Elementary Differential Equations

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Introduction

- Finding the general solution of a linear differential equation depends on determining a fundamental set of solutions of the corresponding homogeneous equation;
- We have given a systematic procedure for constructing fundamental solutions if the equation has constant coefficients;
- To deal with equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus;
- The principal tool is the representation of a given function by a power series;
- The basic idea is similar to that in the method of undetermined coefficients: We assume that the solutions of a given differential equation have power series expansions, and then we attempt to determine the coefficients so as to satisfy the differential equation;

Subsection 1

Review of Power Series

Convergence and Absolute Convergence of Power Series

- A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to **converge** at a point x if $\lim_{m\to\infty}\sum_{n=0}^{m} a_n(x-x_0)^n$ exists for that x; The series certainly converges for $x=x_0$; It may converge for all x, or it may converge for some values of x and not for others;
- The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to **converge absolutely** at a point x if the series $\sum_{n=0}^{\infty}|a_n(x-x_0)^n| = \sum_{n=0}^{\infty}|a_n||x-x_0|^n$ converges;
- It can be shown that if the series converges absolutely, then the series also converges;
- The converse is not necessarily true;

The Ratio Test for Absolute Convergence

The Ratio Test

If $a_n \neq 0$, and if, for a fixed x, $\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L$, then the power series converges absolutely at x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$; If $|x-x_0|L = 1$, then the test is inconclusive.

- Example: For which values of x does $\sum_{n=1}^{\infty}(-1)^{n+1}n(x-2)^n$ converge?

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1}n(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\frac{n+1}{n} = |x-2|;$$

The series converges absolutely for $|x-2| < 1$, or $1 < x < 3$, and diverges for $|x-2| > 1$; The values of x corresponding to $|x-2| = 1$ are $x = 1$ and $x = 3$; The series diverges for each of these values of x since the n-th term of the series does not approach zero as $n \to \infty$;

Radius of Convergence

- If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at $x = x_1$, it converges absolutely for $|x - x_0| < |x_1 - x_0|$; and if it diverges at $x = x_1$, it diverges for $|x - x_0| > |x_1 - x_0|$;
- There is a nonnegative number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$; For a series that converges only at x_0, we define ρ to be zero; For a series that converges for all x, we say that ρ is infinite; If $\rho > 0$, then the interval $|x - x_0| < \rho$ is called the **interval of convergence**; The series may either converge or diverge when $|x - x_0| = \rho$;

Example

- Determine the radius of convergence of $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}$;

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{(x+1)^n} \right| = \frac{|x+1|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x+1|}{2};$$

  Thus the series converges absolutely for $|x+1| < 2$, or $-3 < x < 1$, and diverges for $|x+1| > 2$; The radius of convergence of the power series is $\rho = 2$;
  Finally, we check the endpoints of the interval of convergence.
  - At $x = 1$ the series becomes the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges;
  - At $x = -3$ we have $\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which converges but does not converge absolutely; The series is said to **converge conditionally** at $x = -3$;

Operations on Power Series: Addition and Multiplication

- Suppose that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x-x_0|<\rho$, with $\rho>0$;
- The series can be added or subtracted termwise, and $f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-x_0)^n$; The resulting series converges at least for $|x-x_0|<\rho$;
- The series can be formally multiplied, and $f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right] = \sum_{n=0}^{\infty} c_n(x-x_0)^n$, where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$; The resulting series converges at least for $|x-x_0|<\rho$;

Operations on Power Series: Division

- Suppose that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge to $f(x)$ and $g(x)$, respectively, for $|x-x_0| < \rho$, with $\rho > 0$;
- If $g(x_0) \neq 0$, the series can be formally divided, and $\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x-x_0)^n$; The coefficients d_n can be most easily obtained by equating coefficients in $\sum_{n=0}^{\infty} a_n(x-x_0)^n =$

$$\left[\sum_{n=0}^{\infty} d_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right] = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_k b_{n-k}\right)(x-x_0)^n;$$

  In the case of division, the radius of convergence of the resulting power series may be less than ρ;

Differentiation and Coefficients

- The function f is continuous and has derivatives of all orders for $|x - x_0| < \rho$; Further, $f', f'', \ldots$ can be computed by differentiating the series termwise; That is,

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x - x_0) + \cdots + na_n(x - x_0)^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}, \\ f''(x) &= 2a_2 + 6a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}, \\ &\cdots \end{aligned}$$

  Each of the series converges absolutely for $|x - x_0| < \rho$;
- The value of a_n is given by $a_n = \frac{f^{(n)}(x_0)}{n!}$; The series is called the **Taylor series** for the function f **about** $x = x_0$;

Equality and Analyticity

- If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$, for x in some open interval with center x_0, then $a_n = b_n$, for all n; Thus, if $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$, for each such x, then $a_0 = a_1 = \cdots = a_n = \cdots = 0$;
- A function f that has a Taylor series expansion about $x = x_0$ $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$, with a radius of convergence $\rho > 0$, is said to be **analytic at** $x = x_0$;
- All of the familiar functions of calculus are analytic except perhaps at certain easily recognized points;
- For example, $\sin x$ and e^x are analytic everywhere, $\frac{1}{x}$ is analytic except at $x = 0$, and $\tan x$ is analytic except at odd multiples of $\frac{\pi}{2}$;
- If f and g are analytic at x_0, then $f \pm g$, $f \cdot g$, and $\frac{f}{g}$ (provided that $g(x_0) \neq 0$) are also analytic at $x = x_0$;

Shifting the Index of Summation: Two Examples

- Write $\sum_{n=2}^{\infty} a_n x^n$ as a series whose first term corresponds to $n = 0$ rather than $n = 2$;
  Let $m = n - 2$; Then $n = m + 2$, and $n = 2$ corresponds to $m = 0$;
  Hence $\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2} = \sum_{n=0}^{\infty} a_{n+2} x^{n+2}$;
  The index was shifted upward by 2 and o compensate counting starts at a level 2 lower than originally;
- Write the series $\sum_{n=2}^{\infty} (n+2)(n+1) a_n (x - x_0)^{n-2}$ as a series whose generic term involves $(x - x_0)^n$ rather than $(x - x_0)^{n-2}$;
  Again, we shift the index by 2 so that n is replaced by $n + 2$ and start counting 2 lower; We obtain $\sum_{n=0}^{\infty} (n+4)(n+3) a_{n+2} (x - x_0)^n$;

Further Manipulation of the Index of Summation

- Write the expression $x^2\sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}$ as a series whose generic term involves x^{r+n};

  First, take the x^2 inside the summation, obtaining
  $\sum_{n=0}^{\infty}(r+n)a_nx^{r+n+1}$;
  Next, shift the index down by 1 and start counting 1 higher;
  Thus, $\sum_{n=0}^{\infty}(r+n)a_nx^{r+n+1}=\sum_{n=1}^{\infty}(r+n-1)a_{n-1}x^{r+n}$;

Rewriting a Series

- Assume that $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$, for all x, and determine what this implies about the coefficients a_n;

  We equate corresponding coefficients in the two series; To do this, first rewrite the equation so that the series display the same power of x in their generic terms: $\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$; Therefore, $(n+1)a_{n+1} = a_n$, for all n, or $a_{n+1} = \frac{a_n}{n+1}$, for all n; This yields

$$\begin{aligned} a_n &= \frac{a_{n-1}}{n} = \frac{a_{n-2}}{n(n-1)} = \frac{a_{n-3}}{n(n-1)(n-2)} \\ &= \frac{a_{n-4}}{n(n-1)(n-2)(n-3)} = \cdots = \frac{a_0}{n!}, \end{aligned}$$

  for all n; Thus all the coefficients may be determined in terms of a_0;

  Using this relationship, we obtain $\sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$;

Subsection 2

Series Solutions Near an Ordinary Point, Part I

The Framework: Ordinary Points

- Consider the homogeneous equation $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$;
- Examples from physics include the **Bessel equation** $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, where ν is a constant, and the **Legendre equation** $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, where α is a constant;
- We primarily consider the case in which the functions P, Q, and R are polynomials, having no common factors;
- If we wish to solve the equation in the neighborhood of a point x_0, its solution is closely associated with the behavior of P in that interval;
- A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**; Since P is continuous, there is an interval about x_0 in which $P(x)$ is never zero;
- In that interval, dividing by $P(x)$, we get $y'' + p(x)y' + q(x)y = 0$, where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are continuous functions;
- According to the Existence and Uniqueness Theorem, there exists in that interval a unique solution of the differential equation that also satisfies any given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$;

Singular Points; Power Series Solutions

- We handle, first, solutions in the neighborhood of an ordinary point;
- If $P(x_0) = 0$, then x_0 is called a **singular point**; In this case at least one of $Q(x_0)$ and $R(x_0)$ is not zero; Thus, at least one of the coefficients $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ becomes unbounded as $x \to x_0$, and, therefore, the Existence and Uniqueness does not apply in this case; In the latter sections, we will deal with finding solutions in the neighborhood of a singular point;
- In the neighborhood of an ordinary point x_0, we look for solutions of the form

$$y = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

  assuming the series converges in $|x - x_0| < \rho$, for some $\rho > 0$;
- The most practical way to determine the coefficients a_n is to substitute the series and its derivatives in the equation;

Example I

- Find a series solution of the equation $y'' + y = 0$, $-\infty < x < \infty$;
  Since $P(x) = 1$, $Q(x) = 0$ and $R(x) = 1$, every point is ordinary; We look for a solution $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$, assuming the series converges for some $|x| < \rho$; Differentiating, we get $y' = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$,
  $y'' = 2a_2 + \cdots + n(n-1) a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$;
  Substituting in the differential equation:
  $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$; Shifting the index in first sum:
  $$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0;$$

Example I (Cont'd)

We got $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+a_n]x^n=0$; Setting the coefficient of each power of x equal to zero, we get $(n+2)(n+1)a_{n+2}+a_n=0$, for all n; The even-numbered coefficients $(a_0,a_2,a_4,\ldots)$ and the odd-numbered ones $(a_1,a_3,a_5,\ldots)$ are determined separately;

- For the even-numbered coefficients we have $a_2=-\frac{a_0}{2\cdot 1}=-\frac{a_0}{2!}$, $a_4=-\frac{a_2}{4\cdot 3}=+\frac{a_0}{4!}$, $a_6=-\frac{a_4}{6\cdot 5}=-\frac{a_0}{6!}$, etc. In general, if $n=2k$, then $a_n=a_{2k}=\frac{(-1)^k}{(2k)!}a_0$;
- For the odd-numbered coefficients $a_3=-\frac{a_1}{2\cdot 3}=-\frac{a_1}{3!}$, $a_5=-\frac{a_3}{5\cdot 4}=+\frac{a_1}{5!}$, $a_7=-\frac{a_5}{7\cdot 6}=-\frac{a_1}{7!}$, etc. In general, if $n=2k+1$, then $a_n=a_{2k+1}=\frac{(-1)^k}{(2k+1)!}a_1$;

Example I (Cont'd)

Substituting into the equation

$$\begin{aligned} y &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 \\ &\quad + \cdots + \frac{(-1)^n a_0}{(2n)!}x^{2n} + \frac{(-1)^n a_1}{(2n+1)!}x^{2n+1} + \cdots \\ &= a_0[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots] \\ &\quad + a_1[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots] \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}. \end{aligned}$$

- We can test the series solutions for convergence;
- The ratio test shows that each of the series converges for all x;
- We recognize that the first series is exactly the Taylor series for $\cos x$ about $x = 0$ and the second is the Taylor series for $\sin x$ about $x = 0$;
- So, the solution is $y = a_0 \cos x + a_1 \sin x$;
- No conditions are imposed on a_0 and a_1, whence they are arbitrary;

Example II

- Find a series solution in powers of x of Airy's equation $y'' - xy = 0$, $-\infty < x < \infty$;
  For this equation $P(x) = 1$, $Q(x) = 0$,and $R(x) = -x$, whence every point is an ordinary point; Let $y = \sum_{n=0}^{\infty} a_n x^n$, convergent in some $|x| < \rho$; We get $y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$; Substituting, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = x\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}a_n x^{n+1}$; Rewrite the right side $2 \cdot 1a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty}a_{n-1}x^n$; Thus, we get $a_2 = 0$, and we obtain the recurrence relation $(n+2)(n+1)a_{n+2} = a_{n-1}$, for all n; Since a_{n+2} is given in terms of a_{n-1}, the a's are determined in steps of three;

Example II (Cont'd)

- Recall $a_2 = 0$ and $(n+2)(n+1)a_{n+2} = a_{n-1}$, for all n;
  - Since $a_2 = 0$, we get $a_5 = a_8 = a_{11} = \cdots = 0$;
  - For $a_0, a_3, a_6, a_9, \ldots$, we set $n = 1, 4, 7, 10, \ldots$ in the recurrence relation: $a_3 = \frac{a_0}{2\cdot 3}$, $a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}$, $a_9 = \frac{a_6}{8\cdot 9} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$, etc. The general formula is $a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}$, $n \geq 4$;
  - For the sequence $a_1, a_4, a_7, a_{10}, \ldots$ we set $n = 2, 5, 8, 11, \ldots$ in the recurrence relation: $a_4 = \frac{a_1}{3\cdot 4}$, $a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}$, $a_{10} = \frac{a_7}{9\cdot 10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$, etc. In general, we have $a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}$, $n \geq 4$;

  Thus the general solution of Airy's equation is

$$\begin{aligned} y &= a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots + \frac{x^{3n}}{2\cdot 3\cdots(3n-1)(3n)} + \cdots\right] \\ &\quad + a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots + \frac{x^{3n+1}}{3\cdot 4\cdots(3n)(3n+1)} + \cdots\right] \end{aligned}$$

Example II (Cont'd)

- We can now investigate the convergence;
- Use the ratio test to show that both these series converge for all x;
- Let y_1 and y_2 denote the functions defined by the expressions in the first and second sets of brackets, respectively;
- By choosing first $a_0 = 1, a_1 = 0$ and then $a_0 = 0, a_1 = 1$, it follows that y_1 and y_2 are individually solutions;
- y_1 satisfies the initial conditions $y_1(0) = 1, y_1'(0) = 0$ and y_2 satisfies the initial conditions $y_2(0) = 0, y_2'(0) = 1$;
- Thus, $W(y_1, y_2)(0) = 1 \neq 0$, and consequently y_1 and y_2 are a fundamental set of solutions;
- Hence, the general solution of Airy's equation is $y = a_0 y_1(x) + a_1 y_2(x)$, $-\infty < x < \infty$;

Subsection 3

Series Solutions Near an Ordinary Point, Part II

Justifying the Power Series Solution Process I

- We looked at $P(x)y'' + Q(x)y' + R(x)y = 0$ where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0;
- If we have a solution $y = \phi(x)$ with a Taylor series
$$y = \phi(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n \text{ converging for } |x - x_0| < \rho, \text{ where } \rho > 0,$$
we can find it by substituting in the differential equation;
- How is the statement that, if x_0 is an ordinary point of the equation, then there exist solutions of this form justified?
- Moreover, what is the radius of convergence of such a series?
- To investigate these questions, assume that there is a power series solution of the differential equation;
- By differentiating m times and setting x equal to x_0, we obtain $m!a_m = \phi^{(m)}(x_0)$;
- To compute a_n, we must show that we can determine $\phi^{(n)}(x_0)$ from the differential equation;

Justifying the Power Series Solution Process II

- Suppose that $y = \phi(x)$ is a solution satisfying the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$; Then $a_0 = y_0$ and $a_1 = y_0'$;
- To find $\phi^{(n)}(x_0)$ and a_n, $n \geq 2$, we turn to the original equation; Since ϕ is a solution, $P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0$; We can rewrite $\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x)$, where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$;
- For, $x = x_0$, $\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0)$; Hence a_2 is given by $2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0$;
- To determine a_3, we differentiate and then set $x = x_0$, obtaining $3!a_3 = \phi'''(x_0) = -[\phi'' + (p' + q)\phi' + q'\phi]|_{x=x_0} = -2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q'(x_0)a_0$; Substituting for a_2 gives a_3 in terms of a_1 and a_0;
- Since all the derivatives of p and q exist at x_0, we can continue to differentiate indefinitely, determining after each differentiation the successive coefficients $a_4, a_5, \ldots$ by setting $x = x_0$;

Power Series Solutions and Radius of Convergence

- If the functions $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ are analytic at x_0, then x_0 is said to be an **ordinary point**; Otherwise, it is a **singular point**;
- The question concerning the interval of convergence of the series solution can be answered at once for a wide class of problems:

Theorem

If x_0 is an ordinary point of $P(x)y'' + Q(x)y' + R(x)y = 0$, i.e., if $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ are analytic at x_0, then the general solution is

$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y_2(x)$, where a_0 and a_1 are arbitrary,

and y_1 and y_2 are two power series solutions that are analytic at x_0; The solutions y_1 and y_2 form a fundamental set of solutions; Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

Example I

- What is the radius of convergence of the Taylor series for $(1+x^2)^{-1}$ about $x=0$?
  - **Method 1**: Find the Taylor series in question, namely,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + + (-1)^n x^{2n} + \cdots .$$

    Then apply the ratio test to show that $\rho = 1$;

  - **Method 2**: The zeros of $1+x^2$ are $x = \pm i$; Since the distance from 0 to i or to $-i$ is 1, the radius of convergence of the power series about $x=0$ is 1;

Example II

- What is the radius of convergence of the Taylor series for $(x^2 - 2x + 2)^{-1}$ about $x = 0$? How about $x = 1$?

  First notice that $x^2 - 2x + 2 = 0$ has solutions $x = 1 \pm i$; The distance in the complex plane from $x = 0$ to either $x = 1 + i$ or $x = 1 - i$ is $\sqrt{2}$; Hence, the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} a_n x^n$ about $x = 0$ is $\sqrt{2}$;

  The distance in the complex plane from $x = 1$ to either $x = 1 + i$ or $x = 1 - i$ is 1; Hence the radius of convergence of the Taylor series expansion $\sum_{n=0}^{\infty} b_n (x-1)^n$ about $x = 1$ is 1;

Example III

- Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, where α is a constant;

  Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of P, namely, $x = \pm 1$, are a distance 1 from $x = 0$; Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and possibly for larger values of x;

  Indeed, it can be shown that if α is a positive integer, one of the series solutions terminates after a finite number of terms and hence converges not just for $|x| < 1$ but for all x;

  For example, if $\alpha = 1$, the polynomial solution is $y = x$;

Example IV

- Determine a lower bound for the radius of convergence of series solutions of the differential equation $(1+x^2)y''+2xy'+4x^2y=0$ about the point $x=0$; Also, about the point $x=-\frac{1}{2}$;

  Again P,Q and R are polynomials, and P has zeros at $x=\pm i$; The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\frac{1}{2}$ to $\pm i$ is $\sqrt{1+\frac{1}{4}}=\frac{\sqrt{5}}{2}$; Hence, in the first case the series $\sum_{n=0}^{\infty}a_nx^n$ converges at least for $|x|<1$, and in the second case the series $\sum_{n=0}^{\infty}b_n(x+\frac{1}{2})^n$ converges at least for $|x+\frac{1}{2}|<\frac{\sqrt{5}}{2}$;

  If initial conditions $y(0)=y_0$ and $y'(0)=y_0'$ are given, since $1+x^2\neq 0$, for all x, there exists a unique solution of the initial value problem on $-\infty<x<\infty$; On the other hand, a series solution of the form $\sum_{n=0}^{\infty}a_nx^n$ (with $a_0=y_0,a_1=y_0'$) is only guaranteed for $-1<x<1$; The unique solution on the interval $-\infty<x<\infty$ may not have a power series about $x=0$ that converges for all x;

Example V

- Can we determine a series solution about $x = 0$ for the differential equation $y'' + (\sin x)y' + (1 + x^2)y = 0$, and if so, what is the radius of convergence?

  For this differential equation, $p(x) = \sin x$ and $q(x) = 1 + x^2$;
  Recall from calculus that $\sin x$ has a Taylor series expansion about $x = 0$ that converges for all x;
  Further, q also has a Taylor series expansion about $x = 0$, namely, $q(x) = 1 + x^2$, that converges for all x;

  Thus there is a series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 arbitrary, and the series converges for all x;

Subsection 4

Euler Equations; Regular Singular Points

Euler Equations

- Consider equations of the form $P(x)y'' + Q(x)y' + R(x)y = 0$ in the neighborhood of a singular point x_0;
- If P, Q and R are polynomials having no common factors, then the **singular points** are those for which $P(x) = 0$;
- A relatively simple differential equation that has a singular point is the **Euler equation** $L[y] = x^2y'' + \alpha xy' + \beta y = 0$, α, β real;
- In this case $P(x) = x^2$, so $x = 0$ is the only singular point;
- Consider, first, the interval $x > 0$;
- Observe that $(x^r)' = rx^{r-1}$ and $(x^r)'' = r(r-1)x^{r-2}$;
- So, if the equation has a solution of the form $y = x^r$, then $L[x^r] = x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = x^r[r(r-1) + \alpha r + \beta] = 0$;
- If r is a root of the quadratic equation $F(r) = r(r-1) + \alpha r + \beta = 0$, then $L[x^r]$ is zero, and $y = x^r$ is a solution of the differential equation;
- The roots of the quadratic are $r_1, r_2 = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$, and $F(r) = (r - r_1)(r - r_2)$;

Case I: Real Distinct Roots

- We consider separately the cases in which the roots are real and different, real but equal, and complex conjugates;
- **Real, Distinct Roots**: If $F(r) = 0$ has real roots r_1 and r_2, with $r_1 \neq r_2$, then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are solutions;
- Since $W(x^{r_1}, x^{r_2}) = (r_2 - r_1)x^{r_1+r_2-1} \neq 0,$ for $r_1 \neq r_2$ and $x > 0$, it follows that the general solution is $y = c_1 x^{r_1} + c_2 x^{r_2}, x > 0$;
- If r is not a rational number, then x^r is defined by $x^r = e^{r \ln x}$;
- Example: Solve $2x^2 y'' + 3xy' - y = 0$, $x > 0$;
  Substituting $y = x^r$ gives $2x^2 r(r-1)x^{r-2} + 3xrx^{r-1} - x^r = x^r[2r(r-1) + 3r - 1] = x^r(2r^2 + r - 1) = x^r(2r-1)(r+1) = 0$;
  Hence $r_1 = \frac{1}{2}$ and $r_2 = -1,$ so the general solution is $y = c_1 x^{1/2} + c_2 x^{-1}, x > 0$;

Case II: Equal Roots

- If the roots r_1 and r_2 are equal, then we obtain only $y_1(x) = x^{r_1}$;
- Since $r_1 = r_2$, it follows that $F(r) = (r - r_1)^2$;
- Thus, in this case not only does $F(r_1) = 0$ but also $F'(r_1) = 0$;
- We differentiate the equation with respect to r and set $r = r_1$;
- Differentiating with respect to r gives$L[x^r \ln x] = L[\frac{\partial(x^r)}{\partial r}] = \frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r}[x^r F(r)] = \frac{\partial}{\partial r}[x^r (r - r_1)^2] = (r - r_1)^2 x^r \ln x + 2(r - r_1) x^r$;
- The right side is zero for $r = r_1$, whence, $y_2(x) = x^{r_1} \ln x$, $x > 0$, is a second solution;
- By evaluating the Wronskian, we find that $W(x^{r_1}, x^{r_1} \ln x) = x^{2r_1 - 1}$, so x^{r_1} and $x^{r_1} \ln x$ are a fundamental set of solutions for $x > 0$;
- The general solution is $y = (c_1 + c_2 \ln x) x^{r_1},$ for $x > 0$;
- Example: Solve $x^2 y'' + 5xy' + 4y = 0$, $x > 0$;
  Substituting $y = x^r$, we get $x^r[r(r-1) + 5r + 4] = x^r(r^2 + 4r + 4) = 0$; Hence, $r_1 = r_2 = -2,$ and $y = x^{-2}(c_1 + c_2 \ln x),$ $x > 0$ is the general solution;

Case III: Complex Conjugate Roots

- Suppose $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, with $\mu \neq 0$;
- If $x > 0$, r real, then $x^r = e^{r \ln x}$; Define $x^r = e^{r \ln x}$, r complex;
- Then, since $e^{i\mu \ln x} = \cos(\mu \ln x) + i \sin(\mu \ln x)$, we obtain $x^{\lambda + i\mu} = e^{(\lambda + i\mu)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x} = x^\lambda e^{i\mu \ln x} = x^\lambda [\cos(\mu \ln x) + i \sin(\mu \ln x)]$;
- Using algebra, it can be checked that x^{r_1} and x^{r_2} are indeed solutions and the general solution is $y = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu}$;
- The real and imaginary parts of $x^{\lambda + i\mu}$, namely, $x^\lambda \cos(\mu \ln x)$ and $x^\lambda \sin(\mu \ln x)$ are also solutions;
- Since $W[x^\lambda \cos(\mu \ln x), x^\lambda \sin(\mu \ln x)] = \mu x^{2\lambda - 1}$, the solutions form a fundamental set for $x > 0$, and the general solution is
  $y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x), \ x > 0$;
- Example: Solve $x^2 y'' + xy' + y = 0$;
  Substituting $y = x^r$, we get $x^r[r(r-1) + r + 1] = x^r(r^2 + 1) = 0$;
  Hence $r = \pm i$ ($\lambda = 0$ gives $x^\lambda = 1$), and the general solution is
  $y = c_1 \cos(\ln x) + c_2 \sin(\ln x), x > 0$;

Solving the Euler Equation for $x < 0$

- Consider again $L[y] = x^2 y'' + \alpha x y' + \beta y = 0$ with $x < 0$;
- One issue is the meaning of x^r for x negative and r not an integer; Similarly, $\ln x$ has not been defined for $x < 0$;
- The solutions given for $x > 0$ can be shown to be valid for $x < 0$, but in general they are complex valued;
- It is always possible to obtain real valued solutions of the Euler equation for $x < 0$ by setting $x = -\xi$, where $\xi > 0$, and $y = u(\xi)$;
- Then $\frac{dy}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = -\frac{du}{d\xi}$, $\frac{d^2y}{dx^2} = \frac{d}{d\xi}(-\frac{du}{d\xi})\frac{d\xi}{dx} = \frac{d^2u}{d\xi^2}$;
- Thus, for $x < 0$, we get $\xi^2 \frac{d^2u}{d\xi^2} + \alpha\xi\frac{du}{d\xi} + \beta u = 0$, $\xi > 0$;
- So $u(\xi) = \begin{cases} c_1\xi^{r_1} + c_2\xi^{r_2} \\ (c_1 + c_2 \ln\xi)\xi^{r_1} \\ c_1\xi^{\lambda}\cos(\mu\ln\xi) + c_2\xi^{\lambda}\sin(\mu\ln\xi) \end{cases}$ depending on whether the zeros of $F(r) = r(r-1) + \alpha r + \beta$ are real and different, real and equal, or complex conjugates;
- To obtain u in terms of x, we replace ξ by $-x$ in the ξ-solutions;

Unifying the Solutions for $x < 0$ and $x > 0$

- We combine the cases $x > 0$ and $x < 0$ using $|x|$;
- The general solution of the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation $F(r) = r(r-1) + \alpha r + \beta = 0$ as follows:
  - If the roots are real and different, then $y = c_1|x|^{r_1} + c_2|x|^{r_2}$;
  - If the roots are real and equal, then $y = (c_1 + c_2 \ln|x|)|x|^{r_1}$;
  - If the roots are complex conjugates, then $y = |x|^\lambda[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)]$, where $r_1, r_2 = \lambda \pm i\mu$;
- The solutions of an Euler equation of the form $(x - x_0)^2y'' + \alpha(x - x_0)y' + \beta y = 0$ are similar;
- If we look for solutions of the form $y = (x - x_0)^r$, then the general solution is given by the equation above with x replaced by $x - x_0$;
- Alternatively, we can reduce this form to the original by performing a change of variable $t = x - x_0$;

Discussion of Solutions for Singular Points

- Consider $P(x)y'' + Q(x)y' + R(x)y = 0$ where x_0 is singular, i.e., $P(x_0) = 0$ and at least one of Q and R is not zero at x_0;
- Since those points are few in number, can we simply ignore them and just consider solutions about ordinary points?
- This is not feasible because the singular points determine to a large extent the principal features of the solution; In the neighborhood of a singular point the solution often becomes large in magnitude or experiences rapid changes in magnitude;
- Some information on the behavior of $\frac{Q}{P}$ and $\frac{R}{P}$ in the neighborhood of the singular point is needed to understand the behavior of the solutions near $x = x_0$;
- It may be that there are two distinct solutions that remain bounded as $x \to x_0$ or only one, with the other becoming unbounded, or they may both become unbounded as $x \to x_0$;
- If there are solutions that become unbounded as $x \to x_0$, it is often important to determine how these solutions behave as $x \to x_0$;

Extending the Method to Cover Singular Points

- To extend the method used for ordinary points to a singular point x_0, it is necessary to restrict to singularities that are not too severe;
- We might call these "weak singularities"; The conditions needed are that $\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)}$ is finite and $\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$ is finite;
- This means that the singularity in $\frac{Q}{P}$ can be no worse than $(x - x_0)^{-1}$ and the singularity in $\frac{R}{P}$ can be no worse than $(x - x_0)^{-2}$; Such a point is called a **regular singular point**;
- For equations with more general coefficients than polynomials, x_0 is a **regular singular point** if it is a singular point and $(x - x_0)\frac{Q(x)}{P(x)}$ and $(x - x_0)^2 \frac{R(x)}{P(x)}$ have convergent Taylor series about x_0;
- Any singular point that is not a regular singular point is called an **irregular singular point**;
- The singularity in an Euler equation is a regular singular point; Indeed, we will see that all general equations behave very much like Euler equations near a regular singular point;

Example I

- Determine the singular points of the Legendre equation $(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0$ and determine whether they are regular or irregular;

  $P(x)=1-x^2$, so the singular points are $x=1$ and $x=-1$; Divide by $1-x^2$ to get the coefficients of y' and y: They are $-\frac{2x}{1-x^2}$ and $\frac{\alpha(\alpha+1)}{1-x^2}$;

  - $\lim_{x\to 1}(x-1)\frac{-2x}{1-x^2}=\lim_{x\to 1}\frac{(x-1)(-2x)}{(1-x)(1+x)}=\lim_{x\to 1}\frac{2x}{1+x}=1$;
    $\lim_{x\to 1}(x-1)^2\frac{\alpha(\alpha+1)}{1-x^2}=\lim_{x\to 1}\frac{(x-1)^2\alpha(\alpha+1)}{(1-x)(1+x)}=\lim_{x\to 1}\frac{(x-1)(-\alpha)(\alpha+1)}{1+x}=0$;
    Since these limits are finite, the point $x=1$ is a regular singular point;
  - It can be shown in a similar manner that $x=-1$ is also a regular singular point;

Example II

- Determine the singular points of the differential equation $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$ and classify them as regular or irregular;

  Dividing the differential equation by $2x(x-2)^2$, we have $y'' + \frac{3}{2(x-2)^2}y' + \frac{1}{2x(x-2)}y = 0$, so $p(x) = \frac{Q(x)}{P(x)} = \frac{3}{2(x-2)^2}$ and $q(x) = \frac{R(x)}{P(x)} = \frac{1}{2x(x-2)}$; The singular points are $x = 0$ and $x = 2$;
  - $\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{3}{2(x-2)^2} = 0$; $\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2\frac{1}{2x(x-2)} = 0$; Since these limits are finite, $x = 0$ is a regular singular point;
  - $\lim_{x\to 2}(x-2)p(x) = \lim_{x\to 2}(x-2)\frac{3}{2(x-2)^2} = \lim_{x\to 2}\frac{3}{2(x-2)}$; Thus, the limit does not exist; Hence $x = 2$ is an irregular singular point;

Example III

- Determine the singular points of $(x - \frac{\pi}{2})^2 y'' + (\cos x)y' + (\sin x)y = 0$ and classify them as regular or irregular.

  The only singular point is $x = \frac{\pi}{2}$; To study it, we consider the functions $(x - \frac{\pi}{2})p(x) = (x - \frac{\pi}{2})\frac{Q(x)}{P(x)} = \frac{\cos x}{x - \pi/2}$ and
  $(x - \frac{\pi}{2})^2 q(x) = (x - \frac{\pi}{2})^2 \frac{R(x)}{P(x)} = \sin x$;
  Starting from the Taylor series for $\cos x$ about $x = \frac{\pi}{2}$, we find that
  $\frac{\cos x}{x - \pi/2} = -1 + \frac{(x - \pi/2)^2}{3!} - \frac{(x - \pi/2)^4}{5!} + \cdots$ which converges for all x;
  Similarly, $\sin x$ is analytic at $x = \frac{\pi}{2}$;
  Therefore, we conclude that $\frac{\pi}{2}$ is a regular singular point for this equation;

Subsection 5

Series Solutions Near a Regular Singular Point, Part I

The General Equation at Regular Singular Points

- How do we solve the general second order linear equation $P(x)y'' + Q(x)y' + R(x)y = 0$ in the neighborhood of a regular singular point $x = x_0$?
- Suppose $x_0 = 0$; Otherwise, we can apply $t = x - x_0$;
- Regular singularity means that $x\frac{Q(x)}{P(x)} = xp(x)$ and $x^2\frac{R(x)}{P(x)} = x^2 q(x)$ have finite limits as $x \to 0$ and are analytic at $x = 0$;
- Thus, they have convergent power series expansions of the form
$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \ x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \text{ for } |x| < \rho, \ \rho > 0;$$
- Divide by $P(x)$ and multiply by x^2:
$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0$, or $x^2 y'' + x(p_0 + p_1 x + \cdots + p_n x^n + \cdots)y' + (q_0 + q_1 x + \cdots + q_n x^n + \cdots)y = 0$;
- If all of the coefficients p_n and q_n are zero, except possibly $p_0 = \lim_{x\to 0} x\frac{Q(x)}{P(x)}$ and $q_0 = \lim_{x\to 0} x^2\frac{R(x)}{P(x)}$, then we get the Euler equation $x^2 y'' + p_0 x y' + q_0 y = 0$;

The General Equation: The General Case

- $x^2y'' + x(p_0 + p_1x + \cdots)y' + (q_0 + q_1x + \cdots)y = 0;$
- In general, some of the p_n and q_n are not zero; Still, the character of solutions is identical to those of the Euler equation;
- Again, let us look at $x > 0$;
- Since the coefficients are "Euler coefficients" times power series, we seek solutions in the form of "Euler solutions" times power series;
- Assume $y = x^r(a_0 + a_1x + \cdots + a_nx^n + \cdots) =$
  $x^r \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{r+n}$, with $a_0 \neq 0$;
- We would like to determine:
  1. The values of r for which we get a solution of this form;
  2. The recurrence relation for the coefficients a_n;
  3. The radius of convergence of the series $\sum_{n=0}^{\infty} a_nx^n$;
- To simplify matters, we assume that there exists a solution of the stated form and we show how to determine the coefficients;

Example

- Solve the differential equation $2x^2y'' - xy' + (1+x)y = 0$;

  It is easy to show that $x = 0$ is a regular singular point; Further, $xp(x) = -\frac{1}{2}$ and $x^2q(x) = \frac{1+x}{2}$; Thus $p_0 = -\frac{1}{2}$, $q_0 = \frac{1}{2}$, $q_1 = \frac{1}{2}$ and all other p's and q's are zero; Then, the corresponding Euler equation is $2x^2y'' - xy' + y = 0$; We assume that there is a solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$; Then $y' = \sum_{n=0}^{\infty} a_n(r+n)x^{r+n-1}$ and $y'' = \sum_{n=0}^{\infty} a_n(r+n)(r+n-1)x^{r+n-2}$;

  Therefore, $2x^2y'' - xy' + (1+x)y = \sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1}$; The last term can be written as $\sum_{n=1}^{\infty} a_{n-1}x^{r+n}$, so we obtain
  $2x^2y'' - xy' + (1+x)y = a_0[2r(r-1) - r + 1]x^r$
  $+\sum_{n=1}^{\infty}[[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}]x^{r+n} = 0$;

Example (Cont'd)

- We had $2x^2y'' - xy' + (1+x)y = a_0[2r(r-1) - r + 1]x^r$
  $+ \sum_{n=1}^{\infty}[[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1}]x^{r+n} = 0;$
- The coefficient of each power of x must be zero;
- Since $a_0 \neq 0$, the coefficient of x^r yields
  $2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (r-1)(2r-1) = 0;$ This is called the **indicial equation**; It is exactly the polynomial equation obtained for the associated Euler equation;
- The roots of the indicial equation are $r_1 = 1,\ r_2 = \frac{1}{2};$ These values are called the **exponents at the singularity** for $x = 0$; They determine the qualitative behavior of the solution close to $x = 0$;
- Now we set the coefficient of x^{r+n} equal to zero:
  $[2(r+n)(r+n-1) - (r+n) + 1]a_n + a_{n-1} = 0$ or
  $a_n = -\frac{a_{n-1}}{2(r+n)^2 - 3(r+n) + 1} = -\frac{a_{n-1}}{[(r+n)-1][2(r+n)-1]},\ n \geq 1;$
- For each root r_1 and r_2 of the indicial equation, we use this recurrence relation to determine a set of coefficients $a_1, a_2, \ldots;$

Example (Cont'd): The Root $r_1 = 1$

- For $r = r_1 = 1$, $a_n = -\frac{a_{n-1}}{(2n+1)n}$;
- Thus $a_1 = -\frac{a_0}{3 \cdot 1}$, $a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{(3 \cdot 5)(1 \cdot 2)}$, and $a_3 = -\frac{a_2}{7 \cdot 3} = -\frac{a_0}{(3 \cdot 5 \cdot 7)(1 \cdot 2 \cdot 3)}$;
- In general, we have $a_n = \frac{(-1)^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!} a_0$, $n \geq 4$;
- Multiplying the numerator and denominator of the right side by $2 \cdot 4 \cdot 6 \cdots 2n = 2^n n!$, we get $a_n = \frac{(-1)^n 2^n}{(2n+1)!} a_0$;
- Hence, if we omit the constant multiplier a_0, we get the solution
$$y_1(x) = x[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!} x^n], x > 0;$$
- To determine the radius of convergence of the series we use the ratio test: $\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \frac{2|x|}{(2n+2)(2n+3)} = 0$; Thus the series converges for all x;

Example (Cont'd): The Root $r_2 = 1/2$

- Corresponding to the second root $r = r_2 = \frac{1}{2}$, we proceed similarly; $a_n = -\frac{a_{n-1}}{2n(n-\frac{1}{2})} = -\frac{a_{n-1}}{n(2n-1)}$;
- Hence $a_1 = -\frac{a_0}{1\cdot 1}$, $a_2 = -\frac{a_1}{2\cdot 3} = \frac{a_0}{(1\cdot 2)(1\cdot 3)}$, $a_3 = -\frac{a_2}{3\cdot 5} = -\frac{a_0}{(1\cdot 2\cdot 3)(1\cdot 3\cdot 5)}$;
- In general, $a_n = \frac{(-1)^n}{n![1\cdot 3\cdot 5\cdots(2n-1)]}a_0$, $n \geq 4$;
- Multiply the numerator and denominator by $2\cdot 4\cdot 6\cdots 2n = 2^n n!$: $a_n = \frac{(-1)^n 2^n}{(2n)!}a_0$, $n \geq 1$;
- Thus, $y_2(x) = x^{1/2}[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(2n)!}x^n]$, $x > 0$;
- As before, we can show that the series converges for all x;
- Since y_1 and y_2 behave like x and $x^{1/2}$, respectively, near $x = 0$, they form a fundamental set of solutions, whence the general solution is $y = c_1 y_1(x) + c_2 y_2(x)$, $x > 0$;

Arbitrary Regular Singular Points

- If $x = 0$ is a regular singular point, then sometimes there are two solutions of the form $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ in the neighborhood of $x = 0$;
- If there is a regular singular point at $x = x_0$, then there may be two solutions of form $y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$ near $x = x_0$;
- However, a general equation with a regular singular point may not have two solutions of this form;
- In particular, if the roots r_1 and r_2 of the indicial equation are equal, or differ by an integer, then the second solution normally has a more complicated structure; In all cases, though, it is possible to find at least one solution of this form;
- If r_1 and r_2 differ by an integer, this solution corresponds to the larger value of r; If there is only one such solution, then the second solution involves a logarithmic term;
- If the roots are complex, then they cannot be equal or differ by an integer, so there are always two solutions of this form;

Subsection 6

Series Solutions Near a Regular Singular Point, Part II

The General Equation

- Consider the equation $L[y] = x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$, where $xp(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$ and both series converge in an interval $|x| < \rho$ for some $\rho > 0$;
- The point $x = 0$ is a regular singular point, and the corresponding Euler equation is $x^2y'' + p_0xy' + q_0y = 0$;
- We seek a solution for $x > 0$ and assume that it has the form $y = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}$, where $a_0 \neq 0$, and $y = \phi(r, x)$ indicates that ϕ depends on r as well as on x;
- $$y' = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}, \ \ y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2};$$
- Substituting, we get $a_0r(r-1)x^r + a_1(r+1)rx^{r+1} + \cdots + a_n(r+n)(r+n-1)x^{r+n} + \cdots + (p_0 + p_1x + \cdots + p_nx^n + \cdots) \times [a_0rx^r + a_1(r+1)x^{r+1} + \cdots + a_n(r+n)x^{r+n} + \cdots] + (q_0 + q_1x + \cdots + q_nx^n + \cdots) \times (a_0x^r + a_1x^{r+1} + \cdots + a_nx^{r+n} + \cdots) = 0$;

The Indicial Equation

- Multiplying and collecting terms, we obtain
  $a_0F(r)x^r + [a_1F(r+1) + a_0(p_1r+q_1)]x^{r+1} + [a_2F(r+2) + a_0(p_2r+q_2) + a_1[p_1(r+1)+q_1]]x^{r+2} + \cdots + [a_nF(r+n) + a_0(p_nr+q_n) + a_1[p_{n-1}(r+1) + q_{n-1}] + \cdots + a_{n-1}[p_1(r+n-1)+q_1]]x^{r+n} + \cdots = 0,$
- In a more compact form, $L[\phi](r,x) =$
  $$a_0F(r)x^r + \sum_{n=1}^{\infty}\left[F(r+n)a_n + \sum_{k=0}^{n-1}a_k[(r+k)p_{n-k}+q_{n-k}]\right]x^{r+n} = 0,$$
  where $F(r) = r(r-1) + p_0r + q_0$;
- The term involving x^r yields the equation $F(r) = 0$; This equation is called the **indicial equation**;
- Denote the roots of the indicial equation by r_1 and r_2 with $r_1 \geq r_2$ if the roots are real; If the roots are complex, the designation of the roots is immaterial;
- The roots r_1 and r_2 are called the **exponents at the singularity**;

The Roots and the Solutions

- Setting the coefficient of x^{r+n} equal to zero gives the recurrence
  $$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0;$$
- So a_n depends on the value of r and all the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$;
- We can successively compute $a_1, a_2, \ldots, a_n, \ldots$ in terms of a_0 and the coefficients in the series for $xp(x)$ and $x^2q(x)$, provided that $F(r+1), F(r+2), \ldots, F(r+n), \ldots$ are not zero;
- The only values of r for which $F(r) = 0$ are $r = r_1$ and $r = r_2$; Since $r_1 \geq r_2$, it follows that $r_1 + n$ is not equal to r_1 or r_2 for $n \geq 1$; Consequently, $F(r_1 + n) \neq 0$ for $n \geq 1$; Hence we can always determine one solution of the differential equation, namely,
  $$y_1(x) = x^{r_1}[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n], \ x > 0;$$

The Solution Series

- If $r_2 \neq r_1$, and $r_1 - r_2$ is not a positive integer, then $r_2 + n \neq r_1$ for any $n \geq 1$; Hence, $F(r_2 + n) \neq 0$, and we can also obtain a second solution $y_2(x) = x^{r_2}[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n]$, $x > 0$;
- The series converge at least in the interval $|x| < \rho$, where the series for both $xp(x)$ and $x^2 q(x)$ converge;
- Within their radii of convergence, the power series define functions that are analytic at $x = 0$;
- Thus, the singular behavior, if there is any, of the solutions y_1 and y_2 is due to the factors x^{r_1} and x^{r_2} that multiply these two analytic functions;
- To obtain real-valued solutions for $x < 0$, we can make the substitution $x = -\xi$ with $\xi > 0$;
- It turns out that we need only replace x^{r_1} and x^{r_2} by $|x|^{r_1}$ and $|x|^{r_2}$, respectively;

The Solution for $x > 0$ and the Complex Root Case

- If r_1 and r_2 are complex numbers, then they are necessarily complex conjugates and $r_2 \neq r_1 + N$ for any positive integer N;
- In this case we can always find two series solutions; However, they are complex-valued functions of x; Real-valued solutions can be obtained by taking the real and imaginary parts of the complex solutions;
- We consider $r_1 = r_2$ or $r_1 - r_2 = N$ later;
- To calculate r_1 and r_2, we only have to solve $r(r-1) + p_0 r + q_0 = 0$, with $p_0 = \lim_{x \to 0} xp(x)$, $q_0 = \lim_{x \to 0} x^2 q(x)$; These are exactly the limits that must be evaluated in order to classify the singularity;
- If $x = 0$ is a regular singular point of the equation $P(x)y'' + Q(x)y' + R(x)y = 0$, where P, Q and R are polynomials, then $xp(x) = x\frac{Q(x)}{P(x)}$ and $x^2 q(x) = x^2 \frac{R(x)}{P(x)}$; Thus, $p_0 = \lim_{x \to 0} x\frac{Q(x)}{P(x)}$, $q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{P(x)}$;
- The radii of convergence for the series are at least equal to the distance from the origin to the nearest zero of P other than $x = 0$;

Example

- Discuss the nature of the solutions of the equation $2x(1+x)y'' + (3+x)y' - xy = 0$ near the singular points; $P(x) = 2x(1+x)$, $Q(x) = 3+x$, and $R(x) = -x$; The points $x = 0$ and $x = -1$ are the singular points;
  - The point $x = 0$ is a regular singular point, since $\lim_{x \to 0} x \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \frac{3+x}{2x(1+x)} = \frac{3}{2}$, $\lim_{x \to 0} x^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{-x}{2x(1+x)} = 0$; Further, $p_0 = \frac{3}{2}$ and $q_0 = 0$; Thus the indicial equation is $r(r-1) + \frac{3}{2}r = 0$, and the roots are $r_1 = 0$, $r_2 = -\frac{1}{2}$; Since these roots are not equal and do not differ by an integer, there are two solutions of the form
  $$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n(0)x^n \text{ and } y_2(x) = |x|^{-1/2}[1 + \sum_{n=1}^{\infty} a_n(-\tfrac{1}{2})x^n]; \text{ for}$$
  $0 < |x| < \rho$; A lower bound for the radius of convergence of each series is 1, the distance from $x = 0$ to $x = -1$, the other zero of P(x); y_1 is bounded as $x \to 0$, indeed is analytic there, and y_2 is unbounded as $x \to 0$;

Example (Cont'd)

- Discuss the nature of the solutions of the equation $2x(1+x)y'' + (3+x)y' - xy = 0$ near the singular points; $P(x) = 2x(1+x)$, $Q(x) = 3+x$, and $R(x) = -x$; The points $x = 0$ and $x = -1$ are the singular points;
  - The point $x = -1$ is also a regular singular point, since $\lim_{x \to -1}(x+1)\frac{Q(x)}{P(x)} = \lim_{x \to -1}\frac{(x+1)(3+x)}{2x(1+x)} = -1$, $\lim_{x \to -1}(x+1)^2\frac{R(x)}{P(x)} = \lim_{x \to -1}\frac{(x+1)2(-x)}{2x(1+x)} = 0$; In this case $p_0 = -1$, $q_0 = 0$, so the indicial is $r(r-1) - r = 0$; The roots are $r_1 = 2$ and $r_2 = 0$; Corresponding to the larger root there is a solution of the form $y_1(x) = (x+1)^2[1 + \sum_{n=1}^{\infty} a_n(2)(x+1)^n]$; The series converges at least for $|x+1| < 1$, and y_1 is an analytic function there; Since the two roots differ by a positive integer, there may or may not be a second solution of the form $y_2(x) = 1 + \sum_{n=1}^{\infty} a_n(0)(x+1)^n$; Further analysis needed;

Equal Roots

- Suppose the roots of the indicial equation are equal to r_1;
- Then $L[\phi](r,x) = a_0 F(r)x^r = a_0(r-r_1)^2 x^r$;
- Setting $r = r_1$, $L[\phi](r_1,x) = 0$; So $y_1(x) = x^{r_1}[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n]$ is one solution;
- It also follows that $L[\frac{\partial \phi}{\partial r}](r_1,x) = a_0 \frac{\partial}{\partial r}[x^r(r-r_1)^2]\big|_{r=r_1} = a_0\,[(r-r_1)^2 x^r \ln x + 2(r-r_1)x^r]\big|_{r=r_1} = 0$;
- Hence, a second solution is $y_2(x) = \frac{\partial \phi(r,x)}{\partial r}\Big|_{r=r_1} = \frac{\partial}{\partial r}[x^r[a_0 + \sum_{n=1}^{\infty} a_n(r)x^n]]\big|_{r=r_1} = (x^{r_1}\ln x)[a_0 + \sum_{n=1}^{\infty} a_n(r_1)x^n] + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n$, $x > 0$, where $a_n'(r_1)$ is value of $\frac{da_n}{dr}$ at $r = r_1$;

Equal Roots: Remarks

- It may turn out that it is difficult to determine $a_n(r)$ as a function of r from the recurrence relation and then to differentiate the resulting expression with respect to r;
- An alternative is simply to assume that y has the form
  $y = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} b_n x^n$, $x > 0$, where $y_1(x)$ has already been found; The coefficients b_n are calculated, as usual, by substituting into the differential equation, collecting terms, and setting the coefficient of each power of x equal to zero;
- A third possibility is to use the method of reduction of order to find $y_2(x)$ once $y_1(x)$ is known;

Roots Differing by an Integer

- If the roots r_1 and r_2 differ by an integer N, the derivation of the second solution is considerably more complicated;
- The form of this solution is stated in the following slide;
- The coefficients $c_n(r_2)$ in the solution are given by $c_n(r_2) = \frac{d}{dr}[(r-r_2)a_n(r)]\Big|_{r=r_2}$, where $a_n(r)$ is determined from the recurrence $F(r+n)a_n + \sum_{k=0}^{\infty} a_k[(r+k)p_{n-k} + q_{n-k}] = 0$, with $a_0 = 1$;
- The coefficient a in the solution is $a = \lim_{r \to r_2} (r-r_2)a_N(r)$;
- If $a_N(r_2)$ is finite, then $a = 0$ and there is no logarithmic term in y_2;
- In practice, the best way to determine whether a is zero in the second solution is simply to try to compute the a_n corresponding to the root r_2 and to see whether it is possible to determine $a_N(r_2)$; If so, there is no further problem; If not, we must use the formula with $a \neq 0$;
- When $r_1 - r_2 = N$, there are again three similar ways to find a second solution, as before;

The Main Theorem: Summary of the Results I

- Consider $x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$, with $x = 0$ a regular singular point; Then $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with convergent series expansions $xp(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2q(x) = \sum_{n=0}^{\infty} q_n x^n$, for $|x| < \rho$, where $\rho > 0$ is the minimum of the radii of convergence of the power series for $xp(x)$ and $x^2q(x)$; Let r_1 and r_2 be the roots of the indicial equation $F(r) = r(r-1) + p_0 r + q_0 = 0$, with $r_1 \geq r_2$ if r_1 and r_2 are real; Then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a solution of the form $y_1(x) = |x|^{r_1}[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n]$, where the $a_n(r_1)$ are given by the recurrence relation $F(r_1+n)a_n + \sum_{k=0}^{\infty} a_k[(r_1+k)p_{n-k} + q_{n-k}] = 0$, with $a_0 = 1$;

The Main Theorem: Summary of the Results II

- The following cases may arise:
  - If $r_1 - r_2$ is not zero or a positive integer, then in either the interval $-\rho < x < 0$ or the interval $0 < x < \rho$, there exists a second solution of the form $y_2(x) = |x|^{r_2}[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n]$; The $a_n(r_2)$ are also determined by the same recurrence relation with $a_0 = 1$ and $r = r_2$; The power series in converge at least for $|x| < \rho$;
  - If $r_1 = r_2$, then $y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n$;
  - If $r_1 - r_2 = N$, $y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n]$;
- $a_n(r_1), b_n(r_1), c_n(r_2)$, and a can be determined by substituting the form of the series solutions for y; a may turn out to be zero;
- Each of the series converges at least for $|x| < \rho$ and defines a function that is analytic in some neighborhood of $x = 0$;
- In all three cases the two solutions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of the given differential equation;