Elementary Differential Equations

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LSSU Math 310

Subsection 1

Introduction

From Higher-Order Equations To First-Order Systems

- The motion of a certain spring-mass system is described by the second order differential equation $u'' + 0.125u' + u = 0$; Rewrite this equation as a system of first order equations;

  Let $x_1 = u$ and $x_2 = u'$; Then it follows that $x_1' = x_2$; Further, $u'' = x_2'$; Then, by substituting for u, u' and u'', we obtain

  $$x_2' + 0.125x_2 + x_1 = 0;$$

  Thus x_1 and x_2 satisfy the following system of two first order differential equations:

  $$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 - 0.125x_2 \end{cases}$$

More General Systems

- Consider the general equation of motion of a spring-mass system $mu'' + \gamma u' + ku = F(t)$;
  Let $x_1 = u$ and $x_2 = u'$; Then, we obtain the system
  $\begin{cases} x_1' = x_2 \\ x_2' = -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 + \frac{F(t)}{m} \end{cases}$
- To transform an n-th order equation $y(n) = F(t, y, y', \ldots, y^{(n-1)})$ into a system of n first order equations, we introduce the variables $x_1, x_2, \ldots, x_n$ defined by $x_1 = y,\ x_2 = y',\ x_3 = y'',\ \ldots,\ x_n = y^{(n-1)}$;
  It then follows immediately that

$$\begin{cases} x_1' & = & x_2 \\ x_2' & = & x_3 \\ & \cdots & \\ x_{n-1}' & = & x_n \\ x_n' & = & F(t, x_1, x_2, \ldots, x_n). \end{cases}$$

The Most General System

- The most **general form of a system** is

$$\begin{cases} x_1' = F_1(t, x_1, x_2, \ldots, x_n) \\ x_2' = F_2(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ x_n' = F_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

- A **solution of this system** on $I : \alpha < t < \beta$ is a set of n functions $x_1 = \phi_1(t)$, $x_2 = \phi_2(t)$, ..., $x_n = \phi_n(t)$ that are differentiable in I and that satisfy the system at all points in I;
- We may also have given initial conditions

$$x_1(t_0) = x_1^0,\ x_2(t_0) = x_2^0,\ \ldots,\ x_n(t_0) = x_n^0,$$

where t_0 is a specified value of t in I, and x_1^0, ..., x_n^0 are prescribed numbers;
- The system and the initial conditions form an **initial value problem**;

Guarantees for Existence and Uniqueness of Solution

Theorem

If $F_1, \ldots, F_n$ and $\frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_1}{\partial x_n}, \ldots, \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_n}$ are continuous in $R : \alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, $\ldots$, $\alpha_n < x_n < \beta_n$, and $(t_0, x_1^0, \ldots, x_n^0)$ is in R, then there is an interval $|t - t_0| < h$ in which there exists a unique solution $x_1 = \phi_1(t)$, $\ldots$, $x_n = \phi_n(t)$ of the system of differential equations satisfying the initial conditions.

- If $F_1, \ldots, F_n$ are linear functions of $x_1, \ldots, x_n$, then the system is called **linear**; Otherwise, it is **nonlinear**; The most **general system of n first order linear equations** has the form

$$\left\{\begin{array}{rcl} x_1' & = & p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' & = & p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ & \vdots & \\ x_n' & = & p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{array}\right. \qquad (1)$$

Existence and Uniqueness: Linear Case

- We are looking at the linear system

$$\left\{\begin{array}{rcl} x_1' & = & p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' & = & p_{21}(t)x_1 + \cdots + p_{2n}(t)x_n + g_2(t) \\ & \vdots & \\ x_n' & = & p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{array}\right. \qquad (2)$$

If $g_1(t) = \cdots = g_n(t) = 0$, for all t in I, then the system is **homogeneous**; Otherwise, it is **nonhomogeneous**;

Theorem

If $p_{11}, p_{12}, \ldots, p_{nn}$, $g_1, \ldots, g_n$ are continuous on an open interval $I : \alpha < t < \beta$, then there exists a unique solution

$$x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$$

of (2) satisfying given initial conditions at any t_0 in I; Moreover, the solution exists throughout the interval I.

Subsection 2

Review of Matrices

Matrices and Notation

- Matrices are named by boldfaced capitals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ occasionally using boldfaced Greek capitals $\mathbf{\Phi}, \mathbf{\Psi}, \ldots$;
- A matrix $\mathbf{A}$ consists of a rectangular array of numbers, or **elements**, arranged in m rows and n columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \ldots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix};$$

- We speak of $\mathbf{A}$ as an $m \times n$ **matrix**;
- The element lying in the i-th row and j-th column is designated by a_{ij};
- (a_{ij}) is also used to denote the matrix whose generic element is a_{ij};

Transpose, Conjugate and Adjoint Matrices

- The **transpose** $\mathbf{A}^T$ of $\mathbf{A}$ is obtained from $\mathbf{A}$ by interchanging the rows and columns of $\mathbf{A}$;

  Thus, if $\mathbf{A} = (a_{ij})$, then $\mathbf{A}^T = (a_{ji})$;
- $\bar{a}_{ij}$ is the complex conjugate of a_{ij}, and $\bar{\mathbf{A}}$ the **conjugate** of $\mathbf{A}$ obtained by replacing each element a_{ij} by its conjugate $\bar{a}_{ij}$;
- The **adjoint** $\mathbf{A}^*$ of $\mathbf{A}$ is the transpose of the conjugate:

$$\mathbf{A}^* = \bar{\mathbf{A}}^T;$$

An Example

- Find the transpose, the conjugate and the adjoint of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}$;

$$\begin{aligned} \mathbf{A}^T &= \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix} \\ \bar{\mathbf{A}} &= \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix} \\ \mathbf{A}^* &= \begin{pmatrix} 3 & 4-3i \\ 2+i & -5-2i \end{pmatrix} \end{aligned}$$

Square Matrices and Vectors

- A matrix is called a **square matrix of order** n if $m = n$;
- It is called a **vector** or a **column vector** if it is an $n \times 1$ matrix, i.e., it has only one column; A vector is denoted by $\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$;
- The transpose $\mathbf{x}^T$ of an $n \times 1$ column vector is a $1 \times n$ **row vector**;

If $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then $\boldsymbol{x}^T = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$;

Basic Properties of Matrices

1. **Equality**: Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be **equal** if all corresponding elements are equal, i.e., if $a_{ij} = b_{ij}$, for each i and j;
2. **Zero**: $\mathbf{0}$ is the matrix (or vector) each of whose elements is zero;
3. **Addition**: The sum of two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ is the matrix $\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$; We have that:
   - $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$;
   - $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$;
4. **Multiplication by a Number**: The product of a matrix $\mathbf{A}$ by a complex number α is defined by $\alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij})$; We have:
   - $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$;
   - $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$;
5. The **negative** of $\mathbf{A}$, denoted by $-\mathbf{A}$, is defined by $-\mathbf{A} = (-1)\mathbf{A}$;
6. **Subtraction**: The difference $\mathbf{A} - \mathbf{B}$ of two $m \times n$ matrices is defined by $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$; Thus $\mathbf{A} - \mathbf{B} = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij})$;

Matrix Multiplication

- The **product AB** is defined whenever the number of columns in the first factor is the same as the number of rows in the second;
- If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times r$, then $\mathbf{C} = \mathbf{AB}$ is $m \times r$;
- The element in the i-th row and j-th column of $\mathbf{C}$ is

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj};$$

- We have the properties:
  - $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$;
  - $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$;
- However, even when both products $\mathbf{AB}$ and $\mathbf{BA}$ are defined, in general,

$$\mathbf{AB} \neq \mathbf{BA};$$

Example

- Example: Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 2 & 1 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$; Then:

$$\mathbf{AB} = \begin{pmatrix} 2-2+2 & 1+2-1 & -1+0+1 \\ 0+2-2 & 0-2+1 & 0+0-1 \\ 4+1+2 & 2-1-1 & -2+0+1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & -1 & -1 \\ 7 & 0 & -1 \end{pmatrix};$$

$$\mathbf{BA} = \begin{pmatrix} 2+0-2 & -4+2-1 & 2-1-1 \\ 1+0+0 & -2-2+0 & 1+1+0 \\ 2+0+2 & -4-2+1 & 2+1+1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 0 \\ 1 & -4 & 2 \\ 4 & -5 & 4 \end{pmatrix};$$

So we see that $\mathbf{AB} \neq \mathbf{BA}$;

Inner Product

- The **product** of two n-component vectors $\mathbf{x}$, $\mathbf{y}$: $\mathbf{x}^T\mathbf{y} = \sum_{i=1}^{n} x_i y_i$;
- Then we have:
  - $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x}$;
  - $\mathbf{x}^T(\mathbf{y}+\mathbf{z}) = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}$,;
  - $(\alpha\mathbf{x})^T\mathbf{y} = \alpha(\mathbf{x}^T\mathbf{y}) = \mathbf{x}^T(\alpha\mathbf{y})$;
- The **scalar inner product** of $\mathbf{x}, \mathbf{y}$ is $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \bar{y}_i$;
- The following hold:
  - $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\bar{\mathbf{y}}$;
  - $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$;
  - $(\mathbf{x}, \mathbf{y}+\mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z})$;
  - $(\alpha\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x}, \mathbf{y})$;
  - $(\mathbf{x}, \alpha\mathbf{y}) = \bar{\alpha}(\mathbf{x}, \mathbf{y})$;

Length and Orthogonality

- Even if the vector x has elements with nonzero imaginary parts, the scalar product of x with itself yields a nonnegative real number

$$(\mathbf{x},\mathbf{x}) = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2;$$

- The quantity

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})}$$

  is called the **length**, or **magnitude**, of $\mathbf{x}$;
- If $(\mathbf{x},\mathbf{y}) = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are said to be **orthogonal**;

  Example: The unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ of three-dimensional vector geometry form an orthogonal set;

Example

- If some of the elements of $\mathbf{x}$ are not real, then the product $\mathbf{x}^T\mathbf{x} = \sum_{i=1}^n x_i^2$ may not be a real number; For example, let $\mathbf{x} = \begin{pmatrix} i \\ -2 \\ 1+i \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 2-i \\ i \\ 3 \end{pmatrix}$; Then:

$$\begin{aligned}
\mathbf{x}^T\mathbf{y} &= (i)(2-i)+(-2)(i)+(1+i)(3) \\
&= 2i+1-2i+3+3i = 4+3i; \\
(\mathbf{x},\mathbf{y}) &= (i)(2+i)+(-2)(-i)+(1+i)(3) \\
&= 2i-1+2i+3+3i = 2+7i; \\
\mathbf{x}^T\mathbf{x} &= (i)^2+(-2)^2+(1+i)^2 \\
&= -1+4+1+2i-1 = 3+2i; \\
(\mathbf{x},\mathbf{x}) &= (i)(-i)+(-2)(-2)+(1+i)(1-i) \\
&= 1+4+1+1 = 7;
\end{aligned}$$

Identity Square Matrices and Inverse Matrices

- The **multiplicative identity**, or simply the **identity matrix I**, is given by $\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$;
- We have $\mathbf{AI} = \mathbf{IA} = \mathbf{A},$ for any (square) matrix $\mathbf{A}$;
- The square matrix $\mathbf{A}$ is said to be **nonsingular** or **invertible** if there is a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$ and $\mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ is the identity;

  If there is such a $\mathbf{B}$, it is unique and it is called the **multiplicative inverse**, or simply the **inverse**, of $\mathbf{A}$, and we write $\mathbf{B} = \mathbf{A}^{-1}$;
- Using this notation, $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$;
- Matrices that do not have an inverse are called **singular** or **noninvertible**;

Computing the Inverse Using Determinants

- There are various ways to compute $\mathbf{A}^{-1}$ from $\mathbf{A}$, when it exists;
- One way involves the use of determinants;
- Associated with each element a_{ij} of a given matrix is the **minor** M_{ij}, which is the determinant of the matrix obtained by deleting the i-th row and j-th column of the original matrix, i.e., the row and column containing a_{ij};
- Also associated with each element a_{ij} is the **cofactor** C_{ij} defined by the equation $C_{ij} = (-1)^{i+j} M_{ij}$;
- If $\mathbf{B} = \mathbf{A}^{-1}$, then it can be shown that the general element b_{ij} is given by $b_{ij} = \dfrac{C_{ji}}{\det(\mathbf{A})}$;
- This equation provides a condition that is both necessary and sufficient for invertibility: $\mathbf{A}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$;
- Therefore, if $\det(\mathbf{A}) = 0$, then $\mathbf{A}$ is singular;

Computing the Inverse Using Gaussian Elimination

- Another way to compute $\mathbf{A}^{-1}$ is by means of **elementary row operations**:
  1. Interchange of two rows;
  2. Multiplication of a row by a nonzero scalar;
  3. Addition of any multiple of one row to another row;
- The transformation of a matrix by a sequence of elementary row operations is referred to as **row reduction** or **Gaussian elimination**;
- Any nonsingular matrix $\mathbf{A}$ can be transformed into the identity $\mathbf{I}$ by a systematic sequence of these operations;
- If the same sequence of operations is performed on $\mathbf{I}$, it is transformed into $\mathbf{A}^{-1}$;
- We usually perform the sequence of operations on both matrices at the same time using the **augmented matrix**

$$(\mathbf{A} \mid \mathbf{I}) \ \rightarrow \ (\mathbf{I} \mid \mathbf{A}^{-1}).$$

An Example: Gaussian Elimination

- Find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$;

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 - 2r_1]{r_2 \leftarrow r_2 - 3r_1} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{r_2 \leftarrow (1/2)r_2} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 - 4r_2]{r_1 \leftarrow r_1 + r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{r_3 \leftarrow -\frac{1}{5} r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right) \xrightarrow[r_2 \leftarrow r_2 - \frac{5}{2} r_3]{r_1 \leftarrow r_1 - \frac{3}{2} r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{10} & -\frac{1}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{1}{5} \end{array}\right)$$

Continuous and Differentiable Matrix Functions

- We sometimes need to consider vectors or matrices whose elements are functions of a real variable t;
- We write $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}$;
- The matrix $\mathbf{A}(t)$ is said to be **continuous at** $t = t_0$ or on an interval $\alpha < t < \beta$ if each element of $\mathbf{A}$ is a continuous function at the given point or on the given interval;
- Similarly, $\mathbf{A}(t)$ is said to be **differentiable** if each of its elements is differentiable, and its derivative $\frac{d\mathbf{A}}{dt}$ is defined by $\frac{d\mathbf{A}}{dt} = \left(\frac{da_{ij}}{dt}\right)$;

Integrable Matrix Functions and Rules

- The **integral of a matrix function** is defined as

$$\int_a^b \mathbf{A}(t)dt = \left(\int_a^b a_{ij}(t)dt\right);$$

- The following rules hold:
  - $\frac{d}{dt}(\mathbf{CA}) = \mathbf{C}\frac{d\mathbf{A}}{dt}$, if $\mathbf{C}$ constant matrix;
  - $\frac{d}{dt}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$;
  - $\frac{d}{dt}(\mathbf{AB}) = \mathbf{A}\frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt}\mathbf{B}$;

Example

- Find $\frac{d\mathbf{A}}{dt}$ and $\int_0^\pi \mathbf{A}(t)dt$ if $\mathbf{A}(t) = \begin{pmatrix} \sin t & t \\ 1 & \cos t \end{pmatrix}$;

$$\begin{aligned} \frac{d\mathbf{A}}{dt} &= \begin{pmatrix} \frac{d\sin t}{dt} & \frac{dt}{dt} \\ \frac{d1}{dt} & \frac{d\cos t}{dt} \end{pmatrix} \\ &= \begin{pmatrix} \cos t & 1 \\ 0 & -\sin t \end{pmatrix}; \\ \int_0^\pi \mathbf{A}(t)dt &= \begin{pmatrix} \int_0^\pi \sin t dt & \int_0^\pi t dt \\ \int_0^\pi 1 dt & \int_0^\pi \cos t dt \end{pmatrix} \\ &= \begin{pmatrix} -\cos t|_0^\pi & \frac{t^2}{2}|_0^\pi \\ t|_0^\pi & \sin t|_0^\pi \end{pmatrix} \\ &= \begin{pmatrix} 2 & \frac{\pi^2}{2} \\ \pi & 0 \end{pmatrix}; \end{aligned}$$

Subsection 3

Linear Equations; Linear Independence, Eigenvalues, Eigenvectors

Systems of Linear Algebraic Equations

- A set of n simultaneous linear algebraic equations in n variables

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ & \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{cases}$$

  can be written as $\mathbf{Ax} = \mathbf{b}$, where the $n \times n$ matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are given, and the components of $\mathbf{x}$ are to be determined;
- If $\mathbf{b} = \mathbf{0}$, the system is said to be **homogeneous**;
- If $\mathbf{b} \neq \mathbf{0}$, it is called **nonhomogeneous**;

The Matrix of Coefficients

- Consider the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ & \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{cases}$$

- If the coefficient matrix $\mathbf{A}$ is nonsingular, i.e., if $\det(\mathbf{A}) \neq 0$, then there is a unique solution; Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and the solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$;
- $\mathbf{Ax} = \mathbf{0}$, has only the trivial solution $\mathbf{x} = \mathbf{0}$, if $\mathbf{A}$ is nonsingular;
- If $\mathbf{A}$ is singular, then solutions either do not exist, or do exist but are not unique;
- In this case, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has (infinitely many) nonzero solutions in addition to the trivial solution;

Solving Systems of Linear Algebraic Equations

- Suppose $\boldsymbol{A}$ is singular.
- If $\mathbf{Ax} = \mathbf{b}$ has a solution, then the system has (infinitely many) solutions that are of the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi},$$

  where:
  - $\mathbf{x}^{(0)}$ is a particular solution of the system;
  - $\boldsymbol{\xi}$ is the most general solution of the homogeneous system;
- For solving particular systems, we use row reduction on $(\mathbf{A}|\mathbf{b})$ to transform the system into a much simpler one from which the solution(s), if there are any, can be written down easily:
  - We first perform row operations on the augmented matrix so as to transform $\mathbf{A}$ into an upper triangular matrix, i.e., a matrix whose elements below the main diagonal are all zero;
  - Once this is done, it is easy to see whether the system has solutions, and to find them if it does;

Example I

- Solve the system $\left\{\begin{array}{rcl} x_1 - 2x_2 + 3x_3 & = & 7 \\ -x_1 + x_2 - 2x_3 & = & -5 \\ 2x_1 - x_2 - x_3 & = & 4 \end{array}\right.$;

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 - 2r_1]{r_2 \leftarrow r_2 + r_1} \left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

$$\xrightarrow{r_2 \leftarrow (-1)r_2} \left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -7 & -10 \end{array}\right) \xrightarrow{r_3 \leftarrow r_3 - 3r_2} \left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{array}\right)$$

$$\xrightarrow{r_3 \leftarrow -\frac{1}{4} r_3} \left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The matrix obtained in this manner corresponds to the system of equations $\left\{\begin{array}{rcl} x_1 - 2x_2 + 3x_3 & = & 7 \\ x_2 - x_3 & = & -2 \\ x_3 & = & 1 \end{array}\right.$; So, we get $\left\{\begin{array}{l} x_3 = 1 \\ x_2 = -2 + x_3 = -1 \\ x_1 = 7 + 2x_2 - 3x_3 = 2 \end{array}\right.$

Example II

- Discuss solutions of the system $\left\{\begin{array}{rcl} x_1 - 2x_2 + 3x_3 & = & b_1 \\ -x_1 + x_2 - 2x_3 & = & b_2 \\ 2x_1 - x_2 + 3x_3 & = & b_3 \end{array}\right.$, for various values of b_1, b_2 and b_3;

  The augmented matrix for the system is $\left(\begin{array}{rrr|r} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right)$;

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 - 2r_1]{r_2 \leftarrow r_2 + r_1} \left(\begin{array}{rrr|r} 1 & -2 & 3 & b_1 \\ 0 & -1 & 1 & b_1 + b_2 \\ 0 & 3 & -3 & -2b_1 + b_3 \end{array}\right)$$
$$\xrightarrow[r_2 \leftarrow (-1)r_2]{r_3 \leftarrow r_3 + 3r_2} \left(\begin{array}{rrr|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{array}\right);$$

  The equation corresponding to the third row is $b_1 + 3b_2 + b_3 = 0$;
  The system has no solution unless b_1, b_2 and b_3 satisfy this equation;

Example II (Cont'd)

- To have solutions b_1, b_2 and b_3 must satisfy $b_1 + 3b_2 + b_3 = 0$;
  Assume that $b_1 = 2$, $b_2 = 1$ and $b_3 = -5$;
  Then the first two rows of $\left(\begin{array}{ccc|c} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{array}\right)$
  correspond to the equations $x_1 - 2x_2 + 3x_3 = 2$ and $x_2 - x_3 = -3$;
  We can choose one of the unknowns arbitrarily and then solve for the other two;
  If we let $x_3 = \alpha$, where α is arbitrary, then $x_2 = \alpha - 3$,
  $x_1 = 2(\alpha - 3) - 3\alpha + 2 = -\alpha - 4$;
  In the form $\mathbf{x} = \alpha \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix}$ the second term on the right is a solution of the nonhomogeneous system and the first term is the most general solution of the homogeneous system;

Linear Dependence and Independence

- A set of k vectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ is said to be **linearly dependent** if there exists a set of (complex) numbers $c_1, \ldots, c_k$, at least one of which is nonzero, such that

$$c_1 \mathbf{x}^{(1)} + \cdots + c_k \mathbf{x}^{(k)} = \mathbf{0};$$

- If the only set $c_1, \ldots, c_k$ for which this equation is satisfied is $c_1 = c_2 = \cdots = c_k = 0$, then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are said to be **linearly independent**;

A Condition for Linear Independence

- Consider a set of n vectors, each of which has n components;
- Let $x_{ij} = x_i^{(j)}$ be the i-th component of the vector $\mathbf{x}^{(j)}$;
- Let $\mathbf{X} = (x_{ij})$;
- Then the equation $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0}$ can be written as

$$\begin{pmatrix} x_1^{(1)}c_1 + \cdots + x_1^{(n)}c_n \\ \vdots \\ x_n^{(1)}c_1 + \cdots + x_n^{(n)}c_n \end{pmatrix} = \begin{pmatrix} x_{11}c_1 + \cdots + x_{1n}c_n \\ \vdots \\ x_{n1}c_1 + \cdots + x_{nn}c_n \end{pmatrix} = \mathbf{X}\mathbf{c} = \mathbf{0};$$

  - If $\det \mathbf{X} \neq 0$, then the only solution is $\mathbf{c} = \mathbf{0}$;
  - If $\det \mathbf{X} = 0$, there are nonzero solutions;
- Thus $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent if and only if $\det \mathbf{X} \neq 0$;

Example

- Determine whether the vectors $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$ are linearly independent or linearly dependent; If they are linearly dependent, find a linear relation among them;

  We seek constants c_1, c_2, c_3 such that $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$;

  Equivalently, $\begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;

  We perform row operations on the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{array}\right) \xrightarrow[r_3 \leftarrow r_3 + r_1]{r_2 \leftarrow r_2 - 2r_1} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right) \xrightarrow{r_2 \leftarrow -\frac{1}{3}r_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right) \xrightarrow{r_3 \leftarrow r_3 - 5r_2} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right);$$

Example (Cont'd)

- We obtained $\left(\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$;

  Thus we obtain the equivalent system $\left\{\begin{array}{rcl} c_1 + 2c_2 - 4c_3 & = & 0 \\ c_2 - 3c_3 & = & 0 \end{array}\right.$;

  So, $c_2 = 3c_3$ and $c_1 = 4c_3 - 2c_2 = -2c_3$;

  If we choose $c_3 = -1$ for convenience, then $c_1 = 2$ and $c_2 = -3$. Therefore,

  $$2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \mathbf{0},$$

  and the given vectors are linearly dependent.

Linear Independence of Vector Functions

- Consider a set of vector functions $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ defined on an interval $\alpha < t < \beta$;
- The vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are said to be **linearly dependent on** $\alpha < t < \beta$ if there exists a set of constants $c_1, \ldots, c_k$, not all of which are zero, such that
$$c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0},$$
for all t in the interval;
- Otherwise, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are said to be **linearly independent**;
- Note that if $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are linearly dependent on an interval, they are linearly dependent at each point in the interval;
- However, if $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ are linearly independent on an interval, they may or may not be linearly independent at each point; They may, in fact, be linearly dependent at each point, but with different sets of constants at different points;

Eigenvalues and Eigenvectors

- The equation $\mathbf{Ax} = \mathbf{y}$ can be viewed as a linear transformation that maps $\mathbf{x}$ into $\mathbf{y}$;
- Vectors that are transformed into multiples of themselves are important in many applications;
- To find such vectors, set $\mathbf{y} = \lambda\mathbf{x}$, where λ is a scalar and solve $\mathbf{Ax} = \lambda\mathbf{x}$, or
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0};$$
- The latter equation has nonzero solutions if and only if λ is chosen so that
$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0;$$
- Values of λ that satisfy this equation are called **eigenvalues** of the matrix $\mathbf{A}$;
- The nonzero solutions for $\mathbf{x}$ obtained by using such a $\lambda \neq 0$ are called the **eigenvectors** corresponding to λ;

The 2×2 Case

- If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a 2×2 matrix, then we get

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

  The determinant condition, thus, is

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0;$$

Example

- Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$;

  The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, or $\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; The eigenvalues are the roots of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - \lambda - 2 = 0$; Thus the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$;

  - For $\lambda = 2$ we have $\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; Hence each row of this vector equation leads to the condition $x_1 - x_2 = 0$, so if $x_1 = c$, then $x_2 = c,$ and the eigenvector $\mathbf{x}^{(1)}$ is $\mathbf{x}^{(1)} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ c \neq 0$;
  - Setting $\lambda = -1$, we get $\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; Again we obtain $4x_1 - x_2 = 0$; Thus the eigenvector corresponding to $\lambda_2 = -1$ is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ or any nonzero multiple of this vector;

Normalized Eigenvectors

- Eigenvectors are determined only up to an arbitrary nonzero multiplicative constant;
- If this constant is specified in some way, then the eigenvectors are said to be **normalized**;
- Sometimes it is useful to normalize an eigenvector $\mathbf{x}$ by choosing the constant so that its length

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})} = 1;$$

Multiplicity of Eigenvalues

- Since $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is a polynomial equation of degree n in λ, there are n eigenvalues $\lambda_1, \ldots, \lambda_n$, some of which may be repeated;

  If a given eigenvalue appears m times as a root, then that eigenvalue is said to have **algebraic multiplicity** m;
- Each eigenvalue has at least one associated eigenvector, and an eigenvalue of algebraic multiplicity m may have q linearly independent eigenvectors;

  The number q is called the **geometric multiplicity** of the eigenvalue;
- Tt is always the case that $1 \leq q \leq m$;
- In particular, if each eigenvalue of **A** is **simple** (algebraic multiplicity 1), then each eigenvalue also has geometric multiplicity one;

Linear Independence of Eigenvectors

- It is possible to show that if λ_1 and λ_2 are two eigenvalues of $\mathbf{A}$, with $\lambda_1 \neq \lambda_2$, then their corresponding eigenvectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent;
- This result extends to any set $\lambda_1, \ldots, \lambda_k$ of distinct eigenvalues: their eigenvectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are linearly independent;
- Thus, if each eigenvalue of an $n \times n$ matrix is simple, then the n eigenvectors of $\mathbf{A}$, one for each eigenvalue, are linearly independent;
- On the other hand,if $\mathbf{A}$ has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors associated with $\mathbf{A}$, since for a repeated eigenvalue we may have $q < m$;

Example

- Find the eigenvalues and eigenvectors of the matrix
$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$
The eigenvalues λ and eigenvectors $\mathbf{x}$ satisfy the equation
$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$, or $\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; The eigenvalues are the roots of the equation
$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2 =$
$-(\lambda^3 - 4\lambda + \lambda - 2) = -(\lambda(\lambda^2 - 4) + (\lambda - 2)) =$
$-(\lambda - 2)(\lambda^2 + 2\lambda + 1) = -(\lambda - 2)(\lambda + 1)^2 = 0$; So $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = -1$; Thus 2 is a simple eigenvalue, and -1 is an eigenvalue of algebraic multiplicity 2, or a double eigenvalue;

Example (Cont'd)

- We work to find the eigenvectors:
  - Substitute $\lambda = 2$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; We reduce this via elementary row operations;

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow{r_1 \leftarrow (-1) r_1} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$
$$\xrightarrow[r_3 \leftarrow r_3 - \frac{1}{2} r_1]{r_2 \leftarrow r_2 - \frac{1}{2} r_1} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{2}{2} & -\frac{3}{2} & 0 \end{array}\right) \xrightarrow[r_2 \leftarrow (-\frac{2}{3}) r_2]{r_3 \leftarrow r_3 + r_2} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

  So we have $x_2 = x_3$ and $x_1 = x_3$; We thus get the eigenvector $\mathbf{x}^{(1)} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$;

Example (Cont'd)

- For $\lambda = -1$, we get $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;
  So we have $x_1 + x_2 + x_3 = 0$;
  If $x_1 = c_1$ and $x_2 = c_2$, then $x_3 = -c_1 - c_2$;
  In vector notation we have

$$\mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix};$$

  For example, by choosing $c_1 = 1$ and $c_2 = 0$, we obtain the eigenvector $\mathbf{x}^{(2)} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T$; An independent eigenvector results by choosing $c_1 = 0$ and $c_2 = 1$; So $\mathbf{x}^{(3)} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$;

Hermitian Matrices and Their Special Properties

- A **self-adjoint** or **Hermitian matrix** is one satisfying $\mathbf{A}^* = \mathbf{A}$, i.e., $\bar{a}_{ji} = a_{ij}$;
- Hermitian matrices include as a subclass **real symmetric matrices**, i.e., matrices that have real elements and for which $\mathbf{A}^T = \mathbf{A}$;
- The eigenvalues and eigenvectors of Hermitian matrices always have the following useful properties:
  1. All eigenvalues are real;
  2. There always exists a full set of n linearly independent eigenvectors, regardless of the algebraic multiplicities of the eigenvalues;
  3. If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues, then $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = 0$; Thus, if all eigenvalues are simple, then the associated eigenvectors form an orthogonal set of vectors;
  4. Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m eigenvectors that are mutually orthogonal; Thus the full set of n eigenvectors can always be chosen to be orthogonal as well as linearly independent;

Subsection 4

Basic Theory of Systems of First Order Linear Equations

Systems of First Order Linear Equations

- We consider a system of n first order linear equations

$$\begin{cases} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

- Consider $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ to be components of a vector $\mathbf{x} = \phi(t)$;
- Similarly, $g_1(t), \ldots, g_n(t)$ are components of a vector $\mathbf{g}(t)$;
- Finally $p_{11}(t)$, ..., $p_{nn}(t)$ are elements of an $n \times n$ matrix $\mathbf{P}(t)$;
- Then the system can be written in matrix form:

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t);$$

Solutions

- A vector $\mathbf{x} = \phi(t)$ is said to be a **solution** of
$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t);$$
if its components satisfy the system;
- We assume that $\mathbf{P}$ and $\mathbf{g}$ are continuous on some interval $\alpha < t < \beta$, i.e., each of the scalar functions $p_{11}, \ldots, p_{nn}, g_1, \ldots, g_n$ is continuous there;
- This is sufficient to guarantee the existence of solutions on the interval $\alpha < t < \beta$;

Solving Systems of First Order Linear Equations

- It is convenient to consider first the homogeneous equation
$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x},$$
obtained by setting $\mathbf{g}(t) = 0$;
- Once the homogeneous equation has been solved, there are several methods that can be used to solve the nonhomogeneous equation;
- Write $\mathbf{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, \mathbf{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}$ to designate specific solutions; Note that $x_{ij}(t) = x_i^{(j)}(t)$ refers to the i-th component of the j-th solution $\mathbf{x}^{(j)}(t)$;
- The main facts about the structure of solutions of the system are stated in the following theorems;

Principle of Superposition

Theorem (Linear Combinations of Solutions)

If the vector functions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are solutions of the system, then the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$ is a solution, for any constants c_1, c_2.

- Therefore, if $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are solutions, then $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t)$ is also a solution, for any $c_1, \ldots, c_k$;
- Example: Consider the equation $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$;

  The following vectors are solutions:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \ \mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t};$$

  Therefore, $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t)$ also satisfies the equation;

Space of Solutions

- Every finite linear combination of solutions is also a solution;
- Can all solutions of the equation be found in this way?
- Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be n solutions and consider the matrix $\mathbf{X}(t)$ whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$:

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix};$$

- The columns of $\mathbf{X}(t)$ are linearly independent for a given value of t if and only if $\det \mathbf{X} \neq 0$ for that value of t;
- This determinant is called the **Wronskian** of the n solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ and is also denoted by $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$;
- The solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent at a point if and only if $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ is not zero there;

Solution Space Theorem

Theorem

If the vector functions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent solutions, for each point in $\alpha < t < \beta$, then each solution $\mathbf{x} = \phi(t)$ of the system can be expressed as a linear combination of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$,

$$\phi(t) = c_1 \mathbf{x}^{(1)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t),$$

in exactly one way.

- If the constants $c_1, \ldots, c_n$ are thought of as arbitrary, then $\phi(t) = c_1 \mathbf{x}^{(1)}(t) + \cdots + c_n \mathbf{x}^{(n)}(t)$ is the **general solution**;
- Any set of solutions $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ that is linearly independent at each point in $\alpha < t < \beta$ is said to be a **fundamental set of solutions** for that interval;

Vanishing of the Wronskian and Existence of Solutions

Theorem (Vanishing of the Wronskian)

If $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$, then in this interval $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ either is identically zero or, else, never vanishes.

- Thus, to determine whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions, it suffices to evaluate their Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ at one conveniently chosen point in the interval;

Theorem (Existence of a Fundamental Set of Solutions)

Let $\mathbf{e}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}^T$, $\mathbf{e}^{(2)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^T$, $\ldots$, $\mathbf{e}^{(n)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}^T$; Further let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be solutions that satisfy the initial conditions $\mathbf{x}^{(1)}(t_0) = \mathbf{e}^{(1)}, \ldots, \mathbf{x}^{(n)}(t_0) = \mathbf{e}^{(n)}$, respectively, where t_0 is any point in $\alpha < t < \beta$; Then $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a fundamental set of solutions.

Complex-Valued Solutions and Summary

- If a system whose coefficients are real gives rise to solutions that are complex valued, then the following theorem is handy:

Theorem (Complex-Valued Solutions)

Consider $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ where each $p_{ij}(t)$ is a real-valued continuous function; If $\mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t)$ is a complex valued solution, then its real part $\mathbf{u}(t)$ and its imaginary part $\mathbf{v}(t)$ are also solutions.

- We summarize all our results as follows:
  1. Any set of n linearly independent solutions of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ constitutes a fundamental set of solutions;
  2. Under the conditions given in this section, such fundamental sets always exist;
  3. Every solution of the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ can be represented as a linear combination of any fundamental set of solutions;

Subsection 5

Homogeneous Linear Systems with Constant Coefficients

Homogeneous Systems With Constant Coefficients

- We consider systems of homogeneous linear equations with constant coefficients: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant $n \times n$ matrix; We will assume all the elements of $\mathbf{A}$ are real numbers;
- If $n = 1$, then the system reduces to a single first order equation

$$\frac{dx}{dt} = ax;$$

  Its solution is $x = ce^{at}$;
- For systems of n equations, the situation is somewhat analogous but more complicated;

Example

- Find the general solution of $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}$;

  If we let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we get $x_1' = 2x_1$ and $x_2' = -3x_2$; The first has solution $x_1 = c_1 e^{2t}$ and the second $x_2 = c_2 e^{-3t}$; Therefore
  $\mathbf{x} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} =$
  $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$;
  If $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$, $\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$, the Wronskian
  $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}] = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t} \neq 0$, whence $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental system of solutions;

Solutions of Systems With Constant Coefficients

- Consider again $\mathbf{x}' = \mathbf{A}\mathbf{x}$;
- Equilibrium solutions are found by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$;

  If $\det \mathbf{A} \neq 0$, then $\mathbf{x} = \mathbf{0}$ is the only equilibrium solution;
- To find other solutions, assume that a solution involves e^{rt}; Since solutions are vectors, let us multiply e^{rt} by a constant vector $\boldsymbol{\xi}$; Thus we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$; Substituting, we get $r\boldsymbol{\xi} e^{rt} = \mathbf{A}\boldsymbol{\xi} e^{rt}$; Canceling e^{rt} gives $\mathbf{A}\boldsymbol{\xi} = r\boldsymbol{\xi}$, or

  $$(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0},$$

  where $\mathbf{I}$ is the $n \times n$ identity matrix;
- Thus, the vector $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ is a solution provided that r is an eigenvalue and $\boldsymbol{\xi}$ an associated eigenvector of the coefficient matrix $\mathbf{A}$;

Example

- Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$;

  Assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$ and substitute for $\mathbf{x}$ in the equation:
  $\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; The system has a nontrivial solution if and only if the determinant of coefficients is zero; We set
  $\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1) = 0$,
  which has the roots $r_1 = 3$ and $r_2 = -1$;
  - If $r = 3$, then the system reduces to the single equation $-2\xi_1 + \xi_2 = 0$; Thus $\xi_2 = 2\xi_1$, and the eigenvector corresponding to $r_1 = 3$ can be taken as $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$;

Example (Cont'd)

- We found the eigenvector corresponding to $r_1 = 3$ can be taken as $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$;
  - Similarly, corresponding to $r_2 = -1$, we find that $\xi_2 = -2\xi_1$, so the eigenvector is $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; The corresponding solutions of the differential equation are $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$, $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$;

The Wronskian of these solutions is $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0$, whence the solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set, and the general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$;

Example

- Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$;

  Assume that $\mathbf{x} = \boldsymbol{\xi} e^{rt}$; Then we obtain
  $\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; The eigenvalues satisfy
  $(-3-r)(-2-r)-2 = r^2+5r+4 = (r+1)(r+4) = 0$, so
  $r_1 = -1$ and $r_2 = -4$;
  - For $r=-1$, we get $\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; Hence
    $\xi_2 = \sqrt{2}\xi_1$, and the eigenvector $\boldsymbol{\xi}^{(1)}$ corresponding to the eigenvalue
    $r_1 = -1$ can be taken as $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$;

Example (Cont'd)

- The eigenvector $\boldsymbol{\xi}^{(1)}$ corresponding to the eigenvalue $r_1 = -1$ can be taken as $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$;
  - Similarly, corresponding to the eigenvalue $r_2 = -4$ we have $\xi_1 = -\sqrt{2}\xi_2$, so the eigenvector is $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$;

Thus, a fundamental set of solutions is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$, $\mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$, and the general solution is

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)} = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t};$$

The General Case

- Returning to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we seek the eigenvalues and eigenvectors of $\mathbf{A}$ from $(\mathbf{A} - r\mathbf{I})\xi = \mathbf{0}$;
- The eigenvalues $r_1, \ldots, r_n$ (which need not all be different) are roots of $\det(\mathbf{A} - r\mathbf{I}) = 0$;
- The nature of the eigenvalues and the corresponding eigenvectors determines the nature of the general solution of the system:
- If we assume that $\mathbf{A}$ is a real-valued matrix, there are three possibilities for the eigenvalues of A:
  1. All eigenvalues are real and different from each other;
  2. Some eigenvalues occur in complex conjugate pairs;
  3. Some eigenvalues are repeated;

The General Case: Real and Different Eigenvalues

- If the eigenvalues are all real and different, then associated with each eigenvalue r_i is a real eigenvector $\boldsymbol{\xi}^{(i)}$, and the n eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are linearly independent;
- The corresponding solutions of the differential system are

$$\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{r_1 t}, \ldots, \mathbf{x}^{(n)}(t) = \boldsymbol{\xi}^{(n)} e^{r_n t};$$

- To show that these solutions form a fundamental set, we evaluate their Wronskian:

$$\begin{aligned} W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t) &= \begin{vmatrix} \xi_1^{(1)} e^{r_1 t} & \cdots & \xi_1^{(n)} e^{r_n t} \\ \vdots & & \vdots \\ \xi_n^{(1)} e^{r_1 t} & \cdots & \xi_n^{(n)} e^{r_n t} \end{vmatrix} \\ &= e^{(r_1 + \cdots + r_n)t} \begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{vmatrix}; \end{aligned}$$

The General Case: Real and Different Eigenvalues (Cont'd)

- We computed

$$W[\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}](t) = e^{(r_1+\cdots+r_n)t}\begin{vmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{vmatrix};$$

  First, we observe that the exponential function is never zero;
  Next, since the eigenvectors $\boldsymbol{\xi}^{(1)}$, ..., $\boldsymbol{\xi}^{(n)}$ are linearly independent, the determinant in the last term is nonzero;
  As a consequence, the Wronskian $W[\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}](t)$ is never zero;
  Hence $\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)}$ form a fundamental set of solutions;
- Thus, the general solution is

$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + \cdot + c_n\boldsymbol{\xi}^{(n)}e^{r_n t};$$

The General Case: Real Symmetric Matrices

- If $\mathbf{A}$ is real and symmetric (a special case of Hermitian matrices), all the eigenvalues $r_1, \ldots, r_n$ must be real;
- Even if some of the eigenvalues are repeated, there is always a full set of n eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ that are linearly independent (in fact, orthogonal);
- Hence the corresponding solutions of the differential system form a fundamental set of solutions, and the general solution is also given by

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + \cdot + c_n \boldsymbol{\xi}^{(n)} e^{r_n t};$$

Example

- Find the general solution of $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \mathbf{x}$;

  The coefficient matrix is real and symmetric; We find the eigenvalues and eigenvectors:

$$\begin{aligned}
\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \quad &\Rightarrow \quad -\lambda^3 + 1 + 1 + \lambda + \lambda + \lambda = 0 \\
&\Rightarrow \quad -\lambda^3 + 3\lambda + 2 = 0 \\
&\Rightarrow \quad -\lambda^3 + \lambda + 2\lambda + 2 = 0 \\
&\Rightarrow \quad -\lambda(\lambda+1)(\lambda-1) + 2(\lambda+1) = 0 \\
&\Rightarrow \quad -(\lambda+1)(\lambda^2 - \lambda - 2) = 0 \\
&\Rightarrow \quad -(\lambda+1)(\lambda+1)(\lambda-2) = 0 \\
&\Rightarrow \quad \lambda = -1 \text{ or } \lambda = 2.
\end{aligned}$$

Example (Cont'd)

- For $\lambda = -1$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}.$$

Thus, we get $x_1 + x_2 + x_3 = 0$ or $x_3 = -x_1 - x_2$.

So we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Thus, we take as a basis for the eigenspace the vectors

$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Example (Cont'd)

- For $\lambda = 2$:
$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}.$$
Thus, we get
$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
So we get $x_1 = -x_2 + 2x_3 = x_3$ and $x_2 = x_3$.
So we get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and can take $\xi^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Example (Cont'd)

- We found that eigenvalues $\lambda = -1$ of algebraic (and geometric) multiplicity 2 and $\lambda = 2$ of multiplicity 1.

  We also found corresponding eigenvectors

$$\boldsymbol{\xi}^{(1)} = \left(\begin{array}{r} 1 \\ 0 \\ -1 \end{array} \right), \boldsymbol{\xi}^{(2)} = \left(\begin{array}{r} 0 \\ 1 \\ -1 \end{array} \right), \boldsymbol{\xi}^{(3)} = \left(\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right).$$

  Hence a fundamental set of solutions is

$$\mathbf{x}^{(1)}(t) = \left(\begin{array}{r} 1 \\ 0 \\ -1 \end{array} \right) e^{-t}, \mathbf{x}^{(2)}(t) = \left(\begin{array}{r} 0 \\ 1 \\ -1 \end{array} \right) e^{-t}, \mathbf{x}^{(3)}(t) = \left(\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right) e^{2t}.$$

  The general solution is

$$\mathbf{x} = c_1 \left(\begin{array}{r} 1 \\ 0 \\ -1 \end{array} \right) e^{-t} + c_2 \left(\begin{array}{r} 0 \\ 1 \\ -1 \end{array} \right) e^{-t} + c_3 1 \left(\begin{array}{r} 1 \\ 1 \\ 1 \end{array} \right) e^{2t};$$

Subsection 6

Complex Eigenvalues

System With Complex Eigenvalues

- Consider a system of n linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where the coefficient matrix $\mathbf{A}$ is real-valued;
- If we seek solutions of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, then r must be an eigenvalue and $\boldsymbol{\xi}$ a corresponding eigenvector of the coefficient matrix $\mathbf{A}$;
- Recall that the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ are the roots of the equation $\det(\mathbf{A} - r\mathbf{I}) = 0$ and that the corresponding eigenvectors satisfy $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$;
- If $\mathbf{A}$ is real, then the coefficients in the polynomial equation for r are real, and any complex eigenvalues must occur in conjugate pairs;
- For example, if $r_1 = \lambda + i\mu$, where λ and μ are real, is an eigenvalue of $\mathbf{A}$, then so is $r_2 = \lambda - i\mu$;

Example

- Find a fundamental set of real valued solutions of the system $\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\mathbf{x}$;

  Assuming $\mathbf{x} = \boldsymbol{\xi} e^{rt}$, $\begin{pmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

  The characteristic equation is:

$$\begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = (-\tfrac{1}{2} - r)^2 + 1 = r^2 + r + \tfrac{1}{4} + 1 = r^2 + r + \tfrac{5}{4} = 0.$$

  So we get $r = \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$.

Example (Cont'd)

- For $r_1 = -\frac{1}{2} - i$:
$$\left(\begin{array}{rr} i & 1 \\ -1 & i \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = \mathbf{0} \Rightarrow ix_1 + x_2 = 0 \Rightarrow x_2 = -ix_1;$$
$$\boldsymbol{\xi}^{(1)} = x_1 \left(\begin{array}{c} 1 \\ -i \end{array}\right).$$

  For $r_1 = -\frac{1}{2} + i$:
$$\left(\begin{array}{rr} -i & 1 \\ -1 & -i \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = \mathbf{0} \Rightarrow -ix_1 + x_2 = 0 \Rightarrow x_2 = ix_1;$$
$$\boldsymbol{\xi}^{(2)} = x_1 \left(\begin{array}{c} 1 \\ i \end{array}\right).$$

  Since $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are complex conjugates, a fundamental set of solutions is $\mathbf{x}^{(1)}(t) = \left(\begin{array}{c} 1 \\ -i \end{array}\right) e^{(-\frac{1}{2}-i)t}$, $\mathbf{x}^{(2)}(t) = \left(\begin{array}{c} 1 \\ i \end{array}\right) e^{(-\frac{1}{2}+i)t}$;

Example (Cont'd)

- To obtain a set of real-valued solutions, we find the real and imaginary parts of either $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$;

$$\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-\frac{1}{2}+i)t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t/2}(\cos t + i \sin t)$$
$$= \begin{pmatrix} e^{-t/2}\cos t \\ -e^{-t/2}\sin t \end{pmatrix} + i \begin{pmatrix} e^{-t/2}\sin t \\ e^{-t/2}\cos t \end{pmatrix};$$

Hence $\mathbf{u}(t) = e^{-t/2}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$, $\mathbf{v}(t) = e^{-t/2}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ is a set of real-valued solutions;

To verify that $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are linearly independent, we compute their Wronskian: $W(\mathbf{u},\mathbf{v})(t) = \begin{vmatrix} e^{-t/2}\cos t & e^{-t/2}\sin t \\ -e^{-t/2}\sin t & e^{-t/2}\cos t \end{vmatrix} =$ $e^{-t}(\cos^2 t + \sin^2 t) = e^{-t}$; Since the Wronskian is never zero, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ constitute a fundamental set of solutions;

The General Equation: Complex Conjugate Eigenvalues

- Consider again $\mathbf{x}' = \mathbf{A}\mathbf{x}$;
- Suppose that there is a pair of complex conjugate eigenvalues, $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$;
- Then the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ are also complex conjugates; To see this, recall that r_1 and $\boldsymbol{\xi}^{(1)}$ satisfy $(\mathbf{A} - r_1\mathbf{I})\boldsymbol{\xi}^{(1)} = \mathbf{0}$; Since $\mathbf{A}$ and $\mathbf{I}$ are real valued, $(\mathbf{A} - \bar{r}_1\mathbf{I})\bar{\boldsymbol{\xi}}^{(1)} = \mathbf{0}$;
- The corresponding solutions $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)}e^{r_1 t}$, $\mathbf{x}^{(2)}(t) = \bar{\boldsymbol{\xi}}^{(1)}e^{\bar{r}_1 t}$ are then complex conjugates of each other; We write $\boldsymbol{\xi}^{(1)} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real;
- Then $\mathbf{x}^{(1)}(t) = (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} = (\mathbf{a} + i\mathbf{b})e^{\lambda t}(\cos\mu t + i\sin\mu t)$;
- Separate $\mathbf{x}^{(1)}(t)$ into its real and imaginary parts: $\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + ie^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$;
- If we write $\mathbf{x}^{(1)}(t) = \mathbf{u}(t) + i\mathbf{v}(t)$, then the vectors $\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t)$, $\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t)$ are real valued solutions;

Example

- Find a general solution for the homogeneous system of first-order linear equation $\begin{cases} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= -2y_1 + \frac{3}{2}y_2 \\ y_4' &= \frac{4}{3}y_1 - 3y_2 \end{cases}$

  We can write the system in matrix form as

$$\mathbf{y}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & \frac{3}{2} & 0 & 0 \\ \frac{4}{3} & -3 & 0 & 0 \end{pmatrix} \mathbf{y} = \mathbf{A}\mathbf{y};$$

  We assume that $\mathbf{y} = \boldsymbol{\xi} e^{rt}$, where r must be an eigenvalue of the matrix $\mathbf{A}$ and $\boldsymbol{\xi}$ a corresponding eigenvector;

Example

- The characteristic polynomial of $\mathbf{A}$ is:

$$\begin{aligned}
&\begin{vmatrix} -r & 0 & 1 & 0 \\ 0 & -r & 0 & 1 \\ -2 & \frac{3}{2} & -r & 0 \\ \frac{4}{3} & -3 & 0 & -r \end{vmatrix} \\
&= -r \begin{vmatrix} -r & 0 & 1 \\ \frac{3}{2} & -r & 0 \\ -3 & 0 & -r \end{vmatrix} + \begin{vmatrix} 0 & -r & 1 \\ -2 & \frac{3}{2} & 0 \\ \frac{4}{3} & -3 & -r \end{vmatrix} \\
&= r^2 \begin{vmatrix} -r & 1 \\ -3 & -r \end{vmatrix} + r \begin{vmatrix} -2 & 0 \\ \frac{4}{3} & -r \end{vmatrix} + \begin{vmatrix} -2 & \frac{3}{2} \\ \frac{4}{3} & -3 \end{vmatrix} \\
&= r^2(r^2+3) + 2r^2 + 4 = r^4 + 5r^2 + 4 \\
&= (r^2+1)(r^2+4) = 0.
\end{aligned}$$

So the eigenvalues are $r_1 = i, \ r_2 = -i, \ r_3 = 2i,$ and $r_4 = -2i$;

Example (Cont'd)

- We find an eigenvector for $r_1 = i$:

$$\left(\begin{array}{cccc|c} -i & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 1 & 0 \\ -2 & \frac{3}{2} & -i & 0 & 0 \\ \frac{4}{3} & -3 & 0 & -i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & i & 0 & 0 \\ 0 & 1 & 0 & i & 0 \\ 0 & \frac{3}{2} & i & 0 & 0 \\ 0 & -3 & -\frac{4}{3}i & -i & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & i & 0 & 0 \\ 0 & 1 & 0 & i & 0 \\ 0 & 0 & i & -\frac{3}{2}i & 0 \\ 0 & 0 & -\frac{4}{3}i & 2i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & i & 0 & 0 \\ 0 & 1 & 0 & i & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

- So

$$\left\{\begin{array}{l} x_1 + ix_3 = 0 \\ x_2 + ix_4 = 0 \\ x_3 - \frac{3}{2}x_4 = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x_1 = -ix_3 = -\frac{3}{2}ix_4 \\ x_2 = -ix_4 \\ x_3 = \frac{3}{2}x_4 \end{array}\right. \Rightarrow \xi^{(1)} = \left(\begin{array}{c} -\frac{3}{2}i \\ -i \\ \frac{3}{2} \\ 1 \end{array}\right).$$

Example (Cont'd)

- Note that we have
$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} -\frac{3}{2}i \\ -i \\ \frac{3}{2} \\ 1 \end{pmatrix} = -\frac{1}{2}i \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}.$$
So we may also choose $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}$.
Working along the same lines, we may find that the corresponding eigenvectors for r_1, r_2, r_3 and r_4 are
$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix}, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 3 \\ 2 \\ -3i \\ -2i \end{pmatrix}, \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix}, \boldsymbol{\xi}^{(4)} = \begin{pmatrix} 3 \\ -4 \\ -6i \\ 8i \end{pmatrix};$$

Example (Cont'd)

- The complex-valued solutions $\boldsymbol{\xi}^{(1)}e^{it}$ and $\boldsymbol{\xi}^{(2)}e^{-it}$ are complex conjugates, so two real-valued solutions can be found by finding the real and imaginary parts of either of them; For instance, we have
$$\boldsymbol{\xi}^{(1)}e^{it} = \begin{pmatrix} 3 \\ 2 \\ 3i \\ 2i \end{pmatrix} (\cos t + i \sin t) = \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + i \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} =$$
$\mathbf{u}^{(1)}(t) + i\mathbf{v}^{(1)}(t)$; In a similar way, we obtain
$$\boldsymbol{\xi}^{(3)}e^{2it} = \begin{pmatrix} 3 \\ -4 \\ 6i \\ -8i \end{pmatrix} (\cos 2t + i \sin 2t) = \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + i \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix} =$$
$\mathbf{u}^{(2)}(t) + i\mathbf{v}^{(2)}(t)$; $\mathbf{u}^{(1)}$, $\mathbf{v}^{(1)}$, $\mathbf{u}^{(2)}$, and $\mathbf{v}^{(2)}$ are linearly independent and form a fundamental set of solutions; Thus, the general solution is
$$y = c_1 \begin{pmatrix} 3\cos t \\ 2\cos t \\ -3\sin t \\ -2\sin t \end{pmatrix} + c_2 \begin{pmatrix} 3\sin t \\ 2\sin t \\ 3\cos t \\ 2\cos t \end{pmatrix} + c_3 \begin{pmatrix} 3\cos 2t \\ -4\cos 2t \\ -6\sin 2t \\ 8\sin 2t \end{pmatrix} + c_4 \begin{pmatrix} 3\sin 2t \\ -4\sin 2t \\ 6\cos 2t \\ -8\cos 2t \end{pmatrix};$$

Subsection 7

Fundamental Matrices

The Fundamental Matrix

- Suppose that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions for the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on some interval $\alpha < t < \beta$;
- Then the matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdots & x_1^{(n)}(t) \\ \vdots & & \vdots \\ x_n^{(1)}(t) & \cdots & x_n^{(n)}(t) \end{pmatrix},$$

  whose columns are the vectors $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$, is said to be a **fundamental matrix** for the system;
- A fundamental matrix is nonsingular since its columns are linearly independent vectors;

Example of a Fundamental Matrix

- Find a fundamental matrix for the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$;
- We found that it has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 3$ and corresponding eigenvectors $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- Thus, $\mathbf{x}^{(1)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$, $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ are linearly independent solutions;
- It follows a fundamental matrix is

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix};$$

Expressing a Solution in Terms of a Fundamental Matrix

- The solution of an initial value problem can be written very compactly in terms of a fundamental matrix;
- The general solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ is $\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$; In terms of $\mathbf{\Psi}(t)$, $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector with arbitrary components $c_1, \ldots, c_n$;
- For an initial value problem $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, with $\mathbf{x}(t_0) = \mathbf{x}^0$, where $\alpha < t_0 < \beta$ and $\mathbf{x}^0$ is a given vector, it suffices to choose $\mathbf{c}$, such that $\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0$; Since $\mathbf{\Psi}(t_0)$ is nonsingular, $\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$ and
$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$$
is the solution of the initial value problem;
- In practice we first solve $\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0$ by row reduction and then substitute for $\mathbf{c}$ in $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}$;
- Since each column of the fundamental matrix $\mathbf{\Psi}$ is a solution $\mathbf{\Psi}$ satisfies $\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}$;

A Special Fundamental Matrix

- The **special fundamental matrix**, denoted by $\mathbf{\Phi}(t)$, has columns (the special vectors) $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$;
- Besides the differential equation, these vectors satisfy the initial conditions $\mathbf{x}^{(j)}(t_0) = \mathbf{e}^{(j)}$, where $\mathbf{e}^{(j)}$ is the j-th unit vector;
- Thus, $\mathbf{\Phi}(t)$ has the property that $\mathbf{\Phi}(t_0) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I}$;
- $\mathbf{\Phi}$ will denote the fundamental matrix satisfying this initial condition whereas $\mathbf{\Psi}$ will stand for an arbitrary fundamental matrix;
- Since $\mathbf{\Phi}^{-1}(t_0) = \mathbf{I}$, it follows from $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$ that

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0;$$

Example: Finding a Special Fundamental Matrix

- For the system $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$ in the previous example, find the special fundamental matrix $\mathbf{\Phi}$ such that $\mathbf{\Phi}(0) = \mathbf{I}$;

  The columns of $\mathbf{\Phi}$ are solutions of the equation that satisfy the initial conditions $\mathbf{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{x}^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; Since the general solution is $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$, we can find the solution satisfying the first set of these initial conditions by choosing $c_1 = c_2 = \frac{1}{2}$; We obtain the solution satisfying the second set of initial conditions by choosing $c_1 = \frac{1}{4}$ and $c_2 = -\frac{1}{4}$; Hence $\mathbf{\Phi}(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$;

  Note that the elements of $\mathbf{\Phi}(t)$ are more complicated than those of $\mathbf{\Psi}(t)$; But, it is now easy to determine the solution corresponding to any set of initial conditions via $\mathbf{x} = \mathbf{\Phi}(t)\boldsymbol{x}^0$;

A Matrix Series

- Recall that the solution of the scalar initial value problem $x' = ax, x(0) = x_0$, where a is a constant, is $x = x_0 e^{at}$;
- Now consider the corresponding initial value problem for an $n \times n$ system, namely, $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^0$, where $\mathbf{A}$ is a constant matrix;
- We can write the solution as $\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0$, where $\mathbf{\Phi}(0) = \mathbf{I}$;
- The matrix $\mathbf{\Phi}(t)$ might have an exponential character;
- The scalar exponential function e^{at} can be represented by the power series $e^{at} = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}$, which converges for all t;
- Let us now replace the scalar a by the $n \times n$ constant matrix $\mathbf{A}$ and consider the corresponding series
$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots ;$$

The Matrix $e^{\mathbf{A}t}$ Represented by the Series

- Each element of $\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots$ converges for all t as $n \to \infty$;
- Thus the sum of the series defines a new matrix $e^{\mathbf{A}t} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$;
- By differentiating term by term, we obtain
$$\frac{d}{dt} e^{\mathbf{A}t} = \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^{n-1}}{(n-1)!} = \mathbf{A} \left[\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} \right] = \mathbf{A} e^{\mathbf{A}t};$$
- Thus $e^{\mathbf{A}t}$ satisfies the differential equation $\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}$;
- When $t = 0$, $e^{\mathbf{A}t}$ satisfies the initial condition $e^{\mathbf{A}t}\big|_{t=0} = \mathbf{I}$;
- The fundamental matrix $\mathbf{\Phi}$ satisfies the same initial value problem as $e^{\mathbf{A}t}$, namely, $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$, $\mathbf{\Phi}(0) = \mathbf{I}$;
- By uniqueness (for matrix differential equations), $e^{\mathbf{A}t} = \mathbf{\Phi}(t)$;
- So, the solution of the initial value problem is $\mathbf{x} = e^{\mathbf{A}t} \mathbf{x}^0$;

The Diagonalization Process

- Suppose that the $n \times n$ matrix $\mathbf{A}$ has a full set of n linearly independent eigenvectors; This happens, e.g., if the eigenvalues of $\mathbf{A}$ are all different, or if $\mathbf{A}$ is Hermitian;
- If $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ are the eigenvectors and $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues, let $\mathbf{T} = \begin{pmatrix} \xi_1^{(1)} & \cdots & \xi_1^{(n)} \\ \vdots & & \vdots \\ \xi_n^{(1)} & \cdots & \xi_n^{(n)} \end{pmatrix}$;
- Since the columns of $\mathbf{T}$ are linearly independent vectors, $\det \mathbf{T} \neq 0$; Hence $\mathbf{T}$ is nonsingular and $\mathbf{T}^{-1}$ exists;
- The columns of $\mathbf{AT}$ are $\mathbf{A}\boldsymbol{\xi}^{(1)}, \ldots, \mathbf{A}\boldsymbol{\xi}^{(n)}$; Since $\mathbf{A}\boldsymbol{\xi}^{(k)} = \lambda_k \boldsymbol{\xi}^{(k)}$,
$$\mathbf{AT} = \begin{pmatrix} \lambda_1 \xi_1^{(1)} & \cdots & \lambda_n \xi_1^{(n)} \\ \vdots & & \vdots \\ \lambda_1 \boldsymbol{\xi}_n^{(1)} & \cdots & \lambda_n \boldsymbol{\xi}_n^{(n)} \end{pmatrix} = \mathbf{TD}, \ \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$
is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}$;

Diagonalizable Matrices

- From $\mathbf{AT} = \mathbf{TD}$, we get $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$;
- Thus, if the eigenvalues and eigenvectors of $\mathbf{A}$ are known, $\mathbf{A}$ can be transformed into a diagonal matrix;
- This process is known as a **similarity transformation** and $\mathbf{A}$ is said to be **similar** to the diagonal matrix $\mathbf{D}$;
- We also say that $\mathbf{A}$ is **diagonalizable**;
- If $\mathbf{A}$ is Hermitian, then the determination of $\mathbf{T}^{-1}$ is very simple: The eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$ are known to be mutually orthogonal, so we may choose them so that they are also normalized by $(\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}^{(i)}) = 1$ for each i; Then, it is easy to verify that $\mathbf{T}^{-1} = \mathbf{T}^*$, i.e., the inverse of $\mathbf{T}$ is the same as its adjoint (the transpose of its complex conjugate);
- If $\mathbf{A}$ has fewer than n linearly independent eigenvectors, then there is no matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$; In this case, $\mathbf{A}$ is not similar to a diagonal matrix and is not diagonalizable;

Example of Diagonalization

- Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$; Find the similarity transformation matrix $\mathbf{T}$ and show that $\mathbf{A}$ can be diagonalized.

  The eigenvalues and eigenvectors of $\mathbf{A}$ are $r_1 = -1$, $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; $r_2 = 3$, $\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$;

  Thus the transformation matrix $\mathbf{T}$ and its inverse $\mathbf{T}^{-1}$ are

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}; \quad \mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix};$$

  Consequently, we get that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = \mathbf{D}$;

Rewriting a System Using Diagonalization

- Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant matrix;
- We can solve the system by diagonalizing $\mathbf{A}$; This is possible whenever $\mathbf{A}$ has a full set of n linearly independent eigenvectors;
- Let $\boldsymbol{\xi}^{(1)},\ldots,\boldsymbol{\xi}^{(n)}$ be eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues $r_1, \ldots, r_n$ and form the transformation matrix $\mathbf{T}$ whose columns are $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$;
- Define a new dependent variable $\mathbf{y}$ by $\mathbf{x} = \mathbf{T}\mathbf{y}$;
- Then, $\mathbf{T}\mathbf{y}' = \mathbf{x}' = \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{T}\mathbf{y}$;
- Multiplying by $\mathbf{T}^{-1}$, we obtain

$$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y}, \quad \text{or} \quad \mathbf{y}' = \mathbf{D}\mathbf{y},$$

where $\mathbf{D}$ is the diagonal matrix with the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$ along the diagonal;

Solving the Diagonalized Form

- We obtained

$$\mathbf{y}' = \mathbf{D}\mathbf{y}, \quad \text{where} \quad \mathbf{x} = \mathbf{T}\mathbf{y} \quad \text{and} \quad \mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T};$$

- A fundamental matrix for this system is the diagonal matrix

$$\mathbf{Q}(t) = e^{\mathbf{D}t} = \begin{pmatrix} e^{r_1 t} & 0 & \cdots & 0 \\ 0 & e^{r_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{r_n t} \end{pmatrix};$$

- A fundamental matrix $\mathbf{\Psi}$ for the original system is then found from $\mathbf{Q}$ by applying again the transformation $\mathbf{T}$: $\mathbf{\Psi} = \mathbf{T}\mathbf{Q}$, i.e.,

$$\mathbf{\Psi}(t) = \begin{pmatrix} \xi_1^{(1)} e^{r_1 t} & \cdots & \xi_1^{(n)} e^{r_n t} \\ \vdots & & \vdots \\ \xi_n^{(1)} e^{r_1 t} & \cdots & \xi_n^{(n)} e^{r_n t} \end{pmatrix};$$

Example: Solving By Diagonalizing

- Consider again the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$; Using the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, where $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$, we reduce the system to $\mathbf{y}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{y} = \mathbf{D}\mathbf{y}$; Obtain a fundamental matrix for the system and then transform it to obtain a fundamental matrix for the original system;

  By multiplying $\mathbf{D}$ repeatedly with itself, we find that $\mathbf{D}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$, $\mathbf{D}^3 = \begin{pmatrix} -1 & 0 \\ 0 & 27 \end{pmatrix}, \ldots$; Therefore, $e^{\mathbf{D}t}$ is a diagonal matrix with the entries e^{-t} and e^{3t} on the diagonal, i.e., $e^{\mathbf{D}t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix}$;

  Finally, we obtain the required fundamental matrix $\mathbf{\Psi}(t)$ by multiplying $\mathbf{T}$ and $e^{\mathbf{D}t}$: $\mathbf{\Psi}(t) = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$;

Example

- Consider the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where
  $\mathbf{A} = \begin{pmatrix} -16 & 30 \\ -9 & 17 \end{pmatrix}$;
  We find the eigenvalues:
  $$\begin{vmatrix} -16-\lambda & 30 \\ -9 & 17-\lambda \end{vmatrix} = (-16-\lambda)(17-\lambda)+270 = \lambda^2-\lambda-2 = (\lambda+1)(\lambda-2).$$
  So $\lambda_1 = -1$ and $\lambda_2 = 2$.
- For $\lambda_1 = -1$:
  $$\begin{pmatrix} -15 & 30 \\ -9 & 18 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \Rightarrow \xi_1 = 2\xi_2 \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
- For $\lambda_2 = 2$:
  $$\begin{pmatrix} -18 & 30 \\ -9 & 15 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \Rightarrow -3\xi_1 + 5\xi_2 = 0 \Rightarrow \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Example (Cont'd)

- Therefore,

$$\boldsymbol{T} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad \boldsymbol{T}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix};$$

Moreover $\boldsymbol{D} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

Set $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$. Then a fundamental matrix $\boldsymbol{Q}$ for $\boldsymbol{y}' = \boldsymbol{D}\boldsymbol{y}$ is

$$\boldsymbol{Q}(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix};$$

A fundamental matrix $\boldsymbol{\Psi}(t)$ for $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ is obtained by

$$\boldsymbol{\Psi} = \boldsymbol{T}\boldsymbol{Q} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{-t} & 5e^{2t} \\ e^{-t} & 3e^{2t} \end{pmatrix};$$

Subsection 8

Repeated Eigenvalues

Example

- Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$;

  The eigenvalues r and eigenvectors $\boldsymbol{\xi}$ satisfy the equation $(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$, or $\begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; The eigenvalues are the roots of the equation $\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = r^2 - 4r + 4 = (r-2)^2 = 0$; Thus, the two eigenvalues are $r_1 = 2$ and $r_2 = 2$, i.e., the eigenvalue 2 has algebraic multiplicity 2; To determine the eigenvectors: $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; Hence we obtain the single condition $\xi_1 + \xi_2 = 0$, which determines ξ_2 in terms of ξ_1, or vice versa; Thus the eigenvector corresponding to the eigenvalue $r = 2$ is $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or any nonzero multiple of this vector; There is only one linearly independent eigenvector associated with the double eigenvalue;

The Existing Possibilities

- Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and suppose that $r = \rho$ is a m-fold root of the characteristic equation $\det(\mathbf{A} - r\mathbf{I}) = 0$;
- Then ρ is an eigenvalue of algebraic multiplicity m of the matrix $\mathbf{A}$;
- In this event, there are two possibilities:
  - There are m linearly independent eigenvectors corresponding to the eigenvalue ρ; Let $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(m)}$ be such m linearly independent eigenvectors; Then $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi}^{(1)} e^{\rho t}$, …, $\mathbf{x}^{(m)}(t) = \boldsymbol{\xi}^{(m)} e^{\rho t}$ are m linearly independent solutions; Thus, in this case it makes no difference that the eigenvalue $r = \rho$ is repeated;
  - There are fewer than m such eigenvectors; If so, there will be fewer than m solutions of the form $\boldsymbol{\xi} e^{\rho t}$ associated with this eigenvalue; Therefore, to construct the general solution, it is necessary to find other solutions of a different form;

Example

- Find a fundamental set of solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$;

  We know that $r = 2$ is a double eigenvalue with a single corresponding eigenvector, e.g., $\boldsymbol{\xi}^T = (1, -1)$; Thus, one solution of the system is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$; There is no second solution of the form $\mathbf{x} = \boldsymbol{\xi} e^{rt}$; We attempt to find a second solution of the form $\mathbf{x} = \boldsymbol{\xi} t e^{2t}$, where $\boldsymbol{\xi}$ is a constant vector; Substituting, we get $2\boldsymbol{\xi} t e^{2t} + \boldsymbol{\xi} e^{2t} - \mathbf{A}\boldsymbol{\xi} t e^{2t} = 0$; From the term in e^{2t} we find that $\boldsymbol{\xi} = \mathbf{0}$; Hence there is no nonzero solution of the form $\mathbf{x} = \boldsymbol{\xi} t e^{2t}$; It appears that in addition to $\boldsymbol{\xi} t e^{2t}$, the second solution must contain a term $\boldsymbol{\eta} e^{2t}$, i.e., we need to assume that $\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$, where $\boldsymbol{\xi}, \boldsymbol{\eta}$ are constant;

Example (Cont'd)

- We assume $\mathbf{x} = \boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t}$, $\boldsymbol{\xi}, \boldsymbol{\eta}$ constant;
  Substituting for $\mathbf{x}$, $2\boldsymbol{\xi} t e^{2t} + (\boldsymbol{\xi} + 2\boldsymbol{\eta})e^{2t} = \mathbf{A}(\boldsymbol{\xi} t e^{2t} + \boldsymbol{\eta} e^{2t})$; Equating coefficients of te^{2t} and e^{2t} on each side gives the conditions $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ and $(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$;
  - The first shows that $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = 2$, that is, $\boldsymbol{\xi}^T = (1, -1)$;
  - The augmented matrix of the second is $\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right)$; We have $-\eta_1 - \eta_2 = 1$, so if $\eta_1 = k$, then $\eta_2 = -k - 1$; If we write $\boldsymbol{\eta} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then by substituting for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$;

  The last term is merely a multiple of the first solution $\mathbf{x}^{(1)}(t)$ and may be ignored, but the first two terms constitute a new solution;

Example (Cont'd)

- We got $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$;

  This indicates that $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$,

  $\mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$;

  An elementary calculation shows that $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = -e^{4t} \neq 0$, whence $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ form a fundamental set of solutions;

  The general solution is $\mathbf{x} = c_1 \mathbf{x}^{(1)}(t) + c_2 \mathbf{x}^{(2)}(t) =$
  $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} \right]$;

System of First-Order vs. Single Second-Order Equations

- One difference between a system of two first order equations and a single second order equation becomes clear:
  - For a second order linear equation with a repeated root r_1 of the characteristic equation, a term $ce^{r_1 t}$ in the second solution is not required since it is a multiple of the first solution;
  - For a system of two first order equations, the term $\boldsymbol{\eta} e^{r_1 t}$ is not a multiple of the first solution $\boldsymbol{\xi} e^{r_1 t}$, so the term $\boldsymbol{\eta} e^{r_1 t}$ must be retained;

  This is typical of the general case when there is a double eigenvalue and a single associated eigenvector;
- Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and suppose that $r = \rho$ is a double eigenvalue of $\mathbf{A}$, but that there is only one corresponding eigenvector $\boldsymbol{\xi}$;
  - One solution is $\mathbf{x}^{(1)}(t) = \boldsymbol{\xi} e^{\rho t}$, where $\boldsymbol{\xi}$ satisfies $(\mathbf{A} - \rho\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$;
  - A second solution is $\mathbf{x}^{(2)}(t) = \boldsymbol{\xi} t e^{\rho t} + \boldsymbol{\eta} e^{\rho t}$, where $\boldsymbol{\xi}$ satisfies the same equation and $\boldsymbol{\eta}$ is determined from $(\mathbf{A} - \rho\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$;
    This gives$(\mathbf{A} - \rho\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}$; $\boldsymbol{\eta}$ is called a **generalized eigenvector** corresponding to the eigenvalue ρ;

Subsection 9

Nonhomogeneous Linear Systems

The Nonhomogeneous System

- Consider the nonhomogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, where the $n \times n$ matrix $\mathbf{P}(t)$ and $n \times 1$ vector $\mathbf{g}(t)$ are continuous for $\alpha < t < \beta$;
- The general solution can be expressed as

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) + \mathbf{v}(t),$$

  where
  - $c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$ is the general solution of the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$;
  - $\mathbf{v}(t)$ is a particular solution of the nonhomogeneous system;
- We describe four methods for determining $\mathbf{v}(t)$:
  - diagonalization;
  - undetermined coefficients;
  - variation of parameters;
  - Laplace transforms;

Diagonalization

- Consider $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$, where $\mathbf{A}$ is constant and diagonalizable;
- We solve by diagonalizing the coefficient matrix $\mathbf{A}$;
- Let T be the matrix whose columns are the eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n)}$ of $\mathbf{A}$, and define a new dependent variable $\mathbf{y}$ by $\mathbf{x} = \mathbf{T}\mathbf{y}$;
- Then, substituting for $\mathbf{x}$ we get $\mathbf{T}\mathbf{y}' = \mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \mathbf{A}\mathbf{T}\mathbf{y} + \mathbf{g}(t)$;
- Multiply by $\mathbf{T}^{-1}$:

  $$\mathbf{y}' = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \mathbf{D}\mathbf{y} + \mathbf{h}(t),$$

  where $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$ and where $\mathbf{D}$ is the diagonal matrix whose diagonal entries are the eigenvalues $r_1, \ldots, r_n$ of $\mathbf{A}$;
- In scalar form $y_j'(t) = r_j y_j(t) + h_j(t)$, $j = 1, \ldots, n$, where $h_j(t)$ is a certain linear combination of $g_1(t), \ldots, g_n(t)$;

Diagonalization (Cont'd)

- We found
$$y_j'(t) = r_j y_j(t) + h_j(t), \quad j = 1, \ldots, n;$$
- We solve by the Integrating Factor method:
$$\begin{aligned} & y_j'(t) = r_j y_j(t) + h_j(t) \quad \Rightarrow \quad y_j'(t) - r_j y_j(t) = h_j(t) \\ & e^{-r_j t} y_j'(t) - r_j e^{-r_j t} y_j(t) = e^{-r_j t} h_j(t) \\ & (e^{-r_j t} y(t))' = e^{-r_j t} h(t) \\ & e^{-r_j (t)} y(t) = \int e^{-r_j t} h(t) dt + c_j \\ & y(t) = e^{r_j t} \int e^{-r_j t} h(t) dt + c e^{-r_j t}. \end{aligned}$$
- Thus, we have the solutions
$$y_j(t) = e^{r_j t} \int_{t_0}^{t} e^{-r_j s} h_j(s) ds + c_j e^{r_j t}, \quad j = 1, \ldots, n$$
with c_j constants;
- $\mathbf{x}$ is obtained by multiplying by the matrix $\mathbf{T}$;

Example: Using Diagonalization to Solve a System

- Find the general solution of
  $$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t);$$
  We find the eigenvalues of $\mathbf{A}$:
  $$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = (-2-r)^2 - 1 = r^2 + 4r + 3 = (r+3)(r+1).$$
  So we get $r_1 = -3$ and $r_2 = -1$.
  For the eigenvectors we have:
  $$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \Rightarrow \xi_1 + \xi_2 = 0 \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$
  $$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0} \Rightarrow \xi_1 - \xi_2 = 0 \Rightarrow \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$
  The general solution of the homogeneous is
  $$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t};$$

Using Diagonalization (Cont'd)

- Let $\mathbf{T}$ be matrix of eigenvectors; If they are normalized so that $(\boldsymbol{\xi}, \boldsymbol{\xi}) = 1$, then $\mathbf{T}^{-1}$ is simply the adjoint of $\mathbf{T}$;

  So, if $\mathbf{T} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then $\mathbf{T}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$;

  Letting $\mathbf{x} = \mathbf{T}\mathbf{y}$, we obtain the following system for $\mathbf{y}$:

  $\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{T}^{-1}\mathbf{g}(t) = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} =$
  $\begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} + \frac{1}{\sqrt{2}} \begin{pmatrix} 2e^{-t} - 3t \\ 2e^{-t} + 3t \end{pmatrix}$; Thus,

$$y_1' + 3y_1 = \sqrt{2}e^{-t} - \frac{3}{\sqrt{2}}t, \quad y_2' + y_2 = \sqrt{2}e^{-t} + \frac{3}{\sqrt{2}}t;$$

Using Diagonalization (Cont'd)

- We obtained $y_1' + 3y_1 = \sqrt{2}e^{-t} - \frac{3}{\sqrt{2}}t$, $\quad y_2' + y_2 = \sqrt{2}e^{-t} + \frac{3}{\sqrt{2}}t$;
  Solving those, we obtain

$$\begin{aligned} y_1 &= e^{-3t}(\int(\sqrt{2}e^{2t} - \tfrac{3}{\sqrt{2}}te^{3t})dt + c_1) \\ &= e^{-3t}(\tfrac{\sqrt{2}}{2}e^{2t} - \tfrac{3}{\sqrt{2}}(\tfrac{1}{3}te^{3t} - \tfrac{1}{9}e^{3t}) + c_1) \\ &= \tfrac{\sqrt{2}}{2}e^{-t} - \tfrac{3}{\sqrt{2}}(\tfrac{t}{3} - \tfrac{1}{9}) + c_1e^{-3t}; \\ y_2 &= e^{-t}(\int(\sqrt{2} + \tfrac{3}{\sqrt{2}}te^{t})dt + c_2) \\ &= e^{-t}(\sqrt{2}t + \tfrac{3}{\sqrt{2}}(te^{t} - e^{t}) + c_2) \\ &= \sqrt{2}te^{-t} + \tfrac{3}{\sqrt{2}}(t-1) + c_2e^{-t}; \end{aligned}$$

Using Diagonalization (Cont'd)

- We found $y_1 = \frac{\sqrt{2}}{2}e^{-t} - \frac{3}{\sqrt{2}}(\frac{t}{3} - \frac{1}{9}) + c_1 e^{-3t}$ and $y_2 = \sqrt{2}te^{-t} + \frac{3}{\sqrt{2}}(t-1) + c_2 e^{-t}$;
  Now write

$$\begin{aligned}
\mathbf{x} &= \mathbf{T}\mathbf{y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{c_1}{\sqrt{2}}e^{-3t} + (\frac{c_2}{\sqrt{2}} + \frac{1}{2})e^{-t} + t - \frac{4}{3} + te^{-t} \\ -\frac{c_1}{\sqrt{2}}e^{-3t} + (\frac{c_2}{\sqrt{2}} - \frac{1}{2})e^{-t} + 2t - \frac{5}{3} + te^{-t} \end{pmatrix} \\
&= k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \\
&\quad + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix};
\end{aligned}$$

  The first two terms on the right side form the general solution of the homogeneous system and the remaining terms are a particular solution of the nonhomogeneous system;

Undetermined Coefficients

- To use this method, we assume the form of the solution with some or all of the coefficients unspecified and then seek to determine these coefficients so as to satisfy the equation;
- In practice, this method is applicable only if the coefficient matrix $\mathbf{P}$ is a constant matrix, and if the components of $\mathbf{g}$ are polynomial, exponential, or sinusoidal functions, or sums or products of these;
- In these cases the correct form of the solution can be predicted in a simple and systematic manner;
- The procedure for choosing the form of the solution is substantially the same as for linear second order equations;
- The main difference is illustrated by the case of a nonhomogeneous term of the form $\mathbf{u}e^{\lambda t}$, where λ is a simple root of the characteristic equation; Rather than assuming a solution of the form $\mathbf{a}te^{\lambda t}$, it is necessary to use $\mathbf{a}te^{\lambda t} + \mathbf{b}e^{\lambda t}$, where $\mathbf{a}$ and $\mathbf{b}$ are determined by substituting into the differential equation;

Example: Using Undetermined Coefficients

- Find a particular solution:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t);$$

Note that $\mathbf{g}(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t;$

Recall that the eigenvalues of $\mathbf{A}$ are $r_1 = -3$ and $r = -1$.

So both $\mathbf{a}te^{-t}$ and $\mathbf{b}e^{-t}$ must be included in the assumed solution;

Assume

$$\mathbf{x} = \mathbf{v}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d};$$

By substituting we get $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$

$$(\boldsymbol{a}te^{-t} + \boldsymbol{b}e^{-t} + \boldsymbol{c}t + \boldsymbol{d})' = \mathbf{A}(\mathbf{a}te^{-t} + \mathbf{b}e^{-t} + \mathbf{c}t + \mathbf{d}) + \mathbf{g}(t)$$

$$\mathbf{a}e^{-t} - \mathbf{a}te^{-t} - \mathbf{b}e^{-t} + \mathbf{c} =$$

$$\mathbf{A}\mathbf{a}te^{-t} + \mathbf{A}\mathbf{b}e^{-t} + \mathbf{A}\mathbf{c}t + \mathbf{A}\mathbf{d} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} t;$$

Example: Using Undetermined Coefficients (Cont'd)

- So we have
$$-\mathbf{a} = \mathbf{A}\mathbf{a}, \mathbf{a} - \mathbf{b} = \mathbf{A}\mathbf{b} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{0} = \mathbf{A}\mathbf{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{c} = \mathbf{A}\mathbf{d}.$$
  - From the first $\mathbf{a}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $r = -1$; Thus $\mathbf{a}^T = (\alpha, \alpha)$, where α is any nonzero constant;
  - The second gives:
$$\begin{gathered}(\mathbf{A} + \mathbf{I})\mathbf{b} = \mathbf{a} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 2 \\ \alpha \end{pmatrix} \\ \left\{ \begin{array}{rcl} -b_1 + b_2 & = & \alpha - 2 \\ b_1 - b_2 & = & \alpha \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} b_1 - b_2 & = & \alpha \\ 0 & = & 2\alpha - 2 \end{array} \right.\end{gathered}$$
  So it can be solved only if $\alpha = 1$; In this case $\mathbf{b} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; for any constant k; The simplest choice is $k = 0$, from which $\mathbf{b}^T = (0, -1)$;

Example: Using Undetermined Coefficients (Cont'd)

- We have
$$-\mathbf{a} = \mathbf{A}\mathbf{a}, \mathbf{a} - \mathbf{b} = \mathbf{A}\mathbf{b} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \mathbf{0} = \mathbf{A}\mathbf{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{c} = \mathbf{A}\mathbf{d}.$$

  We found $\mathbf{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;

  We continue:
  - The third yields:
$$\mathbf{A}\mathbf{c} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \mathbf{0}$$
$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \mathbf{0}$$
$$\left\{ \begin{array}{rcr} -2c_1 + c_2 & = & 0 \\ c_1 - 2c_2 & = & -3 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcr} c_1 & = & 1 \\ c_2 & = & 2 \end{array} \right.$$

Example: Using Undetermined Coefficients (Cont'd)

- The fourth yields:

$$\mathbf{Ad} = \mathbf{c} \Rightarrow \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$\left\{ \begin{array}{rcl} -2d_1 + d_2 & = & 1 \\ d_1 - 2d_2 & = & 2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} d_1 & = & -\frac{4}{3} \\ d_2 & = & -\frac{5}{3} \end{array} \right.$$

Finally, the particular solution is

$$\mathbf{v}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix};$$

Variation of Parameters

- Suppose the coefficient matrix is not constant or not diagonalizable;
- Let
$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$
$\mathbf{P}(t)$ and $\mathbf{g}(t)$ continuous on $\alpha < t < \beta$;
- Let $\mathbf{\Psi}(t)$ be a fundamental matrix for $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$;
- The general solution of the homogeneous is $\mathbf{\Psi}(t)\mathbf{c}$;
- So we seek a solution of the nonhomogeneous by replacing $\mathbf{c}$ by a vector function $\mathbf{u}(t)$;
- Assume $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$;
- Then we get
$$\mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{g}(t);$$
- Since $\mathbf{\Psi}(t)$ is a fundamental matrix, $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$;
- Hence $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$;

Variation of Parameters (Cont'd)

- We got $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$;
- $\mathbf{\Psi}(t)$ is nonsingular, whence $\mathbf{\Psi}^{-1}(t)$ exists, and therefore
$$\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{g}(t);$$
- Denote $\mathbf{u}(t)$ by
$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)dt + \mathbf{c}, \quad \mathbf{c} \text{ arbitrary};$$
- Even if the integrals cannot be evaluated, we can still write the general solution in the form
$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_1}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds,$$
where $t_1 \in (\alpha, \beta)$;
- The first term on the right is the general solution of the corresponding homogeneous system and the second term is a particular solution;

Variation of Parameters with Initial Value

- Consider the initial value problem
$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$
with $\mathbf{x}(t_0) = \mathbf{x}^0$;
- Choose as the lower limit of integration t_0; Then the general solution is
$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c} + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds;$$
- For $t = t_0$ the integral is zero, so the initial condition is also satisfied if we choose $\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0$;
- Therefore, the solution is
$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds;$$

Variation of Parameters with Initial Value

- We found
$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 + \mathbf{\Psi}(t)\int_{t_0}^{t}\mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds;$$
- Although it is helpful to use $\mathbf{\Psi}^{-1}$ to write the solutions, it is usually better in particular cases to solve the necessary equations by row reduction than to calculate $\mathbf{\Psi}^{-1}$;
- The solution takes a slightly simpler form if we use the fundamental matrix $\mathbf{\Phi}(t)$ satisfying $\mathbf{\Phi}(t_0) = \mathbf{I}$; In this case we have
$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0 + \mathbf{\Phi}(t)\int_{t_0}^{t}\mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds;$$

Example: Using Variation of Parameters

- Find the general solution:
  $$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t);$$
  The general solution of the homogeneous was
  $$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t};$$
  Thus, $\mathbf{\Psi}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$ is a fundamental matrix;

  Then, $\mathbf{x} = \mathbf{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}(t)$ satisfies $\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$, or
  $$\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix};$$

Example: Using Variation of Parameters (Cont'd)

- We found $\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$;

  We row reduce to get:

$$\left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ -e^{-3t} & e^{-t} & 3t \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} e^{-3t} & e^{-t} & 2e^{-t} \\ 0 & 2e^{-t} & 2e^{-t}+3t \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|c} e^{3t} & e^{-t} & 2e^{-t} \\ 0 & e^{-t} & e^{-t}+\frac{3}{2}t \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} e^{-3t} & 0 & e^{-t}-\frac{3}{2}t \\ 0 & e^{-t} & e^{-t}+\frac{3}{2}t \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & e^{2t}-\frac{3}{2}te^{3t} \\ 0 & 1 & 1+\frac{3}{2}te^{t} \end{array}\right);$$

  So $u_1' = e^{2t} - \frac{3}{2}te^{3t}, \ u_2' = 1 + \frac{3}{2}te^t;$ Hence

$$\begin{aligned} u_1(t) &= \tfrac{1}{2}e^{2t} - \tfrac{1}{2}te^{3t} + \tfrac{1}{6}e^{3t} + c_1; \\ u_2(t) &= t + \tfrac{3}{2}te^t - \tfrac{3}{2}e^t + c_2, \end{aligned}$$

Example: Using Variation of Parameters (Cont'd)

- Now we have

$$\boldsymbol{\Psi}(t)=\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}, \quad \mathbf{u}(t)=\begin{pmatrix} \frac{1}{2}e^{2t}-\frac{1}{2}te^{3t}+\frac{1}{6}e^{3t}+c_1 \\ t+\frac{3}{2}te^{t}-\frac{3}{2}e^{t}+c_2 \end{pmatrix};$$

  We conclude

$$\begin{aligned}
\mathbf{x} &= \boldsymbol{\Psi}(t)\mathbf{u}(t) \\
&= \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2}e^{2t}-\frac{1}{2}te^{3t}+\frac{1}{6}e^{3t}+c_1 \\ t+\frac{3}{2}te^{t}-\frac{3}{2}e^{t}+c_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}e^{-t}-\frac{1}{2}t+\frac{1}{6}+c_1e^{-3t}+te^{-t}+\frac{3}{2}t-\frac{3}{2}+c_2e^{-t} \\ -\frac{1}{2}e^{-t}+\frac{1}{2}t-\frac{1}{6}-c_1e^{-3t}+te^{-t}+\frac{3}{2}t-\frac{3}{2}+c_2e^{-t} \end{pmatrix} \\
&= c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t}+c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}+\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{-t} \\
&\quad +\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}+\begin{pmatrix} 1 \\ 2 \end{pmatrix}t-\frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix};
\end{aligned}$$

Laplace Transforms

- We used the Laplace transform to solve linear equations of arbitrary order;
- It can also be used in very much the same way to solve systems of equations;
- Since the transform is an integral, the transform of a vector is computed component by component;
- Thus $\mathcal{L}\{\mathbf{x}(t)\}$ is the vector whose components are the transforms of the respective components of $\mathbf{x}(t)$, and similarly for $\mathcal{L}\{\mathbf{x}'(t)\}$;
- We will denote $\mathcal{L}\{\mathbf{x}(t)\}$ by $\mathbf{X}(s)$;
- Then, $\mathcal{L}\{\mathbf{x}'(t)\} = s\mathbf{X}(s) - \mathbf{x}(0)$;

Example: Using Laplace Transforms

- Solve the system $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ under the initial condition $\mathbf{x}(0) = \mathbf{0}$;

  Take the Laplace transform:

  $$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{G}(s),$$

  where $\mathbf{G}(s)$ is the transform of $\mathbf{g}(t)$;

  So we have

  $$\mathbf{G}(s) = \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix};$$

  Taking into account the initial condition, we obtain $(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{G}(s)$; Hence, $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{G}(s)$;

Example: Using Laplace Transforms (Cont'd)

- Since $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}$, we get

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+3)} \begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix};$$

Then, substituting and multiplying,

$$\begin{aligned} \mathbf{X}(s) &= \tfrac{1}{(s+1)(s+3)} \begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix} \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}; \end{aligned}$$

Finally, we get $\mathbf{x}(t)$ from $\mathbf{X}(s)$;

For this we need to expand in partial fractions and use tables.

Example: Using Laplace Transforms (Cont'd)

- We found $X_1(s) = \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)}$;
  Expand into partial fractions:

$$\begin{aligned} \frac{2(s+2)}{(s+1)^2(s+3)} &= \frac{s+1}{(s+1)^2} - \frac{1}{s+3} = \frac{1}{s+1} - \frac{1}{s+3}; \\ \frac{3}{s^2(s+1)(s+3)} &= \frac{-\frac{4}{3}}{s} + \frac{1}{s^2} + \frac{\frac{3}{2}}{s+1} + \frac{-\frac{1}{6}}{s+3}; \end{aligned}$$

  So we get

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{2(s+2)}{(s+1)^2(s+3)}\} &= \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{1}{s+3}\} \\ &= e^{-t} - e^{-3t}; \\ \mathcal{L}^{-1}\{\frac{3}{s^2(s+1)(s+3)}\} &= -\frac{4}{3}\mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{1}{s^2}\} \\ &\quad + \frac{3}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{6}\mathcal{L}^{-1}\{\frac{1}{s+3}\} \\ &= -\frac{4}{3} + t + \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}; \end{aligned}$$

  So finally,

$$x_1(t) = -\frac{4}{3} + t + \frac{5}{2}e^{-t} - \frac{7}{6}e^{-3t}.$$

Example: Using Laplace Transforms (Cont'd)

- We found $X_2(s) = \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)}$;
  Expand into partial fractions:

$$\begin{aligned} \frac{2}{(s+1)^2(s+3)} &= \frac{-\frac{1}{2}}{s+1} + \frac{1}{(s+1)^2} + \frac{\frac{1}{2}}{s+3}; \\ \frac{3(s+2)}{s^2(s+1)(s+3)} &= \frac{-\frac{5}{3}}{s} + \frac{2}{s^2} + \frac{\frac{3}{2}}{s+1} + \frac{\frac{1}{6}}{s+3}; \end{aligned}$$

  So we get

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2(s+3)}\right\} &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\ &= -\frac{1}{2}e^{-t} + te^{-t} + \frac{1}{2}e^{-3t}; \\ \mathcal{L}^{-1}\left\{\frac{3(s+2)}{s^2(s+1)(s+3)}\right\} &= -\frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} \\ &\quad + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\ &= -\frac{5}{3} + 2t + \frac{3}{2}e^{-t} + \frac{1}{6}e^{-3t}; \end{aligned}$$

  So finally,

$$x_2(t) = -\frac{5}{3} + 2t + e^{-t} + te^{-t} + \frac{2}{3}e^{-3t}.$$