Solve the differential equation $y'' + 4y' + 4y = 0$;

  The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$, whence $r_1 = r_2 = -2$; Therefore one solution is $y_1(t) = e^{-2t}$; We know that $cy_1(t)$ is also a solution;

  We replace c by a function $v(t)$ and try to determine $v(t)$ so that the $v(t)y_1(t)$ is also a solution:

$$y = v(t)y_1(t) = v(t)e^{-2t};$$

  Then

$$\begin{aligned} y' &= v'(t)e^{-2t} - 2v(t)e^{-2t} \\ y'' &= v''(t)e^{-2t} - 4v'(t)e^{-2t} + 4v(t)e^{-2t}; \end{aligned}$$

  Therefore, since $y'' + 4y' + 4y = 0$, we get

$$[v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)]e^{-2t} = 0,$$

  i.e., $v''(t) = 0$;

Example (Cont'd)

- We set $y(t) = v(t)y_1(t)$ and discovered that $v''(t) = 0$. This yields $v'(t) = c_1$ and $v(t) = c_1 t + c_2$; Thus
$$y = c_1 te^{-2t} + c_2 e^{-2t};$$
The second term corresponds to the original solution $y_1(t) = e^{-2t}$; The first hints at a second solution
$$y_2(t) = te^{-2t};$$
These two solutions form a fundamental set: $W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t} \neq 0$; Thus,
$$y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t}$$
form a fundamental set of solutions;

The General Case

- Suppose the coefficients in $ay'' + by' + cy = 0$ satisfy $b^2 - 4ac = 0$; Then $y_1(t) = e^{-bt/2a}$ is a solution; Assume that

$$y = v(t)y_1(t) = v(t)e^{-bt/2a}$$

is also a solution; We then get

$$\begin{aligned} y' &= v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}; \\ y'' &= v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}; \end{aligned}$$

Therefore, since $ay'' + by' + cy = 0$,

$$\Big[a[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)] \\ + b[v'(t) - \frac{b}{2a}v(t)] + cv(t)\Big]\, e^{-bt/2a} = 0;$$

The General Case (Cont'd)

- Canceling the factor $e^{-bt/2a}$, we obtain

$$av''(t) + (-b+b)v'(t) + (\frac{b^2}{4a} - \frac{b^2}{2a} + c)v(t) = 0;$$

The term involving $v'(t)$ is zero; The coefficient of $v(t)$ is $c - \frac{b^2}{4a}$, which is also zero because $b^2 - 4ac = 0$; Thus, $v''(t) = 0$; So $v(t) = c_1 + c_2 t$; and, therefore,

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a};$$

Thus, y is a linear combination of the two solutions

$$y_1(t) = e^{-bt/2a}, y_2(t) = te^{-bt/2a};$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\frac{b}{2a}e^{-bt/2a} & (1 - \frac{bt}{2a})e^{-bt/2a} \end{vmatrix} = e^{-bt/a} \neq 0,$$

whence the solutions y_1 and y_2 are a fundamental set of solutions.

Example

- Find the solution of the initial value problem

$$y'' - y' + \frac{1}{4}y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{3};$$

The characteristic equation is $r^2 - r + \frac{1}{4} = 0$, So the roots are $r_1 = r_2 = \frac{1}{2}$; Thus the general solution of the differential equation is $y = c_1 e^{t/2} + c_2 te^{t/2}$; The first initial condition requires that $y(0) = c_1 = 2$; To satisfy the second initial condition, we first differentiate and then set $t = 0$; $y'(0) = \frac{1}{2}c_1 + c_2 = \frac{1}{3}$, so $c_2 = -\frac{2}{3}$; Thus the solution of the initial value problem is

$$y = 2e^{t/2} - \frac{2}{3}te^{t/2};$$

Reduction of Order

- Suppose that we know one solution $y_1(t)$ of $y'' + p(t)y' + q(t)y = 0$;
- To find a second solution, let $y = v(t)y_1(t)$;
- Then,

$$\begin{array}{rcl} y' & = & v'(t)y_1(t) + v(t)y_1'(t); \\ y'' & = & v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t) \\ & = & v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t); \end{array}$$

- Thus, since $y'' + py' + qy = 0$,

$$\begin{gathered} [v''y_1 + 2v'y_1' + vy_1''] + p[v'y_1 + vy_1'] + qvy_1 = 0; \\ y_1v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0; \end{gathered}$$

- Since y_1 is a solution, the coefficient of v is zero, so $y_1v'' + (2y_1' + py_1)v' = 0$;

Reduction of Order (Cont'd)

- We set $y = v(t)y_1(t)$ and found
$$y_1 v'' + (2y_1' + py_1)v' = 0;$$
- This is actually a first order equation for the function v' and can be solved either as a first order linear equation or as a separable equation;
- Once v' has been found, then v is obtained by an integration;
- Then, we can determine y;
- The procedure outlined here is called the **method of reduction of order**, because we solve a first order differential equation for v' rather than the second order equation for y;

Example

- Given that $y_1(t) = t^{-1}$ is a solution of $2t^2y'' + 3ty' - y = 0, t > 0$, find a fundamental set of solutions;

  We set $y = v(t)t^{-1}$; Then

$$\begin{array}{rcl} y' & = & v't^{-1} - vt^{-2}; \\ y'' & = & v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3} \\ & = & v''t^{-1} - 2v't^{-2} + 2vt^{-3}; \end{array}$$

  Substituting in the original equation and collecting terms, we obtain:

$$\begin{array}{l} 2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \\ = 2tv'' + (-4+3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v \\ = 2tv'' - v' = 0; \end{array}$$

Example (Cont'd)

- We set $y = v(t)t^{-1}$ and found

$$2tv'' - v' = 0;$$

Separating the variables and solving for $v'(t)$, we find that $v'(t) = ct^{1/2}$; Thus, $v(t) = \frac{2}{3}ct^{3/2} + k$; It follows that

$$y = \frac{2}{3}ct^{1/2} + kt^{-1};$$

The second term on the right side is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution $y_2(t) = t^{1/2}$; The Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2}t^{-1/2} \end{vmatrix} = \frac{1}{2}t^{-3/2} + t^{-3/2} = \frac{3}{2}t^{-3/2};$$

Since $t > 0$, y_1 and y_2 form a fundamental set of solutions;

Subsection 5

Nonhomogeneous Equations; Undetermined Coefficients

The Nonhomogeneous Second Order Differential Equation

- We now return to the nonhomogeneous equation $L[y] = y'' + p(t)y' + q(t)y = g(t)$, where p, q, and g are given (continuous) functions on the open interval I;
- The equation $L[y] = y'' + p(t)y' + q(t)y = 0$ is called the **homogeneous equation corresponding to** the original equation;

Theorem

If Y_1 and Y_2 are two solutions of the nonhomogeneous, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous; If, in addition, y_1 and y_2 are a fundamental set of solutions of the homogeneous, then $Y_1(t) - Y_2(t) = c_1y_1(t) + c_2y_2(t)$ with c_1, c_2 constants.

Theorem

The general solution of the nonhomogeneous can be written in the form $y = \phi(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$, where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous, c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous.

Steps for Solving the Nonhomogeneous Equation

- In somewhat different words, the last theorem states that to solve the nonhomogeneous equation $y'' + p(t)y' + q(t)y = g(t)$, we must do three things:
  1. Find the general solution $c_1y_1(t) + c_2y_2(t)$ of the corresponding homogeneous equation; This solution is called the **complementary solution** and denoted by $y_c(t)$;
  2. Find some solution $Y(t)$ of the nonhomogeneous equation; This solution is referred to as a **particular solution**;
  3. Add together the functions found in the two preceding steps;
- We have already discussed how to find $y_c(t)$, at least when the homogeneous equation has constant coefficients;
- We focus, now, on finding a particular solution $Y(t)$ of the nonhomogeneous equation;
- We study two methods:
  - The **method of undetermined coefficients**;
  - The **method of variation of parameters**;

Method of Undetermined Coefficients

- **Method of undetermined coefficients**:
  - Make an initial assumption about the form of the particular solution $Y(t)$, but with the coefficients left unspecified;
  - Substitute the assumed expression into the equation and attempt to determine the coefficients so as to obtain a solution;
  - If we are successful, then we have found a particular solution $Y(t)$ of the differential equation; If we cannot determine the coefficients, then there is no solution of the form assumed; In this case we may modify the initial assumption and try again;
- The technique is straightforward to execute once the assumption is made as to the form of $Y(t)$;
- Its major limitation is that it is useful primarily for equations for which we can easily write down the correct form of the particular solution in advance;
- We consider only nonhomogeneous terms that consist of polynomials, exponential functions, sines, and cosines;

Example I

- Find a particular solution of $y'' - 3y' - 4y = 3e^{2t}$;

  We seek a function Y such that $Y''(t) - 3Y'(t) - 4Y(t) = 3e^{2t}$;
  The exponential function reproduces itself through differentiation; So, we assume that Y(t) is some multiple of e^{2t}, i.e., $Y(t) = Ae^{2t}$, where the coefficient A is to be determined;
  To find A, we calculate $Y'(t) = 2Ae^{2t}$, $Y''(t) = 4Ae^{2t}$; Then

$$\begin{aligned} & 4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4 \cdot Ae^{2t} = 3e^{2t} \\ \Rightarrow\ & (4A - 6A - 4A)e^{2t} = 3e^{2t} \\ \Rightarrow\ & -6Ae^{2t} = 3e^{2t} \\ \Rightarrow\ & A = -\tfrac{1}{2}; \end{aligned}$$

  Thus, a particular solution is $Y(t) = -\frac{1}{2}e^{2t}$;

Example II

- Find a particular solution of $y'' - 3y' - 4y = 2\sin t$;

  Assume that $Y(t) = A\sin t$, where A is a constant to be determined; We obtain $Y'(t) = A\cos t$, $Y''(t) = -A\sin t$, whence $-A\sin t - 3A\cos t - 4A\sin t = 2\sin t \Rightarrow -5A\sin t - 3A\cos t = 2\sin t \Rightarrow (2+5A)\sin t + 3A\cos t = 0$; We want this hold for all t; Thus, it must hold for $t=0$ and $t=\frac{\pi}{2}$; We get $3A=0$ and $2+5A=0$; There is no choice of the constant A that makes the assumed expression a solution of the differential equation;

  Let us include a cosine term in $Y(t)$ and give it another try, i.e., $Y(t) = A\sin t + B\cos t$, where A and B are to be determined; Then $Y'(t) = A\cos t - B\sin t$, $Y''(t) = -A\sin t - B\cos t$; Therefore, we get $(-A+3B-4A)\sin t + (-B-3A-4B)\cos t = 2\sin t$; Matching the coefficients of $\sin t$ and $\cos t$ on each side of the equation, we get $-5A+3B=2, -3A-5B=0$, obtaining $A = -\frac{5}{17}$ and $B=\frac{3}{17}$;

  Thus, $Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$;

Short Summary

- To summarize our conclusions up to this point:
  - If the nonhomogeneous term $g(t)$ is an exponential function $e^{\alpha t}$, then assume that $Y(t)$ is proportional to the same exponential function;
  - If $g(t)$ is $\sin \beta t$ or $\cos \beta t$, then assume that $Y(t)$ is a linear combination of $\sin \beta t$ and $\cos \beta t$;
  - If $g(t)$ is a polynomial, then assume that $Y(t)$ is a polynomial of like degree.

    Thus, to find a particular solution of $y'' - 3y' - 4y = 4t^2 - 1$ we initially assume that $Y(t)$ is a polynomial of the same degree as the nonhomogeneous term, that is, $Y(t) = At^2 + Bt + C$;
  - The same principle extends to the case where $g(t)$ is a product of any two, or all three, of these types of functions;

Example III

- Find a particular solution of $y'' - 3y' - 4y = -8e^t \cos 2t$;
  We assume that $Y(t)$ is the product of e^t and a linear combination of $\cos 2t$ and $\sin 2t$, that is, $Y(t) = Ae^t \cos 2t + Be^t \sin 2t$; We get

$$\begin{array}{rcl} Y'(t) & = & Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t \\ & = & (A+2B)e^t \cos 2t + (-2A+B)e^t \sin 2t; \\ Y''(t) & = & (A+2B)e^t \cos 2t - 2(A+2B)e^t \sin 2t \\ & & +(-2A+B)e^t \sin 2t + 2(-2A+B) \cos 2t \\ & = & (-3A+4B)e^t \cos 2t + (-4A-3B)e^t \sin 2t; \end{array}$$

  Thus, A and B must satisfy the equation $(-3A+4B)e^t \cos 2t + (-4A-3B)e^t \sin 2t - 3[(A+2B)e^t \cos 2t + (-2A+B)e^t \sin 2t] - 4[Ae^t \cos 2t + Be^t \sin 2t] = -8e^t \cos 2t$, or $(-3A+4B-3A-6B-4A)e^t \cos 2t + (-4A-3B+6A-3B-4B)e^t \sin 2t = -8e^t \cos 2t$; So $10A + 2B = 8$ and $2A - 10B = 0$; These yield $A = \frac{10}{13}$ and $B = \frac{2}{13}$; Therefore, a particular solution is $Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t$;

Decomposition Into a Sum of Differential Equations

- Now suppose that $g(t)$ is the sum of two terms, $g(t) = g_1(t) + g_2(t)$;
- Suppose that

$$\begin{array}{l} Y_1 \text{ is a solution of } ay'' + by' + cy = g_1(t); \\ Y_2 \text{ is a solution of } ay'' + by' + cy = g_2(t). \end{array}$$

- Then $Y_1 + Y_2$ is a solution of the equation

$$ay'' + by' + cy = g(t).$$

- Therefore, for an equation whose nonhomogeneous function g(t) can be expressed as a sum, one can consider instead several simpler equations and then add the results together;

Example IV

- Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t;$$

By splitting up the right side, we obtain the three equations

$$\begin{array}{rcl} y'' - 3y' - 4y & = & 3e^{2t}, \\ y'' - 3y' - 4y & = & 2\sin t, \\ y'' - 3y' - 4y & = & -8e^{t}\cos 2t; \end{array}$$

We have already solved all these three equations; The respective solutions were

$$\begin{array}{rcl} Y_1(t) & = & -\frac{1}{2}e^{2t}, \\ Y_2(t) & = & \frac{3}{17}\cos t - \frac{5}{17}\sin t, \\ Y_3(t) & = & \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t; \end{array}$$

Therefore a particular solution of the given equation is their sum:

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t;$$

Example V

- Find a particular solution of $y'' - 3y' - 4y = 2e^{-t}$;

  Assume that $Y(t) = Ae^{-t}$; Then $Y'(t) = -Ae^{-t}$ and $Y''(t) = Ae^{-t}$; Thus, we get

  $$Ae^{-t} - 3(-Ae^{-t}) - 4Ae^{-t} = 2e^{-t} \Rightarrow 0 = 2e^{-t};$$

  No choice of A satisfies this equation;

  The homogeneous equation $y'' - 3y' - 4y = 0$, has characteristic

  $$r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0 \Rightarrow r = 4 \text{ or } r = -1.$$

  So we get a fundamental set of solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$; Thus the chosen particular solution is actually a solution of the homogeneous equation and it cannot be a solution of the nonhomogeneous equation;

Example V (Cont'd)

- To find a particular solution of $y'' - 3y' - 4y = 2e^{-t}$ consider the form $Y(t) = Ate^{-t}$;

  Then

$$\begin{aligned} Y'(t) &= Ae^{-t} - Ate^{-t}; \\ Y''(t) &= -Ae^{-t} - Ae^{-t} + Ate^{-t} \\ &= -2Ae^{-t} + Ate^{-t}; \end{aligned}$$

  Therefore,

$$\begin{gathered} (-2Ae^{-t} + Ate^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t} = 2e^{-t} \\ (-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t} \\ -5Ae^{-t} = 2e^{-t} \Rightarrow A = -\tfrac{2}{5}; \end{gathered}$$

  Thus a particular solution of the given equation is

$$Y(t) = -\frac{2}{5}te^{-t};$$

Summary

- Steps for finding the solution of $ay'' + by' + cy = g(t)$;
  1. Find the general solution of the corresponding homogeneous equation;
  2. Assume the function $g(t)$ involves only exponential functions, sines, cosines, polynomials, or sums or products of such functions; (If this is not the case, use the method of variation of parameters (next section))
  3. If $g(t) = g_1(t) + \cdots + g_n(t)$, form n subproblems, each containing only one of $g_1(t), \ldots, g_n(t)$; The i-th subproblem consists of the equation $ay'' + by' + cy = g_i(t)$;
  4. For the i-th subproblem assume an appropriate particular solution $Y_i(t)$; If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the homogeneous equation (of Step 1), then multiply $Y_i(t)$ by t, or (if necessary) by t^2;
  5. Find a particular solution $Y_i(t)$ for each of the subproblems. Then the sum $Y_1(t) + \ldots + Y_n(t)$ is a particular solution of original equation;
  6. Form the sum of the general solution of the homogeneous equation and the particular solution of the nonhomogeneous equation; This is the general solution of the nonhomogeneous equation;

Subsection 6

Variation of Parameters

Discussion of Variation of Parameters

- The method of **variation of parameters** complements the method of undetermined coefficients;
- Its main advantage is that it is very general; In principle, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution;
- It can be used to derive a formula for a particular solution of an arbitrary second order linear nonhomogeneous differential equation;
- It eventually requires the evaluation of certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties.

Example I

- Find a particular solution of $y'' + 4y = 3\csc t$;

  The corresponding homogeneous equation is $y'' + 4y = 0$; Its characteristic equation is $r^2 + 4 = 0$; It has solutions $r = \pm 2i$; The general solution of homogeneous is $y_c(t) = c_1 \cos 2t + c_2 \sin 2t$; Replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$, respectively, and try to determine these functions so that $y = u_1(t)\cos 2t + u_2(t)\sin 2t$ is a solution of the nonhomogeneous; Differentiate y:

  $$y' = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t;$$

  Suppose, additionally, that we require the sum of the last two terms on the right to be zero: $u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$; Then $y' = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t$; By differentiating y', we obtain

  $$y'' = -4u_1(t)\cos 2t - 4u_2(t)\sin 2t - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t;$$

Example I (Cont'd)

- We have, under $u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$,

$$y' = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t + u_1'(t)\cos 2t + u_2'(t)\sin 2t;$$
$$y'' = -4u_1(t)\cos 2t - 4u_2(t)\sin 2t - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t;$$

Then, substituting for y and y'' in $y'' + 4y = 3\csc t$, we find

$$\begin{aligned}&-4u_1(t)\cos 2t - 4u_2(t)\sin 2t - 2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t\\ &+4u_1(t)\cos 2t + 4u_2(t)\sin 2t = 3\csc t.\end{aligned}$$

Thus, $u_1(t)$ and $u_2(t)$ must satisfy
$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t;$

Example I (Cont'd)

- We want to choose u_1 and u_2 so that

$$\begin{array}{rcl} u_1'(t)\cos 2t + u_2'(t)\sin 2t & = & 0, \\ -2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t & = & 3\csc t; \end{array}$$

Solve the first for $u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t}$;
Substitute for $u_2'(t)$ in the second and simplify:

$$-2u_1'(t)\sin 2t + 2(-u_1'(t)\frac{\cos 2t}{\sin 2t})\cos 2t = 3\csc t$$

$$\frac{-2u_1'(t)\sin^2 2t - 2u_1'(t)\cos^2 2t}{\sin 2t} = 3\csc t$$

$$-2u_1'(t)(\sin^2 2t + \cos^2 2t) = 3\csc t \sin 2t$$

$$u_1'(t) = \frac{3\csc t 2\sin t\cos t}{-2} = -3\cos t;$$

Back-substituting in the first equation, we get

$$u_2'(t) = \frac{3\cos t\cos 2t}{\sin 2t} = \frac{3(1-2\sin^2 t)}{2\sin t} = \frac{3}{2}\csc t - 3\sin t;$$

Example I (Cont'd)

- We found $u_1'(t) = -3\cos t$, $u_2'(t) = \frac{3}{2}\csc t - 3\sin t$.
  By integration
  $$\begin{aligned} u_1(t) &= -3\sin t + c_1; \\ u_2(t) &= \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2; \end{aligned}$$
  Therefore, we obtain
  $$\begin{aligned} y &= -3\sin t\cos 2t + \frac{3}{2}\ln|\csc t - \cot t|\sin 2t \\ &\quad + 3\cos t\sin 2t + c_1\cos 2t + c_2\sin 2t \\ &= -3\sin t(2\cos^2 t - 1) + \frac{3}{2}\ln|\csc t - \cot t|\sin 2t \\ &\quad + 3\cos t 2\sin t\cos t + c_1\cos 2t + c_2\sin 2t \\ &= 3\sin t + \frac{3}{2}\ln|\csc t - \cot t|\sin 2t \\ &\quad + c_1\cos 2t + c_2\sin 2t; \end{aligned}$$
  The terms involving c_1 and c_2 are the general solution of the homogeneous; The other terms are a particular solution of the nonhomogeneous; Thus, the last expression gives the general solution of the original equation;

Description of Variation of Parameters I

- Consider $y'' + p(t)y' + q(t)y = g(t)$ where p, q, and g are continuous on an open interval I;
- Assume that we know the general solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ of the homogeneous $y'' + p(t)y' + q(t)y = 0$;
- We replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$ to get $y = u_1(t)y_1(t) + u_2(t)y_2(t)$;
- Then we try to determine $u_1(t)$ and $u_2(t)$ so as to get a solution of the nonhomogeneous;
- Differentiate to obtain $y' = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$;
- Set the terms involving $u_1'(t)$ and $u_2'(t)$ equal to zero, i.e., require that $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$;
- Thus, $y' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$;
- By differentiating again, we get $y'' = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$;

Description of Variation of Parameters II

- Under $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$, we found
$$\begin{aligned} y' &= u_1(t)y_1'(t) + u_2(t)y_2'(t), \\ y'' &= u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t); \end{aligned}$$
Substituting into $y'' + p(t)y' + q(t)y = g(t)$, we get
$$\begin{aligned} &(u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)) \\ &\quad + p(t)(u_1(t)y_1'(t) + u_2(t)y_2'(t)) \\ &\quad + q(t)(u_1(t)y_1(t) + u_2(t)y_2(t)) = g(t) \\ &u_1(t)[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)] \\ &\quad + u_2(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] \\ &\quad + u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t); \end{aligned}$$
- Each of the expressions in square brackets is zero because y_1 and y_2 are solutions of the homogeneous, so we get
$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$;
- So we get a system of two linear algebraic equations for the derivatives $u_1'(t)$ and $u_2'(t)$ of the unknown functions;

Description of Variation of Parameters III

- By solving it, we obtain

$$\begin{aligned} u_1'(t) &= -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}, \\ u_2'(t) &= \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}, \end{aligned}$$

  where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2;
- By integrating, we find the desired functions $u_1(t)$ and $u_2(t)$:

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt + c_1, u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt + c_2;$$

- If the integrals can be evaluated in terms of elementary functions, then we substitute back the results to obtain the general solution;

Main Theorem

Theorem

If the functions p, q, and g are continuous on an open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous $y'' + p(t)y' + q(t)y = 0$, then a particular solution of $y'' + p(t)y' + q(t)y = g(t)$ is

$$Y(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds,$$

where t_0 is any conveniently chosen point in I; The general solution is $y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$.

- Difficulties in using the method of variation of parameters:
  - Determination of $y_1(t)$ and $y_2(t)$, a fundamental set of solutions of the homogeneous equation, when the coefficients in that equation are not constants;
  - Evaluation of the integrals appearing in the theorem;
- The advantage: Expression for $Y(t)$ in terms of an arbitrary $g(t)$;