Elementary Differential Equations

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Subsection 1

Definition of the Laplace Transform

Improper Integrals

- The Laplace transform involves an integral from zero to infinity, i.e., an improper integral;
- An **improper integral** over an unbounded interval is defined as a limit of integrals over finite intervals:

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt,$$

where A is a positive real number;

- If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value; Otherwise the integral is said to **diverge**, or to **fail to exist**;

Example

- Let $f(t) = e^{ct}$, $t \geq 0$, where c is a real nonzero constant;

$$\begin{aligned} \int_0^\infty e^{ct} dt &= \lim_{A\to\infty} \int_0^A e^{ct} dt \\ &= \lim_{A\to\infty} \left. \frac{e^{ct}}{c} \right|_0^A \\ &= \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1); \end{aligned}$$

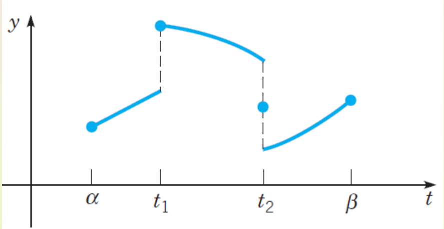
  We draw the following conclusions:
  - If $c < 0$, it converges to the value $-\frac{1}{c}$;
  - If $c > 0$, it diverges;
  - If c $= 0$, $f(t) = 1$, and the integral again diverges;

Example

- Let $f(t) = \frac{1}{t}$, $t \geq 1$;
$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A;$$
Since $\lim\limits_{A\to\infty} \ln A = \infty$, the improper integral diverges;
- Let $f(t) = t^{-p}, t \geq 1$, where p is a real constant and $p \neq 1$;
$$\int_1^\infty t^{-p} dt = \lim_{A\to\infty} \int_1^A t^{-p} dt = \lim_{A\to\infty} \frac{1}{1-p}(A^{1-p} - 1);$$
  - If $p > 1$, $A^{1-p} \to 0$;
  - If $p < 1$, $A^{1-p} \to \infty$;
- Hence $\displaystyle\int_1^\infty t^{-p} dt$ converges to $\frac{1}{p-1}$ for $p > 1$ and diverges for $p \leq 1$;

Piece-wise Continuity

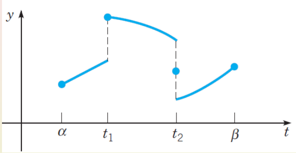
- A function f is said to be **piecewise continuous** on an interval $\alpha \leq t \leq \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ so that
  1. f is continuous on each open subinterval $t_{i-1} < t < t_i$;
  2. f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval;
- Equivalently, f is piecewise continuous on $\alpha < t < \beta$ if it is continuous there except for a finite number of jump discontinuities;



- If f is piecewise continuous on $\alpha < t < \beta$, for every $\beta > \alpha$, then f is said to be **piecewise continuous on** $t \geq \alpha$;

Integrals of Piece-wise Continuous Functions

- The integral of a piecewise continuous function on a finite interval is just the sum of the integrals on the subintervals created by the partition points;
- For instance, for the function



$$\int_\alpha^\beta f(t)dt = \int_\alpha^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \int_{t_2}^{\beta} f(t)dt;$$

- If f is piecewise continuous on $a \le t \le A$, then $\int_a^A f(t)dt$ exists;
- Hence, if f is piecewise continuous for $t \ge a$, then $\int_a^A f(t)dt$ exists for each $A > a$;
- However, piecewise continuity is not enough to ensure convergence of the improper integral $\int_a^\infty f(t)dt$, as the preceding examples show;

The Comparison Theorem

- If f cannot be integrated easily in terms of elementary functions, the definition of convergence of $\int_a^\infty f(t)dt$ may be difficult to apply;
- Frequently, the most convenient way to test the convergence or divergence of an improper integral is by the following comparison theorem (similar to the one for infinite series);

Comparison Theorem

If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\displaystyle\int_M^\infty g(t)dt$ converges, then $\displaystyle\int_a^\infty f(t)dt$ also converges; On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\displaystyle\int_M^\infty g(t)dt$ diverges, then $\displaystyle\int_a^\infty f(t)dt$ also diverges;

- The functions most useful for comparison purposes are e^{ct} and t^{-p}, whose improper integrals we already computed;

The Laplace Transform

- An **integral transform** is a relation $F(s) = \int_\alpha^\beta K(s,t)f(t)dt$, where $K(s,t)$ is a given function, called the **kernel of the transformation**, and the limits of integration α and β are also given;
- It is possible that $\alpha = -\infty$ or $\beta = +\infty$, or both;
- The relation transforms f into another function F, which is called the **transform** of f;
- Let $f(t)$ be given for $t \geq 0$, and suppose that f satisfies certain conditions to be stated later; The **Laplace transform of** f, denoted $\mathcal{L}\{f(t)\}$ or $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt,$$

whenever this improper integral converges;
- The Laplace transform makes use of the kernel $K(s,t) = e^{-st}$;

Laplace Transform: Solving Differential Equations

- Since the solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations;
- The general idea in using the Laplace transform to solve a differential equation is
  1. Transform an initial value problem for an unknown function f in the t-domain into a simpler algebraic problem for F in the s-domain;
  2. Solve this algebraic problem to find F;
  3. Recover the desired function f from its transform F. This last step is known as "inverting the transform".
- In general, s may be complex, and the full power of the Laplace transform becomes available only when we regard $F(s)$ as a function of a complex variable;
- However, for the problems discussed here, it is sufficient to consider only real values of s;

Existence of Laplace Transform for Special Functions

Theorem (Existence of Laplace Transform)

Suppose that

1. f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A;
2. $|f(t)| \leq Ke^{at}$ when $t \geq M$; In this inequality, K, a and M are real constants, K and M necessarily positive;

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

- We deal almost exclusively with functions that satisfy the conditions of the theorem;
- Such functions are described as piecewise continuous and of **exponential order** as $t \to \infty$;
- There do exist functions that are not of exponential order as $t \to \infty$; One such function is $f(t) = e^{t^2}$; As $t \to \infty$, this function increases faster than Ke^{at} regardless of how large K and a may be;

Examples

- Let $f(t) = 1$, $t \geq 0$;

$$\begin{aligned} \mathcal{L}\{1\} &= \int_0^\infty e^{-st} dt = \lim_{A \to \infty} \int_0^A e^{-st} dt \\ &= -\lim_{A \to \infty} \frac{1}{s} e^{-st} \Big|_0^A \\ &= -\frac{1}{s} \lim_{A \to \infty} (e^{-sA} - 1) = \frac{1}{s}, \ s > 0; \end{aligned}$$

- Let $f(t) = e^{at}, t \geq 0$;

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \ s > a;$$

Example

- Let $f(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ k, & \text{if } t = 1, \\ 0, & \text{if } t > 1, \end{cases}$, where k is a constant;

  In engineering contexts $f(t)$ often represents a unit pulse, perhaps of force or voltage;

  Note that f is a piecewise continuous function;

  $$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_0^1 = \frac{1 - e^{-s}}{s};$$

  $\mathcal{L}\{f(t)\}$ does not depend on k;

Example: Applying Double Integration By-Parts

- Let $f(t) = \sin at, t \geq 0$;

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at dt, s > 0;$$

Since $F(s) = \lim_{A \to \infty} \int_0^A e^{-st} \sin at dt$, upon integrating by parts, we obtain $F(s) = \lim_{A \to \infty} \left[\left. \frac{-e^{-st} \cos at}{a} \right|_0^A - \frac{s}{a} \int_0^A e^{-st} \cos at dt \right] = \frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at dt$;

A second integration by parts then yields

$F(s) = \frac{1}{a} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at dt = \frac{1}{a} - \frac{s^2}{a^2} F(s)$;

Hence, solving for $F(s)$, we have $F(s) = \frac{a}{s^2 + a^2}$, $s > 0$;

Linearity of the Laplace Transform

- Now let us suppose that f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively;
- Then, for s greater than the maximum of a_1 and a_2,

$$\begin{aligned}
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt \\
&= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\};
\end{aligned}$$

Hence

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\};$$

Thus, the Laplace transform is a linear operator;

Example

- Find the Laplace transform of

$$f(t) = 5e^{-2t} - 3\sin 4t, \quad t \geq 0;$$

Then for $s > 0$,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{5e^{-2t} - 3\sin 4t\} \\ &= 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\} \\ &= 5\frac{1}{s+2} - 3\frac{4}{s^2+16} \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}; \end{aligned}$$

Subsection 2

Solution of Initial Value Problems

Laplace Transforms of Derivatives

Theorem (Laplace Transform of Derivative)

Suppose that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$; Suppose, further, that there exist constants K, a and M, such that $|f(t)| \leq Ke^{at}$ for $t \geq M$; Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and, moreover,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Corollary (Laplace Transform of Higher Derivatives)

Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \leq t \leq A$; Suppose, further, that there exist constants K, a and M such that $|f(t)| \leq Ke^{at}$, $|f'(t)| \leq Ke^{at}$, $\ldots$, $|f^{(n-1)}(t)| \leq Ke^{at}$, for $t \geq M$; Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Example (Via Characteristic)

- Consider $y'' - y' - 2y = 0$, with $y(0) = 1$, $y'(0) = 0$;

  The characteristic equation is $r^2 - r - 2 = (r-2)(r+1) = 0$; So the general solution is

$$y = c_1 e^{-t} + c_2 e^{2t};$$

  The initial conditions give $c_1 + c_2 = 1$ and $-c_1 + 2c_2 = 0$;

  Therefore, we get $c_1 = \frac{2}{3}$ and $c_2 = \frac{1}{3}$;

  So we get

$$y = \phi(t) = \frac{2}{3} e^{-t} + \frac{1}{3} e^{2t};$$

Example (Via Laplace)

- Assume $y = \phi(t)$ satisfies $y'' - y' - 2y = 0$, with $y(0) = 1$, $y'(0) = 0$; Then, $\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0$; So by the corollary,

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - [s\mathcal{L}\{y\} - y(0)] - 2\mathcal{L}\{y\} = 0,$$

or, writing $Y(s) = \mathcal{L}\{y\}$, $(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0$; Substituting for $y(0)$ and $y'(0)$ and then solving for $Y(s)$, we obtain $Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}$; We expand into partial fractions:

$$Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1} = \frac{a(s+1) + b(s-2)}{(s-2)(s+1)},$$

whence $s - 1 = a(s+1) + b(s-2)$, giving $a + b = 1$ and $a - 2b = -1$; So $a = \frac{1}{3}$ and $b = \frac{2}{3}$; Thus, $Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1}$; Since $\frac{1}{3}e^{2t}$ has the transform $\frac{1}{3}\frac{1}{s-2}$ and $\frac{2}{3}e^{-t}$ has the transform $\frac{2}{3}\frac{1}{s+1}$, the linearity of the Laplace transform gives $y = \phi(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$;

The Method for Solving the General Equation

- Consider $ay'' + by' + cy = f(t)$;
- Assuming that the solution $y = \phi(t)$ satisfies the conditions of the corollary for $n = 2$, we can take the transform

$$a[s^2 Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s),$$

  where $F(s)$ is the transform of $f(t)$;
- By solving for $Y(s)$, we find that

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c};$$

- The problem is then solved, provided that we can find the function $y = \phi(t)$ whose transform is $Y(s)$;

Advantages of the Method

- The transform $Y(s)$ of the unknown function $y = \phi(t)$ is found by solving an algebraic equation rather than a differential equation;
- The solution satisfying given initial conditions is automatically found, so that the task of determining appropriate values for the arbitrary constants in the general solution does not arise;
- Nonhomogeneous equations are handled in exactly the same way as homogeneous ones; it is not necessary to solve the corresponding homogeneous equation first;
- The method can be applied in the same way to higher order equations, as long as we assume that the solution satisfies the conditions of the corollary for the appropriate value of n;

Potential Disadvantages

- The use of partial fractions in determining $\phi(t)$ requires us to factor the polynomial in the denominator of $Y(s)$, whence finding roots of the characteristic equation is not avoided; For equations of higher than second order this may require a numerical approximation;
- The main difficulty lies in determining the function $y = \phi(t)$ corresponding to the transform $Y(s)$; This problem is known as the **inversion problem** for the Laplace transform;
- $\phi(t)$ is called the **inverse transform** corresponding to $Y(s)$, and the process of finding $\phi(t)$ from $Y(s)$ is known as **inverting the transform**; We use $\mathcal{L}^{-1}\{Y(s)\}$ for the inverse transform of $Y(s)$;
- If f and g are continuous functions with the same Laplace transform, then f and g must be identical; Thus, there is essentially a one-to-one correspondence between functions and their Laplace transforms;

Table of Transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\frac{1}{s}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
e^{at}	$\frac{1}{s-a}$	$u_c(t)$	$\frac{e^{-cs}}{s}$
t^n	$\frac{n!}{s^{n+1}}$	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
t^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$e^{ct}f(t)$	$F(s-c)$
$\sin at$	$\frac{a}{s^2+a^2}$	$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$
$\cos at$	$\frac{s}{s^2+a^2}$	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\sinh at$	$\frac{a}{s^2-a^2}$	$\delta(t-c)$	e^{-cs}
$\cosh at$	$\frac{s}{s^2-a^2}$	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$(-1)^n f(t)$	$F^{(n)}(s)$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$		

Example

- Find the solution of $y'' + y = \sin 2t$, with $y(0) = 2$, $y'(0) = 1$;

  Suppose $y = \phi(t)$, satisfying all conditions of the corollary; Then, $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin 2t\}$, whence

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4};$$

  Thus, we obtain

$$\begin{aligned} (s^2+1)Y(s) &= sy(0) + y'(0) + \tfrac{2}{s^2+4} \\ (s^2+1)Y(s) &= \tfrac{(2s+1)(s^2+4)+2}{s^2+4} \\ Y(s) &= \tfrac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}; \end{aligned}$$

  By partial fractions

$$\begin{aligned} Y(s) &= \tfrac{as+b}{s^2+1} + \tfrac{cs+d}{s^2+4} = \tfrac{(as+b)(s^2+4)+(cs+d)(s^2+1)}{(s^2+1)(s^2+4)} \\ &= \tfrac{as^3+bs^2+4as+4b+cs^3+ds^2+cs+d}{(s^2+1)(s^2+4)}; \end{aligned}$$

Example (Cont'd)

- This yields
$2s^3 + s^2 + 8s + 6 = (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)$; So,

$$\left\{\begin{array}{l} a+c=2 \\ b+d=1 \\ 4a+c=8 \\ 4b+d=6 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a=2 \\ b=\frac{5}{3} \\ c=0 \\ d=-\frac{2}{3} \end{array}\right\}.$$

Therefore,

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4};$$

With the help of the table:

$$y = \phi(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t;$$

Example

- Find the solution of $y^{(4)} - y = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$;

  In this problem we need to assume that the solution $y = \phi(t)$ satisfies the conditions of the corollary for $n = 4$; Taking Laplace transforms we get $\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$, whence

  $$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0;$$

  Thus, $s^4 Y(s) - s^2 - Y(s) = 0$, giving $Y(s) = \frac{s^2}{s^4-1}$; A partial fraction expansion of $Y(s)$ is

  $$Y(s) = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1} = \frac{(as+b)(s^2+1)+(cs+d)(s^2-1)}{(s^2-1)(s^2+1)};$$

  It follows that $(as+b)(s^2+1)+(cs+d)(s^2-1) = s^2$;

Example (Cont'd)

- We set $Y(s) = \frac{s^2}{s^4-1} = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}$ and found

$$(as+b)(s^2+1) + (cs+d)(s^2-1) = s^2;$$

For $s=1$ and $s=-1$, we obtain $2(a+b)=1$, $2(-a+b)=1$, whence $a=0$ and $b=\frac{1}{2}$; If we set $s=0$, then $b-d=0$, so $d=\frac{1}{2}$; Finally, equating the coefficients of the cubic terms, $a+c=0$, so $c=0$; Thus,

$$Y(s) = \frac{1/2}{s^2-1} + \frac{1/2}{s^2+1}$$

and, therefore

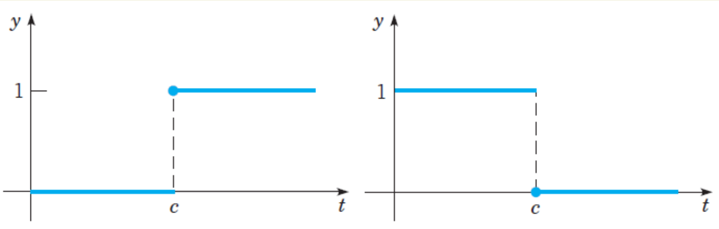
$$y = \phi(t) = \frac{1}{2}\sinh t + \frac{1}{2}\sin t;$$

Subsection 3

Step Functions

Unit Step (Heavyside) Function

- All functions appearing below will be assumed to be piecewise continuous and of exponential order, so that their Laplace transforms exist, at least for s sufficiently large;
- The **unit step function** or **Heaviside function** is denoted by u_c and is defined by $u_c(t) = \begin{cases} 0, & \text{if } t < c, \\ 1, & \text{if } t \geq c. \end{cases}$
- The graph of $y = u_c(t)$ and that of $y = 1 - u_c(t)$ are shown below:



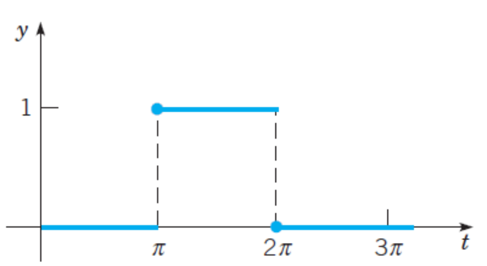
Example (A Rectangular Pulse)

- Sketch the graph of $y = h(t)$, where $h(t) = u_\pi(t) - u_{2\pi}(t)$, $t \geq 0$;

  From the definition of $u_c(t)$, we get

  $$h(t) = \begin{cases} 0 - 0 = 0, & \text{if } 0 \leq t < \pi, \\ 1 - 0 = 1, & \text{if } \pi \leq t < 2\pi, \\ 1 - 1 = 0, & \text{if } 2\pi \leq t < \infty. \end{cases}$$

  Thus the equation $y = h(t)$ has the graph shown here:



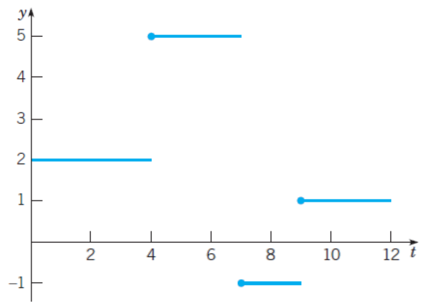
This function can be thought of as a rectangular pulse.

Expressing Step Functions Using $u_c(t)$

- Consider the function

$$f(t) = \begin{cases} 2, & \text{if } 0 \leq t < 4, \\ 5, & \text{if } 4 \leq t < 7, \\ -1, & \text{if } 7 \leq t < 9, \\ 1, & \text{if } t \geq 9, \end{cases}$$

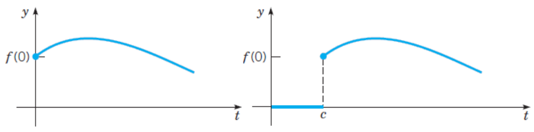
whose graph is shown here: Express $f(t)$ in terms of $u_c(t)$;



We start with the function $f_1(t) = 2$, which agrees with $f(t)$ on $[0, 4)$; To produce the jump of three units at $t = 4$, we add $3u_4(t)$ to $f_1(t)$, obtaining $f_2(t) = 2 + 3u_4(t)$, which agrees with $f(t)$ on $[0, 7)$; The negative jump of six units at $t = 7$ corresponds to adding $-6u_7(t)$, which gives $f_3(t) = 2 + 3u_4(t) - 6u_7(t)$; Finally, we must add $2u_9(t)$ to match the jump of two units at $t = 9$; Thus we obtain $f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t)$.

The Laplace Transform of u_c and of Shifts

- The Laplace transform of u_c is easily determined:
  $\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s}, \; s > 0;$
- For f defined for $t \geq 0$, we define
  $y = g(t) = \begin{cases} 0, & \text{if } t < c, \\ f(t-c), & \text{if } t \geq c, \end{cases}$



- Using u_c we can write $g(t) = u_c(t) f(t-c)$;
- Then, the transform of $f(t)$ and that of its translation $u_c(t)f(t-c)$ are related as follows:

Theorem (Transform of a Shift)

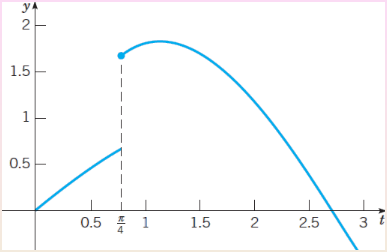
If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$, $s > a$; Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then $u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$.

Example (Laplace Transform)

- If the function f is defined by

  $f(t) = \begin{cases} \sin t, \ \text{if } 0 \leq t < \frac{\pi}{4}, \\ \sin t + \cos\left(t - \frac{\pi}{4}\right), \text{if } t \geq \frac{\pi}{4}, \end{cases}$

  find $\mathcal{L}\{f(t)\}$;



  Note that $f(t) = \sin t + g(t)$, where
  $g(t) = \begin{cases} 0, & \text{if } t < \frac{\pi}{4}, \\ \cos\left(t - \frac{\pi}{4}\right), & \text{if } t \geq \frac{\pi}{4}. \end{cases}$ Thus, $g(t) = u_{\pi/4}(t)\cos\left(t - \frac{\pi}{4}\right)$,
  and $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t)\cos\left(t - \frac{\pi}{4}\right)\} =$
  $\mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}$; Introducing the transforms of $\sin t$ and $\cos t$, we obtain

  $$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{1+se^{-\pi s/4}}{s^2+1};$$

Example (Inverse Laplace Transform)

- Find the inverse transform of $F(s) = \frac{1-e^{-2s}}{s^2}$;

  From the linearity of the inverse transform we have

$$\begin{array}{rcl} f(t) & = & \mathcal{L}^{-1}\{F(s)\} \\ & = & \mathcal{L}^{-1}\{\frac{1}{s^2}\} - \mathcal{L}^{-1}\{\frac{e^{-2s}}{s^2}\} \\ & = & t - u_2(t)(t-2); \end{array}$$

  The function f may also be written as

$$f(t) = \begin{cases} t, & \text{if } 0 \leq t < 2, \\ 2, & \text{if } t \geq 2. \end{cases}$$

Another Property of the Laplace Transform

Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c;$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}.$$

- Example: Find the inverse transform of $G(s) = \frac{1}{s^2-4s+5}$;

  By completing the square in the denominator, we can write $G(s) = \frac{1}{(s-2)^2+1} = F(s-2)$, where $F(s) = \frac{1}{s^2+1}$; Since $\mathcal{L}^{-1}\{F(s)\} = \sin t$, it follows that

  $$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s-2)\} \stackrel{\text{Theorem}}{=} e^{2t}\sin t.$$

Subsection 4

Differential Equations with Discontinuous Forcing Functions

Example I

- Find the solution of the differential equation $2y'' + y' + 2y = g(t)$, where $g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & \text{if } 5 \leq t < 20, \\ 0, & \text{if } 0 \leq t < 5 \text{ and } t \geq 20. \end{cases}$
  Assume that the initial conditions are $y(0) = 0$, $y'(0) = 0$;
  The Laplace transform is

$$\begin{aligned} 2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) \\ = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = \tfrac{e^{-5s} - e^{-20s}}{s}; \end{aligned}$$

Thus, $2s^2Y(s) + sY(s) + 2Y(s) = \frac{e^{-5s} - e^{-20s}}{s}$, giving

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)};$$

So

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \quad \text{where } H(s) = \frac{1}{s(2s^2 + s + 2)};$$

Example I (Cont'd)

- We found $Y(s) = (e^{-5s} - e^{-20s})H(s)$, where $H(s) = \frac{1}{s(2s^2+s+2)}$; We conclude that, for $h(t) = \mathcal{L}^{-1}\{H(s)\}$,

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20);$$

To determine $h(t)$, we use the partial fraction expansion of

$$H(s) = \frac{a}{s} + \frac{bs+c}{2s^2+s+2};$$

We obtain

$$H(s) = \frac{a(2s^2+s+2)+(bs+c)s}{s(2s^2+s+2)};$$
$$(2a+b)s^2 + (a+c)s + 2a = 1;$$
$$a = \frac{1}{2}; b = -1; c = -\frac{1}{2};$$

Thus,

$$H(s) = \frac{1/2}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2};$$

Example I (Cont'd)

- We obtained

$$\begin{aligned} H(s) &= \frac{1/2}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2} \\ &= \frac{1/2}{s} - \frac{1}{2}\frac{(s+\frac{1}{4})+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}} \\ &= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\right. \\ &\qquad \left. +\frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\right]; \end{aligned}$$

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-t/4}\cos(\sqrt{15}t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}t/4)\right];$$

Example II

- Find a solution of the initial value problem $y'' + 4y = g(t)$, $y(0) = 0$, $y'(0) = 0$, where $g(t) = \begin{cases} 0, & \text{if } 0 \leq t < 5, \\ \frac{t-5}{5}, & \text{if } 5 \leq t < 10, \\ 1, & \text{if } t \geq 10, \end{cases}$.

  We write
  $$g(t) = u_5(t)\frac{t-5}{5} + u_{10}(t)(1 - \frac{t-5}{5}) = \frac{u_5(t)(t-5) - u_{10}(t)(t-10)}{5};$$

  Taking Laplace transforms
  $$\begin{array}{l} \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}; \\ s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}; \\ (s^2+4)Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}; \\ Y(s) = \frac{(e^{-5s} - e^{-10s})H(s)}{5}, \quad H(s) = \frac{1}{s^2(s^2+4)}; \end{array}$$

  Thus, since $e^{-cs}H(s)$ has inverse Laplace transform $u_c(t)h(t-c)$, $y = \phi(t) = \frac{u_5(t)h(t-5) - u_{10}(t)h(t-10)}{5}$, where $h(t) = \mathcal{L}^{-1}\{H(s)\}$;

Example II (Cont'd)

- We look at the partial fraction expansion of $H(s) = \frac{1}{s^2(s^+4)}$.

$$\begin{aligned} H(s) &= \tfrac{as+b}{s^2} + \tfrac{cs+d}{s^2+4}; \\ H(s) &= \tfrac{(as+b)(s^2+4)+(cs+d)s^2}{s^2(s^2+4)}; \\ H(s) &= \tfrac{(a+c)s^3+(b+d)s^2+4as+4b}{s^2(s^2+4)}; \\ a+c &= 0,\ b+d=0,\ 4a=0,\ 4b=1; \\ a &= 0,\ b=\tfrac{1}{4},\ c=0,\ d=-\tfrac{1}{4}; \end{aligned}$$

So we get $H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$; This gives $h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$; Therefore,

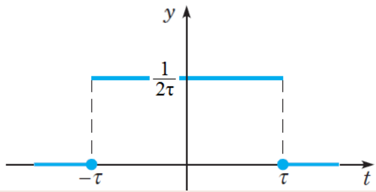
$$\begin{aligned} y(t) &= \frac{u_5(t)[\frac{1}{4}(t-5)-\frac{1}{8}\sin 2(t-5)]-u_{10}(t)[\frac{1}{4}(t-10)-\frac{1}{8}\sin 2(t-10)]}{5} \\ &= \frac{1}{20}u_5(t)(t-5) - \frac{1}{20}u_{10}(t)(t-10) \\ &\quad - \frac{1}{40}u_5(t)\sin(2(t-5)) + \frac{1}{40}u_{10}(t)\sin(2(t-10)). \end{aligned}$$

Subsection 5

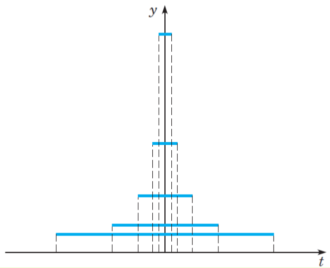
Impulse Functions

Impulse Functions

- $g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & \text{if } -\tau < t < \tau, \\ 0, & \text{if } t \le -\tau \text{ or } t \ge \tau, \end{cases}$ where τ is a small positive constant;



- Then $I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = 1$ independent of the value of τ as long as $\tau \neq 0$;
- Next, we require that $\tau \to 0$: As a result of this limiting operation, we obtain
$$\lim_{\tau \to 0^+} d_\tau(t) = 0, \text{ for all } t \neq 0,$$
$$\lim_{\tau \to 0^+} I(\tau) = 1;$$



Unit Impulse Function δ

- We define a **unit impulse "function"** δ by the properties

$$\delta(t) = 0, t \neq 0; \qquad \int_{-\infty}^{\infty} \delta(t) dt = 1;$$

- There is no ordinary function of the kind studied in elementary calculus that satisfies these equations; The "function" δ is an example of what are known as **generalized functions**; It is usually called the **Dirac delta function**;
- A unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$; It then follows that

$$\delta(t - t_0) = 0, t \neq t_0; \qquad \int_{-\infty}^{\infty} \delta(t - t_0) dt = 1;$$

- Since $\delta(t) = \lim_{\tau \to 0^+} d_\tau(t)$, the Laplace transform of δ is defined as a similar limit of the transform of $d_\tau(t)$;

The Laplace Transform of $d_\tau(t-t_0)$

- Let $t_0>0$ and define $\mathcal{L}\{\delta(t-t_0)\}=\lim\limits_{\tau\to 0^+}\mathcal{L}\{d_\tau(t-t_0)\}$;
- If $\tau<t_0$, which will be the case as $\tau\to 0^+$, then $t_0-\tau>0$; Since $d_\tau(t-t_0)$ is nonzero only in the interval from $t_0-\tau$ to $t_0+\tau$, we have
$$\mathcal{L}\{d_\tau(t-t_0)\}=\int_0^\infty e^{-st}d_\tau(t-t_0)dt=\int_{t_0-\tau}^{t_0+\tau}e^{-st}d_\tau(t-t_0)dt;$$
- Thus,
$$\begin{aligned}\mathcal{L}\{d_\tau(t-t_0)\} &= \frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau}e^{-st}dt\\ &= -\frac{1}{2s\tau}\ e^{-st}\Big|_{t=t_0-\tau}^{t=t_0+\tau}\\ &= \frac{1}{2s\tau}e^{-st_0}(e^{s\tau}-e^{-s\tau})\\ &= \frac{\sinh s\tau}{s\tau}e^{-st_0};\end{aligned}$$

Laplace Transform and Integrals Involving δ

- We have found $\mathcal{L}\{d_\tau(t-t_0)\} = \frac{\sinh s\tau}{s\tau} e^{-st_0}$; Using L'Hospital's rule: $\lim_{\tau \to 0^+} \frac{\sinh s\tau}{s\tau} = \lim_{\tau \to 0^+} \frac{s \cosh s\tau}{s} = 1$; So, we get $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$;
- By letting $t_0 \to 0^+$, $\mathcal{L}\{\delta(t)\} = \lim_{t_0 \to 0^+} e^{-st_0} = 1$;
- To define the integral of the product of the delta function and any continuous function f:

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t-t_0) f(t) dt;$$

  Using the definition of $d_\tau(t)$ and the mean value theorem for integrals, $\int_{-\infty}^{\infty} d_\tau(t-t_0) f(t) dt = \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t) dt = \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*) = f(t^*)$, where $t_0 - \tau < t^* < t_0 + \tau$; Hence $t^* \to t_0$ as $\tau \to 0^+$, and it follows that $\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0)$;

An Initial Value Problem

- Find the solution of the initial value problem

$$2y'' + y' + 2y = \delta(t-5). \quad y(0) = 0,\ y'(0) = 0;$$

Take the Laplace transform

$$\begin{gathered} 2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-5)\}; \\ 2(s^2Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) + 2Y(s) = e^{-5s}; \\ (2s^2 + s + 2)Y(s) = e^{-5s}; \end{gathered}$$

So we get

$$Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2}\frac{1}{s^2+\frac{1}{2}s+1} = \frac{e^{-5s}}{2}\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}};$$

Since

$\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\right\} = \frac{4}{\sqrt{15}}e^{-t/4}\sin\frac{\sqrt{15}}{4}t,$

we have $y = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5);$

Subsection 6

The Convolution Integral

Convolution Theorem

The Convolution Theorem

If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then $H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}$, $s > a$, where

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau.$$

The function h is known as the **convolution** of f and g and the integrals above are known as **convolution integrals**.

- The equality of the two integrals follows by making the change of variable $t - \tau = \xi$ in the first integral;
- According to this theorem, the transform of the convolution of two functions is given by the product of the separate transforms;
- It is conventional to emphasize that the convolution integral can be thought of as a "generalized product" by writing $h(t) = (f * g)(t)$; In particular, we write $(f * g)(t) := \int_0^t f(t-\tau)g(\tau)d\tau$;

Properties of Convolution

- The convolution $f * g$ has many of the properties of ordinary multiplication:
  - $f * g = g * f$ (**commutative law**)
  - $f * (g_1 + g_2) = f * g_1 + f * g_2$ (**distributive law**)
  - $(f * g) * h = f * (g * h)$ (**associative law**)
  - $f * 0 = 0 * f = 0$ (**absorption law**)

  In the last equation the zeros denote not the number 0 but the function that has the value 0 for each value of t;
- There are other properties of ordinary multiplication that the convolution integral does not have;
  - For example, it is not true in general that $f * 1$ is equal to f: $(f * 1)(t) = \int_0^t f(t-\tau) \cdot 1 d\tau = \int_0^t f(t-\tau) d\tau$; If, for example, $f(t) = \cos t$, then $(f * 1)(t) = \int_0^t \cos(t-\tau) d\tau = -\sin(t-\tau)|_0^t = -\sin 0 + \sin t = \sin t$; Clearly, $(f * 1)(t) \neq f(t)$ in this case;
  - Similarly, it may not be true that $f * f$ is nonnegative;

Example I

- Find the inverse transform of $H(s) = \dfrac{a}{s^2(s^2+a^2)} = s^{-2} \cdot \dfrac{a}{s^2+a^2}$;

  Since
  $$\mathcal{L}\{t\} = s^{-2} \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2},$$
  the inverse transform of $H(s)$ is
  $$h(t) = \int_0^t (t-\tau)\sin a\tau d\tau = \frac{at - \sin at}{a^2};$$
- The same result is obtained if $h(t)$ is written
  $$h(t) = \int_0^t \tau \sin a(t-\tau) d\tau;$$
- $h(t)$ can also be found by expanding $H(s)$ into partial fractions
  $$\frac{a}{s^2(s^2+a^2)} = \frac{1}{a}\frac{1}{s^2} - \frac{1}{a}\frac{1}{s^2+a^2};$$

Example II

- Find the solution of the initial value problem $y'' + 4y = g(t)$, $y(0) = 3$, $y'(0) = -1$;

  By taking the Laplace transform,

$$s^2 Y(s) - 3s + 1 + 4Y(s) = G(s)$$
$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4};$$

  Observe that the first and second terms on the right contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively;

  Write
$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s);$$

  Then we obtain
$$y = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin[2(t-\tau)]g(\tau)d\tau;$$

The General Case

- Consider $ay'' + by' + cy = g(t)$, where a, b and c are real constants and g is a given function, together with the initial conditions $y(0) = y_0$, $y'(0) = y'_0$;
  By taking the Laplace transform
  $$\begin{aligned} a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) &= G(s) \\ (as^2 + bs + c)Y(s) - (as + b)y_0 - ay'_0 &= G(s); \end{aligned}$$
  If we let $\Phi(s) = \dfrac{(as+b)y_0 + ay'_0}{as^2 + bs + c}$, $\Psi(s) = \dfrac{G(s)}{as^2 + bs + c}$, we can write $Y(s) = \Phi(s) + \Psi(s)$;
  Consequently, if $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ and $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$, $y = \phi(t) + \psi(t)$;
  - $\phi(t)$ is the solution of the initial value problem $ay'' + by' + cy = 0$, $y(0) = y_0$, $y'(0) = y'_0$, obtained by setting $g(t)$ equal to zero;
  - $\psi(t)$ is the solution of $ay'' + by' + cy = g(t)$, $y(0) = 0$, $y'(0) = 0$, in which the initial values y_0 and y'_0 are each replaced by zero;

The General Case (Cont'd)

- We are considering $ay'' + by' + cy = g(t)$, where a, b and c are real constants and g is a given function, together with the initial conditions $y(0) = y_0$, $y'(0) = y'_0$;
  - Once specific values of a, b and c are given, we can find $\phi(t) = \mathcal{L}^{-1}\{\Phi(s)\}$ by using the table of transforms, possibly in conjunction with a translation or a partial fraction expansion;
  - To find $\psi(t) = \mathcal{L}^{-1}\{\Psi(s)\}$, it is convenient to write $\Psi(s)$ as $\Psi(s) = H(s)G(s)$, where $H(s) = \frac{1}{as^2+bs+c}$; The function H is known as the **transfer function**; H depends only on the properties of the system under consideration whereas $G(s)$ depends only on the external excitation g(t) that is applied to the system;
    By the convolution theorem we can write
    $\psi(t) = \mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)g(\tau)d\tau$, where
    $h(t) = \mathcal{L}^{-1}\{H(s)\}$, and $g(t)$ is the given forcing function;
    Thus, $\psi(t)$ is the convolution of the impulse response $h(t)$ and the forcing function $g(t)$;