Topics in Discrete Mathematics

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LSSU Math 216

1. Partially Ordered Sets (Posets)
   - Fundamentals of Posets
   - Max and Min
   - Linear Orders
   - Linear Extensions
   - Dimension
   - Lattices

Subsection 1

Fundamentals of Posets

Partially Ordered Sets (Posets)

- Consider the following relations defined on sets:
  - The "less-than-or-equal-to" relation $\leq$ defined on the integers $\mathbb{Z}$;
  - The "divides" relation $\mid$ defined on the natural numbers $\mathbb{N}$;
  - The "is-a-subset-of" relation $\subseteq$ defined on 2^A for some set A.
- In all three cases, the relation R captures the flavor of "is smaller than" for the elements of the set X on which it is defined.
- All three relations are reflexive, antisymmetric, and transitive on the sets on which they are defined.

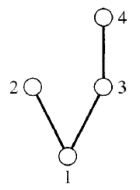
Definition (Partially Ordered Set, Poset)

A **partially ordered set** is a pair $P = (X, R)$ where X is a set and R is a relation on X that satisfies the following:

- R is reflexive: $\forall x \in X, x \; R \; x$;
- R is antisymmetric: $\forall x, y \in X,$ if $x \; R \; y$ and $y \; R \; x$, then $x = y$;
- R is transitive: $\forall x, y, z \in X,$ if $x \; R \; y$, and $y \; R \; z$, then $x \; R \; z$.

Hasse Diagrams

- Example: Let $P = (X, R)$, where $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3), (3,4), (4,4)\}$. It is not hard to see that R is reflexive and antisymmetric. Checking transitivity is tedious. Thus P is a poset.

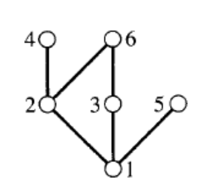


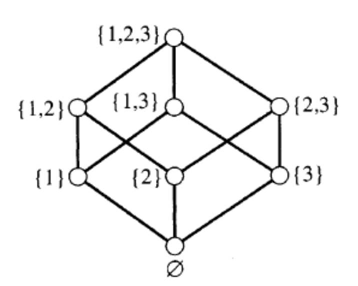
  - The figure shows a diagram for the poset. Each element of X is represented by a dot. If $x \; R \; y$ in the poset, then we draw x's dot below y's and draw a line segment (or curve) from x to y.

  We do not need to draw loops since we know that partial order relations are reflexive. Also, the relationships $(1, 3)$ and $(3, 4)$ are explicit, whereas $(1, 4)$ is implicit. Because partial order relations are transitive, we can infer $1 \; R \; 4$ from the diagram.
- These diagrams of posets are known as **Hasse diagrams**.

Two Examples

- Example: Draw the Hasse diagram of the poset whose ground set is $\{1,2,3,4,5,6\}$ and whose relation is $\mid$ (divides).





- Example: Draw the Hasse diagram for the poset whose ground set is $2^{\{1,2,3\}}$ and whose relation is $\subseteq$.

Refinement

Definition (Refinement)

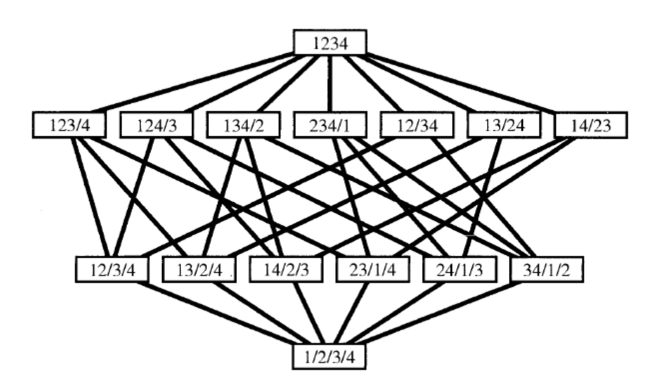
Let P and Q be partitions of a set A. We say that P **refines** Q or that P is **finer than** Q, if every part in P is a subset of some part in Q.

- Example: Let $A = \{1,2,3,4,5,6,7\}$, and let $P = \{\{1,2\},\{3\},\{4\},\{5,6\},\{7\}\}$, and, $Q = \{\{1,2,3,4\},\{5,6,7\}\}$. Every part of P is a subset of a part of Q. Thus we say that P is a refinement of Q or that P is finer than Q.
- It is not hard to see that
  - "refines" is reflexive;
  - "refines" is antisymmetric;
  - "refines" is transitive.

  Therefore, "refines" is a partial order on the set of all partitions of A.

An Example of a Partition Poset

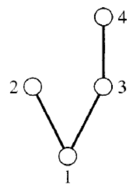
- Draw the Hasse diagram of the "refines" partial order on all partitions of $\{1, 2, 3, 4\}$.
  It is convenient to write $1/2/34$ in lieu of $\{\{1\}, \{2\}, \{3, 4\}\}$ etc.



Notational and Terminology

- Let $P = (X, \leq)$ be a poset (we are using $\leq$ to stand for a generic partial order relation). We define the following:
  - $x < y$ means $x \leq y$ and $x \neq y$;
  - $x \geq y$ means $y \leq x$;
  - $x > y$ means $y \leq x$ and $y \neq x$.

  We may also add a slash to mean that the given relationship does not hold. For example, $x \not\geq y$ means $y \leq x$ is false.
- If we want to discuss two different posets at once, we may attach various decorations to the $\leq$ symbol, e.g., $\leq'$ or $\leq_2$.
- Caution! For an arbitrary poset, $<$ and $\not\geq$ mean different things.



For the poset on the left $2 \not\geq 4$ is true but $2 < 4$ is false. Also, all three $2 < 4$, $2 = 4$ and $2 > 4$ are false. Pairs of elements, such as 2 and 4, that cannot be compared by the relation $\leq$ are called incomparable.

Comparability, Chains and Antichains

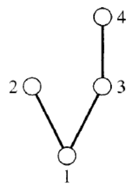
Definition (Comparable, Incomparable)

Let $P = (X, \leq)$ be a poset. Let $x, y \in X$. We call elements x and y **comparable** provided $x \leq y$ or $y \leq x$. We call the elements x and y **incomparable** if $x \not\leq y$ and $y \not\leq x$.

Definition (Chain, Antichain)

Let $P = (X, \leq)$ be a poset and let $C \subseteq X$. We call C a **chain** of P if every pair of elements in C are comparable. Let $A \subseteq X$. We call A an **antichain** of P if every pair of distinct elements of A are incomparable.

- Example: Consider the poset P on the right:
  The following sets are some of the chains of P:
  $\{1\}, \{1,2\}, \{1,4\}, \{1,3,4\}, \emptyset$.
  The following sets are some of the antichains of P:
  $\{3\}, \{2,3\}, \{2,4\}, \emptyset$.

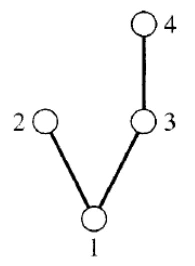


Height and Width

Definition (Height, Width)

Let P be a poset. The **height** of P is the maximum size of a chain. The **width** of P is the maximum size of an antichain.

- Example:



- The largest chain in the poset on the left is $\{1,3,4\}$. So this poset has height equal to 3.
- The largest antichains in this poset are $\{2,3\}$ and $\{2,4\}$. So this poset has width equal to 2.

Subsection 2

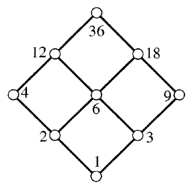
Max and Min

Maximum and Minimum

Definition (Maximum, Minimum)

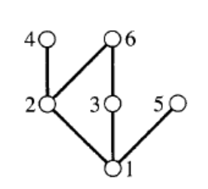
Let $P = (X, \leq)$ be a partially ordered set. An element $x \in X$ is called **maximum** if, for all $a \in X$, we have $a \leq x$. We call x **minimum** if, for all $b \in X$, we have $x \leq b$.

- Example: Consider the poset consisting of the positive divisors of 36 ordered by divisibility:



In this poset, element 1 is minimum because it is strictly below all other elements of the poset. Element 36 is maximum because it is strictly above all other elements.

However, consider the poset consisting of the integers 1 through 6 ordered by divisibility. In this poset, element 1 is minimum, but there is no maximum element.

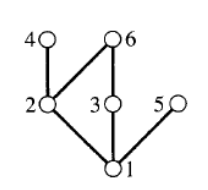


Maximal and Minimal Elements

Definition (Maximal, Minimal)

Let $P = (X, \leq)$ be a partially ordered set. An element $x \in X$ is called **maximal** if there is no $b \in X$ with $x < b$. Element x is called **minimal** if there is no $a \in X$, with $a < x$.

- x is maximal if there is no element strictly above it and minimal if there is no element strictly below it.
- Example:



In the poset consisting of the integers 1 through 6 ordered by divisibility, the elements 4, 5 and 6 are maximal, and element 1 is minimal.

Summary of the Four Types of Elements

- We summarize the four terms:

Term	Meaning
maximum	all other elements are below
maximal	no other element is above
minimum	all other elements are above
minimal	no other element is below

- For not maximal and not minimal:
  - Element x is not maximal if there is some other element y with $y > x$.
  - Element x is not minimal if there is some other element z with $z < x$.
- Is it possible for a poset to have no maximal elements?

  Consider the poset $(Z, \leq)$ - the integers ordered by ordinary "less than or equal to". This poset has no maximal and no minimal elements.

Maximal and Minimal Elements in Finite Posets

- A poset $P = (X, \leq)$ is **finite** and **nonempty**, if X is finite and $X \neq \emptyset$.

Proposition

Let $P = (X, \leq)$ be a finite, nonempty poset. Then P has maximal and minimal elements.

- Let x be any element of P. Let us write $u(x)$ to stand for the number of elements of P that are strictly above x (the **up-degree** of x), i.e., $u(x) = |\{a \in X : a > x\}|$. Because P is finite, $u(x)$ is a natural number (i.e., is finite). Choose an element m such that $u(m)$ is as small as possible (exists since $P \neq \emptyset$).
  Claim: m is a maximal element of P.
  Suppose that m is not maximal. Then, there exists a with $m < a$. By transitivity, every element that is strictly above a is also strictly above m. Furthermore, a is strictly above m. Thus, $u(m) \geq u(a) + 1$, i.e., $u(m) > u(a)$, a contradiction. Therefore, m is maximal.
- Similarly, every finite, nonempty poset has a minimal element.

Subsection 3

Linear Orders

Total or Linear Orders

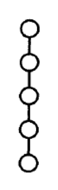
Definition (Total/Linear Order)

Let $P = (X, \leq)$ be a partially ordered set. We call P a **total** or **linear order** provided P does not contain incomparable elements.

- Example: $(\mathbb{Z}, \leq)$ is a total order.
- If x and y are elements of a total order, then either $x \leq y$ or $y \leq x$.
- Total orders satisfy the **trichotomy rule**: For all x and y in the poset, exactly one of the following is true:

$$x < y, \; x = y, \; \text{or } x > y.$$

- Example: Let P be the poset $(\{1, 2, 3, 4, 5\}, \leq)$ - that is, the integers 1 through 5 ordered by ordinary less than or equal to. This is a total order whose Hasse diagram looks like this:



Poset Isomorphism

- Let Q be the partially ordered set consisting of the positive divisors of 81 ordered by divisibility. The elements of Q are $1, 3, 9, 27$ and 81, and they are totally ordered $1 \mid 3 \mid 9 \mid 27 \mid 81$. Notice that this poset has the same Hasse diagram as the one just seen.

Definition (Isomorphism of Posets)

Let $P = (X, \leq)$ and $Q = (Y, \leq')$ be posets. A function $f : X \to Y$ is called a (**poset**) **isomorphism** provided f is a bijection and

$$\forall a, b \in X,\ a \leq b \iff f(a) \leq' f(b).$$

In the case when there is an isomorphism from P to Q, we say that P is **isomorphic to** Q and write $P \cong Q$.

- The condition $a \leq b \iff f(a) \leq' f(b)$ means that the function f is order-preserving and order-reflecting; i.e., an order relation holds between a and b in P, iff the corresponding relation holds between $f(a)$ and $f(b)$ in Q.

Classification of Finite Total Orders

- We show that any two finite total orders with the same number of elements are isomorphic.

Theorem

Let $P = (X, \preceq)$ be a finite total order containing n elements and $Q = (\{1, 2, \ldots, n\}, \leq)$ (integers 1 through n in their standard order). Then $P \cong Q$.

- The proof is by induction on n.
  - Basis Case: The basis case $n = 0$ is trivial.
  - Induction Hypothesis: We assume that the result is true for $n = k$.
  - Induction Step: Suppose $P = (X, \preceq)$ is a total order on $k + 1$ elements. Let $Q = (\{1, 2, \ldots, k + 1\}, \leq)$. We must show that $P \cong Q$. We know that P has a maximal element x. Let P' be the poset $P - x$, the poset formed by deleting x from P. Let Q' be the poset $(\{1, 2, \ldots, k\}, \leq)$. By induction, $P' \cong Q'$ so we can find an order-preserving bijection f' between their ground sets. We define $f : X \to \{1, 2, \ldots, k + 1\}$ by $f(a) = \begin{cases} f'(a), & \text{if } a \neq x \\ k + 1, & \text{if } a = x \end{cases}$. We must show that f is a bijection and is order-preserving.

Proof of Classification (Cont'd)

- To show that f is a bijection:
  - We first check that f is one-to-one. Suppose $f(a) = f(b)$.
    - If neither a nor b equals x, then $f(a) = f'(a)$ and $f(b) = f'(b)$, so $f'(a) = f'(b)$. Since f' is one-to-one, we have $a = b$.
    - If both a and b are x, then clearly $a = b$.
    - Finally, note that if $f(a) = f(b)$, it is impossible for one of a or b to be x and the other one not x; in this case, one of $f(a)$ or $f(b)$ evaluates to $k+1$ and the other does not.

    Therefore f is one-to-one.
  - Next we check that f is onto. Let $b \in \{1, 2, \ldots, k+1\}$, the ground set of Q.
    - If $b = k+1$, then note that $f(x) = b$.
    - If $b \neq k+1$, then, since f' is onto $\{1, \ldots, k\}$, we can find $a \in X - \{x\}$ with $f'(a) = b$. But then $f(a) = f'(a) = b$, as required.

    Thus f is onto.

  Therefore f is a bijection.
- Next we need to show that f is order-preserving and reflecting; i.e., for all $a, b \in X$, $a \preceq b \iff f(a) \leq f(b)$.

Conclusion of the Classification Proof

- Next we need to show that for all $a, b \in X$, $a \preceq b \iff f(a) \leq f(b)$.
  - $(\Rightarrow)$ Suppose $a, b \in X$ and $a \preceq b$. We must show that $f(a) \leq f(b)$.
    - If neither a nor b is equal to x, then $f(a) = f'(a)$ and $f(b) = f'(b)$. Since $f'(a) \leq f'(b)$ (f' is order-preserving), we have $f(a) \leq f(b)$.
    - If both $a = b = x$, then $f(a) = f(b) = k + 1$, so clearly $f(a) \leq f(b)$.
    - If $a \neq x$ and $b = x$, then $f(a) = f'(a) \leq k < k + 1 = f(b)$, so $f(a) \leq f(b)$.
    - Finally, we cannot have $a = x$ and $b \neq x$ because that would give $x \preceq b$, and x is maximal in P.

    Thus, in all possible cases, we have $a \preceq b \Rightarrow f(a) \leq f(b)$.
  - $(\Leftarrow)$ Suppose $f(a) \leq f(b)$. We must show that $a \preceq b$.
    - If neither a nor b is x, then $f(a) = f'(a)$ and $f(b) = f'(b)$. Thus $f'(a) \leq f'(b)$ and so $a \preceq b$ (f' is order-reflecting).
    - If both a and b are x, then $a \preceq b$.
    - We cannot have $a = x$ and $b \neq x$ because then $k + 1 = f(a) \leq f(b) \leq k$, which is a contradiction.
    - The only remaining case is $a \neq x$ and $b = x$. Since $b = x$ is maximal, we know that $a \not\succ b$. Since P is a total order, we must have $a \preceq b$.

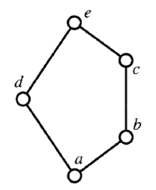
    Thus, in all cases, we have $a \preceq b$.

Subsection 4

Linear Extensions

Incomparability versus Partial Information

- There are two ways to think about a partially ordered set.
  - On the one hand, there may be incomparabilities among the elements of the set, e.g., we cannot compare 8 and 11 with respect to divisibility.
  - On the other hand, we can think of a partially ordered set as representing partial information about an ordered set.
- Example: In the poset on the left below a is a minimum element, e is a maximum element, and we have $a < b < c < e$ and $a < d < e$.

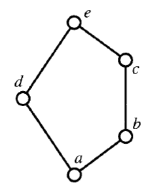


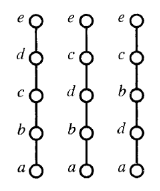
However, d is incomparable to b and c. Perhaps, we simply do not yet know the order relation between b and d (or c and d).

Given that elements $\{a, b, c, d, e\}$ are partially ordered, we can ask: What linear orders are consistent with the partial ordering already given on these elements?

Linear Extensions of Posets

- We want to extend this partial order to a consistent linear order.





For consistency, we must have a below all the other elements and e above all the other elements. We also must have $b < c$.

- d might be above both b and c;
- d might be between b and c;
- d might be below both b and c.

The three linear orderings are called linear extensions of the poset.

Definition (Linear Extension)

Let $P = (X, \preceq)$ be a partially ordered set. A **linear extension of** P is a linear order $L = (X, \leq)$ with the property that

$$\forall x, y \in X,\ x \preceq y \ \Rightarrow\ x \leq y.$$

Remarks on Linear Extensions

- It is important to notice, if L is a linear extension of P:
  - The posets P and L have the same ground set X.
  - The poset L is a linear (total) order.
  - The poset L is an extension of P: if $x \preceq y$ in P, then $x \leq y$ in L.
- The condition $x \preceq y \;\Rightarrow\; x \leq y$ can be written as $\preceq \subseteq \leq$.
- Example: Let $P = (X, \preceq)$ be an antichain containing n elements. Then all possible linear orders on those n elements are linear extensions of P. Thus there are $n!$ possible linear extensions of P.
- Does every poset have a linear extension?

  We prove that, if x and y are incomparable in a finite poset P, we can find a linear extension L in which $x < y$ and, by symmetry, another linear extension L' in which $y <' x$.

Linear Extension of a Partial Order on a Finite Set

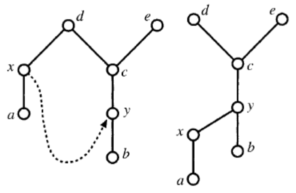
Theorem

Let P be a finite partially ordered set. Then P has a linear extension. Moreover, if x and y are incomparable elements of P, then there is a linear extension L of P in which $x < y$.

- Let $P = (X, \preceq)$ where X is a finite set. If P is a total order, then P is its own linear extension. Assume, now, P is not a total order. Suppose x and y are incomparable in P. We define a new relation, $\preceq'$ on X as follows: Let $s, t \in X$. We have $s \preceq' t$ provided either of the following conditions holds:

  (A) $s \preceq t$ or
  (B) $s \preceq x$ and $y \preceq t$.

  The poset on the right shows the relation $\preceq'$ formed from $\preceq$ on the left.



Linear Extension (Reflexivity and Antisymmetry of $\preceq'$)

- We check $\preceq'$ is a partial order:
  - $\preceq'$ is reflexive: Let $a \in X$ be any element of the poset P. Since $a \preceq a$, we have, by condition (A), $a \preceq' a$. Therefore $\preceq'$ is reflexive.
  - $\preceq'$ is antisymmetric: Suppose $a \preceq' b$ and $b \preceq' a$. We examine four cases:
    - Suppose $a \preceq' b$ because $a \preceq b$ (A), and $b \preceq' a$ because $b \preceq a$ (A). Since $\preceq$ is antisymmetric, and $a \preceq b$ and $b \preceq a$, we have $a = b$.
    - Suppose $a \preceq' b$ because $a \preceq b$ (A), and $b \preceq' a$ because $b \preceq x$ and $y \preceq a$ (B). This case cannot happen: We have $y \preceq a \preceq b \preceq x$, implying that $y \preceq x$. However, x and y are incomparable in P, a contradiction.
    - Suppose $a \preceq' b$ because $a \preceq x$ and $y \preceq b$ (B), and $b \preceq' a$ because $b \preceq a$. This case is just like the previous case and cannot occur.
    - Finally, suppose $a \preceq' b$ because $a \preceq x$ and $y \preceq b$ (B), and $b \preceq' a$ because $b \preceq x$ and $y \preceq a$ (B). In this case, we have $y \preceq b \preceq x$, contradicting the fact that x and y are incomparable. So this case cannot occur either.

  Therefore, $\preceq'$ is antisymmetric.

Linear Extension (Transitivity of $\preceq'$)

- We continue checking that $\preceq'$ is a partial order:
  - $\preceq'$ is transitive: Suppose $a \preceq' b$ and $b \preceq' c$. We must show that $a \preceq' c$. We consider four cases:
    - Suppose $a \preceq' b$ because $a \preceq b$ (A), and $b \preceq' c$ because $b \preceq c$ (A). Then $a \preceq c$ ($\preceq$ is transitive) and so $a \preceq' c$ by (A).
    - Suppose $a \preceq' b$ because $a \preceq b$ (A), and $b \preceq' c$ because $b \preceq x$ and $y \preceq c$ (B). In this case, $a \preceq b \preceq x$, so $a \preceq x$. Also $y \preceq c$, so $a \preceq' c$ by (B).
    - Suppose $a \preceq' b$ because $a \preceq x$ and $y \preceq b$ (B), and $b \preceq' c$ because $b \preceq c$ (A). In this case, $y \preceq b \preceq c$, so $y \preceq c$. Since $a \preceq x$, we have $a \preceq' c$ by (B).
    - Finally, suppose $a \preceq' b$ because $a \preceq x$ and $y \preceq b$ (B), and $b \preceq' c$ because $b \preceq x$ and $y \preceq c$ (B). This case cannot occur: We have $y \preceq b \preceq x$, and so $y \preceq x$. However, x and y are incomparable, a contradiction.

  So $\preceq'$ is transitive.

Therefore $P' = (X, \preceq')$ is a poset.

Linear Extension (Finishing the Proof)

- $P' = (X, \preceq')$ has the following properties:
  - First, $a \preceq b \Rightarrow a \preceq' b$ for all $a, b \in X$.
  - Second, $x \preceq' y$, but x and y are incomparable in P. Thus the number of pairs of elements related by $\preceq'$ is strictly greater than the number of pairs of elements related by $\preceq$.

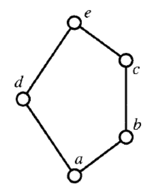
  It is conceivable that $\preceq'$ is a linear order. In this case, P' is the desired linear extension of P. However, if P' is not a linear order, then it contains incomparable elements x' and y'. We can extend $\preceq'$ to form $\preceq''$ in precisely the same way as before. The relation $\preceq''$ will include all relations in $\preceq'$ and will also have the relation $x' \preceq'' y'$. In this way, we create a sequence of partial order relations each containing more pairs than the previous: $\preceq, \preceq', \preceq'', \preceq''', \ldots$. Because X is finite, this process will eventually result in a relation that is a total order. Let that relation be $\leq$. Since $x \preceq' y$, and all subsequent relations are extensions of $\preceq'$, we see that $x \leq y$. Thus we have constructed a linear extension $\leq$ of P in which $x \leq y$.

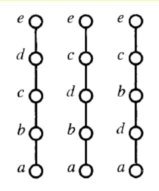
Subsection 5

Dimension

Reconstruction of a Poset from its Linear Extensions

- Consider the partially ordered set and its linear extensions:





- Claim: The three linear extensions of the poset P contain enough information to reconstruct P.
  - Notice that $b < c$ in all three linear extensions. This can happen only if $b < c$ in P itself.
  - On the other hand, in the first linear extension, we have $b < d$, but in the third, we have $b > d$. If it was the case that $b < d$ in P, then we would have $b < d$ in all linear extensions. So we can deduce that b and d are incomparable in P.

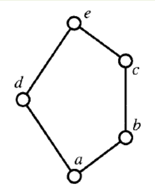
Corollary

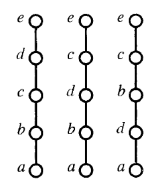
Let P be a finite partially ordered set, and let x and y be distinct elements of P. If $x < y$ in all linear extensions of P, then $x < y$ in P. Conversely, if $x < y$ in one linear extension, but $x > y$ in another, then x and y are incomparable in P.

Computer Representation of a Poset

- We obtain a way to store a partially ordered set in a computer:
  - We can save, as lists, the linear extensions of P.
  - Then, $x < y$ in P, if x is below y in all of the linear extensions.
- Some partially ordered sets have a large number of linear extensions.
- Example: An antichain on ten elements has 10! (over 3 million) linear extensions. However, we do not need all 10! linear extensions to represent this antichain in our computer: We can use just the two linear orders:

$$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10$$
$$10 < 9 < 8 < 7 < 6 < 5 < 4 < 3 < 2 < 1.$$





The same idea works for the five-element poset we considered before. The first and third linear extensions: $a < b < c < d < e$ and $a < d < b < c < e$ are enough for reconstruction.

Realizers

Definition (Realizer)

Let $P = (X, \leq)$ be a partially ordered set. Let $\mathcal{R}$ be a set of linear extensions of P. We call $\mathcal{R}$ a **realizer of** P, provided that for all $x, y \in X$ we have $x \leq y$ in P if and only if $x \leq y$ in all linear extensions in $\mathcal{R}$. We say that $\mathcal{R}$ **realizes** P.

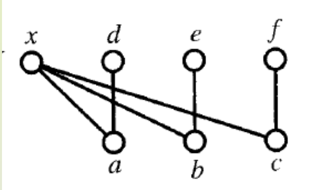
- If $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a realizer for a poset P, then we know that $x \leq y \iff x \leq_i y$, for all $i = 1, 2, \ldots, t$.
  - Half of this statement (the ($\Rightarrow$) implication) always holds by virtue of the fact that the L_i are linear extensions.
  - The other implication (the ($\Leftarrow$) half) is the important feature: it says that if $x \not\leq y$, then we do not have $x \not\leq_i y$ for all i. If $y < x$, this is obvious, for then we have $y <_i x$, for all i. Suppose x and y are incomparable. Since $x \not\leq y$, there is an i with $x >_i y$. And since $y \not\leq x$, there is a j with $x <_j y$.

Characterization of Realizers

Proposition (Characterization of Realizers)

Let P be a poset and let $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ be a set of linear extensions of P. Then $\mathcal{R}$ is a realizer of P if and only if, for all pairs x and y of incomparable elements of P, there exist i and j so that $x <_i y$ and $x >_j y$.

- Example: Let P be the poset whose Hasse diagram is shown here:



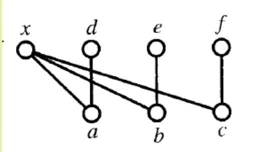
Let L_1, L_2 and L_3 be the following linear extensions of P:

$$L_1: \ b < c < e < f < a < x < d$$
$$L_2: \ a < c < d < f < b < x < e$$
$$L_3: \ a < b < d < e < c < x < f.$$

Let $\mathcal{R} = \{L_1, L_2, L_3\}$. We claim that $\mathcal{R}$ is a realizer of P.

- All three L_i are linear extensions of P. Observe that $a < x$ and $a < d$ in all three L_i. Then check that $b < x$ and $b < e$ in all three. Finally, note that $c < x$ and $c < f$ in all three.

Example (Cont'd)



We saw L_1, L_2 and L_3 are linear extensions of P:

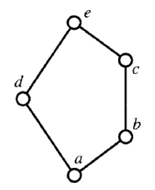
$$L_1: \ b < c < e < f < a < x < d$$
$$L_2: \ a < c < d < f < b < x < e$$
$$L_3: \ a < b < d < e < c < x < f.$$

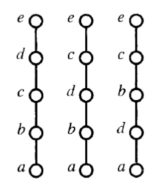
- We check that if u and v are incomparable in P, then $u < v$ in one linear extension and $u > v$ in another:
  - Consider first the incomparabilities among a, b and c. Note that we have $a < b$ in L_3 and $a > b$ in L_1. The incomparabilities between a and c and between b and c are checked in the same way.
  - We also see that $d < e$ in L_2 and $d > e$ in L_1. The other incomparabilities among $\{d, e, f\}$ are checked in the same way.
  - Next, $x < d$ in L_1 and $x > d$ in L_2. The other incomparabilities involving x are checked in the same manner.
  - Finally, notice that $a < e$ in L_2 and $a > e$ in L_1. The incomparabilities $a - f$, $b - d$, $b - f$, $c - d$, and $c - e$ are checked in a similar manner.

Therefore $\mathcal{R}$ is a realizer.

Dimension of a Poset

- Let P be an antichain on ten elements. We can form a realizer of P using all 10! linear extensions, but we can just use two.
- We may realize a poset using all its linear extensions.
- The interesting problem is to use as few linear extensions as possible.
- Example: The following poset can be realized using all three of





its linear extensions or with just two. Clearly, it is not possible to realize this poset with just one linear extension. This poset can be realized with two linear extensions, but no fewer.

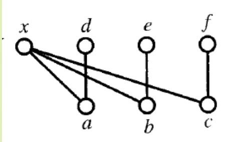
Definition (Dimension)

Let P be a finite poset. The smallest size of a realizer of P is called the **dimension** of P. The dimension of P is denoted $\dim P$.

- Example: An antichain on ten elements and the poset in the figure both have dimension equal to 2.

Establishing a Lower Bound on Dimension

- Consider again:

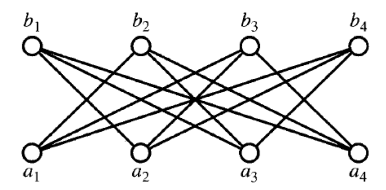


It has a realizer containing three linear extensions. Because P is not a linear order, it cannot be realized by a single linear extension.

- Claim: P cannot be realized using just two linear extensions.
- Suppose that P can be realized with L' and L''. Consider the pairwise incomparable elements a, b and c. By symmetry, we have $a < b < c$ in L' and $a > b > c$ in L''. Since x is above all of a, b and c, we also know that x is above them in L' and L''. So, $a < b < c < x$ in L' and $c < b < a < x$ in L''. Now e and x are incomparable, so $e < x$ in one of L' or L'' and $e > x$ in the other. By symmetry, assume $e > x$ in L' (so in L' we have $a < b < c < x < e$). In L'' we know that $e < x$, but we also know that $e > b$ (because $e > b$ in P). So in L'' we have $c < b < e < x$. In both L' and L'' we have $c < e$. Therefore $\{L', L''\}$ is not a realizer for P, and so there can be no realizer of size 2.

The Posets P_n

- Here is another family of posets whose dimension we calculate:



Let n be an integer, with $n \geq 2$, and let P_n be as follows: The ground set of P_n consists of $2n$ elements:

$$\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}.$$

The only strict order relations in P_n are those of the form $a_i < b_j$, where $i \neq j$.

- The poset P_4 is shown in the figure.

Proposition (Dimension of P_n)

Let n be an integer with $n \geq 2$ and consider the poset P_n. The dimension of P_n is n.

Proving that Dimension of P_n is n (Upper Bound)

- First, we show that P_n has a realizer of size n.
  - Let $1 \leq i \leq n$. Let L_i be (other a's) $< b_i < a_i <$ (other b's). The "other a's" means all a_j (except a_i) are before b_i. Similarly, the "other b's" means all b_j (except b_i) are after a_i.
    Claim: Regardless of the ordering of "other a's" and "other b's", L_i is a linear extension of P_n.
    We just need to check that $a_j < b_k$ whenever $j \neq k$. Indeed, we have $a_j < b_k$ for all j and k except for $j = k = i$. Thus L_i is a linear extension of P, for all $i = 1, 2, \ldots, n$.
    Claim: $\mathcal{R} = \{L_1, L_2, \ldots, L_n\}$ is a realizer for P_n.
    There are three types of incomparable pairs in P_n: two a's, two b's, and $a_i - b_i$ for some i:
    - Incomparable pairs of the form $a_i - a_j$: Notice that $a_i < a_j$ in L_j and $a_i > a_j$ in L_i.
    - Incomparable pairs of the form $b_i - b_j$: Notice that $b_i < b_j$ in L_i and $b_i > b_j$ in L_j.
    - Incomparable pairs of the form $a_i - b_i$: Notice that $a_i > b_i$ in L_i, but $a_i < b_i$ in any other L_k $(k \neq i)$.

Proving that Dimension of P_n is n (Lower Bound)

- Second, we show that P_n cannot have a realizer with fewer than n linear extensions.
  - Suppose, for the sake of contradiction, there is a realizer $\mathcal{R}$ of P_n with $|\mathcal{R}| < n$. For each k (with $1 \leq k \leq n$), there must be a linear extension $L \in \mathcal{R}$ in which $a_k > b_k$ (because a_k and b_k are incomparable). There are n such incomparable pairs, but at most $n-1$ linear extensions in $\mathcal{R}$. Therefore, by the Pigeonhole Principle, there must be a linear extension L and two distinct indices i and j, such that $a_i > b_i$ and $a_j > b_j$ in L. Since $b_j > a_i$ and $b_i > a_j$ in P_n, we must also have these relations in L. Thus in L we have

$$b_j > a_i > b_i > a_j > b_j \;\Rightarrow\; b_j > b_j,$$

    which is a conradiction. Therefore $\mathcal{R}$ is not a realizer of P_n, and so we cannot realize P_n with fewer than n linear extensions.
- Therefore $\dim P_n = n$.

Domination in n-Dimensional Space

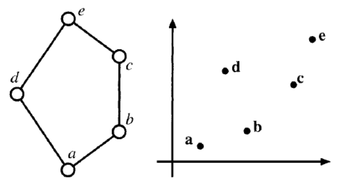
- Every point in the plane can be represented by a pair of real numbers: the (x, y)-coordinates of the point. The plane is denoted $\mathbb{R}^2$.
- Likewise, every point in three-dimensional space can be described as an ordered triple: (x, y, z). Three-dimensional space is denoted $\mathbb{R}^3$.
- In general $\mathbb{R}^n$ stands for the set of all ordered n-tuples of real numbers and represents n-dimensional space.
- We investigate the connection between the two uses (geometry and posets) of the word dimension.
- Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be two points in n-dimensional space $\mathbb{R}^n$. We say that $\boldsymbol{p}$ **dominates** $\boldsymbol{q}$ provided each coordinate of $\boldsymbol{p}$ is greater than or equal to the corresponding coordinate of $\boldsymbol{q}$. In other words, if the coordinates of $\boldsymbol{p}$ and $\boldsymbol{q}$ are $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, then $p_1 \geq q_1$, $p_2 \geq q_2$, $\ldots$, $p_n \geq q_n$. Let us write $\boldsymbol{p} \succeq \boldsymbol{q}$ in the case where $\boldsymbol{p}$ dominates $\boldsymbol{q}$. We also write $\boldsymbol{q} \preceq \boldsymbol{p}$, and we say that $\boldsymbol{q}$ is **dominated by $\boldsymbol{p}$**.

Embeddings in $\mathbb{R}^n$

Definition (Embedding in $\mathbb{R}^n$)

Let $P = (X, \leq)$ be a poset and let n be a positive integer. An embedding of P in $\mathbb{R}^n$ is a one-to-one function $f : X \to \mathbb{R}^n$ such that $x \leq y$ (in P) if and only if $f(x) \preceq f(y)$ (in $\mathbb{R}^n$).

- The following figure shows a poset on the left and an embedding in $\mathbb{R}^2$ on the right.



The chain $a < b < c < e$ corresponds to the sequence of points $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{e}$, where each point is to the northeast of the previous point. Also, since b and d are incomparable, their points $\boldsymbol{b}$ and $\boldsymbol{d}$ are also incomparable in the dominance ($\preceq$) order.

Embedding Theorem

Theorem (Embedding Theorem)

Let P be a finite poset and let n be a positive integer. Then P has a realizer of size n if and only if P embeds in $\mathbb{R}^n$. Thus $\dim P$ is the least positive integer n such that P embeds in $\mathbb{R}^n$.

- ($\Rightarrow$) Suppose that $P = (X, \leq)$ has a realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_n\}$ of size n. For $x \in X$, let $h_i(x)$ denote the **height of** x **in** L_i, i.e., $h_i(x)$ is the number of elements less than or equal to x in L_i. Let $f : P \to \mathbb{R}^n$ be defined by $f(x) = (h_1(x), h_2(x), \ldots, h_n(x))$.
  - f is one-to-one: If $x \neq y$, then $h_1(x) \neq h_1(y)$ (because x and y are at different heights in L_1), and so $f(x) \neq f(y)$.
  - We must show that $x \leq y$ (in P) iff $f(x) \preceq f(y)$.
    - Suppose $x \leq y$ in P. Then $h_i(x) \leq h_i(y)$ (because $x \leq y$ in all the linear extensions L_i). Hence, $f(x)$ is, coordinate by coordinate, less than or equal to $f(y)$, and so $f(x) \preceq f(y)$.
    - Suppose $f(x) \preceq f(y)$. Thus, $h_i(x) \leq h_i(y)$ for all i. So $x \leq y$ in all linear extensions L_i, and, hence, (by definition of realizer) $x \leq y$ in P.

Embedding Theorem (Cont'd)

- ($\Leftarrow$) Suppose $P = (X, \leq)$ can be embedded in $\mathbb{R}^n$. Thus, there is a one-to-one mapping $f : X \to \mathbb{R}^n$ so that for all $x, y \in X$ we have $x \leq y \Longleftrightarrow f(x) \preceq f(y)$. Let i be an integer with $1 \leq i \leq n$.
  We define a linear extension L_i of P as follows: Let $f_i(x)$ be the i-th coordinate of $f(x)$. We form L_i by arranging the elements of X in increasing order of f_i. That is, we have $x \leq_i y$ provided $f_i(x) \preceq f_i(y)$. This would give a total order on the elements X were it not for the possibility of elements with equal i-th coordinate.
  We break such ties as follows: Suppose $f_i(x) = f_i(y)$, for some $x \neq y$. Since f is a one-to-one function, there must be some other coordinate j, where $f_j(x) \neq f_j(y)$. In this case, we declare the order of x and y in L_i to be determined by the lowest index j where $f_j(x) \neq f_j(y)$.

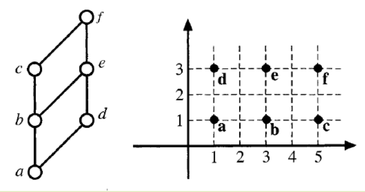
Embedding Theorem (Finishing the Proof)

We finally show that L_i is a linear extension of P and that $\mathcal{R} = \{L_1, \ldots, L_n\}$ is a realizer.

- Claim: L_i is a linear extension of P.
  Clearly L_i is a linear order. Suppose $x < y$ in P. Then $f(x) \preceq f(y)$ and so $f_i(x) \leq f_i(y)$. In case $f_i(x) = f_i(y)$ and $x < y$, we note that for all j, $f_j(x) \leq f_j(y)$ and for some indices j, the inequality is strict. Thus $x < y$ in P implies $x <_i y$, and so L_i is a linear extension of P.
- Claim: $\mathcal{R} = \{L_1, \ldots, L_n\}$ is a realizer.
  We must show that if x and y are incomparable, then there are indices i and j with $x <_i y$ and $x >_j y$. Since $f(x)$ is incomparable to $f(y)$ (f an embedding in $\mathbb{R}^n$), we know there exist i and j with $f_i(x) < f_i(y)$ and $f_j(x) > f_j(y)$, and this gives $x <_i y$ and $x >_j y$.

Example: Using an Embedding to Create a Realizer

- Let P be the poset in the figure on the left:



Let $a \mapsto \boldsymbol{a}$, $b \mapsto \boldsymbol{b}$, $\ldots$, $f \mapsto \boldsymbol{f}$ shown on the right be an embedding of P in $\mathbb{R}^2$. From this embedding we extract the two linear extensions:

$$L_1: \ a < d < b < e < c < f$$
$$L_2: \ a < b < c < d < e < f.$$

- We found L_1 by sorting the six points by their first coordinate and breaking ties using the second coordinate.
- Likewise we found L_2 by sorting the points by their second coordinate and breaking ties using the first coordinate.

Observe that $\mathcal{R} = \{L_1, L_2\}$ is a realizer for P.

Subsection 6

Lattices

Order Theoretic Characterization of Intersection

- We show that the operations $\cap$ (intersection), $\wedge$ (Boolean and) and gcd (greatest common divisor) form "similar" structures.

Proposition

Let A and B be sets. Let Z be a set, such that:

- $Z \subseteq A$ and $Z \subseteq B$;
- if $X \subseteq A$ and $X \subseteq B$, then $X \subseteq Z$.

Then $Z = A \cap B$.

- Suppose $x \in Z$. Since $Z \subseteq A$, we have $x \in A$. Likewise $Z \subseteq B$ implies $X \in B$. Therefore $X \in A \cap B$.
  Conversely, suppose $x \in A \cap B$. This means that $x \in A$ and $x \in B$, and so $X = \{x\}$ is a subset of both A and B. Therefore $X = \{x\}$ is a subset of Z (by the second property). Thus $x \in Z$.
  We have shown that $x \in Z \Leftrightarrow x \in A \cap B$ and so $Z = A \cap B$.

Commonalities of $\cap$ and gcd

- Similarly for the greatest common divisor of two positive integers:

Proposition

Let a, b be positive integers. Let d be a positive integer, such that:

- $d \mid a$ and $d \mid b$;
- if $e \in \mathbb{N}$ with $e \mid a$ and $e \mid b$, then $e \mid d$.

Then $d = \gcd(a, b)$.

- Since $d \mid a$ and $d \mid b$, we have $d \leq \gcd(a, b)$. On the other hand, $\gcd(a, b) \mid a$ and $\gcd(a, b) \mid b$, whence, by the second property, $\gcd(a, b) \mid d$. Therefore, $\gcd(a, b) \leq d$. These show $d = \gcd(a, b)$.
- These propositions suggest defining
  - $A \cap B$ to be the largest set that is below both A and B; "largest" and "below" with respect to $\subseteq$;
  - $\gcd(a, b)$ to be the largest positive integer that is below both a and b; "largest" and "below" with respect to $\mid$.

Greatest Lower Bound and Least Upper Bound

Definition (Lower and Upper Bounds)

Let $P = (X, \leq)$ be a poset and let $a, b \in X$. We say that $x \in X$ is a **lower bound** for a and b provided $x \leq a$ and $x \leq b$. Similarly, we say that $x \in X$ is an **upper bound** for a and b provided $a \leq x$ and $b \leq x$.

Definition (Greatest Lower Bound/Least Upper Bound)

Let $P = (X, \leq)$ be a poset and let $a, b \in X$. We say that $x \in X$ is a **greatest lower bound** for a and b provided

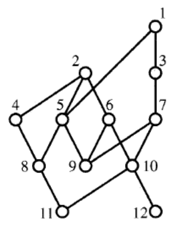
1. x is a lower bound for a and b;
2. if y is lower bound for a and b, then $y \leq x$.

Similarly, we say that $x \in X$ is a **least upper bound** for a and b provided

1. x is an upper bound for a and b;
2. if y is an upper bound for a and b, then $y \geq x$.

An Example

- Let P be the following poset:



- Consider 8 and 9. Notice that 1, 2, and 5 are upper bounds for 8 and 9. Since $5 < 1$ and $5 < 2$, we have that 5 is the least upper bound of 8 and 9. But 8 and 9 have no lower bounds, so no greatest lower bound.
- Elements 4 and 7 have 11 as their only lower bound; thus 11 is their greatest lower bound. Elements 4 and 7 have no upper bound, so no least upper bound.
- Elements 5 and 6 have 2 as the least (and only) upper bound. They have incomparable lower bounds 9 and 11, so they do not have a greatest lower bound.
- 9 and 10 have no greatest lower bound and no least upper bound.
- Elements 4 and 5 have 2 as their least upper bound and 8 as their greatest lower bound.

Meet and Join

- If a greatest lower bound exists, it must be unique.
  Suppose x and y are both greatest lower bounds of a and b. We have $x \leq y$ because y is greatest and we have $y \leq x$ because x is greatest. Therefore $x = y$.
- If a and b have a least upper bound, it must also be unique.

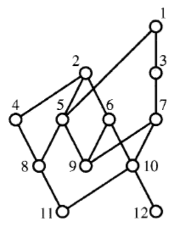
Definition (Meet and Join)

Let $P = (X, \leq)$ be a poset and let $a, b \in X$. If a and b have a greatest lower bound, it is called the **meet** of a and b, and it is denoted $a \wedge b$. If a and b have a least upper bound, it is called the **join** of a and b, and it is denoted $a \vee b$.

- Example: Consider the poset P whose ground set is $\{\mathrm{TRUE}, \mathrm{FALSE}\}$. We define FALSE $<$ TRUE in this poset. Then $\mathrm{T} \wedge \mathrm{F} = \mathrm{F}$ because FALSE is the greatest (and only) lower bound for TRUE and FALSE.
  Also:
  $$\mathrm{T} \wedge \mathrm{T} = \mathrm{T},\ \mathrm{T} \wedge \mathrm{F} = \mathrm{F},\ \mathrm{F} \wedge \mathrm{T} = \mathrm{F},\ \mathrm{F} \wedge \mathrm{F} = \mathrm{F}$$
  $$\mathrm{T} \vee \mathrm{T} = \mathrm{T},\ \mathrm{T} \vee \mathrm{F} = \mathrm{T},\ \mathrm{F} \vee \mathrm{T} = \mathrm{T},\ \mathrm{F} \vee \mathrm{F} = \mathrm{F}.$$

Lattices

- Consider again the poset P:



- We have:
  - $8 \wedge 9 =$ undefined and $8 \vee 9 = 5$.
  - $4 \wedge 7 = 11$ and $4 \vee 7 =$ undefined.
  - $5 \wedge 6 =$ undefined and $5 \vee 6 = 2$.
  - Both $9 \wedge 10$ and $9 \vee 10$ are undefined.
  - $4 \wedge 5 = 8$ and $4 \vee 5 = 2$.

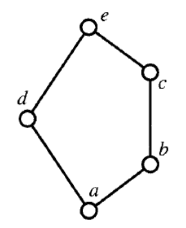
Definition (Lattice)

Let P be a poset. We call P a **lattice** provided that, for all elements x and y of P, $x \wedge y$ and $x \vee y$ are defined.

- Example (Subsets of a Set): Let A be a set and let $P = (2^A, \subseteq)$, i.e., P is the poset of all subsets of A ordered by containment. In this poset we have, for all $x, y \in 2^A$, $x \wedge y = x \cap y$ and $x \vee y = x \cup y$. Therefore P is a lattice.

An Additional Example of a Lattice

- Consider:



- The $\wedge$ and $\vee$ operation tables are given below:

$\wedge$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	a	b
c	a	b	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

$\vee$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	c	c	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

Since $\wedge$ and $\vee$ are defined for every pair of elements, this poset is a lattice.

Natural Numbers with Divisibility and Linear Orders

- Example (Natural Numbers/Positive Integers Ordered by Divisibility): Consider the poset $(\mathbb{N}, |)$, i.e., the set of natural numbers ordered by divisibility. Let $x, y \in \mathbb{N}$. Then, $x \wedge y$ is the greatest common divisor of x and y, and $x \vee y$ is their least common multiple. However, $(\mathbb{N}, |)$ is not a lattice because $0 \wedge 0 = \text{gcd}\,(0, 0)$ is not defined. On the other hand, the poset $(\mathbb{Z}^+, |)$ is a lattice. Here $\mathbb{Z}^+$ stands for the set of positive integers which we order by divisibility. In this case, $\wedge$ and $\vee$ (gcd and lcm) are defined for all pairs of positive integers.
- Example (Linear Orders): Let $P = (X, \leq)$ be a linear (total) order. For any $x, y \in X$, $x \wedge y = \begin{cases} x, & \text{if } x \leq y \\ y, & \text{if } x \geq y \end{cases}$. So $x \wedge y = \min\{x, y\}$ where $\min\{x, y\}$ stands for the smaller of x and y. Likewise $x \vee y = \max\{x, y\}$, i.e., the larger of the pair. Thus all linear orders are lattices.

Properties of Meet and Join

Theorem (Properties of Meet and Join)

Let $P = (X, \leq)$ be a lattice. For all $x, y, z \in X$, the following hold:

- $x \wedge x = x \vee x = x$. (**Idempotency**)
- $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$. (**Commutativity**)
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$. (**Associativity**)
- $x \wedge y = x \iff x \vee y = y \iff x \leq y$.

- We only show $\wedge$ is associative: Let $a = (x \wedge y) \wedge z$, $b = x \wedge (y \wedge z)$.
  - Since $a = (x \wedge y) \wedge z$, a is a lower bound for $x \wedge y$ and for z. Thus $a \leq x \wedge y$ and $a \leq z$. Since $a \leq x \wedge y$ and since $x \wedge y \leq x$ and $x \wedge y \leq y$, we have that $a \leq x$ and $a \leq y$. Thus a is below x, y and z. Since $a \leq y$ and $a \leq z$, we get $a \leq y \wedge z$, since $y \wedge z$ is the greatest lower bound of y and z. Since $a \leq x$ and $a \leq y \wedge z$, we get $a \leq b$, since b is the greatest lower bound for x and $y \wedge z$.
  - By an identical argument, we have $b \leq a$.